Construction of Fuzzy Spaces and
Their Applications to Matrix Models[1]

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Abstract

Quantization of spacetime by means of finite dimensional matrices is the basic idea of fuzzy spaces. There remains an issue of quantizing time, however, the idea is simple and it provides an interesting interplay of various ideas in mathematics and physics. Shedding some light on such an interplay is the main theme of this dissertation. The dissertation roughly separates into two parts. In the first part, we consider rather mathematical aspects of fuzzy spaces, namely, their construction. We begin with a review of construction of fuzzy complex projective spaces $\mathbf{CP}^k$ ($k = 1, 2, \cdots$) in relation to geometric quantization. This construction facilitates defining symbols and star products on fuzzy $\mathbf{CP}^k$. Algebraic construction of fuzzy $\mathbf{CP}^k$ is also discussed. We then present construction of fuzzy S^4, utilizing the fact that $\mathbf{CP}^3$ is an S^2 bundle over S^4. Fuzzy S^4 is obtained by imposing an additional algebraic constraint on fuzzy $\mathbf{CP}^3$. Consequently it is proposed that coordinates on fuzzy S^4 are described by certain block-diagonal matrices. It is also found that fuzzy S^8 can analogously be constructed.

In the second part of this dissertation, we consider applications of fuzzy spaces to physics. We first consider theories of gravity on fuzzy spaces, anticipating that they may offer a novel way of regularizing spacetime dynamics. We obtain actions for gravity on fuzzy S^2 and on fuzzy $\mathbf{CP}^2$ in terms of finite dimensional matrices. Application to M(atrix) theory is also discussed. With an introduction of extra potentials to the theory, we show that it also has new brane solutions whose transverse directions are described by fuzzy S^4 and fuzzy $\mathbf{CP}^3$. The extra potentials can be considered as fuzzy versions of differential forms or fluxes, which enable us to discuss compactification models of M(atrix) theory. In particular, compactification down to fuzzy S^4 is discussed and a realistic matrix model of M-theory in four-dimensions is proposed.

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1 Introduction

Studies of fuzzy spaces cross over a variety of concepts in mathematics and physics. The basic idea of fuzzy spaces is to describe compact spaces in terms of finite dimensional $(N \times N)$-matrices such that they give a concrete realization of noncommutative (NC) spaces [1, 2, 3, 4]. Use of fuzzy spaces in physics was suggested by Madore around 1992 [5]. Since then, fuzzy spaces have been an active area of research. Some of the earlier developments can be found in [6, 7, 8]. For recent reviews on fuzzy spaces, one may refer to [9, 10, 11].

1.1 Matrix realization of NC geometry

Definition of fuzzy spaces can be made from a framework of noncommutative geometry initiated by Connes [1, 12], where it has been shown that the usual differential calculus on a Riemannian manifold $\mathcal{M}$ can be constructed by the so-called spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$; $\mathcal{A}$ is the algebra of smooth bounded functions on $\mathcal{M}$, $\mathcal{H}$ is the Hilbert space of square-integrable spinor functions on $\mathcal{M}$ (or sections of the irreducible spinor bundle) and $\mathcal{D}$ is the Dirac operator on $\mathcal{M}$, carrying the information of metric and Levi-Civita spin connection. With a slight modification of Connes' idea, Fröhlich and Gawędzki have also indicated that the Riemannian geometry can be constructed by the abstract triplet $(\mathcal{A}, \mathcal{H}, \Delta)$, where Δ is the Laplace-Beltrami operator on $\mathcal{M}$ [4]. Fuzzy spaces are then defined by a sequence of triples

$$(Mat_N, \mathcal{H}_N, \Delta_N) \tag{1}$$

where Mat_N is a matrix algebra of $(N \times N)$-matrices which act on the N-dimensional Hilbert space $\mathcal{H}_N$ and Δ_N is a matrix analog of the Laplacian. The inner product of matrix algebra is defined by $\langle A, B \rangle = \frac{1}{N} \text{Tr}(A^\dagger B)$. The Laplacian Δ_N contains information of metrical and other geometrical properties of $\mathcal{M}$. For example, the dimension of the manifold $\mathcal{M}$ relates to the N-dependence of the number of eigenvalues in Δ_N.

Since fuzzy spaces are described by finite dimensional matrices, due to the Cayley-Hamilton theorem, there is a natural cut-off on the number of modes for matrix functions on fuzzy spaces. So one can use fuzzy spaces to construct regularized field theories in much the same way that lattice gauge theories are built. Various interesting features of field theories on fuzzy spaces have been reported; for example, existence of topological solutions such as monopoles and instantons, appearance of the so-called UV-IR mixing, and evasion of the fermion doubling problem which appears in the lattice regularization. For these and other aspects of fuzzy spaces, one may refer to [13]-[30] and, in particular, to [11] for a review.

1.2 Relation to geometric quantization

Construction of fuzzy spaces is closely related to quantization programs in the construction of quantum Hilbert spaces from classical phase spaces. It is known that there exist different quantization schemes such as canonical quantization and functional integral (or path integral) quantization. In either case, the quantum theory is described by a unitary irreducible representation (UIR) of the algebra of symmetry on a Hilbert space.[3] Physical observables are given by hermitian operators which generate unitary transformations on the Hilbert space. In the classical theory, the operators correspond to functions on a phase space which generate canonical transformations. The basic idea of quantization is to have a correspondence between the algebra of Poisson brackets represented by functions on a phase space or a symplectic manifold $\mathcal{M}_s$ and the algebra of commutation rules represented irreducibly by operators on a Hilbert space $\mathcal{H}$. The hermitian operators can be represented by $(N \times N)$-matrices where N is the dimension of the Hilbert space $\mathcal{H}$. From this point of view, the quantization programs are essentially equivalent to the construction of fuzzy spaces. The matrix version of Laplacian Δ_N in (1) can be obtained as a double commutator. This is observed as follows; consider Heisenberg commutation rules of quantum mechanics $[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$, $[\hat{x}, \hat{p}] = i$

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[3]The Hilbert space structure may not be apparent in the path integral approach, where one is interested in computation of correlation functions or S-matrices, however, it is in general possible to define the quantum theory in terms of a UIR of the operator algebra on a Hilbert space.

with $\hat{x}\psi = x\psi$, $\hat{p}\psi = -i\frac{\partial}{\partial x}\psi$ where $\psi(x)$ is a wavefunction, then the Laplacian is expressed as $\Delta f(\hat{x}) = -[\hat{p}, [\hat{p}, f(\hat{x})]]$ where $f(\hat{x})$ is a function of $\hat{x}$. Note that the wavefunction depends only on x instead of (x, p). This is necessary to have an irreducible representation of the operator algebra. It is also related to the notion of polarization or holomorphic condition in a framework of geometric quantization.

Geometric quantization would be a mathematically more rigorous quantization scheme [31, 32, 33, 34]. It turns out to be very useful in quantizing many systems, including the Chern-Simons theory [35, 36]. In geometric quantization, one considers the so-called prequantum line bundle which is a line bundle on a phase space. The curvature or the first Chern class of the line bundle can naturally be chosen as a symplectic two-form Ω_s. By use of the line bundle, one can show an explicit correspondence between the algebra of Poisson brackets and the algebra of commutators. The upshot of geometric quantization is that a quantum Hilbert space is given by sections of a 'polarized' line bundle. The above wavefunction $\psi(x)$ corresponds to this polarized line bundle, while unpolarized one would lead to a function $\psi(x, p)$. Usually, we impose a complex structure on the phase space and identify the symplectic two-form as a Kähler form or some multiple thereof. In this case, the easiest polarization condition to use is a holomorphic condition on a complex line bundle. This gives what is known as the Kähler polarization. The idea of forming a Hilbert space as holomorphic sections of a complex line bundle was in fact exploited in the studies of representation of compact Lie groups by Borel, Weil and Bott. They showed that, for any compact Lie group G, all UIR's of G are realized by holomorphic sections of a complex line bundle on a coset space G/T, where T is the maximal torus of G and G/T is proven to be a Kähler manifold. (The group G acts on the space of holomorphic sections, or a Hilbert space, as right translation.) For detailed description of this Borel-Weil-Bott theory or theorem, one may refer to [31, 32, 33, 36].

Utilizing a quantization program, we can obtain a finite dimensional Hilbert space $\mathcal{H}_N$ in (1) for any compact symplectic manifold $\mathcal{M}_s$. The matrix algebra Mat_N is given by the algebra of operators acting on $\mathcal{H}_N$. As mentioned earlier, the Laplacian Δ_N in (1) is naturally obtained upon the determination of Mat_N. Construction of fuzzy spaces is therefore implemented by quantization of compact symplectic manifolds. A family of such manifolds is given by the so-called co-adjoint orbits of a compact semi-simple Lie group G. (For semi-simple Lie groups, there is no difference between co-adjoint and adjoint orbits.) It is known that the co-adjoint orbits can be quantized when their symplectic two-forms satisfy a Dirac-type quantization condition. For quantization of co-adjoint orbits, one may refer to [31, 32, 33]. The co-adjoint orbit of a compact semi-simple Lie group G, with its Lie algebra being $\underline{G}$, is given by $\{gtg^{-1} : g \in G\}$ where $t \in \underline{G}$. The co-adjoint orbit is then considered as a coset space G/H_t where H_t is a subset of G defined by $H_t = \{g \in G : [g, t] = 0\}$. When H_t coincides with the maximal torus of G, the co-adjoint orbit becomes the above mentioned space G/T, This space, known as a flag manifold, has the maximal dimension of the co-adjoint orbits, $i.e.$, $\dim G - \mathrm{rank} G$. An example of such a space is $\frac{SU(3)}{U(1)\times U(1)}$ where t is given by $t \sim \mathrm{diag}(1, -1, 0)$ corresponding to λ_3 in terms of the Gell-Mann matrices λ_a $(a = 1, 2, \cdots, 8)$ for $SU(3)$. When t has degeneracy, the co-adjoint orbits are called degenerate and their dimensions are given by $\dim G - \dim H_t$. An example is $SU(3)/U(2)$ with $t \sim \lambda_8 \sim \mathrm{diag}(1, 1, -2)$. This coset is equivalent to the four-dimensional complex projective space $\mathbf{CP}^2$. Since we are interested in a finite dimensional UIR of G, the compact group G is to be chosen as $U(n)$ or its subgroup. In this case, the generator t always includes a $U(1)$ element of $U(n)$. Consequently, the subset $H_t \subset G$, known as the stabilizer of t, contains the $\overline{U(1)}$ element of $\overline{U(n)}$. This is a fact of some significance particularly in considering gauge theories on fuzzy spaces.

In quantizing the co-adjoint orbit G/H_t, the Hilbert space is given by holomorphic sections of a complex line bundle over G/H_t. The holomorphic sections correspond to a UIR of G. (The holomorphicity allows the extension of the G-action to a $G^{\mathbf{C}}$-action, where $G^{\mathbf{C}}$ is the complexification of G. Note that any compact group can be complexified; this is known as Chevalley's complexification of compact Lie groups.) We can now make direct use of geometric quantization to construct the fuzzy version of G/H_t. In fact, fuzzy spaces which have been constructed so far, to be consistent with the definition of (1), all fit into this class of coset spaces. Namely, they are fuzzy $S^2 = SU(2)/U(1)$, fuzzy $\mathbf{CP}^2 = SU(3)/U(2)$ and fuzzy $\mathbf{CP}^k = SU(k+1)/U(k)$ $(k = 1, 2, 3, \cdots)$ in general [5, 37, 38].

A detailed construction of fuzzy $\mathbf{CP}^k$ in the same spirit as geometric quantization has been

carried out by Karabali and Nair [39, 40, 41], where the complex line bundle over $\mathbf{CP}^k = SU(k+1)/U(k)$ is expressed in terms of the Wigner $\mathcal{D}$-functions for $SU(k+1)$ which, by definition, give a UIR of $SU(k+1)$. Symbols and star products, notion of functions and their product algebra in commutative space mapped from noncommutative counterparts, are explicitly defined in terms of the $\mathcal{D}$-functions. In the next chapter, we shall recapitulate these results.

