Integrable Aspects of Noncommutative Anti-Self-Dual Yang-Mills Equations

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Based on

- **C.Gilson (Glasgow), MH and J.Nimmo (Glasgow), ``Backlund trfs and the Atiyah-Ward ansatz for NC anti-self-dual (ASD) Yang-Mills (YM) eqs.'' Proc.Roy.Soc.A465 (2009) 2613 [arXiv:0812.1222],**
- **MH**, **NPB 741 (2006) 368 [hep-th/0601206]**
- **MH**, **``Noncommutative Solitons and Quasideterminants (review)'' [arXiv:1001.1001]…**

1. Introduction

NC extension of integrable systems

- **Matrix generalization**
- **Quarternion-valued system**
- **Moyal deformation (=extension to NC spaces=presence of magnetic flux)**

4-dim. Anti-Self-Dual Yang-Mills Eq.

- **plays important roles in QFT**
- **a master eq. of (lower-dim) integrable eqs. (Ward's conjecture)**

NC ASDYM eq. with G=GL(N)

- NC ASDYM eq. (real rep.)

$$F_{01}{}^{*} = -F_{23}{}^{*},$$
$$F_{02}{}^{*} = -F_{31}{}^{*},$$
$$F_{03}{}^{*} = -F_{12}{}^{*}$$

$$(F_{\mu\nu}{}^{*} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_*)$$

$$\theta^{\mu\nu} = \begin{bmatrix} 0 & \theta^1 & & \\ -\theta^1 & 0 & & O \\ & O & 0 & \theta^2 \\ & & -\theta^2 & 0 \end{bmatrix}$$

(Spell: All products are Moyal products.)

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x)$$
$$= f(x)g(x) + i\frac{\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2)$$

Under the spell, we get a theory on NC spaces:

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

NC ASDYM eq. with G=GL(N)

- NC ASDYM eq. (real rep.)

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$$\theta^{\mu\nu} = \begin{bmatrix} 0 & \theta^1 & & \\ -\theta^1 & 0 & & O \\ & O & 0 & \theta^2 \\ & & -\theta^2 & 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}$$

$$[x^\mu, x^\nu]_* = i\theta^{\mu\nu}$$
$$\updownarrow$$
$$[z, \tilde{z}]_* = 2\theta^1, [w, \tilde{w}]_* = 2\theta^2.$$

$$\begin{cases} F_{zw}{}^{*} = 0, \\ F_{\tilde{z}\tilde{w}}{}^{*} = 0, \\ F_{z\tilde{z}}{}^{*} - F_{w\tilde{w}}{}^{*} = 0 \end{cases}$$

(complex rep.)

(Spell: All products are Moyal products.)

NC ASDYM eq. with G=GL(N)

- NC ASDYM eq. (real rep.)

$$F_{01}{}^{*} = -F_{23}{}^{*}, \quad F_{02}{}^{*} = -F_{31}{}^{*}, \quad F_{03}{}^{*} = -F_{12}{}^{*}.$$

$$(F_{\mu\nu}{}^{*} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_*)$$

$$\theta^{\mu\nu} = \begin{bmatrix} 0 & \theta^1 & & \\ -\theta^1 & 0 & & O \\ & O & 0 & \theta^2 \\ & & -\theta^2 & 0 \end{bmatrix}$$

$$[x^\mu, x^\nu]_* = i\theta^{\mu\nu}$$

$$\Updownarrow \qquad \begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix} \qquad \updownarrow$$

$$[z, \tilde{z}]_* = 2\theta^1, [w, \tilde{w}]_* = 2\theta^2.$$

$$\begin{cases} F_{zw}{}^{*} = 0, \\ F_{\tilde{z}\tilde{w}}{}^{*} = 0, \\ F_{z\tilde{z}}{}^{*} - F_{w\tilde{w}}{}^{*} = 0 \end{cases}$$

**From now on,
we would omit the symbol '*'
(Products can be (general)
NC and associative ones.)**

Reduction to NC KdV from NC ASDYM

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

:NC ASDYM eq.
G=GL(2)

$$(z, \tilde{z}, w, \tilde{w}) \to (t, x) = (z, w + \tilde{w})$$

Reduction conditions

$$A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_w = \begin{pmatrix} q & -1 \\ q' + q^2 & -q \end{pmatrix}, \quad A_z = \begin{pmatrix} \frac{1}{2}q'' + q'q & -q' \\ \frac{1}{4}q''' + \frac{1}{2}(q'^2 + qq'' + q''q) + qq'q & -\frac{1}{2}q'' - qq' \end{pmatrix}$$

NOT traceless → U(1) part is crucial!

