非可換Anti-Self-Dual Yang-Mills方程式の数理と可積分系

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場の理論と弦理論@基研
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Based mainly on

- C.Gilson (Glasgow), MH, J.Nimmo (Glasgow), ``Backlund transforms and the Atiyah-Ward ansatz for non-commutative (NC) anti-self-dual (ASD) Yang-Mills (YM) eqs.'' Proc. Roy. Soc. A 465 (2009) 2631, arXiv:0812.1222 [hep-th].

「非可換空間上」の意味

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

($\theta^{\mu\nu}$:非可換パラメータ(ここでは実定数))

1. Introduction

非可換理論の発展 (特に2000年前後)

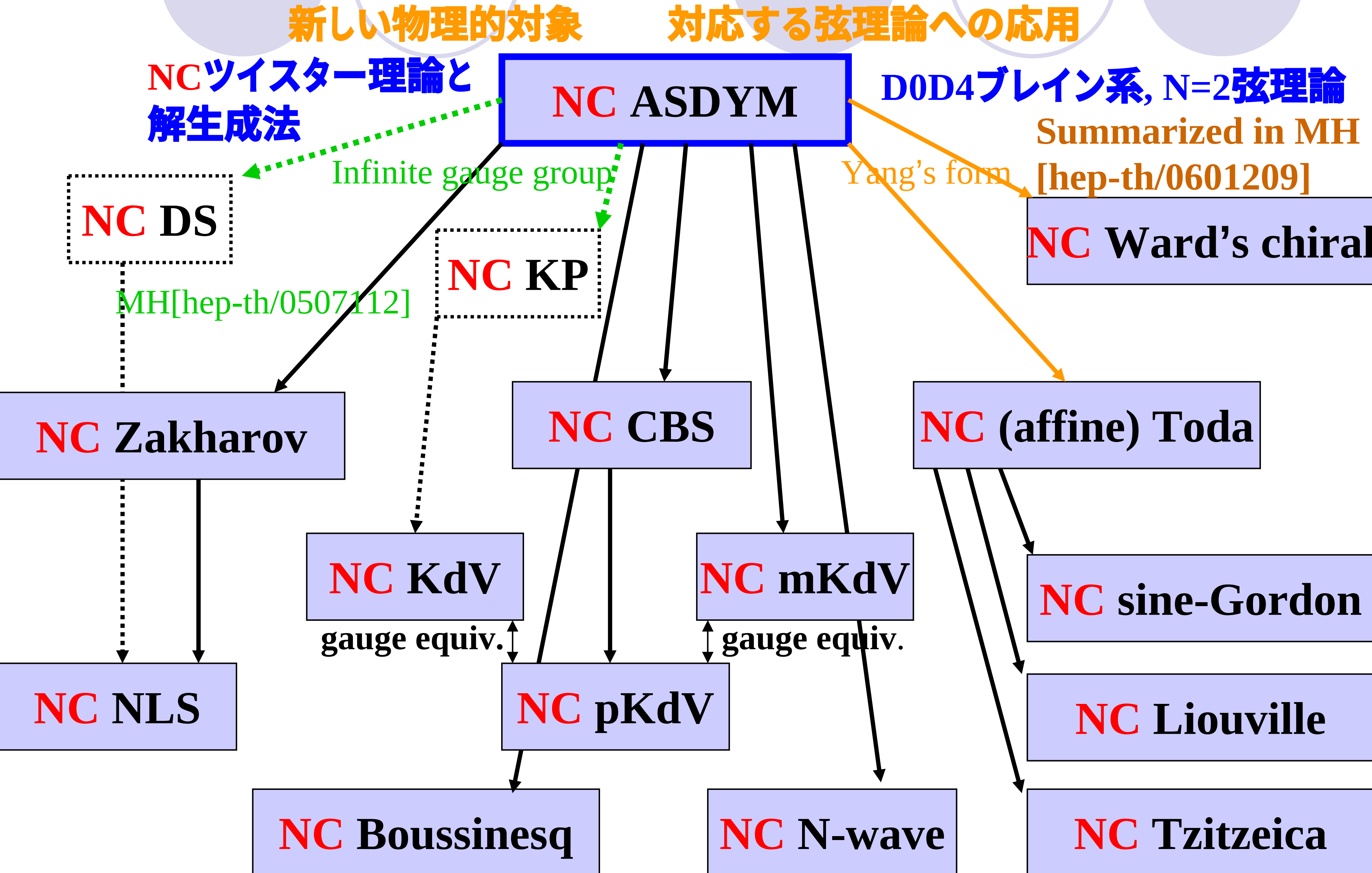
- 新しい物理的対象 (特異点の解消による)
- (ゲージ理論では)背景磁場中の物理を記述
- Dブレーン力学にさまざまな応用 (タキオン凝縮等)