For those manifolds that do not have a symplectic structure, there exist no quantization schemes. This is the main reason for the difficulty encountered in construction of odd-dimensional fuzzy spaces and fuzzy spheres of dimension higher than two. Construction of higher dimensional fuzzy spheres has been proposed in [42, 43, 44, 45, 46, 47]. Each proposal starts from a co-adjoint orbit such as $\frac{SO(2k+1)}{U(k)}$, $\frac{SO(k+2)}{SO(k)\times SO(2)}$ $(k = 1, 2, \cdots)$. Factors irrelevant to the sphere in such a co-adjoint orbit are projected out in a sort of brute-force way. As a result, the resulting fuzzy spheres break either associativity or closure of the algebra. These fuzzy spheres are therefore not compatible with the definition (1) where fuzzy spaces are defined by the matrix algebra on a finite dimensional Hilbert space. One way to avoid this problem is to impose an extra constraint on a Lie algebra of G so that the co-adjoint orbit G/H_t (or its multiple) globally defines a sphere under the algebraic extra constraint. This is a natural prescription for proper construction of fuzzy spheres because functions on fuzzy spaces are described by matrix representation of the algebra $\underline{G}$. The fuzzy spheres are embedded in $\mathbf{R}^{\dim G}$ and its algebra is a subset of $\underline{G}$, preserving closure and associativity. It is by use of this idea of introducing an extra constraint that fuzzy $S^3/\mathbf{Z}_2$ is constructed from fuzzy $S^2 \times S^2$ in [48]. The same idea proves to be applicable to construction of fuzzy S^4 from fuzzy $\mathbf{CP}^3$ [49]. This construction utilizes the fact that $\mathbf{CP}^3$ is an S^2-bundle over S^4, or a Hopf fibration of S^7 as an S^3-bundle over S^4. Utilizing a Hopf fibration of S^{15} as an S^7-bundle over S^8, one can similarly construct fuzzy S^8 from fuzzy $\mathbf{CP}^7$ with some algebraic constraint. In chapter 3, we shall discuss construction of higher dimensional spheres along these lines, focusing on the case of fuzzy S^4.

1.3 Applications to physics

The fact that co-adjoint orbits are given by coset spaces is important in application of fuzzy spaces to physics. The coset space G/H naturally gives rise to an interpretation of G as an H-bundle over G/H or more generally a sum of $H^{(i)}$-bundles over G/H, with H being a direct product of $H^{(i)}$'s $(i = 1, 2, \cdots)$. As mentioned earlier, H always contains a $U(1)$ group, so at least one of the $H^{(i)}$'s can be identified as $U(1)$. The corresponding $U(1)$-bundle gives a complex line bundle whose holomorphic sections are, as discussed earlier, regarded as a Hilbert space $\mathcal{H}_N$. There is an interesting correspondence between $\mathcal{H}_N$ and the Hilbert space of the lowest Landau level, which is a restricted energy level for charged particles in a strong magnetic field. Physical observables in such a system are projected onto the lowest Landau level. As a result, they acquire noncommutativity and it is possible to identify them with the observables on fuzzy spaces. (For further description of this correspondence, see a recent review [9]; for the Landau problem and its relation to fuzzy sphere and more general Riemann surfaces, see [50, 51, 52].) In this context, the $U(1)$-bundle is understood as a magnetic monopole-bundle over G/H whose holomorphic sections give wavefunctions on the lowest Landau level in G/H. Note that the Landau problem was originally considered on $\mathbf{R}^2$ but it can naturally be extended to higher dimensional curved (coset) spaces. When $H^{(i)}$ is a non-abelian group, we have a non-abelian vector bundle over G/H. Physically this corresponds to the presence of a non-abelian background magnetic field.

There is a series of remarkable results in the study of the edge excitations of quantum Hall droplets on the lowest Landau level in $\mathbf{CP}^k$ [39, 40, 41]. Here we simply state these results. In [40] it is shown that an effective action for the edge excitations in a $U(1)$ background magnetic field is given by a chiral bosonic action in the limit of a large number of edge states. The action can be interpreted as a generalization of a chiral abelian Wess-Zumino-Witten (WZW) theory. With a non-ablian $U(k)$ background magnetic field, the effective action for the edge excitations leads to a chiral and gauged WZW theory generalized to higher dimensions, also in the limit of a large number of edge states [41]. (For uses of fuzzy spaces in the quantum Hall systems, see also [53].)

The gauge principle is probably the most important concept in physics in a sense that it provides a unified view of all physical interactions, including gravity. The gauge principle means

the invariance of physical quantities under local frame transformations. As is well-known, fibre bundles are the mathematical framework for the local or gauge symmetries. The concept of fibre bundles is then useful in understanding the geometrical and topological properties of gauge theories. Fibre bundles also provide a natural setting for all physical fields. Matter fields are sections of various vector bundles over spacetime manifold, with the fibre being complex numbers or spinors of the Lorentz group. Gauge fields are connections on these vector bundles. Connections of tangent bundles over spacetime give Christoffel symbols, which lead to the metric and spin connections and, eventually, the theory of gravity.

As mentioned above, bundle structures naturally arise in fuzzy spaces. Like in ordinary commutative spaces, gauge fields on fuzzy spaces are defined by 'fuzzy' covariant derivatives. In a fuzzy G/H-space, derivative and coordinate operators obey the same algebra $\underline{G}$, so they are identical. This is related to the fact that co-adjoint and adjoint orbits are equivalent for a compact semi-simple group G. One can then regard the covariant derivatives as 'covariant' coordinates on fuzzy spaces. In this sense, the gauge fields are considered as fluctuations from fuzzy spaces. Gauge theories on noncommutative spaces in general have been received a lot of interest [11, 54, 55, 56]. This is partly motivated by the discovery that noncommutative spaces can arise as solutions in string and M-theories. The solutions are known as D-branes or simply branes, corresponding to non-perturbative objects in string theories [57]. Later we shall consider such objects in relation to fuzzy spaces. Application of noncommutative geometry to gauge theories was in fact initiated by Connes and others [1, 2, 3]. Part of their motivation is to understand the standard model of particle physics (and the involving Higgs mechanism) in a more mathematical framework, namely, in terms of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [58, 59]. There is also a series of developments in construction of gravitational theories in terms of the spectral triple [60, 61, 62, 63, 64, 65].

Gauge fields which describe gravitational degrees of freedom (*i.e.*, frame fields and spin connections) on fuzzy spaces are particularly interesting, since they would offer a regularized gravity theory as a novel alternative to the Regge calculus or triangulation of spaces, which is essentially the only finite-mode truncation of gravity, preserving the notion of diffeomorphism. Gravitational fields on a fuzzy G/H-space are given by hermitian $(N \times N)$ matrices. The matrices have an invariance under $U(N)$ transformations which is usually imposed in any hermitian matrix models. The matrix elements of functions on a fuzzy space correspond to the coefficients in a harmonic expansion of truncated functions on the corresponding commutative space. This so-called matrix-function correspondence implies the $U(N)$ invariance as a fuzzy analog of the coordinate invariance or the diffeomorphism.

The gauge group of the gravitational fields on commutative Euclidean spacetime is given by a combination of translational and rotational space-time symmetries on the tangent frame. In ordinary flat space, this group is the Poincaré group. However, in the G/H-space, the Poincaré group is replaced by the compact semi-simple group G. The stabilizer H, which we consider as a subgroup of G in what follows, corresponds to the Lorentz group so that the translations on G/H are represented by $\underline{G} - \underline{H}$. Theories of gravity on such an even-dimensional (coset) space have been studied in connection with topological gauge theories. For example, an action for two-dimensional gravity is given by the Jackiw-Teitelboim action [66]. One can also construct a physically more interesting case, *i.e.*, an action for gravity on four-dimensional spacetime, following Chang, MacDowell and Mansouri [67]. As mentioned earlier in the context of the construction of fuzzy spaces, the group G is seen as a compact group embedded in $U(n)$. So one can consider the existence of $U(k)$ ($k \leq n$) such that $G \subseteq U(k) \subset U(n)$. In noncommutative spaces, gauge groups should contain a $U(1)$ element, otherwise one cannot properly define a noncommutative version of curvature or field strength. A natural choice of the gauge group on the fuzzy G/H-space is therefore the $U(k)$ group. It is based on these arguments that a Chang-MacDowell-Mansouri (CMM) type action for gravity on even-dimensional noncommutative spaces has been proposed by Nair in [68]. (For some of the other approaches to noncommutative gravity, one may refer to [69]-[75]; for a matrix model of gravity on fuzzy S^2 in particular, see [76, 77].) In [76] the CMM type action is applied to fuzzy S^2 as well as fuzzy $\mathbf{CP}^2$ and actions for gravity in terms of $(N \times N)$ matrices are obtained. The action on fuzzy S^2 reduces to the Jackiw-Teitelboim action on S^2 in the large N limit. We shall present these results in chapter 4.

Fuzzy spaces are in principle constructed for any even-dimensional symplectic manifolds. Re-

striction to the number of dimensions should come from physical reasonings. One convincing reason is the matrix model of M-theory or the M(atrix) theory proposed by Banks, Fischler, Shenker and Susskind [78]. For a review of M(atrix) theory, one may refer to [79]. In M(atrix) theory, nine dimensions out of eleven are described by $(N \times N)$ matrices, being referred to the transverse directions. Brane solutions are then described by fuzzy spaces as far as the transverse directions are concerned. Solutions with matrix configurations of fuzzy S^2, fuzzy S^4 and fuzzy $\mathbf{CP}^2$ are known to exist [80, 81, 82, 83]; they are respectively called spherical membranes, spherical longitudinal five-branes and longitudinal five branes of $\mathbf{CP}^2 \times S^1$ geometry. Note that when the solutions involve the longitudinal directions, as opposed to the transverse ones, they are called longitudinal branes. It is known that there exist longitudinal five-brane solutions in M(atrix) theory [84, 85, 86]. But brane solutions of dimension higher than five are excluded due to energy consideration [86]. Details of these points are discussed in chapter 5, where we also consider the emergence of longitudinal seven-branes of $\mathbf{CP}^3 \times S^1$ geometry, introducing extra potentials to the M(atrix) theory Lagrangian. For related analyses on fuzzy spaces as brane solutions, one may refer to [87, 88].

There is another version of matrix model corresponding to type IIB string theory proposed by Ishibashi, Kawai, Kitazawa and Tsuchiya [89]. For a reveiw of this model, see [90]. This IIB matrix model also has solutions described by fuzzy spaces [91, 92, 93, 94, 95]. Fuzzy spaces, or finite dimensional matrix realization of spacetime, are suitable for numerical simulations. There is a series of numerical studies on certain fuzzy spaces appearing in a generalized IIB matrix model [96]. For a different type of simulation, see also [97].

In terms of M(atrix) theory, the number of dimensions for fuzzy spaces arising as transverse branes is restricted to 2, 4, 6 and 8. (We omit odd dimensions here because they do not lead to a symplectic structure, but they may be possible as shown in [48].) When the dimension is higher than four, we are faced with higher dimensional brane solutions. These can be interpreted either as extended physical objects along the lines of a brane-world scenario [98], or as bundles over four-dimensional spacetime. In the former case, extra dimensions are somehow allowed to exist and one can use Kaluza-Klein type compactification to discuss their effects on spacetime. In the latter case, the extra dimensions are relevant to internal symmetries or a fibre. A typical example is Penrose's twsitor space $\mathbf{CP}^3$ which is an S^2-bundle over (compact) spacetime S^4 [99]. In this context, fuzzy $\mathbf{CP}^3$ is quite interesting in application to physics. (It has also been useful in construction of fuzzy S^4 [49].) For a recent development in connection with this idea, one may refer to [100].

