Recent Development of Noncommutative Soliton Theory

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Nihon university on July 18th

Based on

- **MH**, ``NC Ward's conjecture and integrable systems,'' NPB741 (2006) 368, [hep-th/0601209]
- **MH**, ``Notes on exact multi-soliton solutions of NC integrable hierarchies ,''JHEP02(07)94 [hepth/0610006]
- **C.Gilson, MH and J.Nimmo et al,** forthcoming papers…
  (Glasgow) (Glasgow)

1. Introduction

Successful points in NC theories

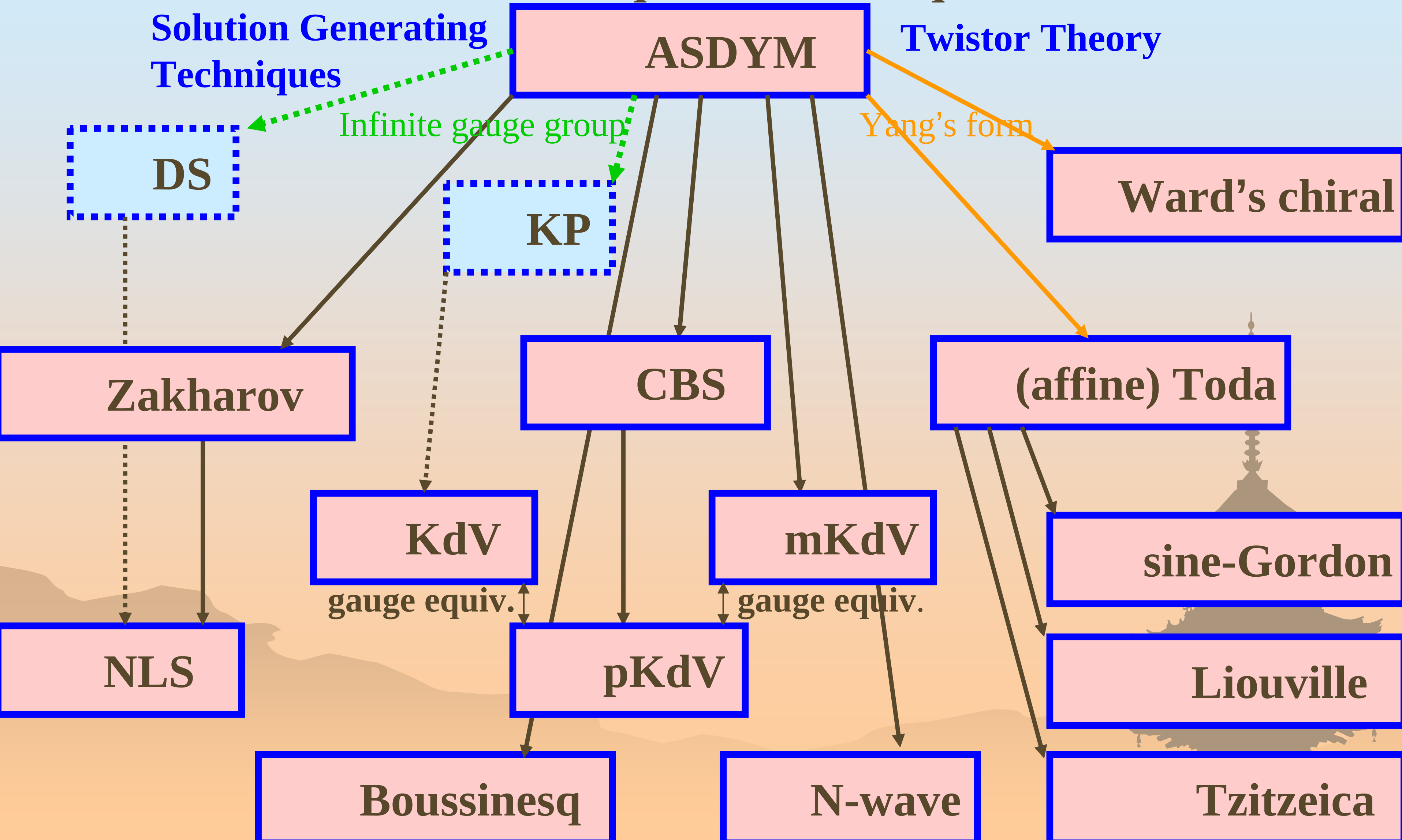
- Appearance of new physical objects
- Description of real physics (in gauge theory)
- Various successful applications to D-brane dynamics etc.

Construction of exact solitons are important. (partially due to their integrablity)

Final goal: NC extension of all soliton theories (Soliton eqs. can be embedded in gauge theories !)

Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

ASDYM eq. is a master eq. !



NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

NC ASDYM eq. is a master eq. ?

Solution Generating Techniques

NC ASDYM

NC Twistor Theory

New physical objects

Application to D-branes

Reductions

NC DS

NC KP

NC Ward's chiral

NC Zakharov

NC CBS

NC (affine) Toda

NC KdV

NC mKdV

NC sine-Gordon

NC NLS

NC pKdV

NC Liouville

NC Boussinesq

NC N-wave

NC Tzitzeica

Program of NC extension of soliton theories

- **(i) Confirmation of NC Ward's conjecture**
  - **NC twistor theory → geometrical origin**
  - **D-brane interpretations → applications to physics**
- **(ii) Completion of NC Sato's theory**
  - **Existence of ``hierarchies'' → various soliton eqs.**
  - **Existence of infinite conserved quantities**
    → infinite-dim. hidden symmetry
  - **Construction of multi-soliton solutions**
  - **Theory of tau-functions → structure of the solution spaces and the symmetry**

(i),(ii) → complete understanding of the NC soliton theories

Plan of this talk

1. Introduction
2. NC ASDYM eqs.
3. NC Ward's conjecture

   --- reduction to (1+1)-dim.
4. Exact Soliton Solutions of NC KdV eq.