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u'u + uu') \qquad u = 2q'$$

:NC KdV eq.!

Reduction to NC NLS from NC ASDYM

$$\begin{cases} F_{zw} = 0, \\ F_{\tilde{z}\tilde{w}} = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0 \end{cases}$$

:NC ASDYM eq.
G=GL(2)

$$(z, \tilde{z}, w, \tilde{w}) \to (t, x) = (z, w + \tilde{w})$$

Reduction conditions

$$A_{\tilde{z}} = \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

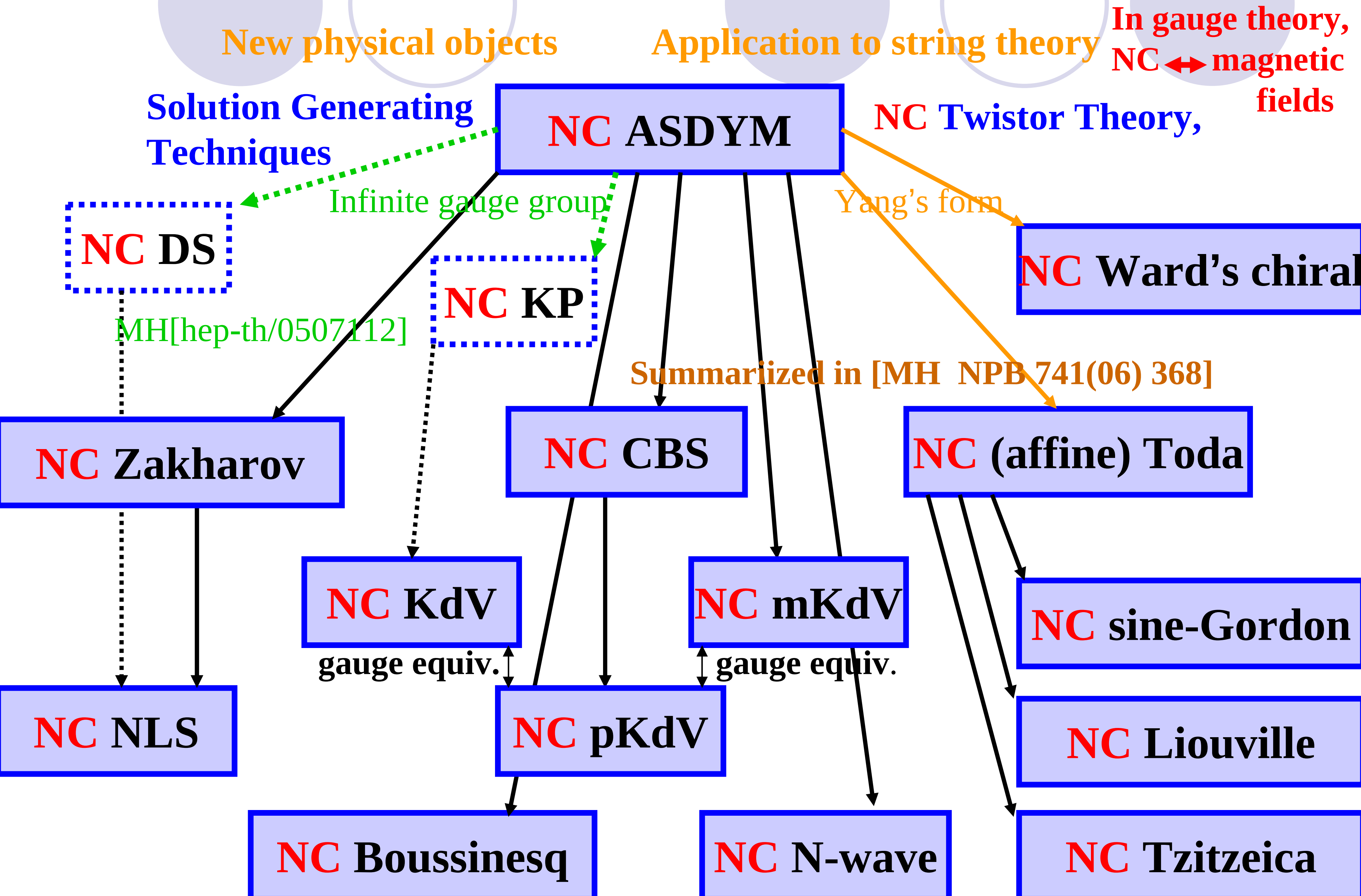
$$A_w = \begin{pmatrix} 0 & \psi \\ \overline{\psi} & 0 \end{pmatrix}, A_z = \sqrt{-1}\begin{pmatrix} -\psi\overline{\psi} & -\psi' \\ \psi' & \overline{\psi}\psi \end{pmatrix}$$

