Backlund Transformations for Non-Commutative (NC) Integrable Eqs.

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(visiting Glasgow until Feb. 13
and IHES Feb13 - Mar13 2009)

Based on

- **Claire R.Gilson (Glasgow), MH and Jonathan J.C.Nimmo (Glasgow), ``Backlund trfs for NC anti-self-dual (ASD) Yang-Mills (YM) eqs.'' arXiv:0709.2069 (to appear in GMJ) & 08mm.nnnn.**
- **MH, NPB 741 (2006) 368, JHEP 02 (07) 94, PLB 625 (05) 324, JMP46 (05) 052701…**

1. Introduction

NC extension of integrable systems

- Matrix generalization
- Quarternion-valued system
- Moyal deformation (=extension to NC spaces =presence of magnetic flux)

All variables belong to NC ring, which implies associativity.

4-dim. Anti-Self-Dual Yang-Mills Eq.

- plays important roles in QFT
- a master eq. of (lower-dim) integrable eqs. (Ward's conjecture)

4-dim. NC ASDYM eq. (G=GL(N))

$$F_{\mu\nu} = -*F_{\mu\nu} \qquad \mu,\nu = 0,1,2,3$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

$$\Updownarrow \qquad \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$$

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

Reduction to NC KdV from NC ASDYM

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

:NC ASDYM eq.
G=GL(2)

$$(z, \tilde{z}, w, \tilde{w}) \rightarrow (t, x) = (z, w + \tilde{w})$$

$$A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Reduction conditions

$$A_w = \begin{pmatrix} q & -1 \\ q' + q^2 & -q \end{pmatrix}, A_z = \begin{pmatrix} \frac{1}{2}q'' + q'q & -q' \\ \frac{1}{4}q''' + \frac{1}{2}(q'^2 + qq'' + q''q) + qq'q & -\frac{1}{2}q'' - qq' \end{pmatrix}$$

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u'u + uu') \qquad u = 2q'$$

:NC KdV eq.!

Reduction to NC NLS from NC ASDYM

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

:NC ASDYM eq.
G=GL(2)

