Noncommutative Solitons and Quasideterminants

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Goal

- **Extension of all soliton theories and integrable systems to non-commutative (NC) spaces, including the KdV eq. etc.**

$$[x^\mu, x^\nu] = i\underline{\theta^{\mu\nu}}$$

NC parameter (real const.)

1. Introduction

Successful points in NC theories

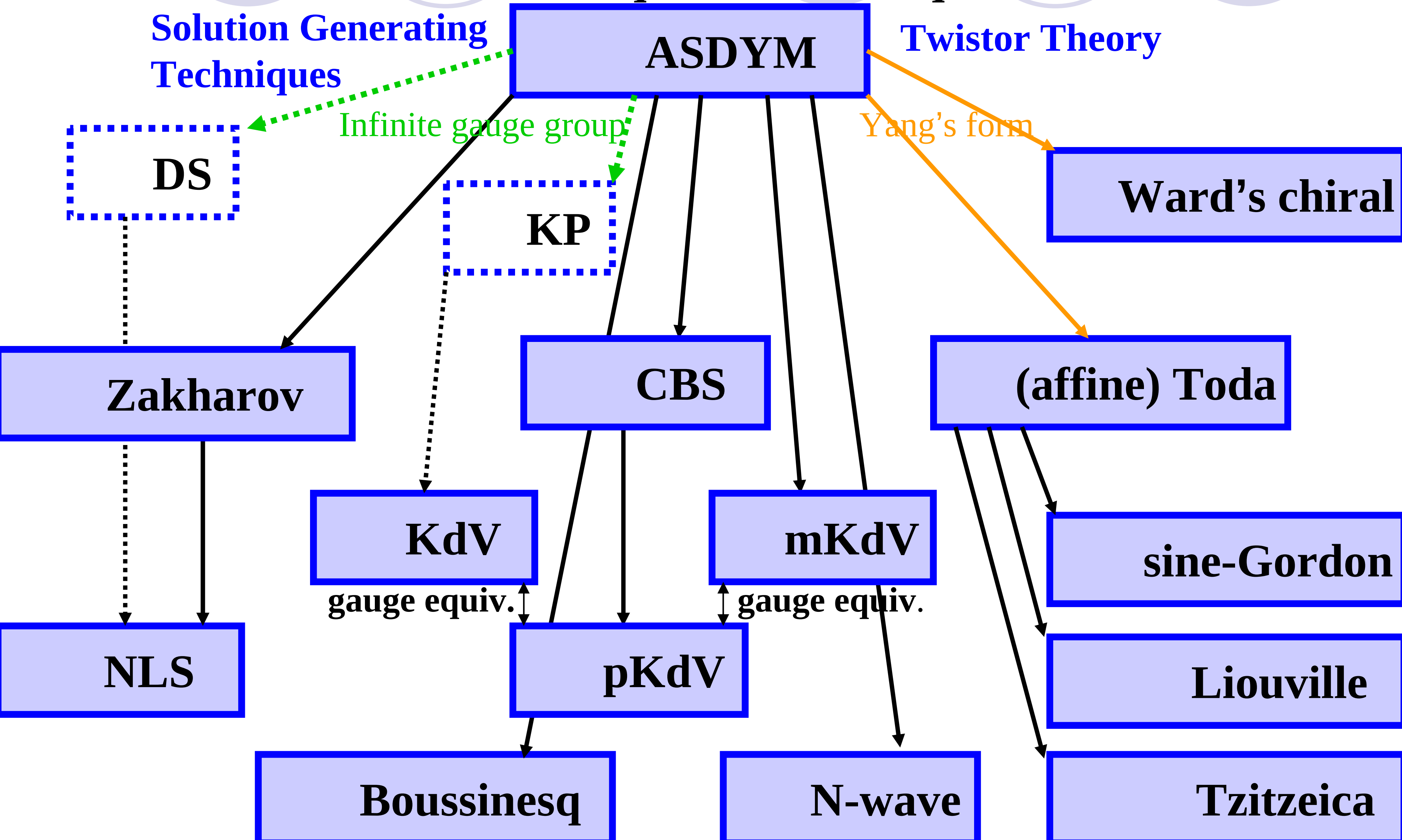
- Appearance of new physical objects
- Description of real physics (in gauge theory)
- Various successful applications
  to D-brane dynamics etc.

Construction of exact solitons are important.
(partially due to their integrablity)

Final goal: NC extension of all soliton theories
(Soliton eqs. can be embedded in gauge theories via Ward's conjecture ! [R. Ward, 1985])