NOT traceless → U(1) part is crucial!

$$\sqrt{-1}\dot{\psi} = \psi'' - 2\psi\overline{\psi}\psi$$

:NC NLS eq.!

NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.



Plan of this talk

1. Introduction
2. Quasi-determinants
3. Backlund Transforms for the NC ASDYM eqs. (and NC Atiyah-Ward ansatz solutions in terms of quasideterminants)
4. Origin of the Backlund trfs from NC twistor theory
5. Conclusion and Discussion

2. Quasi-determinants

[For a review, see Gelfand et al., math.QA/0208146]

- Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.
- [Def1] For an n by n matrix $X=(x_{ij})$ and the inverse $Y=(y_{ij})$ of X, quasi-determinant of X is directly defined by

$$\left|X\right|_{ij}=y_{ji}^{-1} \quad \left(\xrightarrow{\text{In commutative limit}} \frac{(-1)^{i+j}}{\det X^{ij}}\det X\right)$$

X^{ij} : the matrix obtained from X deleting i-th row and j-th column

proportional to the det. or ratio of dets.

- [Def2] (Iterative definition)

$$\left|X\right|_{ij}=x_{ij}-\sum_{i',j'}x_{ii'}((X^{ij})^{-1})_{i'j'}x_{j'j}=x_{ij}-\sum_{i',j'}x_{ii'}(\left|X^{ij}\right|_{j'i'})^{-1}x_{j'j}$$

n+1 × n+1

n × n

A comment on Def 2

Formula for inverse matrix:

$$X = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \Rightarrow$$

n+1 × n+1

$$Y = X^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(d - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(d - CA^{-1}B)^{-1} \\ -(d - CA^{-1}B)^{-1}CA^{-1} & (d - C\underline{A^{-1}}B)^{-1} \end{pmatrix}$$

n × n

A convenient notation:

$$\left|X\right|_{ij} = \begin{vmatrix} & \vdots & \\ \cdots & \boxed{x_{ij}} & \cdots \\ & \vdots & \end{vmatrix}$$

Examples of quasi-determinants

$$n=1: \quad |X|_{ij} = x_{ij}$$

$$n=2: \quad |X|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11} - x_{12}\cdot x_{22}^{-1}\cdot x_{21}, \quad |X|_{12} = \begin{vmatrix} x_{11} & \boxed{x_{12}} \\ x_{21} & x_{22} \end{vmatrix} = x_{12} - x_{11}\cdot x_{21}^{-1}\cdot x_{22},$$

$$|X|_{21} = \begin{vmatrix} x_{11} & x_{12} \\ \boxed{x_{21}} & x_{22} \end{vmatrix} = x_{21} - x_{22}\cdot x_{12}^{-1}\cdot x_{11}, \quad |X|_{22} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & \boxed{x_{22}} \end{vmatrix} = x_{22} - x_{21}\cdot x_{11}^{-1}\cdot x_{12},$$