$$(z, \tilde{z}, w, \tilde{w}) \to (t, x) = (z, w + \tilde{w})$$

Reduction conditions

$$A_{\tilde{z}} = \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

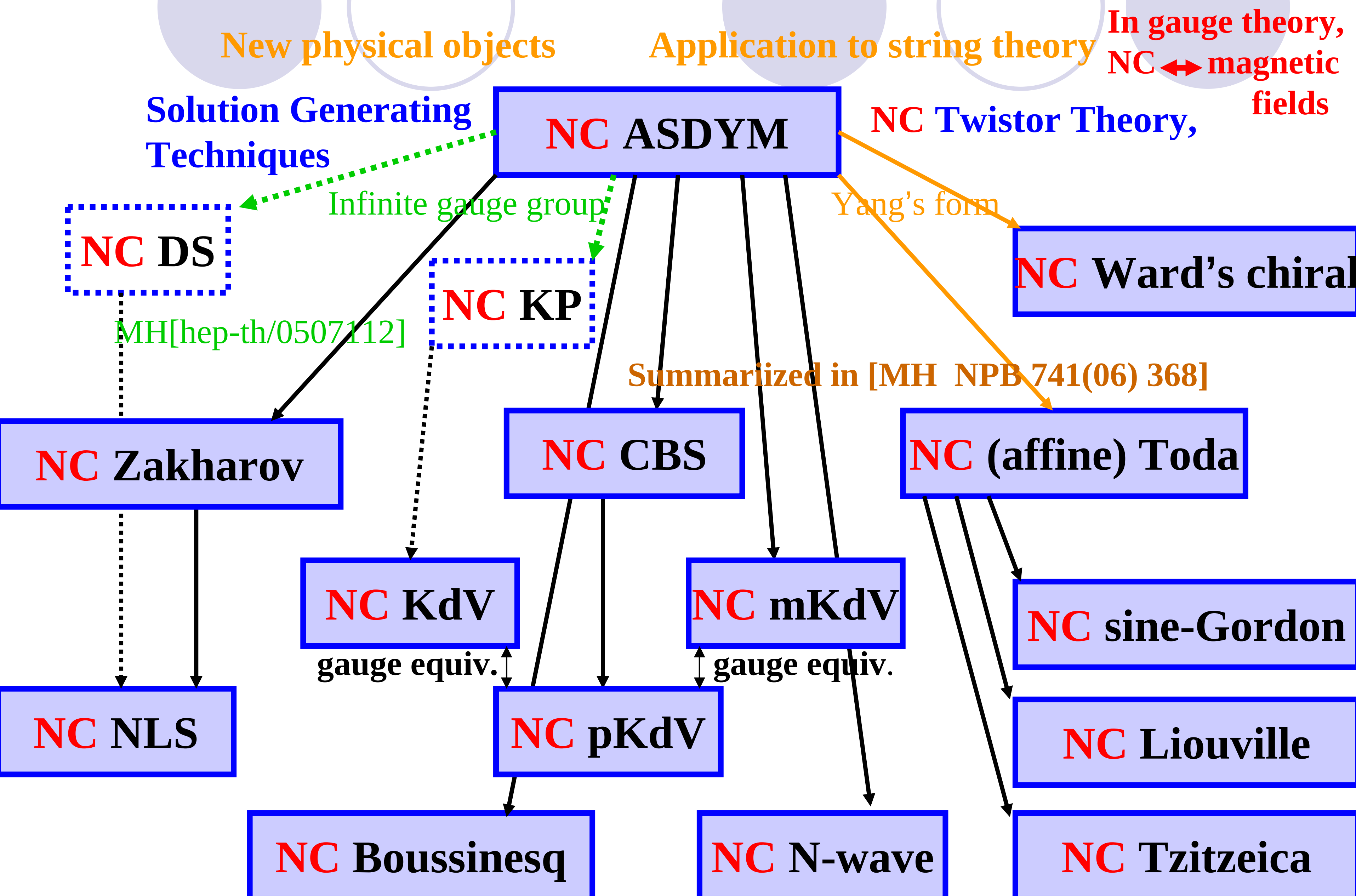
$$A_w = \begin{pmatrix} 0 & \psi \\ \overline{\psi} & 0 \end{pmatrix}, A_z = \sqrt{-1}\begin{pmatrix} -\psi\overline{\psi} & -\psi' \\ \psi' & \overline{\psi}\psi \end{pmatrix}$$

$$\sqrt{-1}\dot{\psi} = \psi'' - 2\psi\overline{\psi}\psi$$

:NC NLS eq.!

NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

[MH &K.Toda, PLA316(03)77]



Plan of this talk

1. Introduction
2. Backlund Transforms for the NC ASDYM eqs. (and NC Atiyah-Ward ansatz solutions in terms of quasideterminants)
3. Origin of the Backlund trfs from NC twistor theory
4. Conclusion and Discussion

2. Backlund transform for NC ASDYM eqs.

- In this section, we derive (NC) ASDYM eq. from the viewpoint of linear systems, which is suitable for discussion on integrability.
- We define NC Yang's equations which is equivalent to NC ASDYM eq. and give a Backlund transformation for it.
- The generated solutions are NC Atiyah-Ward ansatz solutions in terms of quasideterminants, which contain not only finite-action solutions (NC instantons) but also infinite-action solutions (non-linear plane waves and so on.)

A derivation of NC ASDYM equations

We discuss G=GL(N) NC ASDYM eq. from the viewpoint of NC linear systems with a (commutative) spectral parameter ζ .

- **Linear systems:**

$$\begin{cases} L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0, \\ M\psi = (D_z - \zeta D_{\tilde{w}})\psi = 0. \end{cases}$$

- **Compatibility condition of the linear system:**

$$[L, M] = [D_w, D_z] + \zeta([D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}]) + \zeta^2 [D_{\tilde{z}}, D_{\tilde{w}}] = 0$$

$$\Leftrightarrow \begin{cases} F_{zw} = [D_z, D_w] = 0, \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0 \end{cases}$$

:NC ASDYM eq.