The rest of this dissertation is organized as follows. In chapter 2, we briefly review construction of fuzzy $\mathbf{CP}^k$ ($k = 1, 2, \cdots$), following [9, 10]. We rephrase known results such that relation to geometric quantization is transparent. In this chapter, we also present algebraic construction of fuzzy $\mathbf{CP}^k$ by use of creation and annihilation operators on a Hilbert space [11]. We follow the presentation given in an appendix of [49]. In chapter 3, we review construction of fuzzy S^4, following also [49]. Chapter 4 is devoted to application of fuzzy spaces to theories of gravity. We shall obtain Chang-MacDowell-Mansouri type matrix models for gravity, following the work of [68, 76]. Chapter 5 deals with application of fuzzy spaces to M(atrix) theory based on a recent work [123]. Finally, in chapter 6 we present brief conclusions.

2 Construction of fuzzy $\mathbf{CP}^k$

2.1 Hilbert space

A finite dimensional Hilbert space $\mathcal{H}_N$ for fuzzy $\mathbf{CP}^k = SU(k+1)/U(k)$ ($k = 1, 2, \cdots$) is given by holomorphic sections of a complex line bundle over $\mathbf{CP}^k$. As discussed in section 1.2, the holomorphic sections of the complex line bundle should correspond to a unitary irreducible representation (UIR) of $G = SU(k+1)$. Representation of $SU(k+1)$ ($k \geq 2$) is given by a general form (p, q) ($p, q = 0, 1, 2, \cdots$) if we use a standard tensor method. Notion of holomorphicity in the representation of G can be realized by totally symmetric part of the representation, $i.e.$, $(n, 0)$, where n is the rank of the representation ($n = 1, 2, \cdots$). The other totally symmetric representation $(0, n)$ corresponds to antiholomorphic part of the $SU(k+1)$ representation and the (p, p)-representation

gives real representation. For $SU(2)$ (corresponding to $k = 1$), the representation is given by a single component, say (p), so there is no real representation. (Because of this, the $SU(2)$ representation is sometimes called pseudo-real.) The dimension of $\mathcal{H}_N$ is then determined by that of the $(n,0)$-representation for $SU(k+1)$;

$$N^{(k)} \equiv \dim(n,0) = \frac{(n+k)!}{k! \, n!}. \tag{2}$$

Consequently, matrix algebra of fuzzy $\mathbf{CP}^k$ is realized by $N^{(k)} \times N^{(k)}$-matrices. Operators or matrix functions on fuzzy $\mathbf{CP}^k$ are expressed by linear combinations of $N^{(k)} \times N^{(k)}$-matrix representations of the algebra of $SU(k+1)$ in the $(n,0)$-representation. Let L_A, with $A = 1,2,\cdots,k^2+2k = \dim SU(k+1)$, denote such matrix representations. We need to impose extra constraints on them otherwise the Hilbert space is defined simply on R^{k^2+2k} without any information of $\mathbf{CP}^k$. As we shall discuss later, such extra constraints can be imposed at an algebraic level in terms of L_A but, in order to construct $\mathcal{H}_N$ along a program of geometric quantization, we would rather consider a holomorphic line bundle on $\mathbf{CP}^k$ first and implement the relevant extra constraints in it.

To begin with, we write down a holomorphic $U(1)$ bundle $\Psi_m^{(n)}$ as

$$\Psi_m^{(n)}(g) = \sqrt{N^{(k)}} \mathcal{D}_{mN^{(k)}}^{(n,0)}(g), \tag{3}$$

$$\mathcal{D}_{mN^{(k)}}^{(n,0)}(g) = \langle (n,0),m|\hat{g}|(n,0),N^{(k)} \rangle \tag{4}$$

where $|(n,0),m\rangle$ $(m = 1,2,\cdots,N^{(k)})$ denote the states on the Hilbert space $\mathcal{H}_N$, $|(n,0),N^{(k)}\rangle$ is the highest or lowest weight state, g is an element of $G = SU(k+1)$ and $\hat{g}$ is a corresponding operator acting on these states. $\mathcal{D}_{mN^{(k)}}^{(n,0)}(g)$ is known as Wigner $\mathcal{D}$-functions for $SU(k+1)$ in the $(n,0)$-representation. The lower indices label the states of this representation, allowing us to interpret the $\mathcal{D}$-functions as matrix elements. As mentioned in chapter 1, G acts on the Hilbert space as right translation. Let R_A denote the right-translation operator on g;

$$R_A \, g = g \, t_A \tag{5}$$

where t_A are the generator of G in the fundamental representation $(1,0)$. The element g is given by $g = \exp(it_A\theta^A)$ with continuous parameters θ^A. We now consider the splitting of t_A's to those of $U(k) = SU(k) \times U(1)$ subalgebra and the rest of them, i.e., those relevant to $\mathbf{CP}^k$. Let t_j $(j = 1,2,\cdots,k^2)$ and t_{k^2+2k} denote the generators of $U(k) \subset SU(k+1)$, t_{k^2+2k} being a $U(1)$ element of the $U(k)$, and let $t_{\pm i}$ $(i = 1,2,\cdots,k)$ denote the rest of t_A's. One can consider $t_{\pm i}$ as a combination of rasing-type (t_{+i}) and lowering-type (t_{-i}) operators acting on the states of $\mathcal{H}_N$. Choosing $|(n,0),N^{(k)}\rangle$ to be the lowest weight state, we then find

$$R_j \mathcal{D}_{mN^{(k)}}^{(n,0)}(g) = 0 \quad (j = 1,2,\cdots,k^2), \tag{6}$$

$$R_{k^2+2k} \mathcal{D}_{mN^{(k)}}^{(n,0)}(g) = -\frac{nk}{\sqrt{2k(k+1)}} \mathcal{D}_{mN^{(k)}}^{(n,0)}(g), \tag{7}$$

$$R_{-i} \mathcal{D}_{mN^{(k)}}^{(n,0)}(g) = 0. \tag{8}$$

Equations (6) and (7) indicate that $\Psi_m^{(n)}(g) \sim \mathcal{D}_{mN^{(k)}}^{(n,0)}(g)$ is a $U(1)$ bundle over $\mathbf{CP}^k$. One can also check that under the $U(1)$ transformations, $g \to gh$ with $h = \exp(it_{k^2+2k}\theta)$, $\theta \equiv \theta^{k^2+2k}$, $\Psi_m^{(n)}(g)$ transforms as

$$\Psi_I^{(n)}(g) \to \Psi_m^{(n)}(gh) = \Psi_m^{(n)}(g) \exp\left(-i\frac{nk}{\sqrt{2k(k+1)}}\theta\right). \tag{9}$$

Note that we use the fact that the states in the $(n,0)$-representation is constructed by products of the states in the $(1,0)$-representation. We also use a conventional choice of t_{k^2+2k} as

$$t_{k^2+2k} = \frac{1}{\sqrt{2k(k+1)}} \mathrm{diag}(1,1,\cdots,1,-k). \tag{10}$$

In terms of geometric quantization, equation (8) corresponds to the polarization condition on a prequantum $U(1)$ bundle. The Hilbert space is therefore constructed as sections of the holomorphic $U(1)$ bundle $\Psi_m^{(n)}$. The square-integrability of $\mathcal{H}_N$ is guaranteed by the orthogonality condition of the Wigner $\mathcal{D}$ function;

$$\int d\mu(g)\mathcal{D}_{m,k}^{*(R)}(g)\mathcal{D}_{m',k'}^{(R')}(g) = \delta^{RR'}\frac{\delta_{mm'}\delta_{kk'}}{\dim R} \tag{11}$$

where $\mathcal{D}_{m,k}^{*(R)}(g) = \mathcal{D}_{k,m}^{(R)}(g^{-1})$, R denotes the representation of $G = SU(k+1)$, and $d\mu(g)$ is the Haar measure of $G = SU(k+1)$ normalized to unity; $\int d\mu(g) = 1$. The orthogonality condition of our interest is given by

$$\int d\mu(g)\mathcal{D}_{m,N^{(k)}}^{*(n,0)}(g)\mathcal{D}_{m',N^{(k)}}^{(n,0)}(g) = \frac{\delta_{mm'}}{N^{(k)}} . \tag{12}$$

The normalization factor $\sqrt{N^{(k)}}$ in (3) is determined by this relation, which also provides a natural definition of the inner product of $\Psi_m^{(n)}$. Note that the integrand is invariant under $U(k)$, so we may use the Haar measure of $SU(k+1)$ for the integration over $\mathbf{CP}^k$.

The Kähler two-form (or, equivalently, the symplectic structure) of $\mathbf{CP}^k$ in terms of g is obtained as follows. As in (5), $g \in G = SU(k+1)$ is considered as a $(k+1) \times (k+1)$ matrix. In order to obtain coordinates on $\mathbf{CP}^k = SU(k+1)/U(k)$ out of g, we need to impose the identification $g \sim gh$ where $h \in H = U(k)$. Such subgroup elements h can be represented by t_{k^2+2k} in (10) and

$$h_{SU(k)} = \begin{pmatrix} h_k & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

where h_k is a $(k \times k)$-matrix. The coordinates on $\mathbf{CP}^k$ are then defined by matrix elements $g_{\alpha,k+1}$ ($\alpha = 1, 2, \cdots, k+1$). Since $g^\dagger g = 1$, we have $g_{k+1,\alpha}^* g_{\alpha,k+1} = 1$. We now introduce the notation $u_\alpha \equiv g_{\alpha,k+1}$, $\bar{u} \cdot u = 1$. In terms of u_α's, the Wigner $\mathcal{D}$-functions (4) are written in a form of $\mathcal{D}^{(n)} \sim u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n}$. Homogeneous complex coordinates of $\mathbf{CP}^k$ are defined by $Z = (z_1, z_2, \cdots, z_k)^T$ with $Z \sim \lambda Z$, where λ is a nonzero complex number and T denotes transposition of the vector or $(1 \times k)$-matrix. u_α's are related to Z by

$$u_\alpha = \frac{1}{\sqrt{1 + \bar{z} \cdot z}}(1, z_1, z_2, \cdots, z_k)^T . \tag{14}$$

Using u_α, one can construct a one form

$$\begin{aligned} A &= -i\, u_\alpha^* du_\alpha \\ &= -\frac{i}{2}\left(\frac{\bar{u} \cdot du - d\bar{u} \cdot u}{\bar{u} \cdot u}\right) \end{aligned} \tag{15}$$

where $\bar{u} \cdot du = \bar{u}^\alpha du_\alpha$, etc. The Kähler two-form can be identified with $dA \sim du_\alpha^* du_\alpha$, since it is closed but it is not exact. Explicitly, the Kähler form Ω is written as

$$\begin{aligned} \Omega &= -i\left(\frac{d\bar{u} \cdot du}{\bar{u} \cdot u} - \frac{d\bar{u} \cdot u\bar{u} \cdot du}{(\bar{u} \cdot u)^2}\right) \\ &= -i\left(\frac{d\bar{z}_i dz_i}{1 + \bar{z} \cdot z} - \frac{d\bar{z} \cdot z\bar{z} \cdot dz_i}{(1 + \bar{z} \cdot z)^2}\right) . \end{aligned} \tag{16}$$

The one-form (15) is also expressed as

$$A = i\,\sqrt{\frac{2k}{k+1}}\,\operatorname{tr}(t_{k^2+2k}g^{-1}dg) . \tag{17}$$

This form suggests a general way to obtain a symplectic structure for a co-adjoint orbit defined by $\{gtg^{-1} : g \in G\}$ where $t \in \underline{G}$, with $\underline{G}$ denoting the algebra of a compact and semi-simple group G. Namely, we start from a one-form, $A \sim \operatorname{tr}(tg^{-1}dg)$, and then the symplectic two-form Ω_s is given by $\Omega_s = dA$. When t has degeneracy as in (10), the co-adjoint orbit is called degenerate. In our case, the stabilizer H_t, defined by $[H_t, t] = 0$, $H_t \subset G$ as in chapter 1, becomes the $U(k)$ subgroup of $G = SU(k+1)$. While t does not have degeneracy, the stabilizer becomes the maximal torus of G. In this case, the co-adjoint orbit also has Kähler structure and dA gives its Kähler form.