$$n=3: \quad |X|_{11} = \begin{vmatrix} \boxed{x_{11}} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11} - (x_{12}, x_{13}) \begin{pmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{pmatrix}^{-1} \begin{pmatrix} x_{21} \\ x_{31} \end{pmatrix}$$

$$= x_{11} - x_{12}\cdot(x_{22} - x_{23}\cdot x_{33}^{-1}\cdot x_{32})^{-1}\cdot x_{21} - x_{13}\cdot(x_{32} - x_{33}\cdot x_{23}^{-1}\cdot x_{22})^{-1}\cdot x_{21}$$

$$- x_{12}\cdot(x_{23} - x_{22}\cdot x_{32}^{-1}\cdot x_{33})^{-1}\cdot x_{31} - x_{13}\cdot(x_{33} - x_{32}\cdot x_{22}^{-1}\cdot x_{23})^{-1}\cdot x_{31}$$

$$\cdots$$

Note:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow Y = X^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DB^{-1}A)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

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Some identities of quasideterminants

- **Homological relation**

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix}$$

e.g.

$$C - DB^{-1}A = \begin{vmatrix} A & B \\ \boxed{C} & D \end{vmatrix} = \begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix} \begin{vmatrix} A & B \\ \boxed{0} & 1 \end{vmatrix} = (D - CA^{-1}B)(-B^{-1}A)$$

The identity in the previous page!

Some identities of quasideterminants

- **NC Jacobi's identitiy**

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}$$

$``D - CA^{-1}B"$

$$\xrightarrow{\theta \to 0} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & i \end{vmatrix} |A| = \begin{vmatrix} A & C \\ E & i \end{vmatrix} \begin{vmatrix} A & B \\ D & f \end{vmatrix} - \begin{vmatrix} A & B \\ E & h \end{vmatrix} \begin{vmatrix} A & C \\ D & g \end{vmatrix}$$

:(commutative) Jacobi's id.

3. Backlund transform for NC ASDYM eqs.

- **In this section, we give Backlund transformations for the NC ASDYM equation.**
- **The proof is made very simply by using identities of quasideterminants.**
- **The generated solutions are represented in terms of quasideterminants, which contain not only finite-action solutions (NC instantons) but also infinite-action solutions (non-linear plane waves and so on.)**

NC Yang's form and Yang's equation

- **NC ASDYM eq. can be rewritten as follows**

$$
\begin{cases}
F_{zw} = [D_z, D_w] = 0, \quad \Rightarrow \quad \exists h,\ D_z h = 0,\ \ D_w h = 0 \quad (A_z = -h_z h^{-1}, etc.) \\
F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0, \quad \Rightarrow \quad \exists \tilde{h},\ D_{\tilde{z}} \tilde{h} = 0,\ \ D_{\tilde{w}} \tilde{h} = 0 \quad (A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1}, etc.) \\
F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0
\end{cases}
$$

If we define Yang's matrix: $J := \tilde{h}^{-1} h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} \partial_{\tilde{z}} J) - \partial_w (J^{-1} \partial_{\tilde{w}} J) = 0$$

:NC Yang's eq.

↓ **The solution J reproduce the gauge fields as**

$$A_z = -h_z h^{-1},\ \ A_w = h_w h^{-1},\ A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \tilde{h}^{-1},\ \ A_{\tilde{w}} = \tilde{h}_{\tilde{w}} \tilde{h}^{-1}$$

Backlund trf. for NC ASDYM eq. G=GL(2)

