Noncommutative Solitons and Quasideterminants

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Theoretical Physics Seminar in Hannover on Feb.28th

Based on

- **MH, ``NC Ward's conjecture and integrable systems,'' NPB741 (2006) 368, [hep-th/0601209]**
- **MH, ``Notes on exact multi-soliton solutions of NC integrable hierarchies ,''JHEP02(07)94[hepth/0610006]**
- **And forthcoming papers…**

1. Introduction

Successful points in NC theories

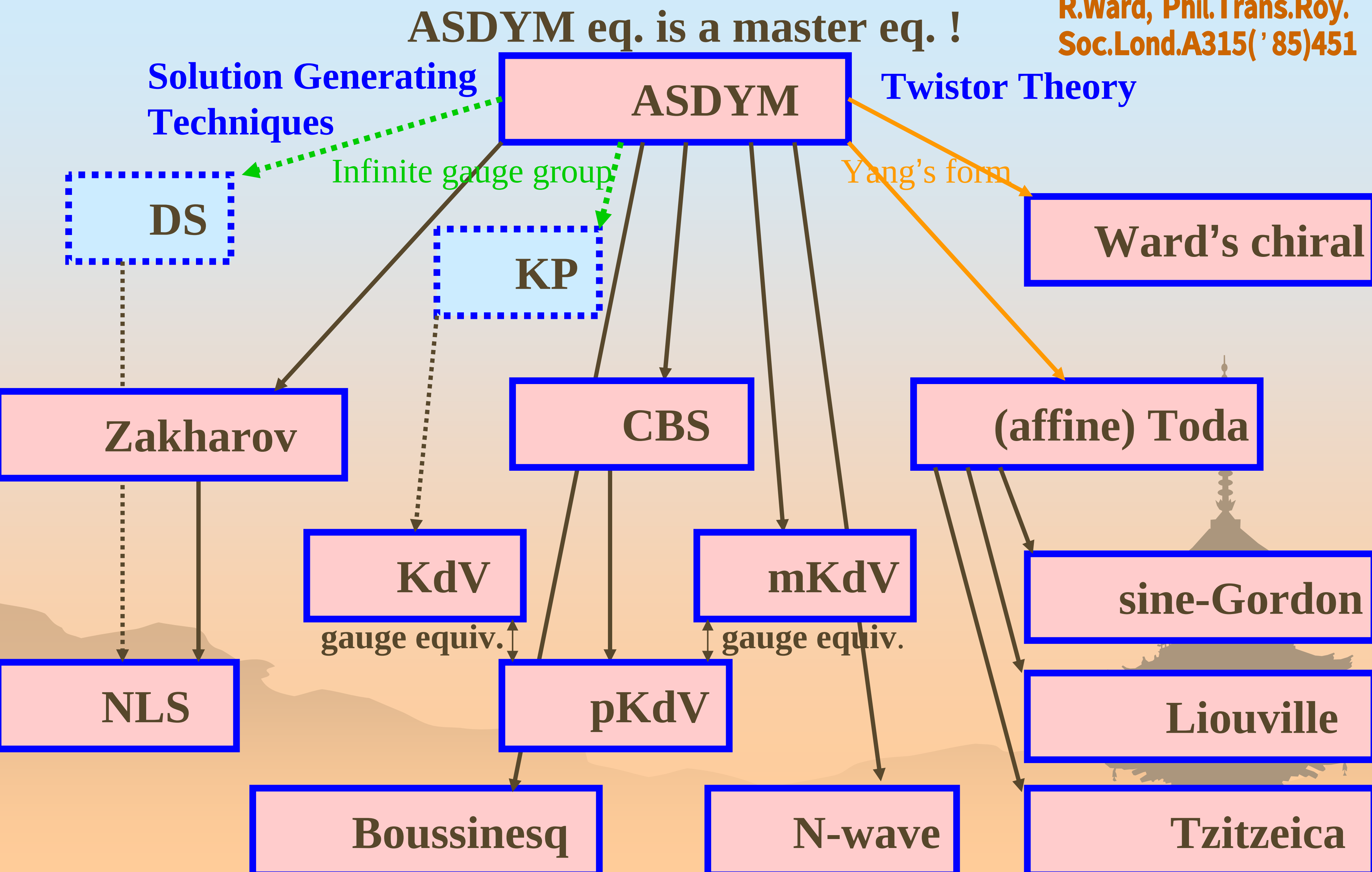
- Appearance of new physical objects
- Description of real physics (in gauge theory)
- Various successful applications to D-brane dynamics etc.