   (In terms of quasideterminants)
5. Conclusion and Discussion

2. (NC) ASDYM equations

Here we discuss G=GL(N) (NC) ASDYM eq. from the viewpoint of linear systems with a spectral parameter ζ .

- **Linear systems (commutative case):**

$$L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0,$$
$$M\psi = (D_z - \zeta D_{\tilde{w}})\psi = 0.$$

e.g. $$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$$

- **Compatibility condition of the linear system:**

$$[L, M] = [D_w, D_z] + \zeta([D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}]) + \zeta^2 [D_{\tilde{z}}, D_{\tilde{w}}] = 0$$

$$\Leftrightarrow \begin{cases} F_{zw} = [D_z, D_w] = 0, \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0 \end{cases}$$

:ASDYM equation

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

Yang's form and Yang's equation

- **ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w] = 0, \quad \Rightarrow \quad \exists h,\ D_z h = 0,\ \ D_w h = 0 \quad (A_z = -h_z h^{-1}, etc.) \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \quad \Rightarrow \quad \exists \tilde{h},\ D_{\tilde{z}} \tilde{h} = 0,\ \ D_{\tilde{w}} \tilde{h} = 0 \quad (A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1}, etc.) \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0 \end{cases}$$

If we define Yang's matrix: $J := \tilde{h}^{-1} h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} \partial_{\tilde{z}} J) - \partial_w (J^{-1} \partial_{\tilde{w}} J) = 0$$ **:Yang's eq.**

The solution J reproduce the gauge fields as

$$A_z = -h_z h^{-1},\ \ A_w = h_w h^{-1},\ A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1},\ \ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} \tilde{h}^{-1}$$

J **is gauge invariant. The decomposition into** h **and** $\tilde{h}$ **corresponds to a gauge fixing**

(Q) How we get NC version of the theories?

(A) We have only to replace all products of fields in ordinary commutative gauge theories

with star-products: $f(x)g(x) \to f(x) * g(x)$

- **The star product: (NC and associative)**

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x) = f(x)g(x) + i\frac{\theta^{\mu\nu}}{2}\partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)$$

A deformed product

Presence of background magnetic fields

Note: coordinates and fields themselves are usual c-number functions. But commutator of coordinates becomes…

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

NC !

Here we discuss G=GL(N) NC ASDYM eq. from the viewpoint of linear systems with a spectral parameter ζ .

(All products are star-products.)

- **Linear systems (NC case):**

$$L * \psi = (D_w - \zeta D_{\tilde{z}}) * \psi = 0,$$
$$M * \psi = (D_z - \zeta D_{\tilde{w}}) * \psi = 0.$$

e.g. $$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$$

- **Compatibility condition of the linear system:**

$$[L, M]_* = [D_w, D_z]_* + \zeta([D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_*) + \zeta^2 [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0$$

$$\Leftrightarrow \begin{cases} F_{zw} = [D_z, D_w]_* = 0, \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_* = 0 \end{cases}$$

:NC ASDYM equation

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_*)$$

$$\theta^{\mu\nu} = \begin{bmatrix} 0 & \theta^1 & & \\ -\theta^1 & 0 & & O \\ & & 0 & \theta^2 \\ O & & -\theta^2 & 0 \end{bmatrix}$$

Yang's form and NC Yang's equation

- **NC ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w]_* = 0, & \Rightarrow \quad \exists h,\ D_z * h = 0,\ D_w * h = 0 \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0, & \Rightarrow \quad \exists \tilde{h},\ D_{\tilde{z}} * \tilde{h} = 0,\ D_{\tilde{w}} * \tilde{h} = 0 \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_* = 0 & \end{cases}$$

If we define Yang's matrix: $J := \tilde{h}^{-1} * h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} * \partial_{\tilde{z}} J) - \partial_w (J^{-1} * \partial_{\tilde{w}} J) = 0$$ **:NC Yang's eq.**

The solution J reproduces the gauge fields as

$$A_z = -h_z * h^{-1},\ \ A_w = h_w * h^{-1},\ A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} * \tilde{h}^{-1},\ \ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} * \tilde{h}^{-1}$$

J **is gauge invariant. The decomposition into** h **and** $\tilde{h}$ **corresponds to a gauge fixing**

Backlund transformation for NC Yang's eq.

- **Yang's J matrix can be decomposed as follows**

$$J=\begin{pmatrix} A^{-1}-\widetilde{B}*A*B & -\widetilde{B}*\widetilde{A} \\ \widetilde{A}*B & \widetilde{A} \end{pmatrix}$$