- **The J matrix can be reparametrized as follows**

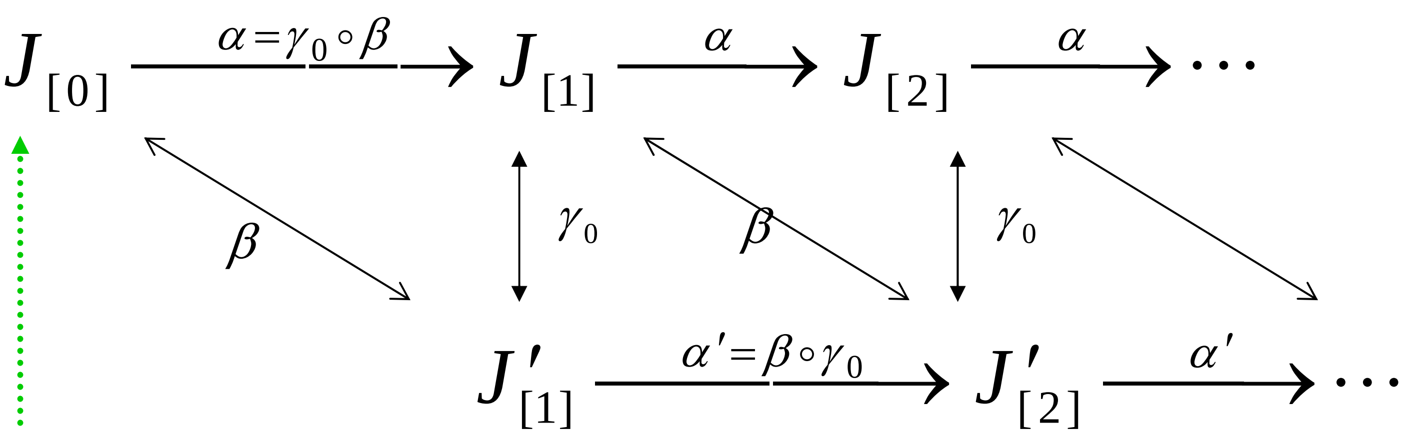
$$J=\begin{pmatrix} f-gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{pmatrix}$$

- **Then the following two kind of trfs. leave NC ASDYM eq.** $\partial_z(J^{-1}\partial_{\tilde{z}}J)-\partial_w(J^{-1}\partial_{\tilde{w}}J)=0$ **as it is:**

$$\beta:\begin{cases}\partial_z e^{new}=-f^{-1}g_{\tilde{w}}b^{-1},\ \partial_w e^{new}=-f^{-1}g_{\tilde{z}}b^{-1}, \\ \partial_{\tilde{z}}g^{new}=-b^{-1}e_w f^{-1},\ \partial_{\tilde{w}}g^{new}=-b^{-1}e_z f^{-1}, \\ f^{new}=b^{-1},\ b^{new}=f^{-1}\end{cases}$$

$$\gamma_0:\begin{pmatrix} f^{new} & g^{new} \\ e^{new} & b^{new}\end{pmatrix}=\begin{pmatrix}(b-ef^{-1}g)^{-1} & (g-fe^{-1}b)^{-1} \\ (e-bg^{-1}f)^{-1} & (f-gb^{-1}e)^{-1}\end{pmatrix}$$

- **Both trfs. are involutive ($\beta \circ \beta = id, \gamma_0 \circ \gamma_0 = id$), but the combined trf. $\gamma_0 \circ \beta$ is non-trivial.)**
- **Then we could generate various (non-trivial) solutions of NC ASDYM eq. from a (trivial) seed solution (so called, NC Atiyah-Ward ansatz solutions)**



A seed solution:

$$b_{[0]} = f_{[0]} = e_{[0]} = g_{[0]} = \Delta_0^{-1},$$

$$\partial^2 \Delta_0 = (\partial_z \partial_{\tilde{z}} - \partial_w \partial_{\tilde{w}}) \Delta_0 = 0$$

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

Explicit Atiyah-Ward ansatz solutions of NC ASDYM eq. G=GL(2)

[Gilson-MH-Nimmo, arXiv:0709.2069]

$$b_{[0]} = f_{[0]} = e_{[0]} = g_{[0]} = \Delta_0^{-1}, \quad \partial^2 \Delta_0 = 0$$

$$b_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \; f_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \; e_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \; g_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1},$$

$$\partial_z \Delta_0 = \partial_{\tilde{w}} \Delta_1, \; \partial_z \Delta_{-1} = \partial_{\tilde{w}} \Delta_0, \; \partial_w \Delta_0 = \partial_{\tilde{z}} \Delta_1, \; \partial_w \Delta_{-1} = \partial_{\tilde{z}} \Delta_0$$

$$f_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}, \; b_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1},$$

$$e_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}, \; g_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \Delta_0 \end{vmatrix}^{-1}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