2.2 Symbols and star products

We define the symbol of a matrix operator A_{ms} $(m, s = 1, 2, \cdots, N^{(k)})$ on the Hilbert space of fuzzy $\mathbf{CP}^k$ by

$$
\begin{aligned}
\langle \hat{A} \rangle &\equiv \sum_{ms} \mathcal{D}^{(n,0)}_{m,N^{(k)}}(g) \, A_{ms} \, \mathcal{D}^{*(n,0)}_{s,N^{(k)}}(g) \\
&= \langle (n,0), N^{(k)} | \hat{g}^T \hat{A} \hat{g}^* | (n,0), N^{(k)} \rangle
\end{aligned}
\tag{18}
$$

The star product of fuzzy $\mathbf{CP}^k$ is defined by $\langle \hat{A}\hat{B} \rangle \equiv \langle \hat{A} \rangle * \langle \hat{B} \rangle$. From (18), $\langle \hat{A}\hat{B} \rangle$ can be written as

$$
\begin{aligned}
\langle \hat{A}\hat{B} \rangle &= \sum_{msr} A_{mr} B_{rs} \, \mathcal{D}^{(n,0)}_{m,N^{(k)}}(g) \mathcal{D}^{*(n,0)}_{s,N^{(k)}}(g) \\
&= \sum_{msrr'p} \mathcal{D}^{(n,0)}_{m,N^{(k)}}(g) A_{mr} \mathcal{D}^{*(n,0)}_{r,p}(g) \, \mathcal{D}^{(n,0)}_{r',p}(g) B_{r's} \mathcal{D}^{*(n,0)}_{s,N^{(k)}}(g)
\end{aligned}
\tag{19}
$$

where we use the relation $\sum_p \mathcal{D}^{*(n,0)}_{r,p}(g) \mathcal{D}^{(n,0)}_{r',p}(g) = \delta_{rr'}$. In the sum over $p = 1, 2, \cdots, N^{(k)}$ on the right hand side of (19), the term corresponding to $p = N^{(k)}$ gives the product $\langle \hat{A} \rangle \langle \hat{B} \rangle$. The terms corresponding to $p < N^{(k)}$ may be expressed in terms of the raising operators R_{+i} $(i = 1, 2, \cdots, k)$ as

$$
\mathcal{D}^{(n,0)}_{r',p}(g) = \sqrt{\frac{(n-s)!}{n! \, i_1! i_2! \cdots i_k!}} R_{+1}^{i_1} R_{+2}^{i_2} \cdots R_{+k}^{i_k} \mathcal{D}^{(n,0)}_{r',N^{(k)}}(g)
\tag{20}
$$

where $s = i_1 + i_2 + \cdots + i_k$ and the state $|(n,0), p\rangle$ is specified by

$$
R_{k^2+2k} \mathcal{D}^{(n,0)}_{r',p}(g) = \frac{-nk + sk + s}{\sqrt{2k(k+1)}} \, \mathcal{D}^{(n,0)}_{r',p}(g) \,.
\tag{21}
$$

Since $R_{+i} \mathcal{D}^{*(n)}_{s,-n} = 0$, we can also write

$$
\begin{aligned}
\sum_{r's} \left[R_{+i} \mathcal{D}^{(n,0)}_{r',N^{(k)}}(g) \right] B_{r's} \mathcal{D}^{*(n,0)}_{s,N^{(k)}}(g) &= \sum_{r's} \left[R_{+i} \mathcal{D}^{(n,0)}_{r',N^{(k)}}(g) B_{r's} \mathcal{D}^{*(n,0)}_{s,N^{(k)}}(g) \right] \\
&= R_{+i} \langle \hat{B} \rangle \,.
\end{aligned}
\tag{22}
$$

The conjugate of (20) can be written in terms of R_{-i} by use of the relation $R^*_{+i} = -R_{-i}$. Combining (20)-(22), we can express (19) as

$$
\begin{aligned}
\langle \hat{A}\hat{B} \rangle &= \sum_{s=0}^{n} (-1)^s \frac{(n-s)!}{n! s!} \sum_{i_1+i_2+\cdots+i_k=s}^{n} \frac{s!}{1! i_2! \cdots i_k!} \\
&\quad \times R_{-1}^{i_1} R_{-2}^{i_2} \cdots R_{-k}^{i_k} \langle \hat{A} \rangle \, R_{+1}^{i_1} R_{+2}^{i_2} \cdots R_{+k}^{i_k} \langle \hat{B} \rangle \\
&\equiv \langle \hat{A} \rangle * \langle \hat{B} \rangle.
\end{aligned}
\tag{23}
$$

This is a general expression for the star product of matrix functions on fuzzy $\mathbf{CP}^k$. The term corresponding to $s = 0$ gives the ordinary product $\langle \hat{A} \rangle \langle \hat{B} \rangle$ and the successive terms are suppressed by powers of n as $n \to \infty$.

This form of star product, first obtained by Karabali and Nair in [40], is suitable for the discussion of large n (or $N = N^{(k)}$) limit. For example, the symbol of the commutator of matrix functions is given by

$$
\begin{aligned}
\langle [\hat{A}, \hat{B}] \rangle &= -\frac{1}{n} \sum_{i=1}^{k} \left(R_{-i} \langle \hat{A} \rangle R_{+i} \langle \hat{B} \rangle - R_{-i} \langle \hat{B} \rangle R_{+i} \langle \hat{A} \rangle \right) \, + \, \mathcal{O}(1/n^2) \\
&= \frac{i}{n} \{ \langle \hat{A} \rangle, \langle \hat{B} \rangle \} \, + \, \mathcal{O}(1/n^2)
\end{aligned}
\tag{24}
$$

where the term involving the actions of $R_{\pm i}$'s on the symbols can be proven to be the Poisson bracket on $\mathbf{CP}^k$. For detailed description, see [40, 9]. The relation (24) shows an explicit correspondence between the algebra of Poisson brackets for functions on $\mathbf{CP}^k$ and the algebra of commutation relations for functions on fuzzy $\mathbf{CP}^k$ in the large n limit, indicating that the construction of fuzzy spaces is essentially the same as the quantization of symplectic manifolds.

From (12), the trace of a matrix operator A can be expressed as

$$
\begin{aligned}
\mathrm{Tr} A &= \sum_m A_{mm} = N^{(k)} \int d\mu(g) \mathcal{D}^{(n,0)}_{m,N^{(k)}} A_{mm'} \mathcal{D}^{*(n,0)}_{m',N^{(k)}} \\
&= N^{(k)} \int d\mu(g) \langle \hat{A} \rangle .
\end{aligned}
\tag{25}
$$

The trace of the product of two matrices A, B, is also given by

$$
\mathrm{Tr} AB = N^{(k)} \int d\mu(g) \langle \hat{A} \rangle * \langle \hat{B} \rangle .
\tag{26}
$$

2.3 Large N limit

In this subsection, following [9, 41], we briefly review the large n limit of the symbol for an arbitrary matrix function $f(L_A)$, where L_A ($A = 1, 2, \cdots, k^2 + 2k$) are, as before, the $N^{(k)} \times N^{(k)}$-matrix representations of the algebra of $SU(k+1)$ in the $(n,0)$-representation. From (18), the symbol of $L_B A$, A being an arbitrary $N^{(k)} \times N^{(k)}$-matrix, is given by $\langle \hat{L}_B \hat{A} \rangle = \langle N | \hat{g}^T \hat{L}_B A \hat{g}^* | N \rangle$, where $|N\rangle \equiv |(n,0), N^{(k)}\rangle$. We now express the factor $\hat{g}^T \hat{L}_B \hat{g}^*$ as

$$
\begin{aligned}
\hat{g}^T \hat{L}_B \hat{g}^* &= S_{BC}(g) \hat{L}_C \\
&= \frac{1}{2}(S_{B+i} \hat{L}_{-i} + S_{B-i} \hat{L}_{+i}) + S_{Bj} \hat{L}_j + S_{Bk^2+2k} \hat{L}_{k^2+2k} ,
\end{aligned}
\tag{27}
$$

$$
S_{BC}(g) \equiv 2 \, \mathrm{tr}(g^T t_B g^* t_C).
\tag{28}
$$

Note that, in terms of $\hat{L}_C$ acting on $\langle N |$ from the right, the relations (6)-(8) can be expressed as

$$
\langle N | \hat{L}_j = \langle N | \hat{L}_{+i} = 0 , \quad \langle N | \hat{L}_{k^2+2k} = -\frac{nk}{\sqrt{2k(k+1)}} \langle N | .
\tag{29}
$$

The symbol $\langle \hat{L}_B \hat{A} \rangle$ is then written as

$$
\begin{aligned}
\langle \hat{L}_B \hat{A} \rangle &= S_{Bk^2+2k} \langle N | \hat{L}_{k^2+2k} \hat{g}^T \hat{A} \hat{g}^* | N \rangle + \frac{1}{2} S_{B+i} \langle N | \hat{L}_{-i} \hat{g}^T \hat{A} \hat{g}^* | N \rangle \\
&= \mathcal{L}_B \langle \hat{A} \rangle ,
\end{aligned}
\tag{30}
$$

$$
\mathcal{L}_B \equiv -\frac{nk}{\sqrt{2k(k+1)}} S_{Bk^2+2k} + \frac{1}{2} S_{B+i} \tilde{R}_{-i}
\tag{31}
$$

where $\tilde{R}_{-i}$ is defined by $\tilde{R}_{-i} g^T = \hat{L}_{-i} g^T$.

Assuming $\hat{A}$ as the $N^{(k)}$-dimensional identity matrix $\mathbf{1}$, we find that the symbol $\langle L_B \rangle$ is dominated by the quantity $S_{Bk^2+2k}(g)$ in the large n limit. One can in fact check that $-S_{Bk^2+2k}$ satisfy the algebraic constraints for the coordinates of $\mathbf{CP}^k$ which are, as we shall see later, given in (44)-(46).

By taking $\hat{A}$ itself as a product of $\hat{L}_A$'s, we can by iteration express symbols for any products of L_A's as

$$
\langle \hat{L}_{A_1} \hat{L}_{A_2} \cdots \hat{L}_{A_s} \rangle = \langle \mathcal{L}_{A_1} \mathcal{L}_{A_2} \cdots \mathcal{L}_{A_s} \cdot \mathbf{1} \rangle
\tag{32}
$$

where $s = 1, 2, \cdots$. Thus symbols of any matrix functions $f(\hat{L}_A)$ become the corresponding functions of S_{Ak^2+2k}, $\langle f(\hat{L}_A) \rangle \approx f(S_{Ak^2+2k})$, in the large n limit.

2.4 Algebraic construction

In this subsection, we present construction of fuzzy $\mathbf{CP}^k$ ($k = 1, 2, \cdots$) in the framework of the creation-annihilation operators [101, 37]. The coordinates Q_A of fuzzy $\mathbf{CP}^k$ can be defined in terms of L_A as

$$Q_A = \frac{L_A}{\sqrt{C_2^{(k)}}} \,, \tag{33}$$

satisfying the following two constraints

$$Q_A \, Q_A \;=\; \mathbf{1} \,, \tag{34}$$
$$d_{ABC} \, Q_A \, Q_B \;=\; c_{k,n} \, Q_C \tag{35}$$

where d_{ABC} is the totally symmetric symbol of $SU(k+1)$, $C_2^{(k)}$ is the quadratic Casimir of $SU(k+1)$ in the $(n, 0)$-representation

$$C_2^{(k)} = \frac{n \, k \, (n + k + 1)}{2 \, (k + 1)} \tag{36}$$

and $N^{(k)}$ is the dimension of $SU(k + 1)$ in the $(n, 0)$-representation given in (2).