Construction of exact solitons are important.
(partially due to their integrablity)

Final goal: NC extension of all soliton theories
(Soliton eqs. can be embedded in gauge theories !)

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

R.Ward, Phil.Trans.Roy. Soc.Lond.A315('85)451



NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

[Lechtenfeld-Popov]... **NC ASDYM eq. is a master eq. ?** MH &K.Toda, PLA316 (03)77 [hep-th/0211148]

Solution Generating Techniques

NC ASDYM

NC Twistor Theory

New physical objects **Application to D-branes**

Reductions

NC DS

NC KP

NC Ward's chiral

NC Zakharov

NC CBS

NC (affine) Toda

NC KdV

NC mKdV

NC sine-Gordon

NC NLS

NC pKdV

NC Liouville

NC Boussinesq

NC N-wave

NC Tzitzeica

Plan of this talk

1. Introduction

2. Backlund transforms for NC ASDYM eqs.
   (Exact solutions for Atiyah-Ward ansatz)
3. Backlund transforms for NC KdV eqs.
   (Exact N-soliton solutions)
4. Conclusion and Discussion

In terms of
Quasideterminants

2. Backlund transforms for NC ASDYM eqs.

- In this section, we derive (NC) ASDYM eq. from the viewpoint of linear systems, which is suitable for discussion on integrable aspects.
- We define NC Yang's equations which is equivalent to NC ASDYM eq. and give a Backlund transformation for it.
- The generated solutions would contain not only finite-action solutions (NC instantons) but also infinite-action solutions (non-linear plane waves and so on.)
- This Backlund transformation would be applicable for lower-dimensional integrable eqs. via Ward's conjecture.

Here we discuss G=GL(N) (NC) ASDYM eq. from the viewpoint of linear systems with a spectral parameter ζ .

- **Linear systems (commutative case):**

$$L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0,$$
$$M\psi = (D_z - \zeta D_{\tilde{w}})\psi = 0.$$

e.g. $\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$

- **Compatibility condition of the linear system:**

$$[L, M] = [D_w, D_z] + \zeta([D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}]) + \zeta^2 [D_{\tilde{z}}, D_{\tilde{w}}] = 0$$

$$\Leftrightarrow \begin{cases} F_{zw} = [D_z, D_w] = 0, \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0 \end{cases}$$

:ASDYM equation

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

Yang's form and Yang's equation

- **ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w] = 0, & \Rightarrow \quad \exists h,\ D_z h = 0,\ D_w h = 0 \quad (A_z = -h_z h^{-1}, etc.) \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0, & \Rightarrow \quad \exists \tilde{h},\ D_{\tilde{z}} \tilde{h} = 0,\ D_{\tilde{w}} \tilde{h} = 0 \quad (A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1}, etc.) \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0 & \end{cases}$$

If we define Yang's matrix: $J := \tilde{h}^{-1} h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} \partial_{\tilde{z}} J) - \partial_w (J^{-1} \partial_{\tilde{w}} J) = 0$$

:Yang's eq.

The solution J reproduce the gauge fields as

$$A_z = -h_z h^{-1},\ A_w = h_w h^{-1},\ A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1},\ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} \tilde{h}^{-1}$$

J **is gauge invariant. The decomposition into** h **and** $\tilde{h}$ **corresponds to a gauge fixing**

(Q) How we get NC version of the theories?

(A) We have only to replace all products of fields in ordinary commutative gauge theories with star-products:

$$f(x)g(x) \to f(x) * g(x)$$

- **The star product: (NC and associative)**

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) g(x) = f(x)g(x) + i \frac{\theta^{\mu\nu}}{2} \partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)$$

A deformed product

Presence of background magnetic fields

Note: coordinates and fields themselves are usual c-number functions. But commutator of coordinates becomes…

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

NC !

Here we discuss G=GL(N) NC ASDYM eq. from the viewpoint of linear systems with a spectral parameter ζ .

(All products are star-products.)