Yang's form and NC Yang's equation

- **NC ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w] = 0, \quad \Rightarrow \quad \exists h,\ D_z h = 0,\ \ D_w h = 0 \quad (A_z = -h_z h^{-1}, etc.) \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \quad \Rightarrow \quad \exists \tilde{h},\ D_{\tilde{z}} \tilde{h} = 0,\ \ D_{\tilde{w}} \tilde{h} = 0 \quad (A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1}, etc.) \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0 \end{cases}$$

If we define Yang's J-matrix: $J := \tilde{h}^{-1} h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} \partial_{\tilde{z}} J) - \partial_w (J^{-1} \partial_{\tilde{w}} J) = 0$$

:NC Yang's eq.

$\downarrow$ **The solution** J **reproduce the gauge fields as**

$$A_z = -h_z h^{-1},\ \ A_w = h_w h^{-1},\ A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1},\ \ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} \tilde{h}^{-1}$$

J **is gauge invariant. The decomposition into** h **and** $\tilde{h}$ **corresponds to a gauge fixing**

Backlund trf. for NC Yang's eq. G=GL(2)

- **Yang's J matrix can be reparametrized as follows**

$$J=\begin{pmatrix} f-gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{pmatrix}$$

- **Then NC Yang's eq. becomes**

$$\partial_z(f^{-1}g_{\tilde{z}}b^{-1})-\partial_w(f^{-1}g_{\tilde{w}}b^{-1})=0,\quad \partial_{\tilde{z}}(b^{-1}e_z f^{-1})-\partial_{\tilde{w}}(b^{-1}e_w f^{-1})=0,$$

$$\partial_z(b_{\tilde{z}}b^{-1})-\partial_w(b_{\tilde{w}}b^{-1})+e_z f^{-1}g_{\tilde{z}}b^{-1}-e_w f^{-1}g_{\tilde{w}}b^{-1}=0,$$

$$\partial_z(f^{-1}f_{\tilde{z}})-\partial_w(f^{-1}f_{\tilde{w}})+f^{-1}g_{\tilde{z}}b^{-1}e_z-f^{-1}g_{\tilde{w}}b^{-1}e_w=0.$$

- **The following trf. leaves NC Yang's eq. as it is:**

$$\beta:\begin{cases} \partial_z e^{new}=f^{-1}g_{\tilde{w}}b^{-1},\ \partial_w e^{new}=f^{-1}g_{\tilde{z}}b^{-1}, \\ \partial_{\tilde{z}}g^{new}=b^{-1}e_w f^{-1},\ \partial_{\tilde{w}}g^{new}=b^{-1}e_z f^{-1}, \\ f^{new}=b^{-1},\ b^{new}=f^{-1} \end{cases}$$

Backlund transformation for NC Yang's eq.

- **Yang's J matrix can be reparametrized as follows**

$$J = \begin{pmatrix} f - gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{pmatrix}$$

- **Then NC Yang's eq. becomes**

$$\partial_z(f^{-1}g_{\tilde{z}}b^{-1}) - \partial_w(f^{-1}g_{\tilde{w}}b^{-1}) = 0, \quad \partial_{\tilde{z}}(b^{-1}e_z f^{-1}) - \partial_{\tilde{w}}(b^{-1}e_w f^{-1}) = 0,$$

$$\partial_z(b_{\tilde{z}}b^{-1}) - \partial_w(b_{\tilde{w}}b^{-1}) + e_z f^{-1}g_{\tilde{z}}b^{-1} - e_w f^{-1}g_{\tilde{w}}b^{-1} = 0,$$

$$\partial_z(f^{-1}f_{\tilde{z}}) - \partial_w(f^{-1}f_{\tilde{w}}) + f^{-1}g_{\tilde{z}}b^{-1}e_z - f^{-1}g_{\tilde{w}}b^{-1}e_w = 0.$$