厳密解の構成が重要な役割を果たす
(可積分性→厳密な取り扱い)

目標: すべての可積分系(KdV等)の非可換化
(実は多くの可積分系はゲージ理論に埋め込める！
：Ward予想 [R. Ward, 1985])

ゲージ理論では
NC化 ←→ 背景磁場

非可換Ward予想: (ほとんど)全ての非可換可積分方程式は非可換ASDYM方程式から得られる！ [MH&Toda,hep-th/0211148]



4-dim. ASDYM eq. (G=GL(N))

$$F_{\mu\nu} = -*F_{\mu\nu} \qquad \mu,\nu = 0,1,2,3$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])$$

$$\Updownarrow \qquad \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$$

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

Reduction to NC KdV from NC ASDYM

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

:NC ASDYM eq.
G=GL(2)

$$\downarrow$$

$$(z, \tilde{z}, w, \tilde{w}) \to (t, x) = (z, w + \tilde{w})$$

$$A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Reduction conditions

$$A_w = \begin{pmatrix} q & -1 \\ q' + q^2 & -q \end{pmatrix}, \quad A_z = \begin{pmatrix} \frac{1}{2}q'' + q'q & -q' \\ \frac{1}{4}q''' + \frac{1}{2}(q'^2 + qq'' + q''q) + qq'q & -\frac{1}{2}q'' - qq' \end{pmatrix}$$

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u'u + uu')$$

$$u = 2q'$$

:NC KdV eq.!

The NC KdV eq. has integrable-like properties:

- **possesses infinite conserved densities:**

$$\sigma_n = res_{-1}L^n + \frac{3}{4}\theta((res_{-1}L^n)\lozenge u'' - 2(res_{-2}L^n)\lozenge u')$$

MH, JMP46 (2005) [hep-th/0311206]

$res_r L^n$: coefficient of ∂_x^r in L^n

$\lozenge$: Strachan′s product (commutative and non-associative)

$$f(x)\lozenge g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

$$[t, x] = i\theta$$

- **has exact N-soliton solutions:**

Etingof-Gelfand-Retakh, MRL [q-alg/9701008]
MH, JHEP [hep-th/0610006]
cf. Paniak, [hep-th/0105185]

$$u = 2\partial_x \sum_{i=1}^{N}(\partial_x W_i) * W_i^{-1}$$

$$W_i := \left|W(f_1, \ldots, f_i)\right|_{i,i}$$:quasi-determinant of Wronski matrix

$$f_i = \exp\xi(x,\alpha_i) + a_i \exp(-\xi(x,\alpha_i)) \qquad \xi(x,\alpha) = x\alpha + t\alpha^3$$

Reduction to NC NLS from NC ASDYM

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

:NC ASDYM eq.
G=GL(2)

$$(z, \tilde{z}, w, \tilde{w}) \to (t, x) = (z, w + \tilde{w})$$

$$A_{\tilde{z}} = \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_w = \begin{pmatrix} 0 & \psi \\ \overline{\psi} & 0 \end{pmatrix}, \; A_z = \sqrt{-1}\begin{pmatrix} -\psi\overline{\psi} & -\psi' \\ \psi' & \overline{\psi}\psi \end{pmatrix}$$

Reduction conditions

$$\sqrt{-1}\dot{\psi} = \psi'' - 2\psi\overline{\psi}\psi$$

:NC NLS eq.!