MH, NPB [hep-th/0601209]
Book of Mason and Woodhouse

- **Then NC Yang's eq. becomes**

$$\partial_z(A*\widetilde{B}_{\tilde{z}}*\widetilde{A})-\partial_w(A*\widetilde{B}_{\tilde{w}}*\widetilde{A})=0,\quad \partial_{\tilde{z}}(\widetilde{A}*B_z*A)-\partial_{\tilde{w}}(\widetilde{A}*B_w*A)=0,$$

$$\partial_z(\widetilde{A}^{-1}*\widetilde{A}_{\tilde{z}})*\widetilde{A}^{-1}-\partial_w(\widetilde{A}^{-1}*\widetilde{A}_{\tilde{w}})*\widetilde{A}^{-1}+B_z*A*\widetilde{B}_{\tilde{z}}-B_w*A*\widetilde{B}_{\tilde{w}}=0,$$

$$A^{-1}*\partial_z(A_{\tilde{z}}*A^{-1})-A^{-1}*\partial_w(A_{\tilde{w}}*A^{-1})+\widetilde{B}_{\tilde{z}}*\widetilde{A}*B_z-\widetilde{B}_{\tilde{w}}*\widetilde{A}*B_w=0.$$

- **The following trf. leaves NC Yang's eq. as it is:**

$$\beta:\begin{cases} \partial_z B^{new}=A*\widetilde{B}_{\tilde{w}}*\widetilde{A},\ \partial_w B^{new}=A*\widetilde{B}_{\tilde{z}}*\widetilde{A}, \\ \partial_{\tilde{z}}\widetilde{B}^{new}=\widetilde{A}*B_w*A,\ \partial_{\tilde{w}}\widetilde{B}^{new}=\widetilde{A}*B_z*A, \\ A^{new}=\widetilde{A}^{-1},\ \widetilde{A}^{new}=A^{-1} \end{cases}$$

We could generate various (non-trivial) solutions of NC Yang's eq. from a (trivial) seed solution by using the previous Backlund trf. together with a simple trf.

$$\gamma_0 : J^{new} = C^{-1} J C, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Leftrightarrow \gamma_0 : \begin{pmatrix} A^{-1\,new} & \tilde{B}^{new} \\ B^{new} & \tilde{A}^{-1\,new} \end{pmatrix} = \begin{pmatrix} \tilde{A}^{-1} & B \\ \tilde{B} & A^{-1} \end{pmatrix}^{-1}$$

This combined trf. would generate a group of hidden symmetry of NC Yang's eq., which would be also applied to lower-dimension.

For G=GL(2), we can present the transforms more explicitly and give an explicit form of a class of solutions (Atiyah-Ward ansatz).

Backlund trf. for NC Yang's eq. G=GL(2)

- **Let's consider the following Backlund trf.**

$$J_{[1]} \xrightarrow{\alpha = \gamma_0 \circ \beta} J_{[2]} \xrightarrow{\alpha} J_{[3]} \xrightarrow{\alpha} \cdots$$

$$J_{[n]} = \begin{pmatrix} A_{[n]}{}^{-1} - \widetilde{B}_{[n]} * A_{[n]} * B_{[n]} & -\widetilde{B}_{[n]} * \widetilde{A}_{[n]} \\ \widetilde{A}_{[n]} * B_{[n]} & \widetilde{A}_{[n]} \end{pmatrix}$$

Collaboration with Gilson-san and Nimmo-san **(Very Hot)**

All ingredients in AW ansatz can be determined from Δ_0 only

- **If we take a seed sol.** $A_{[1]} = \widetilde{A}_{[1]} = B_{[1]}^{-1} = \widetilde{B}_{[1]}^{-1} = \Delta_0, \ \partial^2 \Delta_0 = 0$

the generated solutions would be

$$A_{[n]} = \left| D_{[n]} \right|_{11}, \widetilde{A}_{[n]} = \left| D_{[n]} \right|_{nn}, B_{[n]} = \left| D_{[n]} \right|_{1n}^{-1}, \widetilde{B}_{[n]} = \left| D_{[n]} \right|_{n1}^{-1}$$

$$\frac{\partial \Delta_r}{\partial z} = -\frac{\partial \Delta_{r+1}}{\partial \widetilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = -\frac{\partial \Delta_{r+1}}{\partial \widetilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

NC Atiyah-Ward ansatz

Quasideterminants !

Quasi-determinants

- **Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.**
- **For an n by n matrix** $X=(x_{ij})$ **and the inverse** $Y=(y_{ij})$ **of X, quasi-determinant of X is directly defined by**

$$|X|_{ij}=y_{ji}^{-1} \qquad \left(\xrightarrow{\theta\to 0} \frac{(-1)^{i+j}}{\det X^{ij}}\det X\right)$$

some factor

X^{ij}: **the matrix obtained from X deleting i-th row and j-th column**

- **Recall that**

$$X=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \Rightarrow$$

$$Y=X^{-1}=\begin{pmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

→ **We can also define quasi-determinants recursively**

Quasi-determinants

- **Defined inductively as follows**

[For a review, see Gelfand et al., math.QA/0208146]

$$|X|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'}((X^{ij})^{-1})_{i'j'} x_{j'j}$$

$$= x_{ij} - \sum_{i',j'} x_{ii'}(|X^{ij}|_{j'i'})^{-1} x_{j'j}$$

$n=1: \quad |X|_{ij} = x_{ij}$

$n=2: \quad |X|_{11} = x_{11} - x_{12} \cdot x_{22}^{-1} \cdot x_{21}, \ |X|_{12} = x_{12} - x_{11} \cdot x_{21}^{-1} \cdot x_{22},$

$\quad |X|_{21} = x_{21} - x_{22} \cdot x_{12}^{-1} \cdot x_{11}, \ |X|_{22} = x_{22} - x_{21} \cdot x_{11}^{-1} \cdot x_{12},$

$n=3: \quad |X|_{11} = x_{11} - x_{12} \cdot (x_{22} - x_{23} \cdot x_{33}^{-1} \cdot x_{32})^{-1} \cdot x_{21} - x_{13} \cdot (x_{32} - x_{33} \cdot x_{23}^{-1} \cdot x_{22})^{-1} \cdot x_{21}$