- **Another trf. also leaves NC Yang's eq. as it is:**

$$\gamma_0 : \begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new} \end{pmatrix} = \begin{pmatrix} b & e \\ g & f \end{pmatrix}^{-1} = \begin{pmatrix} (b - ef^{-1}g)^{-1} & (g - fe^{-1}b)^{-1} \\ (e - bg^{-1}f)^{-1} & (f - gb^{-1}e)^{-1} \end{pmatrix}$$

- Both trfs. are involutive ($\beta \circ \beta = id, \gamma_0 \circ \gamma_0 = id$), but the combined trf. $\gamma_0 \circ \beta$ is non-trivial.)
- Then we could generate various (non-trivial) solutions of NC Yang's eq. from a (trivial) seed solution (so called, NC Atiyah-Ward solutions)

$$J_{[0]} \xrightarrow{\alpha = \gamma_0 \circ \beta} J_{[1]} \xrightarrow{\alpha} J_{[2]} \xrightarrow{\alpha} \cdots$$

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

$$\beta : \begin{cases} \partial_z e^{new} = f^{-1} g_{\tilde{w}} b^{-1}, \ \partial_w e^{new} = f^{-1} g_{\tilde{z}} b^{-1}, \\ \partial_{\tilde{z}} g^{new} = b^{-1} e_w f^{-1}, \ \partial_{\tilde{w}} g^{new} = b^{-1} e_z f^{-1}, \\ f^{new} = b^{-1}, \ b^{new} = f^{-1} \end{cases}$$

$$\gamma_0 : \begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new} \end{pmatrix} = \begin{pmatrix} b & e \\ g & f \end{pmatrix}^{-1}$$

Generated solutions (NC Atiyah-Ward sols.)

- **Let's consider the combined Backlund trf.**

$$J_{[0]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[1]} \xrightarrow{\alpha} J_{[2]} \xrightarrow{\alpha} \cdots$$

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

with a seed solution: $b_{[0]} = -f_{[0]} = e_{[0]} = -g_{[0]} = \Delta_0^{-1}, \ \partial^2 \Delta_0 = 0$

- **Then, the generated solutions are :**

$$b_{[n-1]} = \left|D_{[n]}\right|_{nn}^{-1}, \ f_{[n-1]} = -\left|D_{[n]}\right|_{11}^{-1}, e_{[n-1]} = \left|D_{[n]}\right|_{1n}^{-1}, g_{[n-1]} = -\left|D_{[n]}\right|_{n1}^{-1}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \ \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

Quasideterminants !
(a kind of NC determinants)

[Gelfand-Retakh]

Quasi-determinants

- Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.
- For an n by n matrix $X=(x_{ij})$ and the inverse $Y=(y_{ij})$ of X, quasi-determinant of X is directly defined by

$$|X|_{ij}=y_{ji}^{-1} \quad \left(\xrightarrow[\text{linmit}]{\text{commutative}} \underbrace{\frac{(-1)^{i+j}}{\det X^{ij}}}_{\text{some factor}} \det X \right)$$

X^{ij}: the matrix obtained from X deleting i-th row and j-th column

- Recall that

$$X=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \Rightarrow$$

$$Y=X^{-1}=\begin{pmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

→ We can also define quasi-determinants recursively

Quasi-determinants

[For a review, see Gelfand et al., math.QA/0208146]

- **Defined inductively as follows**

$$\left|X\right|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'}\left((X^{ij})^{-1}\right)_{i'j'} x_{j'j} = x_{ij} - \sum_{i',j'} x_{ii'}\left(\left|X^{ij}\right|_{j'i'}\right)^{-1} x_{j'j} = \begin{vmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{vmatrix}$$

n+1 by n+1 n by n

convenient notation

$$n = 1: \quad \left|X\right|_{ij} = x_{ij}$$

$$n = 2: \quad \left|X\right|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12} \cdot x_{22}^{-1} \cdot x_{21}, \quad \left|X\right|_{12} = \begin{vmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{vmatrix} = x_{12} - x_{11} \cdot x_{21}^{-1} \cdot x_{22},$$