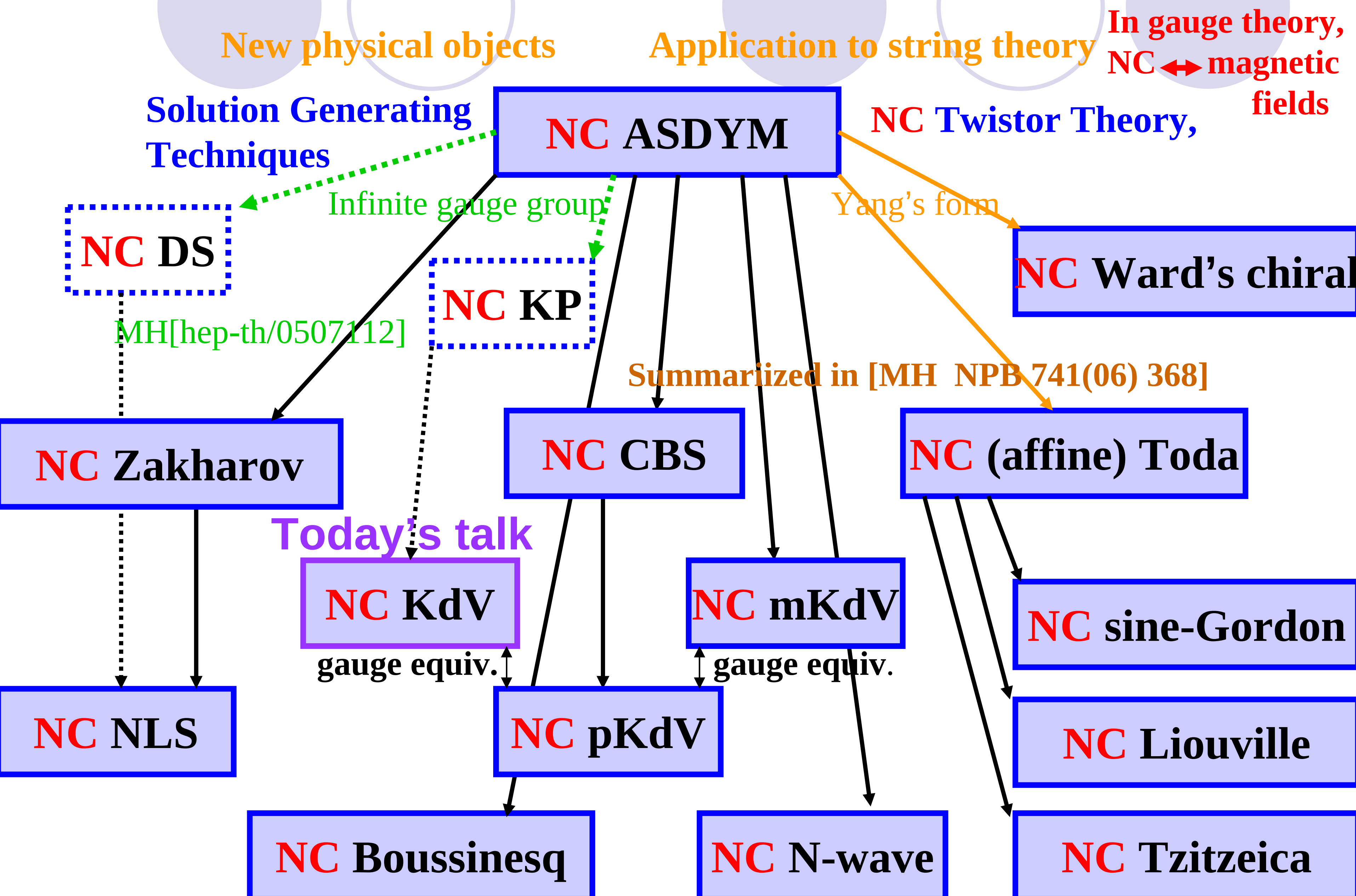
Ward's conjecture: Many (perhaps all?) integrable equations are reductions of the ASDYM eqs.

ASDYM eq. is a master eq. !



NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.

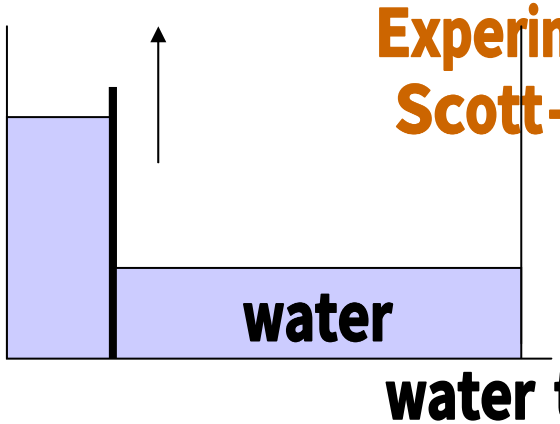
MH &K.Toda, PLA316 (03)77 [hepth/0211148]

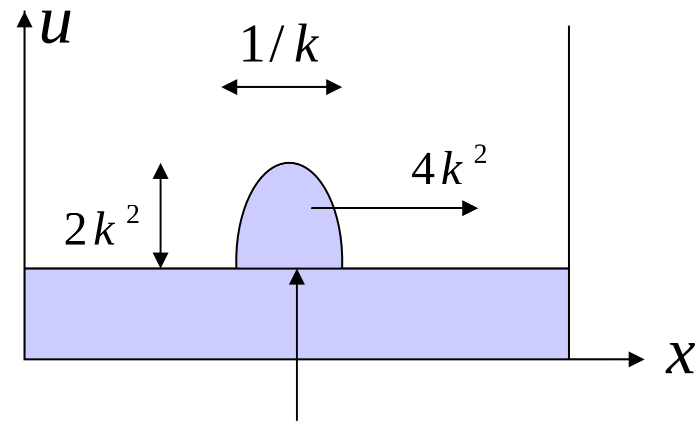


2. Review of Soliton Theories

- KdV equation : describe shallow water waves

Experiment by Scott-Russel, 1834





solitary wave = soliton

This configuration satisfies

$$u = 2k^2 \cosh^{-2}(kx - 4k^3 t)$$

$$\dot{u} + u''' + 6u'u = 0$$: **KdV eq.** [Korteweg-de Vries, 1895]

This is a typical integrable equation.