In order to determine the coefficient $c_{k,n}$ in (35), we now notice that the $SU(k+1)$ generators in the $(n, 0)$-representation can be written by

$$\Lambda_A = a_i^\dagger \, (t_A)_{ij} \, a_j \tag{37}$$

where t_A ($A = 1, 2, \cdots, k^2 + 2k$) are the $SU(k+1)$ generators in the fundamental representation with normalization $\mathrm{tr}(t_A t_B) = \frac{1}{2}\delta_{AB}$ and $a_i^\dagger$, a_i ($i = 1, \cdots, k+1$) are the creation and annihilation operators acting on the states of $\mathcal{H}_N$ which are spanned by

$$| \, n_1, n_2, \cdots, n_{k+1} \, \rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2} \cdots (a_{k+1}^\dagger)^{n_{k+1}} \, | \, 0 \, \rangle \tag{38}$$

with the following relations

$$a_i^\dagger a_i \, | \, n_1, n_2, \cdots, n_{k+1} \, \rangle \;=\; (n_1 + n_2 + \cdots + n_{k+1}) \, | \, n_1, n_2, \cdots, n_{k+1} \, \rangle$$
$$=\; n \, | \, n_1, n_2, \cdots, n_{k+1} \, \rangle \,, \tag{39}$$
$$a_i \, | \, 0 \, \rangle \;=\; 0 \,. \tag{40}$$

Notice that the condition (40) corresponds to the polarization condition in the context of geometric quantization.

Using the completeness relation for t_A's

$$(t_A)_{ij} \, (t_A)_{kl} = \frac{1}{2} \left(\delta_{il} \, \delta_{jk} \; - \; \frac{1}{k+1} \, \delta_{ij} \, \delta_{kl} \right) \tag{41}$$

and the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$, we can check $\Lambda_A \Lambda_A = C_2^{(k)}$, where the creation and annihilation operators act on the states of the form (38) from the left. We also find

$$d_{ABC} \, \Lambda_B \, \Lambda_C \;=\; (k - 1) \left(\frac{n}{k+1} + \frac{1}{2} \right) \, a_i^\dagger \, (t_A)_{ij} \, a_j$$
$$=\; (k - 1) \left(\frac{n}{k+1} + \frac{1}{2} \right) \, \Lambda_A \,. \tag{42}$$

Representing Λ_A by L_A, we can determine the coefficient $c_{k,n}$ in (35) by

$$c_{k,n} \;=\; \frac{(k - 1)}{\sqrt{C_2^{(k)}}} \left(\frac{n}{k+1} + \frac{1}{2} \right). \tag{43}$$

For $k \ll n$, we have

$$c_{k,n} \;\longrightarrow\; c_k \;=\; \sqrt{\frac{2}{k(k+1)}} \, (k - 1) \tag{44}$$

and this leads to the constraints for the coordinates q_A of $\mathbf{CP}^k$

$$q_A \, q_A \;=\; 1 \,, \tag{45}$$

$$d_{ABC} \, q_A \, q_B \;=\; c_k \, q_C \,. \tag{46}$$

The second constraint (46) restricts the number of coordinates to be $2k$ out of $k^2 + 2k$. For example, in the case of $\mathbf{CP}^2 = SU(3)/U(2)$ this constraint around the pole of $A = 8$ becomes $d_{8BC}q_8 q_B = \frac{1}{\sqrt{3}}q_C$. Normalizing the 8-coordinate to be $q_8 = -2$, we find the indices of the coordinates are restricted to 4, 5, 6, and 7 with the conventional choice of the generators of $SU(3)$ as well as with the definition $d_{ABC} = 2\mathrm{tr}(t_A t_B t_C + t_A t_C t_B)$.

2.4.1 Matrix-Function Correspondence

The matrix-function correspondence for fuzzy $\mathbf{CP}^k$ can be expressed by

$$N^{(k)} \times N^{(k)} = \sum_{l=0}^{n} \dim(l,l) \tag{47}$$

where $\dim(l,l)$ is the dimension of $SU(k+1)$ in the (l,l)-representation. This expression indicates that the number of matrix elements coincides with the number of coefficients in an expansion series of truncated functions on $\mathbf{CP}^k = SU(k+1)/U(k)$. We need the real (l,l)-representation in order to have an expansion of scalar functions on $\mathbf{CP}^k$. Symbolically the correspondence is written as

$$(n,0) \bigotimes (0,n) = \bigoplus_{l=0}^{n} (l,l) \tag{48}$$

in terms of the dimensionality of $SU(k+1)$. The left-hand-side of (48) can be interpreted from the fact that $\Lambda_A = a_i^\dagger (t_A)_{ij} a_j \sim a_i^\dagger a_j$ transforms like $(n,0) \otimes (0,n)$. The right-hand-side of (48), on the other hand, can be interpreted by a usual tensor analysis, i.e., $\dim(l,l)$ is the number of ways to construct tensors of the form $T^{i_1, i_2, \cdots, i_l}_{j_1, j_2, \cdots, j_l}$ such that the tensor is traceless and totally symmetric in terms of $i, j = 1, 2, \cdots, k+1$.

3 Construction of fuzzy S^4

3.1 Introduction to fuzzy S^4

As we have witnessed for more than a decade, the idea of fuzzy S^2 [5] has been one of the guiding forces for us to investigate fuzzy spaces. For example, as discussed in the previous chapter, fuzzy complex projective spaces $\mathbf{CP}^k$ ($k = 1, 2, \cdots$) are successfully constructed in the same spirit as the fuzzy S^2. From physicists' point of view, it is of great interest to obtain a four-dimensional fuzzy space. The well-defined fuzzy $\mathbf{CP}^2$ is not suitable for this purpose, since $\mathbf{CP}^2$ does not have a spin structure [37]. Construction of fuzzy S^4 is then physically well motivated. (Notice that fuzzy spaces are generally obtained for compact spaces and that S^4 is the simplest four-dimensional compact space that allows a spin structure.) Since S^4 naturally leads to $\mathbf{R}^4$ at a certain limit, the construction of fuzzy S^4 would also shed light on the studies of noncommutative Euclidean field theory.

There have been several attempts to construct fuzzy S^4 from a field theoretic point of view [102, 45, 46] as well as from a rather mathematical interest [43, 42, 103], however, it would be fair to say that the construction of fuzzy S^4 has not yet been satisfactory. In [43, 42], the construction is carried out with a projection from some matrix algebra (which in fact coincides with the algebra of fuzzy $\mathbf{CP}^3$) and, owing to this forcible projection, it is advocated that fuzzy S^4 obeys a non-associative algebra. Associativity is recovered in the commutative limit, however, non-associativity limits the use of fuzzy S^4 for physical models. (Non-associativity is not compatible with unitarity of the algebra for symmetry operations in any physical models.) Further, non-associativity is not compatible with the definition of fuzzy spaces (1) in which the algebra of fuzzy spaces is given by the algebra of finite dimensional matrices. In [45, 46], fuzzy S^4 is alternatively considered in a way

of constructing a scalar field theory on it, based on the fact that $\mathbf{CP}^3$ is a $\mathbf{CP}^1$ (or S^2) bundle over S^4. While the resulting action leads to a correct commutative limit, it is, as a matter of fact, made of a scalar field on fuzzy $\mathbf{CP}^3$. Its non-S^4 contributions are suppressed by an additional term. (Such a term can be obtained group theoretically.) The action is interesting but the algebra of fuzzy S^4 is still unclear. In this sense, the approach in [45, 46] is related to that in [43, 42]. Either approach uses a sort of brute-force method which eliminates unwanted degrees of freedom from fuzzy $\mathbf{CP}^3$. Such a method gives a correct counting for the degrees of freedom of fuzzy S^4, but it does not clarify the construction of fuzzy S^4 *per se*, as a matrix approximation to S^4. This is precisely what we attempt to do in this chapter. Note that the term "fuzzy S^4" is also used, mainly in the context of M(atrix) theory, *e.g.*, in [104, 105], for the space developed in [82]. This space actually obeys the constraints for fuzzy $\mathbf{CP}^3$. We shall discuss this point later in section 5.5.

In [103], the construction of fuzzy S^4 is considered through fuzzy $S^2 \times S^2$. This allows one to describe fuzzy S^4 with some concrete matrix configurations. However, the algebra is still non-associative and one has to deal with non-polynomial functions on fuzzy S^4. Since those functions do not naturally become polynomials on S^4 in the commutative limits, there is no proper matrix-function correspondence. The matrix-function correspondence is a correspondence between functions on fuzzy spaces and truncated functions on the corresponding commutative spaces. In the case of fuzzy $\mathbf{CP}^k$, the fuzzy functions are represented by full $(N \times N)$-matrices, so the product of them is given by matrix multiplication which leads to associativity of the algebra for fuzzy $\mathbf{CP}^k$. As we have seen in (23), the star products of fuzzy $\mathbf{CP}^k$ reduce to ordinary commutative products of functions (or symbols) on $\mathbf{CP}^k$ in the large N limit. In this case, one may check the matrix-function correspondence by the matching between the number of matrix elements and that of truncated functions. This matching, however, is not enough to warrant the matrix-function correspondence of fuzzy S^4; further we need to confirm the correspondence between the product of fuzzy functions and that of truncated functions. In order to do so, it is important to construct fuzzy S^4 with a clear matrix configuration (which should be different from the proposal in [103]).

The plan of this chapter is as follows. In section 3.2, following Medina and O'Connor in [45], we propose construction of fuzzy S^4 by use of the fact that $\mathbf{CP}^3$ is an S^2 bundle over S^4. We shall obtain fuzzy S^4, imposing a further constraint on fuzzy $\mathbf{CP}^3$. The extra constraint is expressed as an algebraic constraint such that it enables us to describe the algebra of fuzzy S^4 in terms of the algebra of $SU(4)$ in the $(n, 0)$-representation. The emerging algebra is not a subalgebra of fuzzy $\mathbf{CP}^3$ since the algebra of fuzzy $\mathbf{CP}^3$ is defined globally by $\underline{SU(4)}$ with the algebraic constraints given in (34) and (35) for $k = 3$. The algebra of fuzzy S^4 is obtained from $\underline{SU(4)}$ as well with the extra constraint on top of these fuzzy $\mathbf{CP}^3$ constraints. As mentioned in chapter 1, the algebra of fuzzy S^4 is consequently given by a subset of $\underline{SU(4)}$, preserving closure and associativity of the algebra. The structure of algebra becomes clearer in the commutative limit which will be considered in terms of homogeneous coordinates of $\mathbf{CP}^3$. With these coordinates we shall explicitly show that the extra constraint for fuzzy S^4 has a correct commutative limit. The idea of constructing fuzzy spaces from another by means of an additional constraint was in fact first proposed by Nair and Randjbar-Daemi in obtaining fuzzy $S^3/\mathbf{Z}_2$ from fuzzy $S^2 \times S^2$ [48]. Our construction of fuzzy S^4 provides another example of such construction.

In section 3.3, we show the matrix-function correspondence of fuzzy S^4. After a brief review of the case of fuzzy S^2, we shall present different calculations of the number of truncated functions on S^4. We then show that this number agrees with the number of degrees of freedom for fuzzy S^4. This number turns out to be *a sum of absolute squares*, and hence we can choose a block-diagonal matrix configuration for functions on fuzzy S^4. This block-diagonal form is also induced from the structure of fuzzy functions. The star products are determined by matrix products of such functions and naturally reduce to commutative products, similarly to what happens in fuzzy $\mathbf{CP}^3$. This leads to the precise matrix-function correspondence of fuzzy S^4. Of course, such a matrix realization of fuzzy S^4 is not the only one that leads to the correspondence; there are a number of ways related to the ways of allocating the absolute squares to form any block-diagonal matrices. Our construction is, however, useful in comparison with the fuzzy $\mathbf{CP}^3$.

The fact that $\mathbf{CP}^3$ is an S^2 bundle over S^4 can be seen by a Hopf map, $S^7 \to S^4$ with the fiber being S^3. One can derive the map, noticing that the S^4 is the quaternion projective space. In the same reasoning, octonions define a Hopf map, $S^{15} \to S^8$ with its fiber being S^7, giving us another

fact that $\mathbf{CP}^7$ is a $\mathbf{CP}^3$ bundle over S^8. Following these mathematical facts, in section 3.4, we apply our construction to fuzzy S^8 and outline its construction.