今日の話：この親玉のNC ASDYM方程式の解構成法とツイスター解釈を議論する

- 2章：NC ASDYM eq. のベックルント変換（解を解に写す変換）を構成（自明解から非自明解を生成）
- 3章：ベックルント変換の起源をツイスターの枠組みから解釈（変換の一般化やすべての解を尽くすかどうかを議論したい）
- 4章：まとめと今後の展望（低次元への応用など）

2.Backlund transforms for NC ASDYM eqs.

- NC ASDYM eq. を NC Yang’s equations に書き換え, NC Yang’s equationsを不変に保つ変換(ベックルント変換)を与える
- 種となる自明解(Seed solution)から非自明な解を生成(Quasideterminantsで記述される)
- 生成された解は作用有限の解 (NC instantons) だけでなく作用無限大の解 (non-linear plane waves and so on.：新しい解)も含む
- Jの表式からこの変換がゲージ変換ではなく非自明な変換であることが分かる。(可換空間でも知られていない結果)

NC ASDYM and Yang's equation G=GL(2)

- **NC ASDYM eq. (実表示)**

$$F_{01} = -F_{23}, \quad F_{02} = -F_{31}, \quad F_{03} = -F_{12}.$$

$$(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_*)$$

$$\theta^{\mu\nu} = \begin{bmatrix} 0 & \theta^1 & & \\ -\theta^1 & 0 & & O \\ & O & 0 & \theta^2 \\ & & -\theta^2 & 0 \end{bmatrix}$$

(場同士の積は全てスター積.)

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x)$$

$$= f(x)g(x) + i\frac{\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2)$$

この積の置き換えで非可換空間上の理論が得られる

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

NC ASDYM and Yang's equation G=GL(2)

- **NC ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w]_* = 0, \quad \Rightarrow \quad \exists h, D_z * h = 0, \ D_w * h = 0 \quad (A_z = -h_z * h^{-1}, etc.) \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0, \quad \Rightarrow \quad \exists \tilde{h}, D_{\tilde{z}} * \tilde{h} = 0, \ D_{\tilde{w}} * \tilde{h} = 0 \quad (A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} * \tilde{h}^{-1}, etc.) \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_* = 0 \end{cases}$$

If we define Yang's matrix: $J := \tilde{h}^{-1} * h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} * \partial_{\tilde{z}} J) - \partial_w (J^{-1} * \partial_{\tilde{w}} J) = 0$$ **:Yang's eq.**

The solution J reproduce the gauge fields as

$$A_z = -h_z * h^{-1}, \ A_w = h_w * h^{-1}, A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} * \tilde{h}^{-1}, \ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} * \tilde{h}^{-1}$$

以後、スター積の記号＊は省略 (実は一般の非可換積でもOK)

Backlund trf. for NC Yang's eq. G=GL(2)

- **Yang's J matrix can be reparametrized as follows**

$$J=\begin{pmatrix} f-gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{pmatrix}$$

- **Then the following two kind of trfs. leave NC Yang's eq. as it is:**

$$\beta:\begin{cases} \partial_z e^{new}=f^{-1}g_{\tilde{w}}b^{-1},\ \partial_w e^{new}=f^{-1}g_{\tilde{z}}b^{-1}, \\ \partial_{\tilde{z}}g^{new}=b^{-1}e_w f^{-1},\ \partial_{\tilde{w}}g^{new}=b^{-1}e_z f^{-1}, \\ f^{new}=b^{-1},\ b^{new}=f^{-1} \end{cases}$$

$$\gamma_0:\begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new} \end{pmatrix}=\begin{pmatrix} (b-ef^{-1}g)^{-1} & (g-fe^{-1}b)^{-1} \\ (e-bg^{-1}f)^{-1} & (f-gb^{-1}e)^{-1} \end{pmatrix}$$