Let's solve it now !

- Hirota's method [PRL27(1971)1192]

$$\dot{u} + u''' + 6u'u = 0$$

: naively hard to solve

$$u = 2\partial_x^2 \log \tau$$

$$\tau\dot{\tau}' - \tau'\dot{\tau} + 3\tau''\tau'' - 4\tau'\tau''' + \tau\tau'''' = 0$$

Hirota's bilinear relation : more complicated ?

A solution: $\tau = 1 + e^{2(kx - \omega t)}, \quad \omega = 4k^3$

$$\rightarrow \quad u = 2k^2 \cosh^{-2}(kx - 4k^3 t)$$

: The solitary wave !
(1-soliton solution)

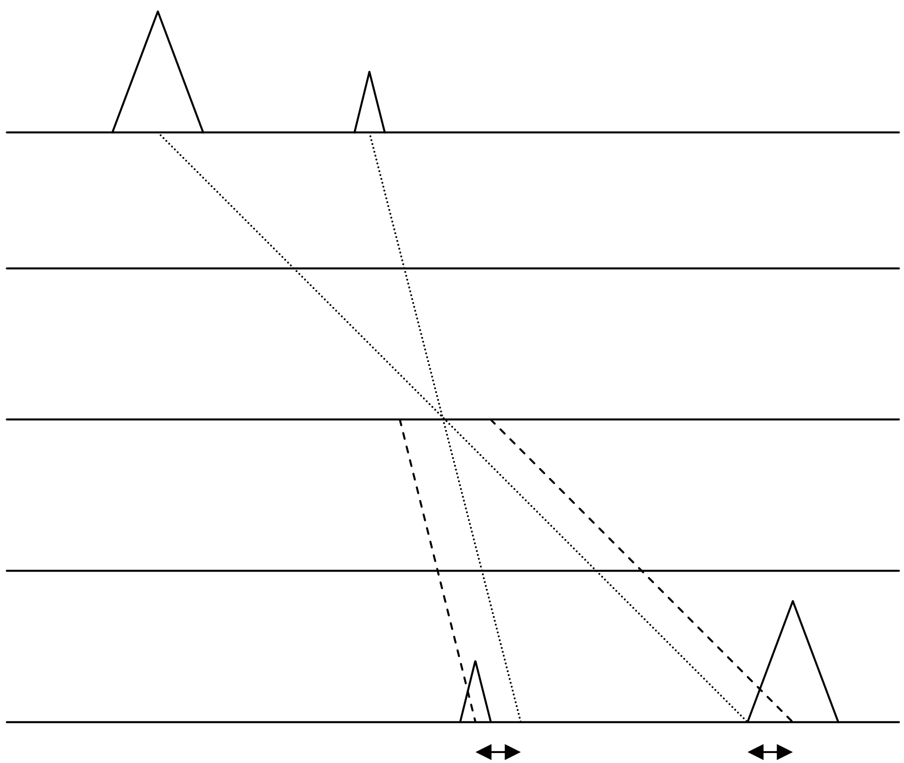
2-soliton solution

$$\tau = 1 + A_1 e^{2\theta_1} + A_2 e^{2\theta_2} + B A_1 A_2 e^{2(\theta_1+\theta_2)}$$

= A determinant of Wronski matrix (general property of soliton sols.)
``tau-functions''

$$\theta_i = k_i x - 4k_i^3 t, \quad B = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

Scattering process



The shape and velocity is preserved ! (stable)

The positions are shifted ! (Phase shift)

3. Darboux transforms for NC KdV eq.

- In this section, we give an exact soliton solutions of NC KdV eq. by a Darboux transformation. [Gilson-Nimmo, JPA40(07) 3839, nlin.si/0701027]
- We see that ingredients of quasi-determinants are naturally generated by iteratation of the Darboux transformation. (an origin of quasi-determinants)
- We also make a comment on asymptotic behavior of soliton scattering process.