Explicit Atiyah-Ward ansatz solutions of NC ASDYM eq. G=GL(2)

[Gilson-MH-Nimmo, arXiv:0709.2069]

$$f'_{[n]} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{1-n} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \boxed{\Delta_0} \end{vmatrix}, \quad b'_{[n]} = \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{1-n} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \Delta_0 \end{vmatrix},$$

$$e'_{[n]} = \begin{vmatrix} \Delta_{-1} & \cdots & \boxed{\Delta_{-n}} \\ \vdots & \ddots & \vdots \\ \Delta_{n-2} & \cdots & \Delta_{-1} \end{vmatrix}, \quad g'_{[n]} = \begin{vmatrix} \Delta_1 & \cdots & \Delta_{2-n} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_n} & \cdots & \Delta_1 \end{vmatrix}$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

$$\beta : \begin{cases} \partial_z e^{new} = -f^{-1} g_{\tilde{w}} b^{-1}, \ \partial_w e^{new} = -f^{-1} g_{\tilde{z}} b^{-1}, \\ \partial_{\tilde{z}} g^{new} = -b^{-1} e_w f^{-1}, \ \partial_{\tilde{w}} g^{new} = -b^{-1} e_z f^{-1}, \\ f^{new} = b^{-1}, \ b^{new} = f^{-1} \end{cases}$$

e.g. $b'_{[1]} = f'_{[1]} = \Delta_0, \ e'_{[1]} = \Delta_{-1}, \ g'_{[1]} = \Delta_1$

$$\partial_z \Delta_0 = \partial_{\tilde{w}} \Delta_1, \ \partial_z \Delta_{-1} = \partial_{\tilde{w}} \Delta_0, \ \partial_w \Delta_0 = \partial_{\tilde{z}} \Delta_1, \ \partial_w \Delta_{-1} = \partial_{\tilde{z}} \Delta_0$$

The proof is in fact very simple!

- Proof is made simply by using only special identities of quasideterminants. (NC Jacobi's identities and homological relations, Gilson-Nimmo's derivative formula etc.)
- In other words, ``NC Backlund trfs are identities of quasideterminants.''
  This is an analogue of the fact in lower-dim. commutative theory: ``Backlund trfs are identities of determinants.''

An example

- γ_0**-transformation**

$$b_{[n]}^{-1} = f'_{[n]} - g'_{[n]} {b'_{[n]}}^{-1} e'_{[n]}$$

$$\Leftrightarrow$$

$$\begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n} \\ \vdots & \ddots & \vdots \\ \Delta_n & \cdots & \boxed{\Delta_0} \end{vmatrix} =$$

$$\begin{vmatrix} \Delta_0 & \cdots & \Delta_{1-n} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \boxed{\Delta_0} \end{vmatrix} - \begin{vmatrix} \Delta_1 & \cdots & \Delta_{2-n} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_n} & \cdots & \Delta_1 \end{vmatrix} \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{1-n} \\ \vdots & \ddots & \vdots \\ \Delta_{n-1} & \cdots & \Delta_0 \end{vmatrix}^{-1} \begin{vmatrix} \Delta_{-1} & \cdots & \boxed{\Delta_{-n}} \\ \vdots & \ddots & \vdots \\ \Delta_{n-2} & \cdots & \Delta_{-1} \end{vmatrix}$$

This is just the NC Jacobi identity!

Some exact solutions

- We could generate various solutions of NC ASDYM eq. from a simple seed solution Δ_0 by using the previous Backlund trf.