- Both trfs. are involutive ($\beta\circ\beta=id,\gamma_0\circ\gamma_0=id$), but the combined trf. $\gamma_0\circ\beta$ is non-trivial.)
- Then we could generate various (non-trivial) solutions of NC Yang's eq. from a (trivial) seed solution (so called, NC Atiyah-Ward solutions)

$$J_{[0]}\xrightarrow{\alpha=\gamma_0\circ\beta}J_{[1]}\xrightarrow{\alpha}J_{[2]}\xrightarrow{\alpha}\cdots$$

$$J_{[n]}=\begin{pmatrix} f_{[n]}-g_{[n]}b_{[n]}^{-1}e_{[n]} & -g_{[n]}b_{[n]}^{-1} \\ b_{[n]}^{-1}e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

$$\beta:\begin{cases}\partial_z e^{new}=f^{-1}g_{\tilde{w}}b^{-1},\ \partial_w e^{new}=f^{-1}g_{\tilde{z}}b^{-1},\\ \partial_{\tilde{z}}g^{new}=b^{-1}e_w f^{-1},\ \partial_{\tilde{w}}g^{new}=b^{-1}e_z f^{-1},\\ f^{new}=b^{-1},\ b^{new}=f^{-1}\end{cases}\qquad \gamma_0:\begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new}\end{pmatrix}=\begin{pmatrix} b & e \\ g & f\end{pmatrix}^{-1}$$

Generated solutions (NC Atiyah-Ward sols.)

- **Let's consider the combined Backlund trf.**

$$J_{[0]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[1]} \xrightarrow{\alpha} J_{[2]} \xrightarrow{\alpha} \cdots$$

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

with a seed solution: $b_{[0]} = f_{[0]} = e_{[0]} = g_{[0]} = \Delta_0^{-1}, \ \partial^2 \Delta_0 = 0$

- **Then, the generated solutions are :**

$$b_{[n-1]} = \left|D_{[n]}\right|_{nn}^{-1}, \ f_{[n-1]} = \left|D_{[n]}\right|_{11}^{-1}, e_{[n-1]} = \left|D_{[n]}\right|_{1n}^{-1}, g_{[n-1]} = \left|D_{[n]}\right|_{n1}^{-1}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

Quasideterminants !
(NC行列式の一種)

[Gelfand-Retakh]

Quasi-determinants

- Quasi-determinantsは、行列式の単なる非可換化ではなくむしろ逆行列に関連
- [直接的定義] n × n 行列 $X=(x_{ij})$ に対して その逆行列を $Y=(y_{ij})$ とおくと、X のquasideterminantは以下のように定義される：

$$|X|_{ij}=y_{ji}^{-1}\quad\left(\xrightarrow{\text{可換極限}}\frac{(-1)^{i+j}}{\det X^{ij}}\det X\right)$$

X^{ij} : Xの i 行目, j 列目を 除いた行列

可換極限で行列式に比例、あるいは行列式の比

- [間接的 (帰納的) 定義]

$$|X|_{ij}=x_{ij}-\sum_{i',j'}x_{ii'}((X^{ij})^{-1})_{i'j'}x_{j'j}=x_{ij}-\sum_{i',j'}x_{ii'}(|X^{ij}|_{j'i'})^{-1}x_{j'j}=\begin{vmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{vmatrix}$$

n+1 × n+1

n × n

convenient notation

Quasi-determinantsの具体的表式

$$n=1: \quad |X|_{ij} = x_{ij}$$

$$n=2: \quad |X|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12}\cdot x_{22}^{-1}\cdot x_{21}, \quad |X|_{12} = \begin{vmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{vmatrix} = x_{12} - x_{11}\cdot x_{21}^{-1}\cdot x_{22},$$

$$|X|_{21} = \begin{vmatrix} x_{11} & x_{12} \\ \boxed{x_{21}} & x_{22} \end{vmatrix} = x_{21} - x_{22}\cdot x_{12}^{-1}\cdot x_{11}, \quad |X|_{22} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & \boxed{x_{22}} \end{vmatrix} = x_{22} - x_{21}\cdot x_{11}^{-1}\cdot x_{12},$$