- **Linear systems (NC case):**

$$L*\psi=(D_w-\zeta D_{\tilde{z}})*\psi=0,$$
$$M*\psi=(D_z-\zeta D_{\tilde{w}})*\psi=0.$$

e.g. $\begin{pmatrix}\tilde{z} & w\\ \tilde{w} & z\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}x^0+ix^1 & x^2-ix^3\\ x^2+ix^3 & x^0-ix^1\end{pmatrix}$

- **Compatibility condition of the linear system:**

$$[L,M]_*=[D_w,D_z]_*+\zeta([D_z,D_{\tilde{z}}]_*-[D_w,D_{\tilde{w}}]_*)+\zeta^2[D_{\tilde{z}},D_{\tilde{w}}]_*=0$$

$$\Leftrightarrow\begin{cases}F_{zw}=[D_z,D_w]_*=0,\\ F_{\tilde{z}\tilde{w}}=[D_{\tilde{z}},D_{\tilde{w}}]_*=0,\\ F_{z\tilde{z}}-F_{w\tilde{w}}=[D_z,D_{\tilde{z}}]_*-[D_w,D_{\tilde{w}}]_*=0\end{cases}$$

:NC ASDYM equation

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]_*)$$

$$\theta^{\mu\nu}=\begin{bmatrix}0 & \theta^1 & & \\ -\theta^1 & 0 & & O\\ & & 0 & \theta^2\\ O & & -\theta^2 & 0\end{bmatrix}$$

Yang's form and NC Yang's equation

- **NC ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w]_* = 0, \quad \Rightarrow \quad \exists h,\ D_z * h = 0,\ \ D_w * h = 0 \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0, \quad \Rightarrow \quad \exists \tilde{h},\ D_{\tilde{z}} * \tilde{h} = 0,\ \ D_{\tilde{w}} * \tilde{h} = 0 \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_* = 0 \end{cases}$$

If we define Yang's matrix: $\quad J := \tilde{h}^{-1} * h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} * \partial_{\tilde{z}} J) - \partial_w (J^{-1} * \partial_{\tilde{w}} J) = 0$$ **:NC Yang's eq.**

The solution J **reproduces the gauge fields as**

$$A_z = -h_z * h^{-1},\ \ A_w = h_w * h^{-1},\ A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} * \tilde{h}^{-1},\ \ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} * \tilde{h}^{-1}$$

J **is gauge invariant. The decomposition into** h **and** $\tilde{h}$ **corresponds to a gauge fixing**

Backlund transformation for NC Yang's eq.

- **Yang's J matrix can be decomposed as follows**

$$J = \begin{pmatrix} A^{-1} - \tilde{B} * A * B & -\tilde{B} * \tilde{A} \\ \tilde{A} * B & \tilde{A} \end{pmatrix}$$

MH, NPB [hep-th/0601209]
and collaboration with Gilson-san
and Nimmo-san et. al.

- **Then NC Yang's eq. becomes**

$$\partial_z (A * \tilde{B}_{\tilde{z}} * \tilde{A}) - \partial_w (A * \tilde{B}_{\tilde{w}} * \tilde{A}) = 0, \quad \partial_{\tilde{z}} (\tilde{A} * B_z * A) - \partial_{\tilde{w}} (\tilde{A} * B_w * A) = 0,$$

$$\partial_z (\tilde{A}^{-1} * \tilde{A}_{\tilde{z}}) * \tilde{A}^{-1} - \partial_w (\tilde{A}^{-1} * \tilde{A}_{\tilde{w}}) * \tilde{A}^{-1} + B_z * A * \tilde{B}_{\tilde{z}} - B_w * A * \tilde{B}_{\tilde{w}} = 0,$$

$$A^{-1} * \partial_z (A_{\tilde{z}} * A^{-1}) - A^{-1} * \partial_w (A_{\tilde{w}} * A^{-1}) + \tilde{B}_{\tilde{z}} * \tilde{A} * B_z - \tilde{B}_{\tilde{w}} * \tilde{A} * B_w = 0.$$

- **The following trf. leaves NC Yang's eq. as it is:**

$$\beta : \begin{cases} \partial_z B^{new} = A * \tilde{B}_{\tilde{w}} * \tilde{A}, \ \partial_w B^{new} = A * \tilde{B}_{\tilde{z}} * \tilde{A}, \\ \partial_{\tilde{z}} \tilde{B}^{new} = \tilde{A} * B_w * A, \ \partial_{\tilde{w}} \tilde{B}^{new} = \tilde{A} * B_z * A, \\ A^{new} = \tilde{A}^{-1}, \ \tilde{A}^{new} = A^{-1} \end{cases}$$

We could generate various (non-trivial) solutions of NC Yang's eq. from a (trivial) seed solution by using the previous Backlund trf. together with a simple trf. $\gamma : J^{new} = C^{-1} J C, \quad C:$ **const.**

This combined trf. would generate a group of hidden symmetry of NC Yang's eq., which would be also applied to lower-dimension.