$\quad - x_{12} \cdot (x_{23} - x_{22} \cdot x_{32}^{-1} \cdot x_{33})^{-1} \cdot x_{31} - x_{13} \cdot (x_{33} - x_{32} \cdot x_{22}^{-1} \cdot x_{23})^{-1} \cdot x_{31}$

$\ldots$

Explicit Atiyah-Ward ansatz solutions of NC Yang's eq. G=GL(2)

$$A_{[1]} = \tilde{A}_{[1]} = B_{[1]}^{-1} = \tilde{B}_{[1]}^{-1} = \Delta_0, \quad \partial^2 \Delta_0 = 0$$

$$A_{[2]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}, \tilde{A}_{[2]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}, B_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \tilde{B}_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1},$$

$$\partial_z \Delta_0 = -\partial_{\tilde{w}} \Delta_1, \partial_z \Delta_{-1} = -\partial_{\tilde{w}} \Delta_0, \partial_w \Delta_0 = -\partial_{\tilde{z}} \Delta_1, \partial_w \Delta_{-1} = -\partial_{\tilde{z}} \Delta_0$$

$$A_{[n]} = \left| D_{[n]} \right|_{11}, \tilde{A}_{[n]} = \left| D_{[n]} \right|_{nn}, B_{[n]} = \left| D_{[n]} \right|_{1n}^{-1}, \tilde{B}_{[n]} = \left| D_{[n]} \right|_{n1}^{-1}$$

$$\frac{\partial \Delta_r}{\partial z} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

$$J_{[n]} = \begin{pmatrix} A_{[n]}^{-1} - \tilde{B}_{[n]} * A_{[n]} * B_{[n]} & -\tilde{B}_{[n]} * \tilde{A}_{[n]} \\ \tilde{A}_{[n]} * B_{[n]} & \tilde{A}_{[n]} \end{pmatrix}$$

We could generate various (complicated) solutions of NC ASDYM eq. from a (simple) seed solution Δ_0 by using the previous Backlund trf. $\alpha = \gamma_0 \circ \beta$

A seed solution:

$$\Delta_0 = 1 + ``\frac{1}{z\tilde{z} - w\tilde{w}}'' \quad \rightarrow \text{NC instantons}$$

$$\Delta_0 = \exp(\mathit{linear\ of}\ z, \tilde{z}, w, \tilde{w}) \quad \rightarrow \text{NC Non-Linear plane-waves}$$

NC CFYG trf. would relate to a Darboux transform for NC ASDYM [Gilson&Nimmo&Ohta et. al] and `weakly non-associative' algebras, (cf. Quasideterminants sols. for NC (m)KP are naturally derived from a Darboux trf. [Gilson-Nimmo] and the `weakly non-associative' algebras. [Dimakis&Mueller-Hoissen])

NC twistor can give an origin of NC CFYG transform.

$$\beta : F^{new} = \Xi^{-1} F \,\Xi, \quad \gamma_0 : F^{new} = C^{-1} F C, \quad \Xi = \begin{pmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{pmatrix}$$

3. NC Ward's conjecture --- reduction to (1+1)-dim.

- **From now on, we discuss reductions of NC ASDYM on (2+2)-dimension to NC KdV, mKdV**
- **Reduction steps are as follows:**

  (1) take a simple dimensional reduction with a gauge fixing.

  (2) put further reduction condition on gauge field.
- **The reduced eqs. coincides with those obtained in the framework of NC KP and GD hierarchies, which possess infinite conserved quantities and exact multi-soliton solutions. (integrable-like)**

Reduction to NC KdV eq.

MH, PLB625, 324 [hep-th/0507112]

- **(1) Take a a dimensional reduction and gauge fixing:**

$$(z,\tilde{z},w,\tilde{w}) \to (t,x) = (z, w+\tilde{w}),\qquad A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The reduced NC ASDYM is:

$$(i)\quad [A_w, A_{\tilde{z}}]_* = 0$$

$$(ii)\quad A_w' - A_{\tilde{w}}' + [A_z, A_{\tilde{z}}]_* - [A_w, A_{\tilde{w}}]_* = 0$$

$$(iii)\quad A_z' - \dot{A}_w + [A_w, A_z]_* = 0$$

- **(2) Take a further reduction condition:**

NOT traceless !

$$A_w = \begin{pmatrix} q & -1 \\ q' + q*q & -q \end{pmatrix},\ A_{\tilde{w}} = O,\ A_z = \begin{pmatrix} \frac{1}{2}q'' + \underline{q'*q} & -q' \\ f(q,q',q'',q''') & -\frac{1}{2}q'' - \underline{q*q'} \end{pmatrix},$$

We can get NC KdV eq. in such a miracle way !

$$(iii) \quad \Rightarrow \quad \dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u'*u + u*u') \qquad u = 2q' \qquad [t,x] = i\theta$$

Note: $A, B, C \in gl(2) \xrightarrow{\theta \to 0} sl(2)$ U(1) part is necessary !