$$\left|X\right|_{21} = \begin{vmatrix} x_{11} & x_{12} \\ \boxed{x_{21}} & x_{22} \end{vmatrix} = x_{21} - x_{22} \cdot x_{12}^{-1} \cdot x_{11}, \quad \left|X\right|_{22} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & \boxed{x_{22}} \end{vmatrix} = x_{22} - x_{21} \cdot x_{11}^{-1} \cdot x_{12},$$

$$n = 3: \quad \left|X\right|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11} - (x_{12}, x_{13}) \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}^{-1} \begin{pmatrix} x_{21} \\ x_{31} \end{pmatrix}$$

$$= x_{11} - x_{12} \cdot (x_{22} - x_{23} \cdot x_{33}^{-1} \cdot x_{32})^{-1} \cdot x_{21} - x_{13} \cdot (x_{32} - x_{33} \cdot x_{23}^{-1} \cdot x_{22})^{-1} \cdot x_{21}$$

$$- x_{12} \cdot (x_{23} - x_{22} \cdot x_{32}^{-1} \cdot x_{33})^{-1} \cdot x_{31} - x_{13} \cdot (x_{33} - x_{32} \cdot x_{22}^{-1} \cdot x_{23})^{-1} \cdot x_{31}$$

$$\cdots$$

Explicit Atiyah-Ward ansatz solutions of NC Yang's eq. G=GL(2)

[Gilson-MH-Nimmo. arXiv:0709.2069]

$$b_{[0]} = -f_{[0]} = e_{[0]} = -g_{[0]} = \Delta_0^{-1}, \quad \partial^2 \Delta_0 = 0$$

$$b_{[1]} = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \quad f_{[1]} = -\begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \boxed{\Delta_0} \end{vmatrix}^{-1}, \quad e_{[1]} = \begin{vmatrix} \Delta_0 & \boxed{\Delta_{-1}} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \quad g_{[1]} = -\begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \boxed{\Delta_1} & \Delta_0 \end{vmatrix}^{-1},$$

$$\partial_z \Delta_0 = \partial_{\tilde{w}} \Delta_1, \quad \partial_z \Delta_{-1} = \partial_{\tilde{w}} \Delta_0, \quad \partial_w \Delta_0 = \partial_{\tilde{z}} \Delta_1, \quad \partial_w \Delta_{-1} = \partial_{\tilde{z}} \Delta_0$$

$$b_{[n]} = \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}, \quad f_{[n]} = -\begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \boxed{\Delta_0} \end{vmatrix}^{-1},$$

$$e_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \boxed{\Delta_{-n}} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}, \quad g_{[n]} = -\begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_n} & \cdots & \Delta_0 \end{vmatrix}^{-1}$$

Yang's matrix J (gauge invariant) [Gilson-Gu, GHN]

$$J_{[n]} = \begin{vmatrix} \boxed{0} & -1 & 0 & \cdots & 0 & \boxed{0} \\ 1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-n} & \Delta_{-n} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-n} & \Delta_{1-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \boxed{0} & \Delta_n & \Delta_{n-1} & \cdots & \Delta_1 & \boxed{\Delta_0} \end{vmatrix}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

The Backlund trf. is not just a gauge trf. but a non-trivial one!

We could generate various solutions of NC ASDYM eq. from a simple seed solution Δ_0 by using the previous Backlund trf. $\alpha = \gamma_0 \circ \beta$

A seed solution:

$$\Delta_0 = 1 + ``\frac{1}{z\tilde{z} - w\tilde{w}}''$$

$$\Delta_0 = \exp(\, linear \;\; of \;\; z, \tilde{z}, w, \tilde{w})$$

→ **NC instantons**

→ **NC Non-Linear plane-waves**

Proof is made simply by using special identities of quasideterminants (NC Jacobi's identities and a homological relation, Gilson-Nimmo's derivative formula etc.),

in other words, ``NC Backlund trfs are identities of quasideterminants.''