For G=GL(2), we can present the transforms more explicitly and give an explicit form of a class of solutions (Atiyah-Ward ansatz).

Backlund trf. for NC Yang's eq. G=GL(2)

- **Let's consider the following Backlund trf.**

$$J_{[1]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[2]} \xrightarrow{\alpha} J_{[3]} \xrightarrow{\alpha} \cdots$$

Collaboration with Gilson-san and Nimmo-san et. al. **(Very Hot)**

$$J_{[n]} = \begin{pmatrix} A_{[n]}{}^{-1} - \tilde{B}_{[n]} * A_{[n]} * B_{[n]} & -\tilde{B}_{[n]} * \tilde{A}_{[n]} \\ \tilde{A}_{[n]} * B_{[n]} & \tilde{A}_{[n]} \end{pmatrix}$$

$$\beta : \begin{cases} \partial_z b_{[n]} = A_{[n]} * \partial_{\tilde{w}} \tilde{B}_{[n]} * \tilde{A}_{[n]}, \ \partial_w b_{[n]} = A_{[n]} * \partial_{\tilde{z}} \tilde{B}_{[n]} * \tilde{A}_{[n]}, \\ \partial_{\tilde{z}} \tilde{b}_{[n]} = \tilde{A}_{[n]} * \partial_w B_{[n]} * A_{[n]}, \ \partial_{\tilde{w}} \tilde{b}_{[n]} = \tilde{A}_{[n]} * \partial_z B_{[n]} * A_{[n]}, \\ a_{[n]} = \tilde{A}_{[n]}{}^{-1}, \ \tilde{a}_{[n]} = A_{[n]}{}^{-1} \end{cases}$$

$$\gamma_0 : \begin{pmatrix} A_{[n+1]} & \tilde{B}_{[n+1]} \\ B_{[n+1]} & \tilde{A}_{[n+1]} \end{pmatrix} = \begin{pmatrix} a_{[n]} & \tilde{b}_{[n]} \\ b_{[n]} & \tilde{a}_{[n]} \end{pmatrix}^{-1}$$

- **If we take a seed sol.** $A_{[1]} = \tilde{A}_{[1]} = B_{[1]} = \tilde{B}_{[1]} = \Delta_0^{-1}, \ \partial^2 \Delta_0 = 0$

the generated solutions would be

$$A_{[n]} = (D_{[n]}{}^{-1})_{11}, \tilde{A}_{[n]} = (D_{[n]}{}^{-1})_{nn}, B_{[n]} = (D_{[n]}{}^{-1})_{n1}, \tilde{B}_{[n]} = (D_{[n]}{}^{-1})_{1n},$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

$$\frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = \frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

NC Atiyah-Ward ansatz

Backlund trf. for NC Yang's eq. G=GL(2)

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$$J_{[1]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[2]} \xrightarrow{\alpha} J_{[3]} \xrightarrow{\alpha} \cdots$$

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Collaboration with Gilson-san and Nimmo-san et. al. **(Very Hot)**

All ingredients in AW ansatz can be determined from Δ_0 only

Various choice of Δ_0 ←→ Various solutions

- **If we take a seed sol.** $A_{[1]} = \tilde{A}_{[1]} = B_{[1]} = \tilde{B}_{[1]} = \Delta_0^{-1}, \ \partial^2 \Delta_0 = 0$

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$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

$$\frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = \frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

NC Atiyah-Ward ansatz

Backlund trf. for NC Yang's eq. G=GL(2)

- **Let's consider the following Backlund trf.**

$$J_{[1]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[2]} \xrightarrow{\alpha} J_{[3]} \xrightarrow{\alpha} \cdots$$

$$J_{[n]} = \begin{pmatrix} A_{[n]}{}^{-1} - \tilde{B}_{[n]} * A_{[n]} * B_{[n]} & -\tilde{B}_{[n]} * \tilde{A}_{[n]} \\ \tilde{A}_{[n]} * B_{[n]} & \tilde{A}_{[n]} \end{pmatrix}$$

Lechtenfeld-Popov
JHEP[hep-th/0109209], …

All ingredients in AW ansatz can be determined from Δ_0 only

e.g. $\Delta_0 = 1 + ``\dfrac{1}{z\tilde{z} - w\tilde{w}}"$ $\quad \Delta_0 = \exp(\mathit{linear\ \ fcn.of}\ z, \tilde{z}, w, \tilde{w})$