A seed solution:

$$\Delta_0 = 1 + ``\frac{1}{z\tilde{z} - w\tilde{w}}''$$

→NC instantons (special to NC spaces)

$$\Delta_0 = \exp(\,linear\ \ of\ \ z, \tilde{z}, w, \tilde{w})$$

→NC Non-Linear plane-waves (new solutions beyond ADHM)

A compact formula of J-matrix

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

$$= \begin{vmatrix} \boxed{0} & -1 & 0 & \cdots & 0 & \boxed{0} \\ 1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-n} & \Delta_{-n} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-n} & \Delta_{1-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \boxed{0} & \Delta_n & \Delta_{n-1} & \cdots & \Delta_1 & \boxed{\Delta_0} \end{vmatrix}$$

- **J:gauge-invariant → The Backlund trf. is not just a gauge trf. but a non-trivial one!**

4. Interpretation from NC twistor theory

- In this section, we give an origin of all ingredients of the Backlund trfs. from the viewpoint of NC twistor theory.
- NC twistor theory has been developed by several authors [Kapustin-Kuznetsov-Orlov, Takasaki, Hannabuss, Lechtenfeld-Popov, Brain-Majid…]
- What we need here is NC Penrose-Ward correspondence between ASD connections and ``NC holomorphic vector bundle’’ on NC twistor space.
- We can see some origin of the quasideterminant solutions.

NC Penrose-Ward correspondence

- Sol. of NC ASDYM ``NC hol. vec. bdl.''

$$\begin{cases} L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0, \\ M\psi = (D_z - \zeta D_{\tilde{w}})\psi = 0. \end{cases}$$

$$\tilde{\zeta} = 1/\zeta$$

$$\begin{cases} \tilde{L}\tilde{\psi} = (\tilde{\zeta} D_w - D_{\tilde{z}})\tilde{\psi} = 0, \\ \tilde{M}\tilde{\psi} = (\tilde{\zeta} D_z - D_{\tilde{w}})\tilde{\psi} = 0. \end{cases}$$

1:1

Patching matrix

1:1 [Takasaki]

$$P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{\psi}^{-1}\psi$$

$$\psi(x;\zeta) = h(x) + O(\zeta)$$

$$\tilde{\psi}(x;\tilde{\zeta}) = \tilde{h}(x) + O(\tilde{\zeta})$$

$$[L, M] = 0 \; or \; [\tilde{L}, \tilde{M}] = 0$$

NC ASDYM eq. (Yang's form)

$$\partial_z (J^{-1}\partial_{\tilde{z}} J) - \partial_w (J^{-1}\partial_{\tilde{w}} J) = 0$$

$$h(x) = \psi(x, \zeta = 0),$$

$$\tilde{h}(x) = \tilde{\psi}(x, \zeta = \infty)$$

$$J := \tilde{h}^{-1} h = \begin{pmatrix} f - g b^{-1} e & -g b^{-1} \\ b^{-1} e & b^{-1} \end{pmatrix}$$

Origin of NC Atiyah-Ward (AW) ansatz sols.

- **The n-th AW ansatz for the Patching matrix**

$$P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta(x;\zeta) \end{pmatrix} \qquad \Delta(x;\zeta) = \Delta(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \sum_i \Delta_i(x)\zeta^i$$

- **The Birkoff factorization** $P = \tilde{\psi}^{-1}\psi$ **leads to:**

$$\tilde{\psi} \leftrightarrow \tilde{h}(x) + O(\tilde{\zeta}) \qquad \psi \leftrightarrow h(x) + O(\zeta)$$

$$h_{11} = h_{12}\left|D_{[n+1]}\right|_{1,n+1}^{-1} - \tilde{h}_{11}\left|D_{[n+1]}\right|_{1,1}^{-1}$$

$$h_{21} = h_{22}\left|D_{[n+1]}\right|_{1,n+1}^{-1} - \tilde{h}_{21}\left|D_{[n+1]}\right|_{1,1}^{-1}$$

$$\tilde{h}_{12} = h_{12}\left|D_{[n+1]}\right|_{n+1,n+1}^{-1} - \tilde{h}_{11}\left|D_{[n+1]}\right|_{n+1,1}^{-1}$$

$$\tilde{h}_{22} = h_{22}\left|D_{[n+1]}\right|_{n+1,n+1}^{-1} - \tilde{h}_{21}\left|D_{[n+1]}\right|_{n+1,1}^{-1}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