$$n=3: \quad |X|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11} - (x_{12}, x_{13}) \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}^{-1} \begin{pmatrix} x_{21} \\ x_{31} \end{pmatrix}$$

$$= x_{11} - x_{12}\cdot(x_{22} - x_{23}\cdot x_{33}^{-1}\cdot x_{32})^{-1}\cdot x_{21} - x_{13}\cdot(x_{32} - x_{33}\cdot x_{23}^{-1}\cdot x_{22})^{-1}\cdot x_{21}$$

$$- x_{12}\cdot(x_{23} - x_{22}\cdot x_{32}^{-1}\cdot x_{33})^{-1}\cdot x_{31} - x_{13}\cdot(x_{33} - x_{32}\cdot x_{22}^{-1}\cdot x_{23})^{-1}\cdot x_{31}$$

$$\cdots$$

Cf.

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow Y = X^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

Explicit Atiyah-Ward ansatz solutions of NC Yang's eq. G=GL(2)

[Gilson-MH-Nimmo. arXiv:0709.2069]

$$b_{[0]} = f_{[0]} = e_{[0]} = g_{[0]} = \Delta_0^{-1}, \ \partial^2 \Delta_0 = 0$$

$$b_{[1]} = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \ f_{[1]} = -\begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \boxed{\Delta_0} \end{vmatrix}^{-1}, \ e_{[1]} = \begin{vmatrix} \Delta_0 & \boxed{\Delta_{-1}} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \ g_{[1]} = -\begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \boxed{\Delta_1} & \Delta_0 \end{vmatrix}^{-1},$$

$$\partial_z \Delta_0 = \partial_{\tilde{w}} \Delta_1, \ \partial_z \Delta_{-1} = \partial_{\tilde{w}} \Delta_0, \ \partial_w \Delta_0 = \partial_{\tilde{z}} \Delta_1, \ \partial_w \Delta_{-1} = \partial_{\tilde{z}} \Delta_0$$

$$b_{[n]} = \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}, \ f_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \boxed{\Delta_0} \end{vmatrix}^{-1},$$

$$e_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \boxed{\Delta_{-n}} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}, \ g_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_n} & \cdots & \Delta_0 \end{vmatrix}^{-1}$$

Yang's matrix J (gauge invariant) [Gilson-Gu, GHN]

$$J_{[n]} = \begin{vmatrix} \boxed{0} & -1 & 0 & \cdots & 0 & \boxed{0} \\ 1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-n} & \Delta_{-n} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-n} & \Delta_{1-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \boxed{0} & \Delta_n & \Delta_{n-1} & \cdots & \Delta_1 & \boxed{\Delta_0} \end{vmatrix}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \ \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

The Backlund trf. is not just a gauge trf. but a non-trivial one!

We could generate various solutions of NC ASDYM eq. from a simple seed solution Δ_0 by using the previous Backlund trf. $\alpha = \gamma_0 \circ \beta$

A seed solution:

$$\Delta_0 = 1 + ``\frac{1}{z\tilde{z} - w\tilde{w}}'' \quad \rightarrow \text{NC instantons}$$

$$\Delta_0 = \exp(\mathit{linear\ \ of}\ \ z, \tilde{z}, w, \tilde{w}) \quad \rightarrow \text{NC Non-Linear plane-waves}$$

Proof is made simply by using special identities of quasideterminants (NC Jacobi's identities and a homological relation, etc.),

in other words, ``NC Backlund trfs are identities of quasideterminants.''

(低次元の可換可積分系と同様のストーリー&本質を捉えている？)

結果のまとめ (ASDYM側) :やや複雑

- **NC ASDYM eq. (Yang's form)**

$$\partial_z(J^{-1}\partial_{\tilde z}J)-\partial_w(J^{-1}\partial_{\tilde w}J)=0 \qquad J=\begin{pmatrix} f-gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1}\end{pmatrix}$$