- **If we take a seed sol.** $A_{[1]} = \tilde{A}_{[1]} = B_{[1]} = \tilde{B}_{[1]} = \Delta_0^{-1}, \ \partial^2 \Delta_0 = 0$

the generated solutions would be

$$A_{[n]} = (D_{[n]}{}^{-1})_{11}, \tilde{A}_{[n]} = (D_{[n]}{}^{-1})_{nn}, B_{[n]} = (D_{[n]}{}^{-1})_{n1}, \tilde{B}_{[n]} = (D_{[n]}{}^{-1})_{1n},$$

$$\frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \ \frac{\partial \Delta_r}{\partial w} = \frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

NC Atiyah-Ward ansatz

Backlund trf. for NC Yang's eq. G=GL(2)

- **Let's consider the following Backlund trf.**

$$J_{[1]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[2]} \xrightarrow{\alpha} J_{[3]} \xrightarrow{\alpha} \cdots$$

$$J_{[n]} = \begin{pmatrix} A_{[n]}^{-1} - \tilde{B}_{[n]} * A_{[n]} * B_{[n]} & -\tilde{B}_{[n]} * \tilde{A}_{[n]} \\ \tilde{A}_{[n]} * B_{[n]} & \tilde{A}_{[n]} \end{pmatrix}$$

All ingredients in AW ansatz can be determined from Δ_0 only

e.g. $\Delta_0 = 1 + ``\dfrac{1}{z\tilde{z} - w\tilde{w}}''$ $\quad \Delta_0 = \exp(\mathit{linear\ \ fcn.of}\ z, \tilde{z}, w, \tilde{w})$

- **If we take a seed sol.** $A_{[1]} = \tilde{A}_{[1]} = B_{[1]} = \tilde{B}_{[1]} = \Delta_0^{-1},\ \partial^2 \Delta_0 = 0$

the generated solutions would be

$$A_{[n]} = \left|D_{[n]}\right|_{11}^{-1}, \tilde{A}_{[n]} = \left|D_{[n]}\right|_{nn}^{-1}, B_{[n]} = \left|D_{[n]}\right|_{1n}^{-1}, \tilde{B}_{[n]} = \left|D_{[n]}\right|_{n1}^{-1}$$

$$\frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = \frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

NC Atiyah-Ward ansatz

Quasideterminants !

Quasi-determinants

- **Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.**
- **For an n by n matrix** $X=(x_{ij})$ **and the inverse** $Y=(y_{ij})$ **of X, quasi-determinant of X is directly defined by**

$$|X|_{ij}=y_{ji}^{-1} \qquad \left(\xrightarrow{\theta\to 0} \frac{(-1)^{i+j}}{\underline{\det X^{ij}}}\det X\right)$$

some factor

- **Recall that**

$$X=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \Rightarrow$$

$$Y=X^{-1}=\begin{pmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

→ **We can also define quasi-determinants recursively**

Quasi-determinants

- **Defined inductively as follows**

[For a review, see Gelfand et al., math.QA/0208146]

$$\left|X\right|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'}((X^{ij})^{-1})_{i'j'} x_{j'j}$$
$$= x_{ij} - \sum_{i',j'} x_{ii'}\left(\left|X^{ij}\right|_{j'i'}\right)^{-1} x_{j'j}$$

X^{ij} **: the matrix obtained from X deleting i-th row and j-th column**

$$n=1: \quad \left|X\right|_{ij} = x_{ij}$$

$$n=2: \quad \left|X\right|_{11} = x_{11} - x_{12}\cdot x_{22}^{-1}\cdot x_{21},\ \left|X\right|_{12} = x_{12} - x_{11}\cdot x_{21}^{-1}\cdot x_{22},$$
$$\left|X\right|_{21} = x_{21} - x_{22}\cdot x_{12}^{-1}\cdot x_{11},\ \left|X\right|_{22} = x_{22} - x_{21}\cdot x_{11}^{-1}\cdot x_{12},$$

$$n=3: \quad \left|X\right|_{11} = x_{11} - x_{12}\cdot(x_{22} - x_{23}\cdot x_{33}^{-1}\cdot x_{32})^{-1}\cdot x_{21} - x_{13}\cdot(x_{32} - x_{33}\cdot x_{23}^{-1}\cdot x_{22})^{-1}\cdot x_{21}$$
$$- x_{12}\cdot(x_{23} - x_{22}\cdot x_{32}^{-1}\cdot x_{33})^{-1}\cdot x_{31} - x_{13}\cdot(x_{33} - x_{32}\cdot x_{22}^{-1}\cdot x_{23})^{-1}\cdot x_{31}$$
$$\cdots$$