The NC KdV eq. has integrable-like properties:

- possesses infinite conserved densities:

$$\sigma_n = res_{-1}L^n + \frac{3}{4}\theta((res_{-1}L^n)\Diamond u'' - 2(res_{-2}L^n)\Diamond u')$$

MH, JMP46 (2005) [hep-th/0311206]

$res_r L^n$: coefficient of ∂_x^r in L^n

$\Diamond$: Strachan's product (commutative and non-associative)

$$f(x)\Diamond g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

$$[t,x] = i\theta$$

- has exact N-soliton solutions:

Etingof-Gelfand-Retakh, MRL [q-alg/9701008]
MH, JHEP [hep-th/0610006]
cf. Paniak, [hep-th/0105185]

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1}$$

$$W_i := \left|W(f_1,\ldots,f_i)\right|_{i,i}$$ **:quasi-determinant of Wronski matrix**

$$f_i = \exp\xi(x,\alpha_i) + a_i \exp(-\xi(x,\alpha_i))$$

$$\xi(x,\alpha) = x\alpha + t\alpha^3$$

Reduction to NC mKdV eq.

MH, NPB741, 368 [hep-th/0601209]

- **(1) Take a dimensional reduction and gauge fixing:**

$$(z,\tilde{z},w,\tilde{w}) \rightarrow (t,x) = (z, w+\tilde{w}), \qquad A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The reduced NC ASDYM is:

$$(i) \quad [A_w, A_{\tilde{z}}]_* = 0$$

$$(ii) \quad A'_w - A'_{\tilde{w}} + [A_z, A_{\tilde{z}}]_* - [A_w, A_{\tilde{w}}]_* = 0$$

$$(iii) \quad A'_z - \dot{A}_w + [A_w, A_z]_* = 0$$

- **(2) Take a further reduction condition:**

$$A_w = \begin{pmatrix} p & -1 \\ 0 & -p \end{pmatrix}, \; A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \; A_z = \begin{pmatrix} c & b \\ 0 & d \end{pmatrix}$$

We get

$$a = -\frac{1}{2}p' - \frac{1}{2}p^2, \; b = -\frac{1}{2}p' + \frac{1}{2}p^2,$$

NOT traceless !

$$c = \frac{1}{4}p'' - \frac{1}{2}p^3 \underline{-\frac{1}{4}[p,p']_*}, \; d = -\frac{1}{4}p'' + \frac{1}{2}p^3 \underline{-\frac{1}{4}[p,p']_*}$$

and $(iii) \quad \Rightarrow \quad \dot{p} = \frac{1}{4}p''' - \frac{3}{4}(p' * p * p + p * p * p')$ **NC mKdV !**

$$[t,x] = i\theta$$

Relation between NC KdV and NC mKdV

- **(1) Take a dimensional reduction and gauge fixing:**

$$(z,\tilde{z},w,\tilde{w}) \to (t,x) = (z, w+\tilde{w}),$$

$$A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Note: There is a residual gauge symmetry:

$$A_\mu \to g^{-1} * A_\mu * g + g^{-1} * \partial_\mu g, \quad g = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$$

- **(2) Take a further reduction condition:**

NCKdV:

$$A_w = \begin{pmatrix} q & -1 \\ q' + q*q & -q \end{pmatrix}, A_{\tilde{w}} = O, A_z = \begin{pmatrix} \frac{1}{2}q'' + q'*q & -q' \\ f(q,q',q'',q''') & -\frac{1}{2}q'' - q*q' \end{pmatrix}$$

Gauge equivalent

The gauge trf. → $\beta = q - p, \quad 2q' = p' - p*p$

NC Miura map !

NCmKdV:

$$A_w = \begin{pmatrix} p & -1 \\ 0 & -p \end{pmatrix}, A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, A_z = \begin{pmatrix} c & b \\ 0 & d \end{pmatrix}$$

MH, NPB741, 368 [hep-th/0601209]

Reduction to NC Tzitzeica eq.

MH, NPB741, 368
[hep-th/0601209]

- **Start with NC Yang's eq.**

$$\partial_z (J^{-1}\partial_{\tilde{z}} J) - \partial_w (J^{-1}\partial_{\tilde{w}} J) = 0$$

- **(1) Take a special reduction condition:**

$$J = \exp(-E_- \tilde{w}) * g(z,\tilde{z}) * \exp(E_+ w)$$

$$E_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

We get a reduced Yang's eq.

$$\partial_z (g^{-1} * \partial_{\tilde{z}} g) - [E_-, g^{-1} * E_+ g]_* = 0$$

$$E_- = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- **(2) Take a further reduction condition:**

$$g = \exp(\rho) * diag\,(\exp(\omega), \exp(-\omega), 1)$$

We get (a set of) NC Tzitzeica eq.:

$$\partial_z (\exp(-\omega) * \partial_{\tilde{z}} \exp(\omega)) + \partial_z (\exp(-\omega) * V * \exp(\omega)) = \exp(\omega) - \exp(-2\omega),$$

$$\partial_z (\exp(\omega) * \partial_{\tilde{z}} \exp(-\omega)) + \partial_z (\exp(\omega) * V * \exp(-\omega)) = \exp(-2\omega) - \exp(\omega),$$

$$\partial_z V = \partial_z (\exp(-\rho) * \partial_{\tilde{z}} \exp(\rho)) = 0$$

$$\left(\xrightarrow{\theta \to 0} \omega_{z\tilde{z}} = \exp(\omega) - \exp(-2\omega) \right)$$

4. Exact Soliton Solutions of NC KdV eq.

- In this section, we give an exact soliton solutions of NC KdV eq. by a Darboux transformation.

  [Gilson-Nimmo, JPA(to appear), nlin.si/0701027]

- We see that ingredients of quasi-determinants are naturally generated by the Darboux transformation. (an origin of quasi-determinants)

- We also make a comment on asymptotic behavior of soliton scattering process [MH, JHEP 02 (2007) 094 [hep-th/0610006].