- **ベックルント変換**

$$\beta:\begin{cases}\partial_z e^{new}=f^{-1}g_{\tilde w}b^{-1},\ \partial_w e^{new}=f^{-1}g_{\tilde z}b^{-1},\\ \partial_{\tilde z}g^{new}=b^{-1}e_w f^{-1},\ \partial_{\tilde w}g^{new}=b^{-1}e_z f^{-1},\\ f^{new}=b^{-1},\ b^{new}=f^{-1}\end{cases} \qquad \gamma_0:\begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new}\end{pmatrix}=\begin{pmatrix} b & e \\ g & f\end{pmatrix}^{-1}$$

- **生成された解 (Quasi-determinant)**

$$J_{[0]}\xrightarrow{\alpha=\gamma_0\circ\beta}J_{[1]}\xrightarrow{\alpha}J_{[2]}\xrightarrow{\alpha}\cdots \qquad D_{[n]}=\begin{pmatrix}\Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0\end{pmatrix}$$

$$b_{[0]}=f_{[0]}=e_{[0]}=g_{[0]}=\Delta_0^{-1},$$
$$\partial^2\Delta_0=0$$

$$b_{[n-1]}=\left|D_{[n]}\right|_{nn}^{-1},\ f_{[n-1]}=\left|D_{[n]}\right|_{11}^{-1},$$
$$e_{[n-1]}=\left|D_{[n]}\right|_{1n}^{-1},\ g_{[n-1]}=\left|D_{[n]}\right|_{n1}^{-1}$$
$$\frac{\partial\Delta_i}{\partial z}=\frac{\partial\Delta_{i+1}}{\partial\tilde w},\quad \frac{\partial\Delta_i}{\partial w}=\frac{\partial\Delta_{i+1}}{\partial\tilde z}$$

3. Interpretation from NC twistor theory

- ここでは, ベックルント変換の起源をNCツイスターの枠組みから明らかにする
- NCツイスター理論はすでに基礎付けが与えられている: [Kapustin-Kuznetsov-Orlov, Takasaki, Hannabuss, Lechtenfeld-Popov, Brain-Majid…]
- ここで必要とするのは、NC Penrose-Ward 対応:4次元時空上の NC ASDYMの解 と ツイスター空間上の「NC 正則ベクトル束」 との一対一対応
- 無限次元解空間 (モジュライ空間) の解明 → QFTにおける新しい非摂動論的効果???

ツイスター理論の舞台

- 5つの複素数に対し次の2方向の射影を考える：

$$(z, w, \tilde{z}, \tilde{w}, \zeta)$$

∂_ζ

$$l = \partial_w - \zeta\partial_{\tilde{z}}, \quad m = \partial_z - \zeta\partial_{\tilde{w}}$$

時空の座標 $(z, w, \tilde{z}, \tilde{w})$

$\bar{z} = \tilde{z}, \bar{w} = \pm\tilde{w}$ といったreality条件を課すと実4次元の空間となる

(λ, μ, ζ) ツイスター座標

$$= (\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta)$$

この上の場の方程式の解 $\longleftrightarrow$ ツイスター座標に依存するある種の（幾何学的）対象

1:1

難　　易

NC Penrose-Ward 対応

- NC ASDYMの(線形系の)解　　``NC正則ベクトル束''

1:1

Patching matrix

$$\begin{cases} L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0, \\ M\psi = (D_z - \zeta D_{\tilde{w}})\psi = 0. \end{cases}$$

$$\tilde{\zeta} = 1/\zeta$$

$$\begin{cases} \tilde{L}\tilde{\psi} = (\tilde{\zeta} D_w - D_{\tilde{z}})\tilde{\psi} = 0, \\ \tilde{M}\tilde{\psi} = (\tilde{\zeta} D_z - D_{\tilde{w}})\tilde{\psi} = 0. \end{cases}$$