- **Under a gauge (** $h_{12} = \tilde{h}_{21} = 0,\ h_{22} = \tilde{h}_{11} = 1$ **), this solution coincides with the quasideterminants sols!**

$$\tilde{h}_{22} = b_{[n]} = \left|D_{[n+1]}\right|_{n+1,n+1}^{-1},\quad h_{11} = f_{[n]} = -\left|D_{[n+1]}\right|_{11}^{-1},$$

$$h_{21} = e_{[n]} = \left|D_{[n+!]}\right|_{1,n+1}^{-1},\quad \tilde{h}_{12} = g_{[n]} = -\left|D_{[n+1]}\right|_{n+1,1}^{-1}$$

OK!

$$J_{[n]} = \tilde{h}^{-1}h = \begin{pmatrix} 1 & g_{[n]} \\ 0 & b_{[n]} \end{pmatrix}^{-1} \begin{pmatrix} f_{[n]} & 0 \\ e_{[n]} & 1 \end{pmatrix} = \begin{pmatrix} f_{[n]} - g_{[n]}b_{[n]}^{-1}e_{[n]} & -g_{[n]}b_{[n]}^{-1} \\ b_{[n]}^{-1}e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

Origin of NC Atiyah-Ward (AW) ansatz sols.

- **The n-th AW ansatz for the Patching matrix**

$$P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta(x;\zeta) \end{pmatrix} \qquad \Delta(x;\zeta) = \underline{\Delta(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta)} = \underline{\sum_i \Delta_i(x)\zeta^i}$$

- **The recursion relation is derived from:**

$$l\Delta(x;\zeta) = (\partial_w - \zeta\partial_{\tilde{z}})\Delta = 0,$$

$$m\Delta(x;\zeta) = (\partial_z - \zeta\partial_{\tilde{w}})\Delta = 0,$$

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}$$

OK!

Origin of the Backlund trfs

- **The Backlund trfs can be understood as the adjoint actions for the Patching matrix:**

$$\beta : P^{new} = B^{-1} P B, \quad \gamma_0 : P^{new} = C_0^{-1} P C_0, \quad B = \begin{pmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

actually:

$$\alpha = \gamma_0 \circ \beta : \quad P_{[n]} = \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta \end{pmatrix} \mapsto C_0^{-1} B^{-1} \begin{pmatrix} 0 & \zeta^{-n} \\ \zeta^{n} & \Delta \end{pmatrix} B C_0 = \begin{pmatrix} 0 & \zeta^{-(n+1)} \\ \zeta^{n+1} & \Delta \end{pmatrix} = P_{[n+1]}$$

- **The γ_0-trf. leads to** $J^{new} = C_0^{-1} J C_0 \Rightarrow$ **The previous γ_0-trf!**
- **The β-trf. is derived with a singular gauge trf.**

$$\beta : \psi \mapsto \psi^{new} = s \psi B, \qquad s = \begin{pmatrix} 0 & \zeta b^{-1} \\ -f^{-1} & 0 \end{pmatrix}$$

$$\downarrow \ \zeta \to 0$$

1-2 component of $L\psi = (D_w - \zeta D_{\tilde{z}})\psi = 0$

$$h^{new} = \begin{pmatrix} f^{new} & 0 \\ e^{new} & 1 \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ -f^{-1}k_{12} & 1 \end{pmatrix} \Rightarrow \quad f^{new} = b^{-1}, \quad \partial_w e^{new} = -\partial_w (f^{-1} k_{12}) = f^{-1} g_{\tilde{z}} b^{-1}$$

OK!

$$\psi = h + k\zeta + O(\zeta^2)$$

The previous β-trf!

5. Conclusion and Discussion

NC integrable eqs (ASDYM) in higher-dim.

- ADHM (OK)
- Twistor (OK)
- Backlund trf (OK), Symmetry (Next)

Quasi-determinants are important !

Profound relation ?? (via Ward conjecture)

NC integrable eqs (KdV) in lower-dims.

- Hierarchy (OK)
- Infinite conserved quantities (OK)
- Exact N-soliton solutions (OK)
- Symmetry (``tau-fcn'', Sato's theory) (Next)
- Behavior of solitons… (Next)

Very HOT !

Quasi-determinants are important !