(common feature in commutative Backlund in lower-dim.!)

3. Interpretation from NC twistor theory

- In this section, we give an origin of the Backlund trfs. from the viewpoint of NC twistor theory.
- NC twistor theory has been developed by several authors [Kapustin-Kuznetsov-Orlov, Takasaki, Hannabuss, Lechtenfeld-Popov, Brain-Majid…]
- What we need here is NC Penrose-Ward correspondence between sol. sp. of ASDYM and ``NC holomorphic vector bundle'' on a NC twistor space.

NC Penrose-Ward correspondence

- Linear systems of ASDYM ``NC hol. Vec. bdl''

$$\begin{cases} L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0, \\ M\psi = (D_z - \zeta D_{\tilde{w}})\psi = 0. \end{cases}$$

$$\tilde{\zeta} = 1/\zeta$$

$$\begin{cases} \tilde{L}\tilde{\psi} = (\tilde{\zeta} D_w - D_{\tilde{z}})\tilde{\psi} = 0, \\ \tilde{M}\tilde{\psi} = (\tilde{\zeta} D_z - D_{\tilde{w}})\tilde{\psi} = 0. \end{cases}$$

$\longleftrightarrow$ **1:1**

[Takasaki]

Patching matrix

$$P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta)$$
$$= \tilde{\psi}^{-1}\psi$$

$$\psi(x;\zeta) = h(x) + O(\zeta)$$
$$\tilde{\psi}(x;\tilde{\zeta}) = \tilde{h}(x) + O(\tilde{\zeta})$$

$$h(x) = \psi(x, \zeta = 0),$$
$$\tilde{h}(x) = \tilde{\psi}(x, \zeta = \infty)$$

$$\downarrow$$

ASDYM gauge fields are reproduced

We have only to factorize a given patching matrix into ψ and $\tilde{\psi}$ to get ASDYM fields.
(Birkhoff factorization or Riemann-Hilbert problem)

Origin of NC Atiyah-Ward (AW) ansatz sols.

- **n-th AW ansatz for the Patching matrix**

$$P_{[n]}=\begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta(x;\zeta) \end{pmatrix} \qquad \Delta(x;\zeta)=\underline{\Delta(\zeta w+\tilde{z},\zeta z+\tilde{w},\zeta)}=\underline{\sum_i \Delta_i(x)\zeta^i}$$

- **The chasing relation is derived from:**

$$l\Delta(x;\zeta)=(\partial_w-\zeta\partial_{\tilde{z}})\Delta=0,$$

$$m\Delta(x;\zeta)=(\partial_z-\zeta\partial_{\tilde{w}})\Delta=0,$$

$$\frac{\partial\Delta_i}{\partial z}=\frac{\partial\Delta_{i+1}}{\partial\tilde{w}},\quad \frac{\partial\Delta_i}{\partial w}=\frac{\partial\Delta_{i+1}}{\partial\tilde{z}}$$

OK!

Origin of NC Atiyah-Ward (AW) ansatz sols.

- **The n-th AW ansatz for the Patching matrix**

$$P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta(x;\zeta) \end{pmatrix} \qquad \Delta(x;\zeta) = \Delta(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \sum_i \Delta_i(x)\zeta^i$$

- **The Birkoff factorization** $P = \tilde{\psi}^{-1}\psi$ **leads to:**

$$\tilde{h}(x) + O(\tilde{\zeta}) \qquad h(x) + O(\zeta)$$

$$h_{11} = h_{12}\left|D_{[n+1]}\right|_{n+1,1}^{-1} - \tilde{h}_{11}\left|D_{[n+1]}\right|_{1,1}^{-1}$$

$$h_{21} = h_{22}\left|D_{[n+1]}\right|_{1,n+1}^{-1} - \tilde{h}_{21}\left|D_{[n+1]}\right|_{1,1}^{-1}$$

$$\tilde{h}_{12} = h_{12}\left|D_{[n+1]}\right|_{n+1,n+1}^{-1} - \tilde{h}_{11}\left|D_{[n+1]}\right|_{n+1,1}^{-1}$$

$$\tilde{h}_{22} = h_{22}\left|D_{[n+1]}\right|_{n+1,n+1}^{-1} - \tilde{h}_{21}\left|D_{[n+1]}\right|_{n+1,1}^{-1}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