Lax pair of NC KdV eq.

- **Linear systems:**

$$L * \psi = (\partial_x^2 + u - \lambda^2) * \psi = 0,$$

$$M * \psi = (\partial_t - \partial_x^3 - (3/2)u\partial_x - (3/4)u_x) * \psi = 0.$$

- **Compatibility condition of the linear system:**

$$[L, M]_* = 0 \quad \Leftrightarrow \quad \dot{u} = \frac{1}{4}u_{xxx} + \frac{3}{4}(u * u_x + u_x * u)$$

:NC KdV equation

- **Darboux transform for NC KdV**

Let us take an eigen function W **of L and define** $\Phi = W * \partial_x W^{-1}$

Then the following trf. leaves the linear systems as it is:

$$\tilde{L} = \Phi * L * \Phi^{-1}, \quad \tilde{M} = \Phi * M * \Phi^{-1}, \quad \tilde{\psi} = \Phi * \psi$$

and $\tilde{u} = u + 2(W_x * W^{-1})_x \quad (\xrightarrow{\theta \to 0} u + 2\partial_x^2 \log W)$

The Darboux transformation can be iterated

- **Let us take eigen fcns.** $(f_1,\cdots,f_N)$ **of** L **and define**

$$\Phi_i = W_i * \partial_x W_i^{-1} = \partial_x - W_{i,x} * W_i^{-1} \qquad (W_1 \equiv f_1, \Phi_1 = f_1 * \partial_x f_1)$$

$$W_{i+1} = \Phi_i * f_{i+1} = f_{i+1,x} - W_{i,x} * W_i^{-1} * f_{i+1} = \left| W(f_1,\cdots,f_{i+1}) \right|_{i+1,i+1}$$

$$\psi_{i+1} = \Phi_i * \psi_i = \psi_{i,x} - W_{i,x} * W_i^{-1} * \psi_i = \left| W(f_1,\cdots,f_i,\psi) \right|_{i+1,i+1}$$

- **Iterated Darboux transform for NC KdV**

The following trf. leaves the linear systems as it is

$$L_{[i+1]} = \Phi_i * L_{[i]} * \Phi_i^{-1}, \quad M_{[i+1]} = \Phi_i * M_{[i]} * \Phi_i^{-1}, \quad \psi_{[i+1]} = \Phi_i * \psi_{[i]}$$

$$(L_{[1]}, M_{[1]}, \psi_{[1]}) \xrightarrow{\Phi_1} (L_{[2]}, M_{[2]}, \psi_{[2]}) \xrightarrow{\Phi_2} \cdots$$

$$|||$$

$$(L, M, \psi)$$

In fact, (W_i, ψ_i) **are quasi-determinants of Wronski matrices !**

and

$$u_{[N+1]} = u + 2\sum_{i=1}^{N} (W_{i,x} * W_i^{-1})_x \quad \left(\xrightarrow{\theta \to 0} u + 2\partial_x^2 \log W(f_1,\cdots,f_N) \right)$$

Exact N-soliton solutions of the NC KdV eq.

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1} \quad \xrightarrow{\theta \to 0} \quad \partial_x^2 \log\det W(f_1, \cdots, f_N)$$

Etingof-Gelfand-Retakh, [q-alg/9701008]

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

$$f_i = \exp\left(\xi(x, \lambda_i)\right) + a_i \exp\left(-\xi(x, \lambda_i)\right)$$

$$\xi(x, t, \lambda) = x_1 \lambda + t \lambda_i^3 \qquad (M * f_i = (\partial_t - \partial_x^3) f_i = 0)$$

Wronski matrix:

$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

Quasi-det solutions can be extended to NC integrable hierarchy

Exact N-soliton solutions of the NC KP hierarchy

$$L = \Phi * \partial_x \Phi^{-1}$$

solves the NC KP hierarchy !

$$= \partial_x + \frac{u}{2}\partial_x^{-1} + \cdots$$

$$\frac{\partial L}{\partial x^m} = [B_m, L]_*$$

quasi-determinant of Wronski matrix

$$\Phi f := \left| W(f_1, \ldots, f_N, f) \right|_{N+1, N+1}$$

Etingof-Gelfand-Retakh, [q-alg/9701008]

$$f_i = \exp \xi(x, \alpha_i) + a_i \exp \xi(x, \beta_i)$$

$$\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots$$

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1} \xrightarrow{\theta \to 0} 2\partial_x^2 \log \det W(f_1, \cdots, f_N)$$

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

Wronski matrix:

$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

Exact N-soliton solutions of NC toroidal lKdV

$$L = \Phi * \partial_x \Phi^{-1}$$
$$= \partial_x + \frac{u}{2}\partial_x^{-1} + \cdots$$

solves the NC toroidal hierarchy !