Review of Quasi-determinants

[For a review, see Gelfand et al., math.QA/0208146]

- Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.
- [Def1] For an n by n matrix $X=(x_{ij})$ and the inverse $Y=(y_{ij})$ of X, quasi-determinant of X is directly defined by

$$\left|X\right|_{ij} = y_{ji}^{-1}$$

X^{ij} : the matrix obtained from X deleting i-th row and j-th column

- [Def2] (Iterative definition)

$$\left|X\right|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'}((X^{ij})^{-1})_{i'j'}x_{j'j} = x_{ij} - \sum_{i',j'} x_{ii'}(\left|X^{ij}\right|_{j'i'})^{-1}x_{j'j}$$

n+1 × n+1

n × n

A comment on Def 2

Formula for inverse matrix:

$$X = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \Rightarrow$$

n+1 × n+1

$$Y = X^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(d - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(d - CA^{-1}B)^{-1} \\ -(d - CA^{-1}B)^{-1}CA^{-1} & (d - \underline{CA^{-1}}B)^{-1} \end{pmatrix}$$

n × n

A convenient notation:

$$|X|_{ij} = \begin{vmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{vmatrix}$$

Examples of quasi-determinants

$$n=1:\quad |X|_{ij}=x_{ij}$$

$$n=2:\quad |X|_{11}=\begin{vmatrix}\boxed{x_{11}} & x_{12}\\ x_{21} & x_{22}\end{vmatrix}=x_{11}-x_{12}\cdot x_{22}^{-1}\cdot x_{21},\quad |X|_{12}=\begin{vmatrix}x_{11} & \boxed{x_{12}}\\ x_{21} & x_{22}\end{vmatrix}=x_{12}-x_{11}\cdot x_{21}^{-1}\cdot x_{22},$$

$$|X|_{21}=\begin{vmatrix}x_{11} & x_{12}\\ \boxed{x_{21}} & x_{22}\end{vmatrix}=x_{21}-x_{22}\cdot x_{12}^{-1}\cdot x_{11},\quad |X|_{22}=\begin{vmatrix}x_{11} & x_{12}\\ x_{21} & \boxed{x_{22}}\end{vmatrix}=x_{22}-x_{21}\cdot x_{11}^{-1}\cdot x_{12},$$

$$n=3:\quad |X|_{11}=\begin{vmatrix}\boxed{x_{11}} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{vmatrix}=x_{11}-(x_{12},x_{13})\begin{pmatrix}x_{22} & x_{23}\\ x_{32} & x_{33}\end{pmatrix}^{-1}\begin{pmatrix}x_{21}\\ x_{31}\end{pmatrix}$$

$$=x_{11}-x_{12}\cdot(x_{22}-x_{23}\cdot x_{33}^{-1}\cdot x_{32})^{-1}\cdot x_{21}-x_{13}\cdot(x_{32}-x_{33}\cdot x_{23}^{-1}\cdot x_{22})^{-1}\cdot x_{21}$$
$$-x_{12}\cdot(x_{23}-x_{22}\cdot x_{32}^{-1}\cdot x_{33})^{-1}\cdot x_{31}-x_{13}\cdot(x_{33}-x_{32}\cdot x_{22}^{-1}\cdot x_{23})^{-1}\cdot x_{31}$$

$$\cdots$$

Note:

$$X=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\ \Rightarrow\ Y=X^{-1}=\begin{pmatrix}A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1}\\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1}\end{pmatrix}$$

$$=\begin{pmatrix}(A-BD^{-1}C)^{-1} & (C-DB^{-1}A)^{-1}\\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1}\end{pmatrix}$$

Some identities of quasideterminants

- **Homological relation**

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix}$$

- **NC Jacobi's identitiy**

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}$$

Lax formalism of commutative KdV eq.

- **Linear systems:**

$$L\psi = (\partial_x^2 + u - \lambda^2)\psi = 0,$$

$$M\psi = (\partial_t - \partial_x^3 - (3/2)u\partial_x - (3/4)u_x)\psi = 0.$$

- **Compatibility condition of the linear system:**

$$[L, M] = 0 \Leftrightarrow \dot{u} = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x$$

:KdV equation

Lax pair of KdV

(Q) How we get NC version of the theories?

(A) We have only to replace all products of fields in ordinary commutative gauge theories with star-products: $f(x)g(x) \to f(x) * g(x)$

- **The star product : (NC and associative)**

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) g(x) = f(x)g(x) + i \frac{\theta^{\mu\nu}}{2} \partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)$$

A deformed product

Presence of background magnetic fields

Note: coordinates and fields themselves are usual c-number functions. But commutator of coordinates becomes…

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

NC !

Lax pair of NC KdV eq.

- **Linear systems:**

$$L*\psi=(\partial_x^2+u-\lambda^2)*\psi=0,$$

$$M*\psi=(\partial_t-\partial_x^3-(3/2)u\partial_x-(3/4)u_x)*\psi=0.$$

- **Compatibility condition of the linear system:**

$$[L,M]_*=0 \quad \Leftrightarrow \quad \dot{u}=\frac{1}{4}u_{xxx}+\frac{3}{4}(u*u_x+u_x*u)$$