In this way, we could generate various (complicated) solutions of NC Yang's eq. from a (simple) seed solution by using the previous Backlund trf. $\alpha = \gamma_0 \circ \beta$ **(NC CFYG trf.)**

A seed solution:

$$\Delta_0 = 1 + ``\frac{1}{z\tilde{z} - w\tilde{w}}"$$ → **NC instantons**

$$\Delta_0 = \exp(\mathit{linear\ \ of}\ \ z, \tilde{z}, w, \tilde{w})$$ → **NC Non-Linear plane-waves**

NC CFYG trf. would relate to a Darboux transform for NC ASDYM [Gilson&Nimmo&Ohta et. al] and `weakly non-associative' algebras, (cf. Quasideterminants sols. for NC KP are naturally derived from a Darboux trf. and the `weakly non-associative' algebras. [GNO, Dimakis&Mueller-Hoissen])

3. Backlund transforms for NC KdV eq.

- In this section, we give an exact soliton solutions of NC KdV eq. by a Darboux transformation.

  [Gilson-Nimmo, JPA(to appear), nlin.si/0701027]
- We see that ingredients of quasi-determinants are naturally generated by the Darboux transformation. (an origin of quasi-determinants)
- We also make a comment on asymptotic behavior of soliton scattering process [MH, JHEP 02 (2007) 094 [hep-th/0610006].

Lax pair of NC KdV eq.

- **Linear systems:**

$$L * \psi = (\partial_x^2 + u - \lambda^2) * \psi = 0,$$

$$M * \psi = (\partial_t - \partial_x^3 - (3/2)u\partial_x - (3/4)u_x) * \psi = 0.$$

- **Compatibility condition of the linear system:**

$$[L, M]_* = 0 \quad \Leftrightarrow \quad \dot{u} = \frac{1}{4}u_{xxx} + \frac{3}{4}(u * u_x + u_x * u)$$

:NC KdV equation

- **Darboux transform for NC KdV**

Let us take an eigen function W **of L and define** $\Phi = W * \partial_x W^{-1}$

Then the following trf. leaves the linear systems as it is:

$$\tilde{L} = \Phi * L * \Phi^{-1}, \quad \tilde{M} = \Phi * M * \Phi^{-1}, \quad \tilde{\psi} = \Phi * \psi$$

and $\tilde{u} = u + 2(W_x * W^{-1})_x \quad (\xrightarrow{\theta \to 0} u + 2\partial_x^2 \log W)$

The Darboux transformation can be iterated

- **Let us take eigen fcns.** $(f_1,\cdots,f_N)$ **of** L **and define**

$$\Phi_i = W_i * \partial_x W_i^{-1} = \partial_x - W_{i,x} * W_i^{-1} \qquad (W_1 \equiv f_1, \Phi_1 = f_1 * \partial_x f_1)$$

$$W_{i+1} = \Phi_i * f_{i+1} = f_{i+1,x} - W_{i,x} * W_i^{-1} * f_{i+1} \qquad (i = 1,2,3,\cdots)$$

$$\psi_{i+1} = \Phi_i * \psi_i = \psi_{i,x} - W_{i,x} * W_i^{-1} * \psi_i$$

- **Iterated Darboux transform for NC KdV**

The following trf. leaves the linear systems as it is

$$L_{[i+1]} = \Phi_i * L_{[i]} * \Phi_i^{-1}, \quad M_{[i+1]} = \Phi_i * M_{[i]} * \Phi_i^{-1}, \quad \psi_{[i+1]} = \Phi_i * \psi_{[i]}$$

$$\begin{array}{c} (L_{[1]}, M_{[1]}, \psi_{[1]}) \xrightarrow{\Phi_1} (L_{[2]}, M_{[2]}, \psi_{[2]}) \xrightarrow{\Phi_2} \cdots \\ ||| \\ (L, M, \psi) \end{array}$$

In fact, (W_i, ψ_i) **are quasi-determinants of Wronski matrices !**

and

$$u_{[N+1]} = u + 2\sum_{i=1}^{N} (W_{i,x} * W_i^{-1})_x \quad \left(\xrightarrow{\theta \to 0} u + 2\partial_x^2 \log W(f_1,\cdots,f_N)\right)$$

Exact N-soliton solutions of the NC KdV eq.