3.2 Construction of fuzzy S^4

We begin with construction of fuzzy $\mathbf{CP}^3$. The algebraic construction of fuzzy $\mathbf{CP}^k$ ($k = 1, 2, \cdots$) is generically given in section 2.4; here we briefly rephrase it in the case of $k = 3$. The coordinates Q_A of fuzzy $\mathbf{CP}^3$ can be defined by

$$Q_A = \frac{L_A}{\sqrt{C_2^{(3)}}} \tag{49}$$

where L_A are $N^{(3)} \times N^{(3)}$-matrix representations of $SU(4)$ generators in the $(n, 0)$-representation. The coordinates satisfy the following constraints:

$$Q_A\, Q_A = \mathbf{1}, \tag{50}$$
$$d_{ABC}\, Q_A\, Q_B = c_{3,n}\, Q_C. \tag{51}$$

As discussed before, in the large n limit these constraints become constraints for the coordinates on $\mathbf{CP}^3$ as embedded in $\mathbf{R}^{15}$. In (49)-(51), $C_2^{(3)}$, $\mathbf{1}$, d_{ABC} and $c_{3,n}$ are all defined in chapter 2, including the relation

$$N^{(3)} = \frac{1}{6}(n + 1)(n + 2)(n + 3). \tag{52}$$

We now consider the decomposition, $SU(4) \to SU(2) \times SU(2) \times U(1)$, where the two $SU(2)$'s and one $U(1)$ are defined by

$$\left(\begin{array}{c|c} SU(2) & 0 \\ \hline 0 & 0 \end{array} \right), \qquad \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underline{SU(2)} \end{array} \right), \qquad \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \tag{53}$$

in terms of the (4×4)-matrix generators of $SU(4)$ in the fundamental representation. (Each $\underline{SU(2)}$ denotes the algebra of $SU(2)$ group in the (2×2)-matrix representation.) As we shall see in subsections 3.2.1 and 3.3.1, functions on S^4 are functions on $\mathbf{CP}^3 = SU(4)/U(3)$ which are invariant under transformations of $H \equiv SU(2) \times U(1)$, H being relevant to the above decomposition of $SU(4)$. In order to obtain functions on fuzzy S^4, we thus need to require

$$[\mathcal{F}, L_\alpha] = 0 \tag{54}$$

where $\mathcal{F}$ denote matrix-functions of Q_A's and L_α are generators of H represented by $N^{(3)} \times N^{(3)}$-matrices. Construction of fuzzy S^4 can be carried out by imposing the additional constraint (54) onto the functions on fuzzy $\mathbf{CP}^3$. What we claim is that the further condition (54) makes the functions $\mathcal{F}(Q_A)$ become functions on fuzzy S^4. This does not mean that fuzzy S^4 is a subset of fuzzy $\mathbf{CP}^3$. Notice that Q_A's are defined in $\mathbf{R}^{15}$ ($A = 1, \cdots, 15$) with the algebraic constraints (50) and (51). While locally, say around the pole of $A = 15$ in (51), one can specify the six coordinates of fuzzy $\mathbf{CP}^3$, globally they are embedded in $\mathbf{R}^{15}$. Equation (54) is a global constraint in this sense. So the algebra of fuzzy S^4 is given by a subset of $\underline{SU(4)}$. The emerging algebraic structure of fuzzy S^4 will be clearer when we consider the commutative limit of our construction.

3.2.1 Commutative limit

As shown in section 2.3, in the large n limit we can approximate Q_A to the commutative coordinates on $\mathbf{CP}^3$;

$$Q_A \approx \phi_A = -2\,\mathrm{tr}(g^\dagger t_A g t_{15}) \tag{55}$$

which indeed obey the following constraints for $\mathbf{CP}^3$

$$\phi_A\, \phi_A = 1, \qquad d_{ABC}\, \phi_A\, \phi_B = \sqrt{\frac{2}{3}}\, \phi_C. \tag{56}$$

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Algebraic constraints for $\mathbf{CP}^k$ are in general given in (44)-(46). In (55), t_A's are the generators of $SU(4)$ in the fundamental representation and g is a group element of $SU(4)$ given as a (4×4)-matrix. Truncated functions on $\mathbf{CP}^3$ are then written as

$$f_{\mathbf{CP}^3}(u, \bar{u}) \sim f^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_l} \bar{u}_{i_1} \bar{u}_{i_2} \cdots \bar{u}_{i_l} u_{j_1} u_{j_2} \cdots u_{j_l} \tag{57}$$

where $l = 0, 1, 2, \cdots, n$, $u_j = g_{j4}$, $\bar{u}_i = (g^\dagger)_{4i}$ and $\bar{u}_i u_i = 1$ $(i, j = 1, 2, 3, 4)$. One can describe $\mathbf{CP}^3$ in terms of four complex coordinates Z_i with the identification $Z_i \sim \lambda Z_i$ where λ is a nonzero complex number $(\lambda \in \mathbf{C} - \{0\})$. Following Penrose and others [99], we now write Z_i in terms of two spinors ω, π as

$$Z_i = (\omega_a, \pi_{\dot{a}}) = (x_{a\dot{a}}\pi_{\dot{a}}, \pi_{\dot{a}}) \tag{58}$$

where $a = 1, 2$, $\dot{a} = 1, 2$ and $x_{a\dot{a}}$ can be defined with the coordinates x_μ on S^4 via $x_{a\dot{a}} = (\mathbf{1}x_4 - i\vec{\sigma}\cdot\vec{x})$, $\vec{\sigma}$ being (2×2) Pauli matrices. The scale invariance $Z_i \sim \lambda Z_i$ leads to the invariance $\pi_{\dot{a}} \sim \lambda\pi_{\dot{a}}$. The $\pi_{\dot{a}}$'s then describe a $\mathbf{CP}^1 = S^2$. This shows the fact that $\mathbf{CP}^3$ is an S^2 bundle over S^4, or Penrose's projective twistor space. Note that, as in (14), we can parametrize u_i of (57) by the homogeneous coordinates Z_i, i.e., $u_i = \frac{Z_i}{\sqrt{Z \cdot \bar{Z}}}$.

Functions on S^4 can be considered as functions on $\mathbf{CP}^3$ which satisfy

$$\frac{\partial}{\partial\pi_{\dot{a}}} f_{\mathbf{CP}^3}(Z, \bar{Z}) = \frac{\partial}{\partial\bar{\pi}_{\dot{a}}} f_{\mathbf{CP}^3}(Z, \bar{Z}) = 0. \tag{59}$$

This implies that $f_{\mathbf{CP}^3}$ are further invariant under transformations of $\pi_{\dot{a}}$, $\bar{\pi}_{\dot{a}}$. In terms of the four-spinor Z, such transformations are expressed by

$$Z \to e^{it_\alpha \theta_\alpha} Z \tag{60}$$

where t_α represent the algebra of $H = SU(2) \times U(1)$ defined previously in regard to the decomposition of $SU(4)$ in (53). The coordinates ϕ_A in (55) can be written by $\phi_A(Z, \bar{Z}) \sim \bar{Z}_i(t_A)_{ij}Z_j$. Under an infinitesimal $(\theta_\alpha \ll 1)$ transformation of (60), the coordinates $\phi_A(Z, \bar{Z})$ transform as

$$\phi_A \to \phi_A + \theta_\alpha f_{\alpha AB} \phi_B \tag{61}$$

where f_{ABC} is the structure constant of $SU(4)$. The constraint (59) is then rewritten as

$$f_{\alpha AB} \phi_B \frac{\partial}{\partial\phi_A} f_{\mathbf{CP}^3} = 0 \tag{62}$$

where $f_{\mathbf{CP}^3}$ are seen as functions of ϕ_A's rather than that of $(Z, \bar{Z})$. Note that ϕ_A's in (62) are defined by (55), i.e., they are globally defined on $\mathbf{R}^{15}$.

From the relation $\phi_A \sim \bar{Z}_i(t_A)_{ij}Z_j$, we find $f_{\alpha AB} \phi_B \sim \bar{Z}_i([t_A, t_\alpha])_{ij}Z_j$ where t_α are the generators of $H = SU(2) \times U(1) \subset SU(4)$ as before. The constraint (59) or (62) is then realized by $[t_A, t_\alpha] = 0$, which can be considered as a commutative implementation of the fuzzy constraint (54). Specifically, we may choose $t_\alpha = \left\{ t_1, t_2, t_3, \sqrt{\frac{2}{3}}t_8 + \sqrt{\frac{1}{3}}t_{15} \right\}$ in the conventional choices of the generators of $SU(4)$ in the fundamental representation. The constraint $[t_A, t_\alpha] = 0$ then restricts A to be $A = 8, 13, 14$, and 15. This is, of course, a local analysis. The constraint $[t_A, t_\alpha] = 0$ does globally define S^4 as embedded in $\mathbf{R}^{15}$ similarly to how we have defined $\mathbf{CP}^3$. The number of $\mathbf{CP}^3$ coordinates ϕ_A is locally restricted to be six because of the algebraic constraints in (56). On top of these, the constraint $[t_A, t_\alpha] = 0$ further restricts the number of coordinates to be four, which is correct for the coordinates on S^4.

Functions on S^4 are polynomials of $\phi_A = -2\text{tr}(g^\dagger t_A g t_{15})$ which obey $[t_A, t_\alpha] = 0$. Products of functions are determined by the products of such t_A's. Extension to the fuzzy case is essentially done by replacing t_A with L_A, where L_A is the matrix representation of the algebra of $SU(4)$ in the totally symmetric $(n, 0)$-representation. The algebra of fuzzy S^4 naturally becomes associative in the commutative limit, while associativity of fuzzy S^4, itself, will be discussed in the next section, where we shall present a concrete matrix configuration of fuzzy S^4 so that the associativity is obviously seen. Even without any such matrix realizations, we can extract another property of the algebra from the condition (54), that is, closure of the algebra; since functions on fuzzy S^4 are represented by matrices, it is easily seen that products of such functions also obey the condition (54). In what follows, we shall clarify these points in some detail.

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3.3 Matrix-function correspondence

In this section, we examine our construction of fuzzy S^4 by confirming its matrix-function correspondence. To show a one-to-one correspondence, one needs to say two things: (a) a matching between the number of matrix elements for fuzzy S^4 and the number of truncated functions on S^4; and (b) a correspondence between the product of functions on fuzzy S^4 and that on S^4. It is now suggestive to take a moment to review how (a) and (b) are fulfilled in the case of fuzzy $S^2 = SU(2)/U(1)$. Let $\mathcal{D}_{mn}^{(j)}(g)$ be Wigner $\mathcal{D}$-functions for $SU(2)$. As we have discussed in section 2.1, these are the spin-j matrix representations of an $SU(2)$ group element g; $\mathcal{D}_{mn}^{(j)}(g) = \langle jm|\hat{g}|jn\rangle$ $(m, n = -j, \cdots, j)$. Functions on S^2 can be expanded in terms of particular Wigner $\mathcal{D}$-functions, $\mathcal{D}_{m0}^{(j)}(g)$, which are invariant under a $U(1)$ right-translation operator acting on g. For definition of such an operator, see (5). Since the state $|j0\rangle$ has no $U(1)$ charge, right action of the $U(1)$ operator, R_3, on g makes $\mathcal{D}_{m0}^{(j)}(g)$ vanish, $R_3\mathcal{D}_{m0}^{(j)}(g) = 0$; in fact one can choose any fixed value $(m = -j, \cdots, j)$ for this $U(1)$ charge. The $\mathcal{D}$-functions are essentially the spherical harmonics, $\mathcal{D}_{m0}^{(l)} = \sqrt{\frac{4\pi}{2l+1}}(-1)^m Y_{-m}^l$, and so a truncated expansion can be written as $f_{S^2} = \sum_{l=0}^n \sum_{m=-l}^l f_m^l \mathcal{D}_{ml}^{(l)}$. The number of coefficients f_m^l are counted by $\sum_{l=0}^n (2l+1) = (n+1)^2$. This relation implements the condition (a) by defining functions on fuzzy S^2 as $(n+1) \times (n+1)$ matrices. The product of truncated functions at the same level of n is also expressed by the same number of coefficients. Therefore, the product corresponds to $(n+1) \times (n+1)$ matrix multiplication. This implies the condition (b). One can show an exact correspondence of products, following the general lines in section 2.2. Let f_{mn} $(m, n = 1, \cdots, n+1)$ be an element of matrix function-operator $\hat{f}$ on fuzzy S^2. As in (18), we define the symbol of the function as