1:1

[Takasaki]

$$P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{\psi}^{-1}\psi$$

$$\psi(x;\zeta) = h(x) + O(\zeta)$$

$$\tilde{\psi}(x;\tilde{\zeta}) = \tilde{h}(x) + O(\tilde{\zeta})$$

両立条件：$[L, M] = 0 \; or \; [\tilde{L}, \tilde{M}] = 0$

$$h(x) = \psi(x, \zeta = 0),$$

$$\tilde{h}(x) = \tilde{\psi}(x, \zeta = \infty)$$

NC ASDYM eq. (Yang's form)

$$\partial_z (J^{-1} \partial_{\tilde{z}} J) - \partial_w (J^{-1} \partial_{\tilde{w}} J) = 0$$

$$J := \tilde{h}^{-1} h = \begin{pmatrix} f - gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{pmatrix}$$

NC Atiyah-Ward (AW) ansatz 解

- **The n-th AW ansatz for the Patching matrix**

$$P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta(x;\zeta) \end{pmatrix} \qquad \Delta(x;\zeta) = \Delta(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \sum_i \Delta_i(x)\zeta^i$$

- **The Birkoff factorization** $P = \tilde{\psi}^{-1}\psi$ **leads to:**

$$\tilde{h}(x) + O(\tilde{\zeta}) \qquad h(x) + O(\zeta)$$

$$h_{11} = h_{12}\left|D_{[n+1]}\right|^{-1}_{1,n+1} - \tilde{h}_{11}\left|D_{[n+1]}\right|^{-1}_{1,1}$$

$$h_{21} = h_{22}\left|D_{[n+1]}\right|^{-1}_{1,n+1} - \tilde{h}_{21}\left|D_{[n+1]}\right|^{-1}_{1,1}$$

$$\tilde{h}_{12} = h_{12}\left|D_{[n+1]}\right|^{-1}_{n+1,n+1} - \tilde{h}_{11}\left|D_{[n+1]}\right|^{-1}_{n+1,1}$$

$$\tilde{h}_{22} = h_{22}\left|D_{[n+1]}\right|^{-1}_{n+1,n+1} - \tilde{h}_{21}\left|D_{[n+1]}\right|^{-1}_{n+1,1}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

- **Under a gauge (** $h_{12} = \tilde{h}_{21} = 0,\ h_{22} = \tilde{h}_{11} = 1$ **), we get a simple quasideterminants sols!**

$$\tilde{h}_{22} = b_{[n]} = \left|D_{[n+1]}\right|^{-1}_{n+1,n+1},\quad h_{11} = f_{[n]} = -\left|D_{[n+1]}\right|^{-1}_{11},$$

$$h_{21} = e_{[n]} = \left|D_{[n+!]}\right|^{-1}_{1,n+1},\quad \tilde{h}_{12} = g_{[n]} = -\left|D_{[n+1]}\right|^{-1}_{n+1,1}$$

$$J_{[n]} = \tilde{h}^{-1}h = \begin{pmatrix} 1 & g_{[n]} \\ 0 & b_{[n]} \end{pmatrix}^{-1} \begin{pmatrix} f_{[n]} & 0 \\ e_{[n]} & 1 \end{pmatrix} = \begin{pmatrix} f_{[n]} - g_{[n]}b_{[n]}^{-1}e_{[n]} & -g_{[n]}b_{[n]}^{-1} \\ b_{[n]}^{-1}e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

NC Atiyah-Ward (AW) ansatz 解

- **n-th AW ansatz for the Patching matrix**

$$P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta(x;\zeta) \end{pmatrix} \qquad \Delta(x;\zeta) = \underline{\Delta(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta)} = \underline{\sum_i \Delta_i(x)\zeta^i}$$

- **The recursion relation is derived from:**

$$l\Delta(x;\zeta) = (\partial_w - \zeta\partial_{\tilde{z}})\Delta = 0,$$

$$m\Delta(x;\zeta) = (\partial_z - \zeta\partial_{\tilde{w}})\Delta = 0,$$

$$\downarrow$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

Δ_0 が決まれば、他のΔ_iはこれで逐次定まる

$$f_{[1]} = b_{[1]} = e_{[1]} = g_{[1]} = \Delta_0^{-1}, \ \partial^2 \Delta_0 = 0$$

ベックルント変換 $J_{[0]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[1]} \xrightarrow{\alpha} J_{[2]} \xrightarrow{\alpha} \cdots$