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1} \xrightarrow{\theta\to 0} \partial_x^2 \log\det W(f_1,\cdots,f_N)$$

Etingof-Gelfand-Retakh,
[q-alg/9701008]

$$W_i := \left| W(f_1,\ldots,f_i) \right|_{i,i}$$

$$f_i = \exp\left(\xi(x,\lambda_i)\right) + a_i \exp\left(-\xi(x,\lambda_i)\right)$$

$$\xi(x,t,\lambda) = x_1\lambda + t\lambda_i^3 \qquad (M * f_i = (\partial_t - \partial_x^3) f_i = 0)$$

Wronski matrix:

$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

Quasi-det solutions can be extended to NC integrable hierarchy

Exact N-soliton solutions of the NC KP hierarchy

$$L=\Phi*\partial_x\Phi^{-1}$$ **solves the NC KP hierarchy !**

$$=\partial_x+\frac{u}{2}\partial_x^{-1}+\cdots$$

$$\Phi f := \left|W(f_1,\ldots,f_N,f)\right|_{N+1,N+1}$$

quasi-determinant of Wronski matrix

Etingof-Gelfand-Retakh, [q-alg/9701008]

$$f_i=\exp\xi(x,\alpha_i)+a_i\exp\xi(x,\beta_i)$$

$$\xi(x,\alpha)=x_1\alpha+x_2\alpha^2+x_3\alpha^3+\cdots$$

$$u=2\partial_x\sum_{i=1}^{N}(\partial_xW_i)*W_i^{-1}\xrightarrow{\theta\to 0}2\partial_x^2\log\det W(f_1,\cdots,f_N)$$

$$W_i := \left|W(f_1,\ldots,f_i)\right|_{i,i}$$

Wronski matrix:

$$W(f_1,f_2,\cdots,f_m)=\begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1}f_1 & \partial_x^{m-1}f_2 & \cdots & \partial_x^{m-1}f_m \end{bmatrix}$$

More generalization is possible.

[MH, hep-th/0610006]

Interpretation of the exact N-soliton solutions

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- Physical interpretations are non-trivial because

  when $f(x), g(x)$ are real, $f(x)*g(x)$ is not in general.
- However, the solutions could be real in some cases.
  - (i) **1-soliton solutions are all the same as commutative ones because of** **Dimakis-Mueller-Hoissen, [hep-th/0007015]**

    $$f(x-vt)*g(x-vt)=f(x-vt)g(x-vt)$$

  - (ii) **In asymptotic region, configurations of multi-soliton solutions could be real in soliton scatterings and the same as commutative ones.**

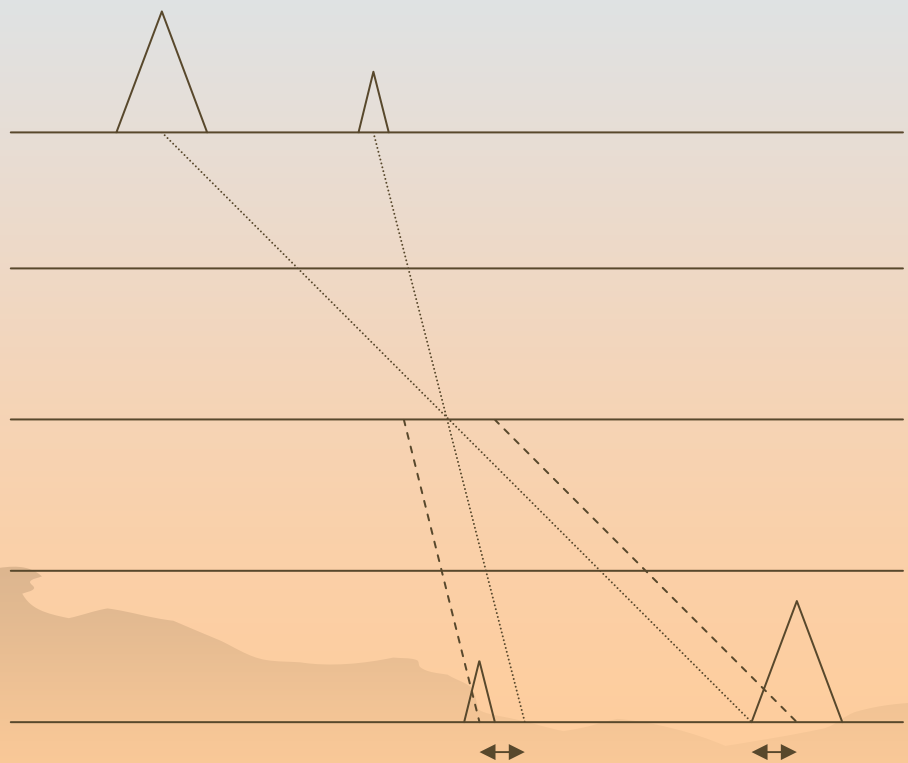
MH, JHEP[hep-th/0610006]