Cf. Commutative ones：[Bogoyavlenskii, Toda-Fukuyama-Yu, Ikeda-Takasaki-(Kakei), Billig, Iohara-Saito-Wakimoto,…]

$$\Phi f := \left| W(f_1, \ldots, f_N, f) \right|_{N+1,N+1}$$

quasi-determinant of Wronski matrix

$$f_i = \exp \xi_r(x, y, \alpha_i) + a_i \exp \xi_r(x, y, \beta_i) \qquad \alpha_i^l = \beta_i^l$$

$$\xi_r(x, y, \alpha) = x_1\alpha + x_2\alpha^2 + \cdots + ry_0 + ry_l\alpha^l + ry_{2l}\alpha^{2l} + \cdots$$

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1} \xrightarrow{\theta \to 0} 2\partial_x^2 \log\det W(f_1, \cdots, f_N)$$

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

Wronski matrix:

$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

他の拡張(戸田階層etc.)も可能

MH, JHEP [hep-th/0610006]

Interpretation of the exact N-soliton solutions

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- Physical interpretations are non-trivial because

  when $f(x), g(x)$ are real, $f(x) * g(x)$ is not in general.
- However, the solutions could be **real** in some cases.
  - (i) **1-soliton solutions are all the same as commutative ones because of** Dimakis-Mueller-Hoissen, [hep-th/0007015]

$$f(x-vt) * g(x-vt) = f(x-vt) g(x-vt)$$

  - (ii) **In asymptotic region, configurations of multi-soliton solutions could be real in soliton scatterings and the same as commutative ones.**

MH, JHEP[hep-th/0610006]

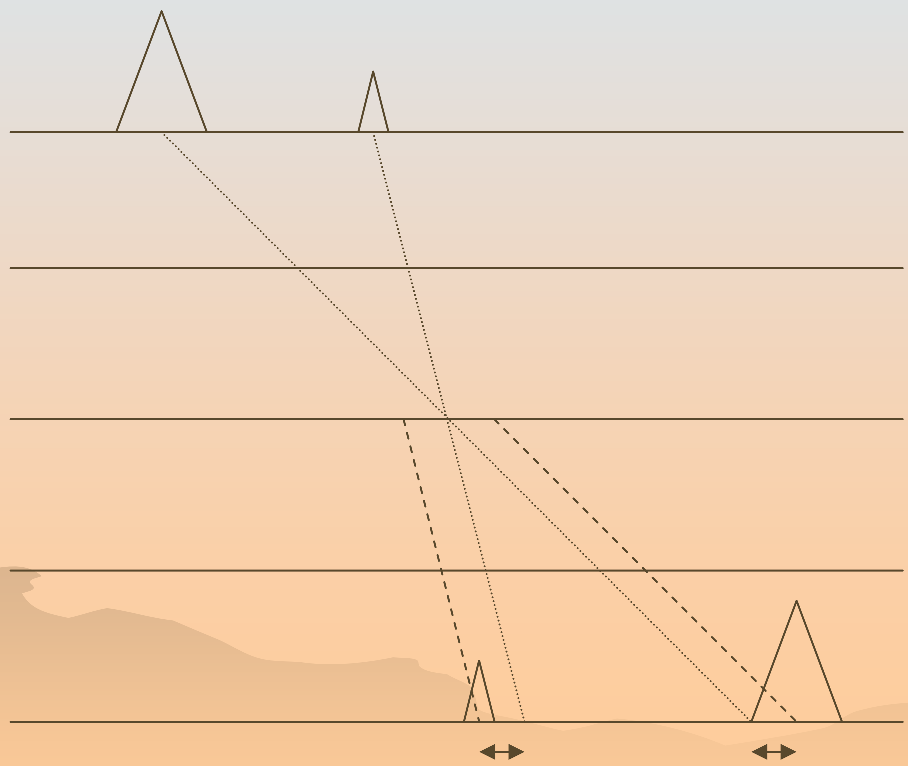
2-soliton solution of KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\,2}\cosh^{-2}(k_i x - 4k_i^{\,3} t), \quad v_i = 4k_i^{\,2}, \quad h_i = 2k_i^{\,2}$$

velocity height

Scattering process (commutative case)



The shape and velocity is preserved ! (stable)

The positions are shifted ! (Phase shift)

MH, JHEP02 (2007) 094
[hep-th/0610006]
cf Paniak, hep-th/0105185

2-soliton solution of NC KdV

each packet has the configuration:

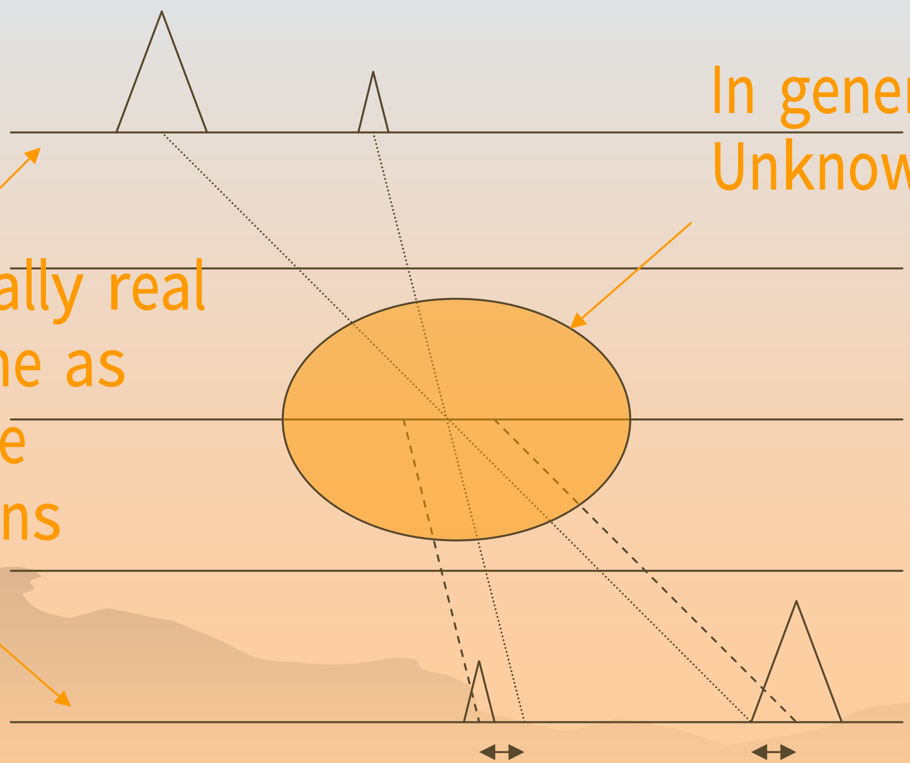
$$u^{(i)} = 2k_i^{\ 2}\cosh^{-2}(k_i x - 4k_i^{\ 3} t), \quad v_i = 4k_i^{\ 2}, \quad h_i = 2k_i^{\ 2}$$