:NC KdV equation $\quad [t,x]=i\theta$

- **Darboux transform for NC KdV** **[Gilson-Nimmo]**

Let us take an eigen function W **of L and define** $\Phi=W*\partial_x W^{-1}$
Then the following trf. leaves the linear systems as it is

$$\tilde{L}=\Phi*L*\Phi^{-1},\quad \tilde{M}=\Phi*M*\Phi^{-1},\quad \tilde{\psi}=\Phi*\psi$$

and $\quad \tilde{u}=u+2(W_x*W^{-1})_x$

The Darboux transformation can be iterated

- **Let us take eigen fcns.** $(f_1, \cdots, f_N)$ **of** L **and define**

$$\Phi_i = W_i * \partial_x W_i^{-1} = \partial_x - W_{i,x} * W_i^{-1} \qquad (W_1 \equiv f_1, \Phi_1 = f_1 * \partial_x f_1)$$

$$W_{i+1} = \Phi_i * f_{i+1} = f_{i+1,x} - W_{i,x} * W_i^{-1} * f_{i+1} \qquad (i = 1,2,3,\cdots)$$

$$\psi_{i+1} = \Phi_i * \psi_i = \psi_{i,x} - W_{i,x} * W_i^{-1} * \psi_i$$

- **Iterated Darboux transform for NC KdV**

The following trf. leaves the linear systems as it is

$$L_{[i+1]} = \Phi_i * L_{[i]} * \Phi_i^{-1}, \quad M_{[i+1]} = \Phi_i * M_{[i]} * \Phi_i^{-1}, \quad \psi_{[i+1]} = \Phi_i * \psi_{[i]}$$

$$(L_{[1]}, M_{[1]}, \psi_{[1]}) \xrightarrow{\Phi_1} (L_{[2]}, M_{[2]}, \psi_{[2]}) \xrightarrow{\Phi_2} \cdots$$

$$|||$$

$$(L, M, \psi)$$

In fact, (W_i, ψ_i) **are quasi-determinants of Wronski matrices !**

and

$$u_{[N+1]} = u + 2\sum_{i=1}^{N} (W_{i,x} * W_i^{-1})_x \quad (\xrightarrow{\theta \to 0} u + 2\partial_x^2 \log W(f_1, \cdots, f_N))$$

The Darboux transformation can be iterated

- **Let us take eigen fcns.** $(f_1,\cdots,f_N)$ **of** L **and define**

$$\Phi_i = W_i * \partial_x W_i^{-1} = \partial_x - W_{i,x} * W_i^{-1} \qquad (W_1 \equiv f_1,\ \Phi_1 = f_1 * \partial_x f_1)$$

$$W_{i+1} = \Phi_i * f_{i+1} = f_{i+1,x} - W_{i,x} * W_i^{-1} * f_{i+1} \qquad (i = 1,2,3,\cdots)$$

$$\psi_{i+1} = \Phi_i * \psi_i = \psi_{i,x} - W_{i,x} * W_i^{-1} * \psi_i$$

- **Examples**

In fact, (W_i, ψ_i) **are quasi-determinants of Wronski matrices !**

$$W_1 \equiv f_1$$

$$W_2 = f_{2,x} - W_{1,x} * W_1^{-1} * f_2 = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}_{2,2} \qquad \textbf{Q-det !}$$

$$W_3 = f_{3,x} - W_{2,x} * W_2^{-1} * f_3 = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}_{3,3} \qquad \textbf{Q-det !}$$

$$u_{[N+1]} = u + 2\sum_{i=1}^{N} (W_{i,x} * W_i^{-1})_x \qquad \left(\xrightarrow{\theta \to 0} u + 2\partial_x^2 \log W(f_1,\cdots,f_N) \right)$$

Exact N-soliton solutions of the NC KdV eq.

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1}$$

$$\left(\xrightarrow{\theta \to 0} \quad \partial_x^2 \log\det W(f_1, \cdots, f_N) \right)$$

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

$$f_i = \exp\left(\xi(x, \lambda_i)\right) + a_i \exp\left(-\xi(x, \lambda_i)\right)$$

$$\xi(x, t, \lambda) = x_1 \lambda + t \lambda_i^3$$

$$\left(L * f_i = (\partial_x^2 - \lambda^2) f_i = 0, M * f_i = (\partial_t - \partial_x^3) f_i = 0 \right)$$

Wronski matrix:

$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

Etingof-Gelfand-Retakh, [q-alg/9701008]

Scattering process of the N-soliton solutions

- **We have found exact N-soliton solutions for the NC KdV eq.**
- **Physical interpretations are non-trivial because when $f(x), g(x)$ are real, $f(x)*g(x)$ is not in general.**
- **However, the solutions could be real in some cases.**
  - (i) **<u>1-soliton solutions</u> are all the same as commutative ones because of**

$$f(x-vt)*g(x-vt) = f(x-vt)g(x-vt)$$

  - (ii) **<u>In asymptotic region</u>, configurations of multi-soliton solutions could be real in soliton scatterings and the same as commutative ones.**

MH JHEP [hep-th/0610006]