- **Under a gauge (** $h_{12} = \tilde{h}_{21} = 0,\ h_{22} = \tilde{h}_{11} = 1$ **), this solution coincides with the quasideterminants sols!**

$$\tilde{h}_{22} = b_{[n]} = \left|D_{[n+1]}\right|_{n+1,n+1}^{-1},\quad h_{11} = f_{[n]} = -\left|D_{[n+1]}\right|_{11}^{-1},$$

$$h_{21} = e_{[n]} = \left|D_{[n+!]}\right|_{1,n+1}^{-1},\quad \tilde{h}_{12} = g_{[n]} = -\left|D_{[n+1]}\right|_{n+1,1}^{-1}$$

OK!

$$J_{[n]} = \tilde{h}^{-1}h = \begin{pmatrix} 1 & g_{[n]} \\ 0 & b_{[n]} \end{pmatrix}^{-1}\begin{pmatrix} f_{[n]} & 0 \\ e_{[n]} & 1 \end{pmatrix} = \begin{pmatrix} f_{[n]} - g_{[n]}b_{[n]}^{-1}e_{[n]} & -g_{[n]}b_{[n]}^{-1} \\ b_{[n]}^{-1}e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

Origin of the Backlund trfs

- **The Backlund trfs can be understood as the adjoint actions for the Patching matrix:**

$$\beta : P^{new} = B^{-1} P B, \quad \gamma_0 : P^{new} = C_0^{-1} P C_0, \quad B = \begin{pmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

actually:

$$\alpha = \gamma_0 \circ \beta : \quad P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta \end{pmatrix} \mapsto C_0^{-1} B^{-1} \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta \end{pmatrix} B C_0 = \begin{pmatrix} 0 & \zeta^{-(n+1)} \\ \zeta^{n+1} & \Delta \end{pmatrix} = P_{[n+1]}$$

- **The γ_0-trf. leads to** $J^{new} = C_0^{-1} J C_0 \Rightarrow$ **The previous γ_0-trf!**

- **The β-trf. is derived with a singular gauge trf.**

$$\beta : \psi \mapsto \psi^{new} = s \psi B, \qquad s = \begin{pmatrix} 0 & \zeta b^{-1} \\ -f^{-1} & 0 \end{pmatrix}$$

$$\downarrow \ \zeta \to 0$$

1-2 component of $L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0$

$$h^{new} = \begin{pmatrix} f^{new} & 0 \\ e^{new} & 1 \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ -f^{-1} k_{12} & 1 \end{pmatrix} \Rightarrow \quad \begin{aligned} & f^{new} = b^{-1} \\ & \partial_w e^{new} = -\partial_w (f^{-1} k_{12}) = f^{-1} g_{\tilde{z}} b^{-1} \end{aligned}$$

OK!

$$\psi = h + k\zeta + O(\zeta^2)$$

The previous β-trf!

4.Conclusion and Discussion

NC integrable eqs (ASDYM) in higher-dim.

- ADHM (OK)
- Twistor (OK)
- Backlund trf (OK), Symmetry (Next)

Quasi-determinants are important !

cf, NC binary Darboux trf.
[Saleem-Hassan-Siddiq]

Profound relation ?? (via Ward conjecture)

NC integrable eqs (KdV) in lower-dims.

- Hierarchy(OK)
- Infinite conserved quantities (OK)
- Exact N-soliton solutions (OK)
- Symmetry (NC Sato's theory) (Next)
- …

Quasideterminants might be a key…

Quasi-determinants are important !

[Etingof,Gelfand,Retakh, Gilson,Nimmo,Ohta,Li,Sooman,Tamizhmani,MacFarlane, Dimakis,Muller-Hoissen, MH,..]