(v_i: velocity, h_i: height)

Scattering process (NC case)

In general, complex.
Unknown in the middle region.

Asymptotically real and the same as commutative configurations

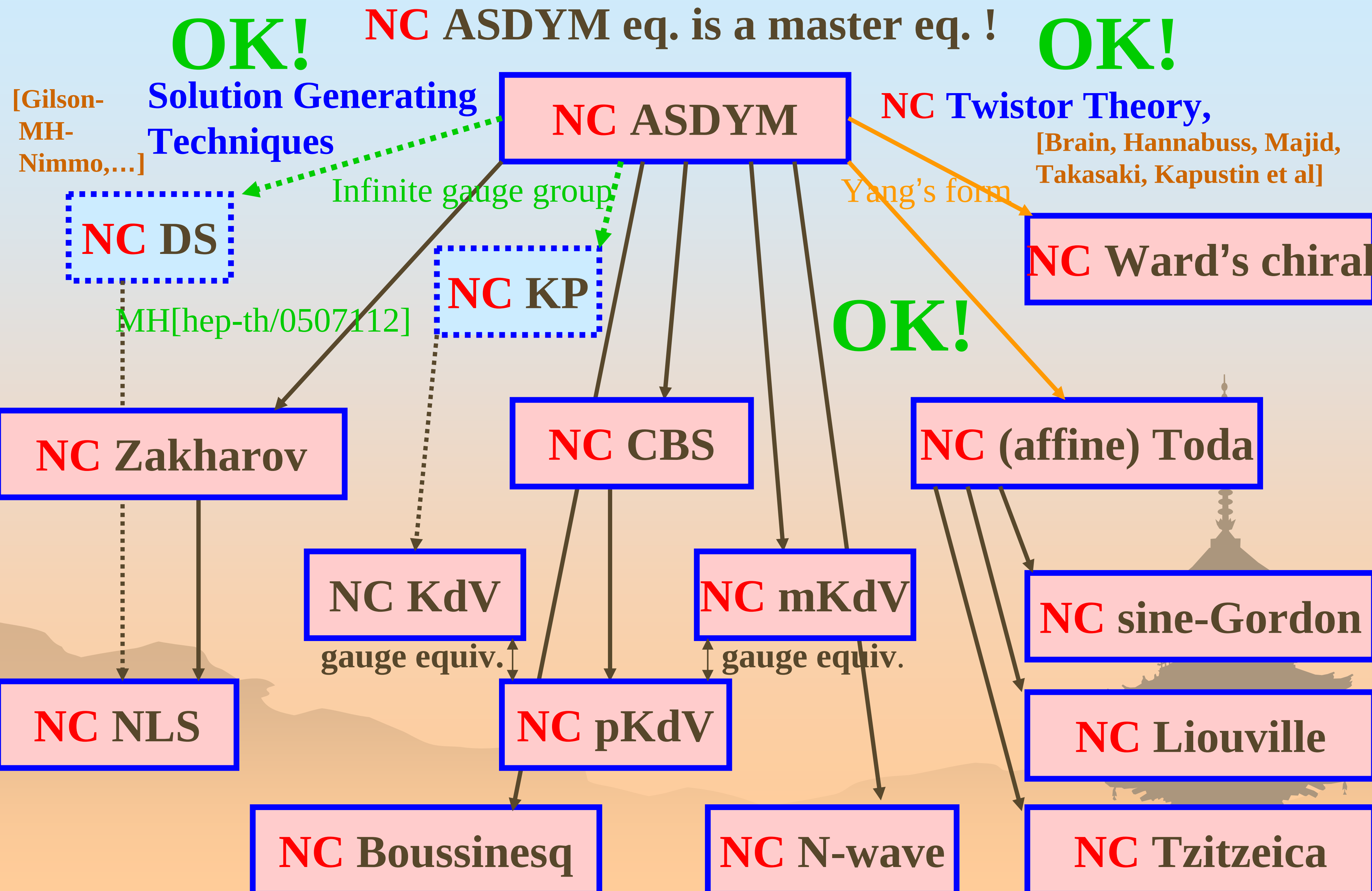


The shape and velocity is preserved ! (stable)

Asymptotically

The positions are shifted ! (Phase shift)

5. Conclusion and Discussion



Current situation

- **Confirmation of NC Ward's conjecture** Solved!
  - **NC twistor theory → geometrical origin**
  - **D-brane interpretations → applications to physics**

  Work in progress

- **Completion of NC Sato's theory**
  - **Existence of ``hierarchies'' →** Solved!
  - **Existence of infinite conserved quantities** Successful

    → infinite-dim. hidden symmetry?
  - **Construction of multi-soliton solutions** Successful
  - **Theory of tau-functions → description of the symmetry and the soliton solutions** Q-det plays key roles ?