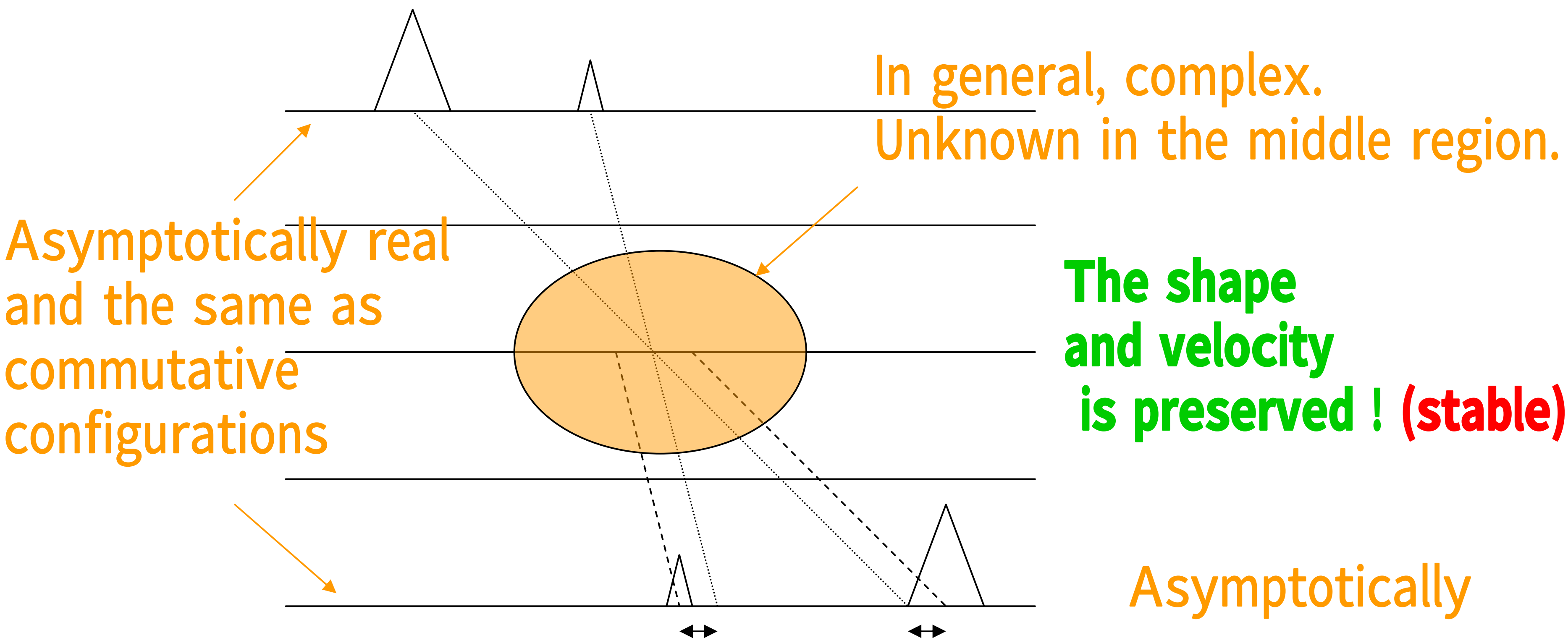
MH, JHEP02 (2007) 094 [hep-th/0610006]
cf Paniak, hep-th/0105185

2-soliton solution of NC KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\ 2}\cosh^{-2}(k_i x - 4k_i^{\ 3} t), \quad \underbrace{v_i = 4k_i^2}_{\text{velocity}}, \quad \underbrace{h_i = 2k_i^2}_{\text{height}}$$

Scattering process (NC case)



4. Toward NC Sato's Theory

- Sato's Theory : one of the most beautiful theory of solitons
  - Based on the exsitence of

hierarchies and tau-functions

A set of infinite soliton equations (in terms of u)	$\xrightarrow{\quad u = 2\partial_x^2 \log \tau \quad}$	A set of infinite bilinear equations (in terms of τ)

Infinite evolution eqs. whose flows are all commuting

$\downarrow$

Infinite conserved quantities

Plucker embedding maps which define an infinite-dim. Grassmann manifold. (=the solution space)

$\downarrow$

Infinite dimensional symmetry

Derivation of soliton equations

- Prepare a Lax operator which is a pseudo-differential operator

$$L := \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots$$

$$u_k = u_k(\underbrace{x^1, x^2, x^3, \cdots})$$

Noncommutativity is introduced here:

$$[x^i, x^j] = i\theta^{ij}$$

- Introduce a differential operator

$$B_m := (\underbrace{L * \cdots * L}_{m \text{ times}})_{\geq 0}$$

- Define NC (KP) hierarchy:

$$\boxed{\frac{\partial L}{\partial x^m} = [B_m, L]_*}$$

$$\partial_m u_2 \partial_x^{-1} + \partial_m u_3 \partial_x^{-2} + \partial_m u_4 \partial_x^{-3} + \cdots$$

$$f_{m2}(u) \partial_x^{-1} + f_{m3}(u) \partial_x^{-2} + f_{m4}(u) \partial_x^{-3} + \cdots$$

Here all products are star product:

Each coefficient yields a differential equation.