- **The Backlund trfs can be understood as the adjoint actions for the Patching matrix:**

$$\beta : P^{new} = B^{-1} P B, \quad \gamma_0 : P^{new} = C_0^{-1} P C_0, \quad B = \begin{pmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

actually:

$$\alpha = \gamma_0 \circ \beta : \quad P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta \end{pmatrix} \mapsto C_0^{-1} B^{-1} \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta \end{pmatrix} B C_0 = \begin{pmatrix} 0 & \zeta^{-(n+1)} \\ \zeta^{n+1} & \Delta \end{pmatrix} = P_{[n+1]}$$

- **The γ_0-trf. leads to** $J^{new} = C_0^{-1} J C_0 \Rightarrow \quad \gamma_0 : \begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new} \end{pmatrix} = \begin{pmatrix} b & e \\ g & f \end{pmatrix}^{-1}$
- **The β-trf. is derived with a singular gauge trf.**

$$\beta : \psi \mapsto \psi^{new} = s \psi B, \qquad s = \begin{pmatrix} 0 & \zeta b^{-1} \\ -f^{-1} & 0 \end{pmatrix}$$

$$\downarrow \ \zeta \to 0$$

1-2 component of $L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0$

$$h^{new} = \begin{pmatrix} f^{new} & 0 \\ e^{new} & 1 \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ -f^{-1}k_{12} & 1 \end{pmatrix} \Rightarrow \qquad \begin{aligned} & f^{new} = b^{-1} \\ & \partial_w e^{new} = -\partial_w (f^{-1} k_{12}) = f^{-1} g_{\tilde{z}} b^{-1} \end{aligned}$$

$$\psi = h + k\zeta + O(\zeta^2)$$

結果のまとめ (ツイスター側) :シンプル

- 正則ベクトル束

$$P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta(x;\zeta) \end{pmatrix} \qquad \Delta(x;\zeta) = \Delta(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \sum_i \Delta_i(x)\zeta^i$$

- ベックルント変換

$$\beta : P^{new} = B^{-1} P B, \quad \gamma_0 : P^{new} = C_0^{-1} P C_0, \quad B = \begin{pmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- 生成された解 (Quasi-determinant)

$$b_{[n-1]} = \left| D_{[n]} \right|_{nn}^{-1}, \; f_{[n-1]} = \left| D_{[n]} \right|_{11}^{-1},$$

$$e_{[n-1]} = \left| D_{[n]} \right|_{1n}^{-1}, \; g_{[n-1]} = \left| D_{[n]} \right|_{n1}^{-1}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

$$J = \begin{pmatrix} f - g b^{-1} e & -g b^{-1} \\ b^{-1} e & b^{-1} \end{pmatrix}$$

4. Conclusion and Discussion

NC ASDYMのBacklund変換

- 線形方程式の解(seed sol.)から非自明な解を生成
- 新しい解(作用無限大)、新しい表式も含まれる
- Quasi-determinantsが自然に現れ、証明が簡略化
- 起源はツイスターからすべて説明可能(見通しがよい。一般化も容易)

今後の方向・課題

解の振る舞い(可換と比べてどう変形されるのか)

解の弦理論的 (Dブレーン) 解釈(NL Plane waveは何?)

解空間の構造,対称性(可積分系の分類,QFTへの応用)

低次元へのリダクション (高次元BHへの応用もあり)

可積分系の非可換空間への拡張プログラムの現状

高次元可積分系(ASDYM)の非可換空間への拡張

- ASDYMの厳密解 (OK)
- ツイスター (OK)　　Quasi-determinantsが重要な役割！
- ベックルント変換群、解空間の構造解明 (次)

深遠な関係？？(via Ward予想)

低次元可積分系(KdV等)の非可換空間への拡張

- 階層　(OK)
- 無限個の保存量　(OK)
- Nソリトン解　(OK)　　Quasi-determinantsが重要な役割！
- 解空間の構造解明（タウ関数？、佐藤理論）　(次)
- ハミルトン形式（SWマップで解決？）…　(次)