2-soliton solution of KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\,2} \cosh^{-2}(k_i x - 4k_i^{\,3} t), \quad \underset{\text{velocity}}{v_i = 4k_i^{\,2}}, \quad \underset{\text{height}}{h_i = 2k_i^{\,2}}$$

Scattering process (commutative case)



The shape and velocity is preserved ! (stable)

The positions are shifted ! (Phase shift)

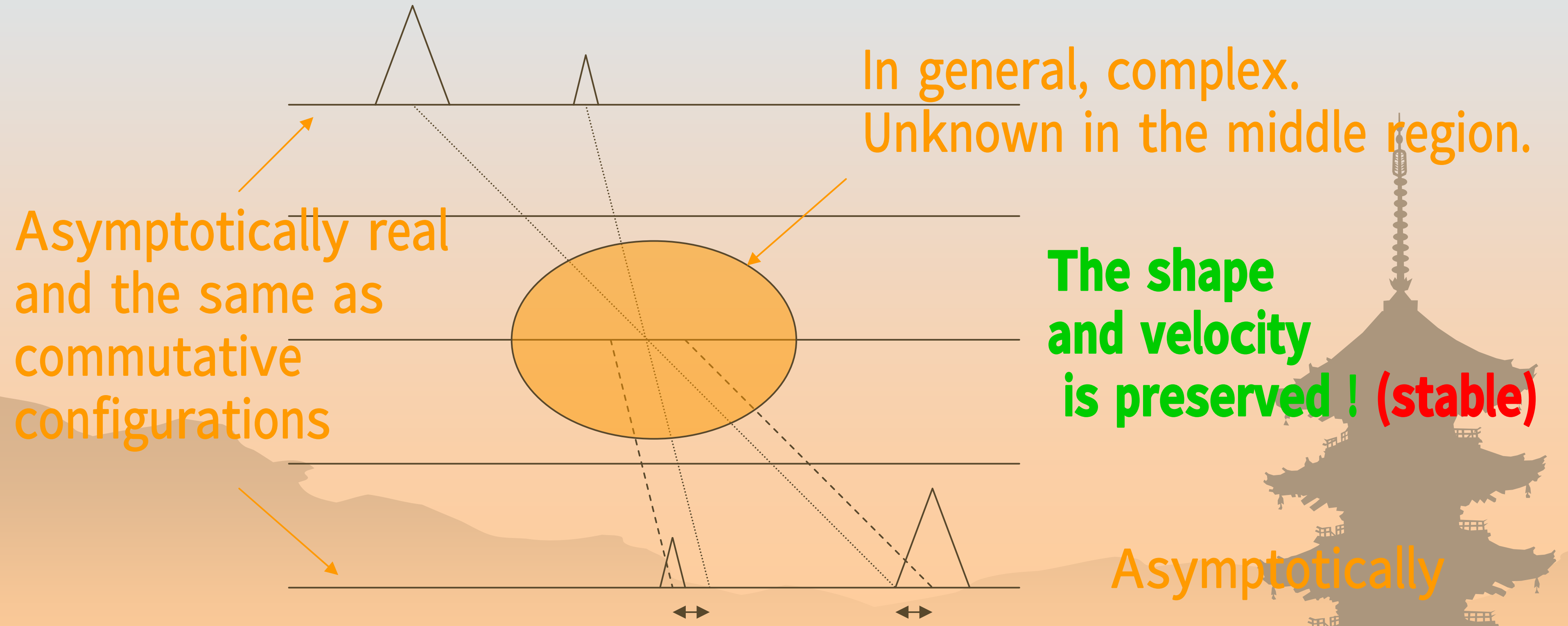
MH, JHEP02 (2007) 094 [hep-th/0610006]
cf Paniak, hep-th/0105185

2-soliton solution of NC KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\,2}\cosh^{-2}(k_i x - 4k_i^{\,3}t), \quad \underbrace{v_i = 4k_i^{\,2}}_{\text{velocity}}, \quad \underbrace{h_i = 2k_i^{\,2}}_{\text{height}}$$

Scattering process (NC case)



The positions are shifted ! (Phase shift)

4. Conclusion and Discussion