Negative powers of differential operators

$$\partial_x^n \circ f := \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \partial_x^{n-j}$$

$$\binom{n}{j} \equiv \frac{n(n-1)(n-2)\cdots(n-(j-1))}{j(j-1)(j-2)\cdots 1}$$

: **binomial coefficient which can be extended to negative n**
→ negative power of differential operator (well-defined !)

$$\partial_x^3 \circ f = f\partial_x^3 + 3f'\partial_x^2 + 3f''\partial_x^1 + f'''$$

$$\partial_x^2 \circ f = f\partial_x^2 + 2f'\partial_x + f''$$

$$\partial_x^{-1} \circ f = f\partial_x^{-1} - f'\partial_x^{-2} + f''\partial_x^{-3} - \cdots$$

$$\partial_x^{-2} \circ f = f\partial_x^{-2} - 2f'\partial_x^{-3} + 3f''\partial_x^{-4} - \cdots$$

Closer look at NC KP hierarchy

For m=2

$\partial_x^{-1})\quad \partial_2 u_2 = \underline{2u_3'} + u_2''$

$\partial_x^{-2})\quad \partial_2 u_3 = \underline{2u_4'} + u_3'' + 2u_2 * u_2' + 2[u_2, u_3]_*$

$\partial_x^{-3})\quad \partial_2 u_4 = \underline{2u_5'} + u_4'' + 4u_3 * u_2' - 2u_2 * u_2'' + 2[u_2, u_4]_*$

$\vdots$

Infinite kind of fields are represented in terms of one kind of field $u_2 \equiv u$

MH&K.Toda, [hep-th/0309265]

$u_x := \dfrac{\partial u}{\partial x}$

$\partial_x^{-1} := \int^x dx'$

For m=3

$\partial_x^{-1})\quad \partial_3 u_2 = u_2''' + 3u_3'' + 3u_4'' + 3u_2' * u_2 + 3u_2 * u_2'$

etc.

$\vdots$

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x * u + u * u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_{yy}]_*$$

(2+1)-dim. NC KP equation

and other NC equations (NC KP hierarchy equations)

$u = u(x^1, x^2, x^3, \cdots)$

$x \quad y \quad t$

reductions

(KP hierarchy) → (various hierarchies.)

- (Ex.) KdV hierarchy

Reduction condition

$$L^2 = B_2 (=: \partial_x^2 + u)$$: 2-reduction

gives rise to NC KdV hierarchy

which includes (1+1)-dim. NC KdV eq.:

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{4} (u_x * u + u * u_x)$$

Note $\dfrac{\partial u}{\partial x_{2N}} = 0$: dimensional reduction in x_{2N} directions

$$\begin{array}{lll} \textbf{KP :} & u(x^1, x^2, x^3, x^4, x^5, \ldots) & \textbf{: (2+1)-dim.} \\ & \quad x \quad y \quad t & \\ \downarrow & & \downarrow \\ \textbf{KdV :} & u(x^1, \quad x^3, \quad x^5, \ldots) & \textbf{: (1+1)-dim.} \\ & \quad x \qquad t & \end{array}$$

l-reduction of NC KP hierarchy yields wide class of other NC (GD) hierarchies

- No-reduction → NC KP $\quad (x,y,t)=(x^1,x^2,x^3)$
- 2-reduction → NC KdV $\quad (x,t)=(x^1,x^3)$
- 3-reduction → NC Boussinesq $\quad (x,t)=(x^1,x^2)$
- 4-reduction → NC Coupled KdV $\quad \ldots$
- 5-reduction → …
- 3-reduction of BKP → NC Sawada-Kotera
- 2-reduction of mKP → NC mKdV
- Special 1-reduction of mKP → NC Burgers
- … **Noncommutativity should be introduced into space-time coords**

5. Conservation Laws

- Conservation laws: $\partial_t \sigma = \partial_i J^i$ $\quad \sigma$: Conserved density

time space

Then $Q := \int_{space} dx\, \sigma$ is a conserved quantity.

$$\because \partial_t Q = \int_{space} dx \partial_t \sigma = \int_{\substack{spatial \\ \inf inity}} dS_i J^i = 0$$

Conservation laws for the hierarchies

$$\partial_m res_{-1} L^n = \partial_x J + \underline{\theta^{ij} \partial_j \Xi_i}$$

time space

I have succeeded in the evaluation explicitly !

$res_{-r} L^n$: coefficient of ∂_x^{-r} in L^n

$t \equiv x^m$

Noncommutativity should be introduced in space-time directions only. →

∂_j **should be space or time derivative**
→ ordinary conservation laws !

Infinite conserved densities for the NC soliton eqs. (n=1,2,… , ∞)

$$\sigma_n = res_{-1}L^n + \theta^{im}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\binom{k}{l}(\partial_x^{k-l} res_{-(l+1)}L^n)\lozenge(\partial_i res_k L^m)$$

$t \equiv x^m$ $\qquad$ $res_r L^n$: coefficient of ∂_x^r in L^n

$\lozenge$: Strachan's product (commutative and non-associative)

$$f(x)\lozenge g(x) := f(x)\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\left(\frac{1}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)^{2s}\right)g(x)$$

MH, JMP46 (2005) [hep-th/0311206]

This suggests infinite-dimensional symmetries would be hidden.

We can calculate the explicit forms of conserved densities for the wide class of NC soliton equations. (existence of negative power of derivatives is crucial!)