$$\langle \hat{f} \rangle = \sum_{m,n} f_{mn} \mathcal{D}_{mj}^{*(j)}(g) \mathcal{D}_{nj}^{(j)}(g) \tag{63}$$

where $\mathcal{D}_{mj}^{*(j)}(g) = \mathcal{D}_{jm}^{(j)}(g^{-1})$. We here consider $|jj\rangle$ as the highest weight state. The star product of fuzzy S^2 is defined by $\langle \hat{f}\hat{g} \rangle = \langle \hat{f} \rangle * \langle \hat{g} \rangle$. From (63), we can write

$$
\begin{aligned}
\langle \hat{f}\hat{g} \rangle &= \sum_{mnl} f_{mn} g_{nl} \mathcal{D}_{mj}^{*(j)}(g) \mathcal{D}_{lj}^{(j)}(g) \\
&= \sum_{mnkrl} f_{mn} g_{kl} \mathcal{D}_{mj}^{*(j)}(g) \mathcal{D}_{nr}^{(j)}(g) \mathcal{D}_{kr}^{*(j)}(g) \mathcal{D}_{lj}^{(j)}(g)
\end{aligned} \tag{64}
$$

where we use the orthogonality of $\mathcal{D}$-functions $\sum_r \mathcal{D}_{nr}^{(j)}(g) \mathcal{D}_{kr}^{*(j)}(g) = \delta_{nk}$. Let R_- be the lowering operator in right action, we then find $R_-\mathcal{D}_{mn}^{(j)}(g) = \sqrt{(j+n)(j-n+1)}\mathcal{D}_{mn-1}^{(j)}(g)$. By iteration, (64) may be rewritten as

$$\langle \hat{f}\hat{g} \rangle = \sum_{s=0}^{2j} (-1)^s \frac{(2j-s)!}{s!(2j)!} R_-^s \langle \hat{f} \rangle R_+^s \langle \hat{g} \rangle \equiv \langle \hat{f} \rangle * \langle \hat{g} \rangle \tag{65}$$

where we use the relation $R_-^* = -R_+$. In the large j limit, the term with $s = 0$ in (65) dominates and this leads to an ordinary commutative product of $\langle \hat{f} \rangle$ and $\langle \hat{g} \rangle$. By construction, the symbols of functions on fuzzy S^2 can be regarded as commutative functions on S^2. The expression (65) therefore shows a one-to-one correspondence between the product of fuzzy functions and the product of truncated functions on S^2.

From (64) and (65), it is easily seen that the square-matrix configuration, in addition to the orthogonality of the $\mathcal{D}$-functions or of the states $|jm\rangle$, is the key ingredient for the condition (b) in the case of fuzzy S^2. Associativity of the star product is direct consequence of this matrix configuration. Suppose the number of truncated functions on some compact space is given by an absolute square. Then, following the above procedure, one may establish the matrix-function correspondence. As shown in (47), this is true for fuzzy $\mathbf{CP}^k$ in general. In the case of fuzzy $\mathbf{CP}^3$, the absolute square appears from

$$N^{(3)} \times N^{(3)} = \sum_{l=0}^n \dim(l,l), \tag{66}$$

$$\dim(l,l) \quad = \quad \frac{1}{12}(2l+3)(l+1)^2(l+2)^2 \tag{67}$$

where $\dim(l,l)$ is the dimension of $SU(4)$ in the real (l,l)-representation. This arises from the fact that functions on $\mathbf{CP}^3 = SU(4)/U(3)$ can be expanded by $\mathcal{D}_{M\mathbf{0}}^{(l,l)}(g)$, Wigner $\mathcal{D}$-functions of $SU(4)$ in the (l,l)-representation ($l = 0, 1, 2, \cdots$). Here, g is an element of $SU(4)$. The lower index M ($M = 1, \cdots, \dim(l,l)$) labels the state in the (l,l)-representation, while the index $\mathbf{0}$ represents any suitably fixed state in this representation. Like in (63), the symbol of fuzzy $\mathbf{CP}^3$ is defined in terms of $\mathcal{D}_{IN^{(3)}}^{(n,0)}(g)$ and its complex conjugate, where $\mathcal{D}_{IN^{(3)}}^{(n,0)}(g) = \langle (n,0), I|g|(n,0), N^{(3)}\rangle$ are the $\mathcal{D}$-functions belonging to the symmetric $(n,0)$-representation. The states of fuzzy $\mathbf{CP}^3$ are expressed by $|(n,0), I\rangle$. The index I ($I = 1, 2, \cdots, \dim(n,0) = N^{(3)}$) labels these states and the index $N^{(3)}$ indicates the highest or lowest weight state. Notice that one can alternatively express the states by $\phi_{i_1 i_2 \cdots i_n}$ where the sequence of $i_m = \{1, 2, 3, 4\}$ ($m = 1, \cdots, n$) is in a totally symmetric order.

We now return to the conditions (a) and (b) of fuzzy S^4. In the following subsections, we present (i) different ways of counting the number of truncated functions on S^4, (ii) a one-to-one matrix-function correspondence for fuzzy S^4, and (iii) a concrete matrix configuration for functions on fuzzy S^4. In (ii), the condition (a) is shown; we find the number of matrix elements for fuzzy S^4 agrees with the number calculated in (i). The condition (b) is also shown in (ii) by considering the symbols and star products on fuzzy S^4 in the commutative limit. In (iii), we confirm the one-to-one correspondence by proposing a block-diagonal matrix realization of fuzzy S^4. Along these arguments, it will become clear that the algebra of fuzzy S^4 is closed and associative.

3.3.1 Ways of Counting

A direct counting of the number of truncated functions on S^4 can be made in terms of the spherical harmonics $Y_{l_1 l_2 l_3 m}$ on S^4 with a truncation at $l_1 = n$ [103];

$$N^{S^4}(n) = \sum_{l_1=0}^{n} \sum_{l_2=0}^{l_1} \sum_{l_3=0}^{l_2} (2l_3+1) = \frac{1}{12}(n+1)(n+2)^2(n+3). \tag{68}$$

Alternatively, one can count $N^{S^4}(n)$ by use of a tensor analysis. The number of truncated functions on $\mathbf{CP}^3$ is given by the totally symmetric and traceless tensors $f_{j_1 \cdots j_l}^{i_1 \cdots i_l}$ ($i, j = 1, \cdots, 4$) in (57). Now we split the indices into $i = a, \dot{a}$ ($a = 1, 2$, $\dot{a} = 3, 4$), and similarly for $j = b, \dot{b}$. The additional constraint (59) for the extraction of S^4 from $\mathbf{CP}^3$ means that the tensors are independent of any combinations of $\dot{a}$'s in the sequence of i's. In other words, in terms of the transformation (60), $Z \to e^{it_\alpha \theta_\alpha}Z$, functions on S^4 are invariant under the transformations involving $(t_\alpha)_{\dot{a}_1 \dot{a}_2}$ where t_α are the (4×4) matrix representations of the algebra of $H = SU(2) \times U(1)$. There are $N^{(2)}(l) = \frac{1}{2}(l+1)(l+2)$ ways of having a symmetric order $i_1, i_2, \cdots, i_l$ for $i = \{1, 2, \dot{a}\}$ ($\dot{a} = 3, 4$). This can be regarded as an $N^{(2)}(l)$-degeneracy due to an S^2 internal symmetry for the extraction of S^4 out of $\mathbf{CP}^3 \sim S^4 \times S^2$. This internal symmetry is relevant to the above $(t_\alpha)_{\dot{a}_1 \dot{a}_2}$-transformations. Since the number of truncated functions on $\mathbf{CP}^3$ is given by (67), the number of those on S^4 may be calculated by

$$N^{S^4}(n) = \sum_{l=0}^{n} \frac{\dim(l,l)}{N^{(2)}(l)} = \sum_{l=0}^{n} \frac{1}{6}(l+1)(l+2)(2l+3) = \frac{1}{12}(n+1)(n+2)^2(n+3) \tag{69}$$

which reproduces (68). This is also in accordance with a corresponding calculation in the context of $S^4 = SO(5)/SO(4)$ [45, 46].

3.3.2 One-to-one matrix-function correspondence

As mentioned earlier in this section, the states of fuzzy $\mathbf{CP}^3$ can be denoted by $\phi_{i_1 i_2 \cdots i_n}$ where the sequence of $i_m = \{1, 2, 3, 4\}$ ($m = 1, \cdots, n$) is in a totally symmetric order. Let $(\hat{F})_{IJ}$ ($I, J = 1, 2, \cdots, N^{(3)}$) denote a matrix-function on fuzzy $\mathbf{CP}^3$. Matrix elements of the function $\hat{F}$ on fuzzy $\mathbf{CP}^3$ can be defined by $\langle I|\hat{F}|J\rangle$, where we denote $\phi_{i_1 \cdots i_n} = |i_1 \cdots i_n\rangle \equiv |I\rangle$. We need to find an

analogous matrix expression $(\hat{F}^{S^4})_{IJ}$ for a function on fuzzy S^4. We now consider the states on fuzzy S^4 in terms of $\phi_{i_1 i_2 \cdots i_n}$. Splitting each i into a and $\dot{a}$, we may express $\phi_{i_1 i_2 \cdots i_n}$ as

$$\phi_{i_1 i_2 \cdots i_n} = \{\phi_{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_n} \ , \ \phi_{a_1 \dot{a}_1 \cdots \dot{a}_{n-1}} \ , \ \cdots \ , \ \phi_{a_1 \cdots a_{n-1} \dot{a}_1} \ , \ \phi_{a_1 a_2 \cdots a_n}\} \ . \tag{70}$$

From the analysis in the previous section, one can obtain the states corresponding to fuzzy S^4 by imposing an additional condition on (70), *i.e.*, the invariance under the transformations involving any $\dot{a}_m$ ($m = 1, \cdots, n$). Transformations of the states on fuzzy S^4, under this particular condition, can be considered as follows. On the set of states $\phi_{\dot{a}_1 \dot{a}_2 \cdots \dot{a}_n}$, which are $(n+1)$ in number, the transformations must be diagonal because of (59), but we can have an independent transformation for each state. (The number of the states is $(n+1)$, since the sequence of $\dot{a}_m = \{3, 4\}$ is in a totally symmetric order.) Thus we get $(n+1)$ different functions proportional to identity. On the set of states $\phi_{a_1 \dot{a}_1 \cdots \dot{a}_{n-1}}$, we can transform the a_1 index (to $b_1 = \{1, 2\}$ for instance), corresponding to a matrix function f_{a_1, b_1} which have 2^2 independent components. But we can also choose the matrix f_{a_1, b_1} to be different for each choice of $(\dot{a}_1 \cdots \dot{a}_{n-1})$ giving $2^2 \times n$ functions in all, at this level. We can represent these as $f_{a_1, b_1}^{(\dot{a}_1 \cdots \dot{a}_{n-1})}$, the extra composite index $(\dot{a}_1 \cdots \dot{a}_{n-1})$ counting the multiplicity. Continuing in this way, we find that the set of all functions on fuzzy S^4 is given by

$$\begin{aligned}(\hat{F}^{S^4})_{IJ} \ = \ & \{f^{(\dot{a}_1 \cdots \dot{a}_n)} \ \hat{\delta}_{\dot{a}_1 \cdots \dot{a}_n, \dot{b}_1 \cdots \dot{b}_n}, \ f_{a_1, b_1}^{(\dot{a}_1 \cdots \dot{a}_{n-1})} \ \hat{\delta}_{\dot{a}_1 \cdots \dot{a}_{n-1}, \dot{b}_1 \cdots \dot{b}_{n-1}}, \\ & f_{a_1 a_2, b_1 b_2}^{(\dot{a}_1 \cdots \dot{a}_{n-2})} \ \hat{\delta}_{\dot{a}_1 \cdots \dot{a}_{n-2}, \dot{b}_1 \cdots \dot{b}_{n-2}}, \ \cdots\cdots, \ f_{a_1 \cdots a_n, b_1 \cdots b_n}\}\end{aligned} \tag{71}$$

where we split i_m into a_m, $\dot{a}_m$ and j_m into b_m, $\dot{b}_m$. Each of the operators $\hat{\delta}_{\dot{a}_1 \cdots \dot{a}_m, \dot{b}_1 \cdots \dot{b}_m}$ indicates an identity operator such that the corresponding matrix is invariant under transformations from $\{\dot{a}_1 \cdots \dot{a}_m\}$ to $\{\dot{b}_1 \cdots \dot{b}_m\}$. The structure in (71) shows that $\hat{F}^{S^4}$ is composed of $(l+1) \times (l+1)$-matrices ($l = 0, 1, \cdots, n$), with the number of these matrices for fixed l being $(n+1-l)$. Thus the number of matrix elements for fuzzy S^4 is counted by

$$N^{S^4}(n) = \sum_{l=0}^{n} (l+1)^2 (n+1-l) = \frac{1}{12}(n+1)(n+2)^2(n+3) \ . \tag{72}$$