- Space-Space noncommutativity:

  NC deformation is slight: $\sigma_n = res_{-1} L^n$

  involutive (integrable in Liouville's sense)

- Space-time noncommutativity

  NC deformation is drastical:

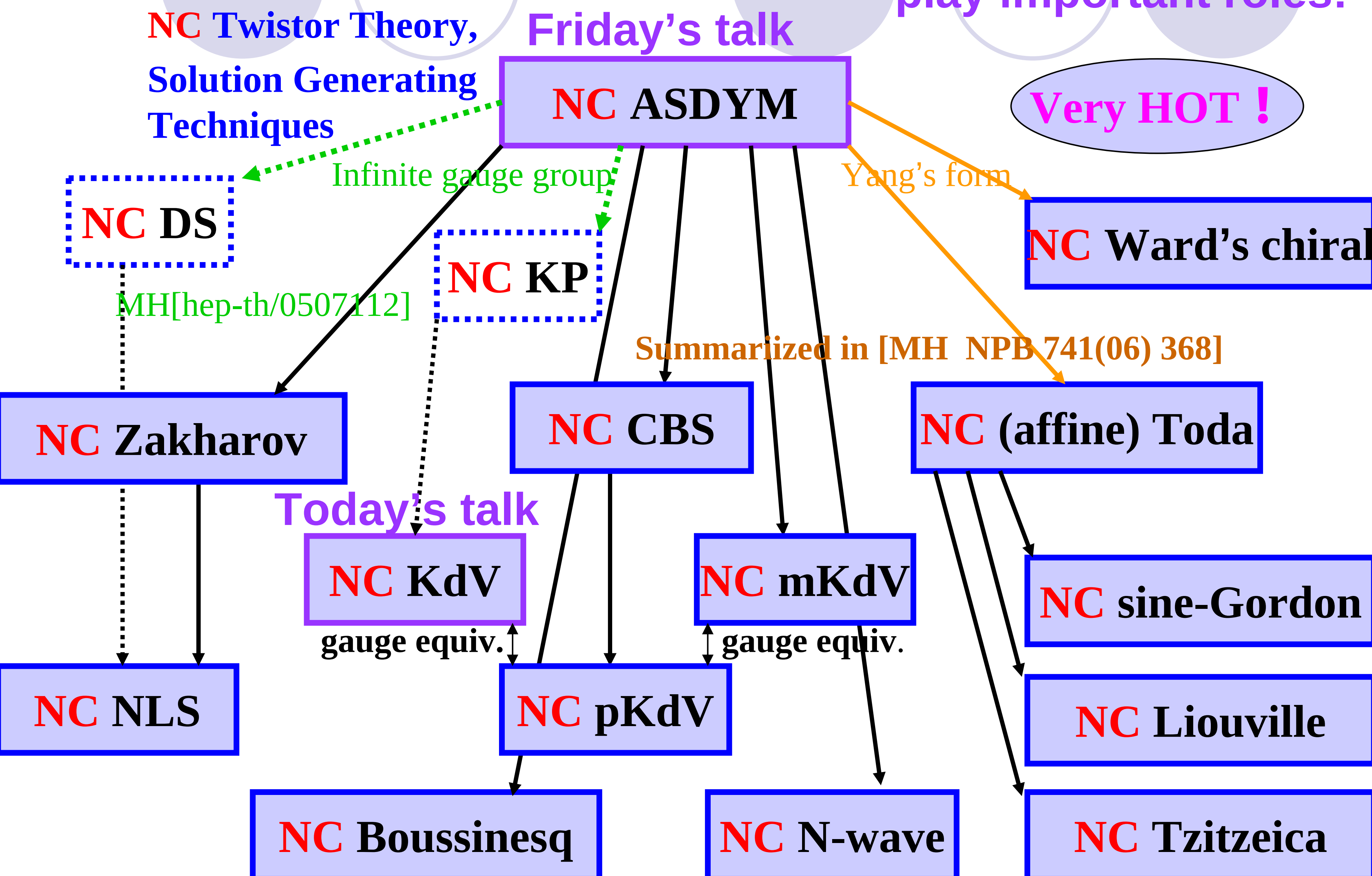
  - Example: NC KP and KdV equations $([t, x] = i\theta)$

$$\sigma_n = res_{-1} L^n - 3\theta((res_{-1} L^n) \Diamond u_3' + (res_{-2} L^n) \Diamond u_2')$$

meaningful ?

6. Conclusion and Discussion

In every situation,
Quasideterminants
play important roles!



NC Ward's conjecture (NC KdV eq.)

- Reduced ASDYM eq.: $x^\mu \to (t,x)$

MH, PLB625, 324 [hep-th/0507112]

$$(i)\quad B' = 0$$

$$(ii)\quad C' + \dot{A} + [A,C]_* = 0$$

$$(iii)\quad A' - \dot{B} + [C,B]_* = 0$$

A, B, C: 2 times 2 matrices (gauge fields)

Further Reduction:

$$A = \begin{pmatrix} q & -1 \\ q' + q^2 & -q \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2}q'' + q' * q & -q' \\ f(q,q',q'',q''') & -\frac{1}{2}q'' - q * q' \end{pmatrix}$$

NOT Traceless !

$$(ii) \Rightarrow \begin{pmatrix} \oplus & 0 \\ \otimes & -\oplus \end{pmatrix} = 0 \quad \Rightarrow \dot{q} = \frac{1}{4}q''' + \frac{3}{4}q' * q'$$ **: NC pKdV eq. !!!**

$u = q'$ → **NC KdV**

Note: $A, B, C \in gl(2) \xrightarrow{\theta \to 0} sl(2)$ **U(1) part is necessary !**