This relation satisfies the condition (a). In order to show the precise matrix-function correspondence, we further need to show the condition (b), the correspondence of products. We carry out this part in analogy with the case of fuzzy S^2 in (63)-(65). The symbol of the function $\hat{F}$ on fuzzy $\mathbf{CP}^3$ can be defined as

$$\langle \hat{F} \rangle \ = \ \sum_{I,J} \langle N|g|I \rangle \ (\hat{F})_{IJ} \ \langle J|g|N \rangle \tag{73}$$

where $|N\rangle \equiv |(n, 0), N^{(3)}\rangle$ is the highest or lowest weight state of fuzzy $\mathbf{CP}^3$ and $\langle J|g|N \rangle$ denotes the previous $\mathcal{D}$-function, $\mathcal{D}_{JN^{(3)}}^{(n,0)}(g)$. The symbol of a function on fuzzy S^4 is defined in the same way except that $(\hat{F})_{IJ}$ is replaced with $(\hat{F}^{S^4})_{IJ}$ in (73). We now consider the product of two functions on fuzzy S^4. As we discussed above, a function on fuzzy S^4 can be described by $(l+1) \times (l+1)$-matrices. From the structure of $\hat{F}^{S^4}$ in (71), we are allowed to treat these matrices independently. The product is then considered as a set of matrix multiplications. This leads to a natural definition of the product preserving closure, since the product of functions also becomes a function, retaining the same structure as in (71). The star product of fuzzy S^4 is written as

$$\langle \hat{F}^{S^4} \hat{G}^{S^4} \rangle = \sum_{IJK} (\hat{F}^{S^4})_{IJ} (\hat{G}^{S^4})_{JK} \langle N|g|I \rangle \langle K|g|N \rangle \equiv \langle \hat{F}^{S^4} \rangle * \langle \hat{G}^{S^4} \rangle \tag{74}$$

where the product $(\hat{F}^{S^4})_{IJ}(\hat{G}^{S^4})_{JK}$ is given by the set of matrix multiplications. This fact, along with the orthogonality of the $\mathcal{D}$-functions, leads to associativity of the star products.

The symbols and star products of fuzzy S^4 can be obtained from those of fuzzy $\mathbf{CP}^3$ by simply replacing the function operator $\hat{F}$ with $\hat{F}^{S^4}$. So the correspondence between fuzzy and commutative

products on S^4 can be shown in the large n limit as we have seen in section 2.2. We can in fact directly check this correspondence even at the level of finite n from the following discussion.

Let us consider functions on S^4 in terms of the homogeneous coordinates on $\mathbf{CP}^3$, $Z_i = (\omega_a, \pi_{\dot{a}}) = (x_{a\dot{a}}\pi_{\dot{a}}, \pi_{\dot{a}})$, as in (58). Functions on S^4 can be constructed from $x_{a\dot{a}}$ under the constraint (59), which implies that the functions are independent of $\pi_{\dot{a}}$ and $\bar{\pi}_{\dot{a}}$. Expanding in powers of $x_{a\dot{a}}$, we may express the functions by the following set of terms; $\{1, x_{a\dot{a}}, x_{a_1\dot{a}_1}x_{a_2\dot{a}_2}, x_{a_1\dot{a}_1}x_{a_2\dot{a}_2}x_{a_3\dot{a}_3}, \cdots\}$, where the indices a's and $\dot{a}$'s are symmetric in their order. Owing to the extra constraint (59), one can consider that all the factors involving $\pi_{\dot{a}}$ and $\bar{\pi}_{\dot{a}}$ can be absorbed into the coefficients of these terms. By iterative use of the relations, $x_{a\dot{a}}\pi_{\dot{a}} = \omega_a$ and its complex conjugation, the above set of terms can be expressed in terms of ω's and $\bar{\omega}$'s as

$$
1 \ , \ \underbrace{\begin{pmatrix} \bar{\omega}_{a_1} \\ \omega_{b_1} \end{pmatrix}}_{2\times 2} \ , \ \underbrace{\begin{pmatrix} \bar{\omega}_{a_1}\bar{\omega}_{a_2} \\ \bar{\omega}_{a_1}\omega_{b_1} \\ \omega_{b_1}\omega_{b_2} \end{pmatrix}}_{3\times 3} \ , \ \underbrace{\begin{pmatrix} \bar{\omega}_{a_1}\bar{\omega}_{a_2}\bar{\omega}_{a_3} \\ \bar{\omega}_{a_1}\bar{\omega}_{a_2}\omega_{b_1} \\ \bar{\omega}_{a_1}\omega_{b_1}\omega_{b_2} \\ \omega_{b_1}\omega_{b_2}\omega_{b_3} \end{pmatrix}}_{4\times 4} \ , \ \cdots\cdots \tag{75}
$$

where the indices a and b are used to distinguish $\bar{\omega}$ and ω. Because the indices are symmetric, the number of independent terms in each column should be counted as indicated in (75).

Notice that even though functions on S^4 can be parametrized by ω's and $\bar{\omega}$'s, the overall variables of the functions should be given by the coordinates on S^4, x_μ, instead of $\omega_a = \pi_{\dot{a}}x_{a\dot{a}}$. The coefficients of the terms in (75) need to be chosen accordingly. For instance, the term ω_a with a coefficient c_a will be expressed as $c_a\omega_a = c_a\pi_{\dot{a}}x_{a\dot{a}} \equiv h_{a\dot{a}}x_{a\dot{a}}$, where $h_{a\dot{a}}$ is considered as some arbitrary set of constants. We now define truncated functions on S^4 in the present context. Functions on S^4 are generically expanded in powers of $\bar{\omega}_a$ and ω_b ($a = 1, 2$ and $b = 1, 2$)

$$
f_{S^4}(\omega, \bar{\omega}) \sim f^{a_1 a_2 \cdots a_\alpha}_{b_1 b_2 \cdots b_\beta} \bar{\omega}_{a_1}\bar{\omega}_{a_2}\cdots\bar{\omega}_{a_\alpha}\omega_{b_1}\omega_{b_2}\cdots\omega_{b_\beta} \tag{76}
$$

where $\alpha, \beta = 0, 1, 2, 3, \cdots$ and the coefficients $f^{a_1 a_2 \cdots a_\alpha}_{b_1 b_2 \cdots b_\beta}$ should be understood as generalizations of the above-mentioned c_a. The truncated functions on S^4 may be obtained by putting an upper bound for the value $(\alpha + \beta)$. We choose this by setting $\alpha + \beta \leq n$. In (75), this choice corresponds to a truncation at the column which is to be labelled by $(n+1) \times (n+1)$. In order to count the number of truncated functions in (76), we have to notice the following relation between ω_a and $\bar{\omega}_a$

$$
\bar{\omega}_a \omega_a \sim x_\mu x_\mu = x^2. \tag{77}
$$

Using this relation, we can contract $\bar{\omega}_a$'s in (75). For example, we begin with the contractions involving $\bar{\omega}_{a_1}$ with all terms in (75), which yield the following new set of terms

$$
1 \ , \ \begin{pmatrix} \bar{\omega}_{a_2} \\ \omega_{b_1} \end{pmatrix} \ , \ \begin{pmatrix} \bar{\omega}_{a_2}\bar{\omega}_{a_3} \\ \bar{\omega}_{a_2}\omega_{b_1} \\ \omega_{b_1}\omega_{b_2} \end{pmatrix} \ , \ \cdots\cdots \tag{78}
$$

The coefficients for the terms in (78) are independent of those for (75), due to the scale invariance $\bar{\pi}_{\dot{a}}\pi_{\dot{a}} \sim |\lambda|^2$ ($\lambda \in \mathbf{C} - \{0\}$) in the contracting relation (77). Consecutively, we can make similar contractions at most n-times. The total number of truncated functions on S^4 is then counted by

$$
N^{S^4}(n) \equiv \sum_{l=0}^{n} \left[1^2 + 2^2 + \cdots + (l+1)^2 \right] = \frac{1}{12}(n+1)(n+2)^2(n+3) \tag{79}
$$

which indeed equals to the previously found results in (68) and (69).

From (75)-(79), we find that all the coefficients in $f_{S^4}(\omega, \bar{\omega})$ correspond to the number of the matrix elements for $\hat{F}^{S^4}$ given in (72). Further, since any products of fuzzy functions do not alter their structure in (71), such products correspond to commutative products of $f_{S^4}(\omega, \bar{\omega})$'s. This leads to the precise correspondence between the functions on fuzzy S^4 and the truncated functions on S^4 at any level of truncation.

3.3.3 Block-diagonal matrix realization of fuzzy S^4

We have analyzed the structure of functions on fuzzy S^4 and their products in some detail, however, we have not presented an explicit matrix configuration for those fuzzy functions. But, by now, it is obvious that we can use a block-diagonal matrix to represent them, which naturally leads to associativity of the algebra of fuzzy S^4. Let us write down the equation (72) in the following form:

$$
\begin{aligned}
N^{S^4}(n) \;=\; & 1 \\
& +1 + 2^2 \\
& +1 + 2^2 + 3^2 \\
& +1 + 2^2 + 3^2 + 4^2 \\
& + \quad \cdots\cdots\cdots \\
& +1 + 2^2 + 3^2 + 4^2 + \cdots + (n+1)^2 .
\end{aligned}
\tag{80}
$$

If we locate all the squared elements block-diagonally, then the dimension of an embedding matrix is given by

$$
\sum_{l=0}^{n} [1 + 2 + \cdots + (l+1)] = \frac{1}{6}(n+1)(n+2)(n+3) = N^{(3)}.
\tag{81}
$$

Coordinates of fuzzy S^4 are then represented by these $N^{(3)} \times N^{(3)}$ block-diagonal matrices, X_A, which satisfy

$$
X_A X_A \sim \mathbf{1}
\tag{82}
$$

where $\mathbf{1}$ is the $N^{(3)} \times N^{(3)}$ identity matrix and $A = 1, 2, 3, 4$ and 5, four of which are relevant to the coordinates of fuzzy S^4. The fact that N^{S^4} is a sum of absolute squares does not necessarily warrant associativity of the algebra. (Every integer is a sum of squares, $1 + 1 + \cdots + 1$, but this does not mean any linear space of any dimension is an algebra.) It is the structure of $\hat{F}^{S^4}$ as well as the matching between (79) and (72) that lead to these block-diagonal matrices X_A.

Of course, X_A are not the only matrices that describe fuzzy S^4. Instead of diagonally locating every block one by one, we can also put the same-size blocks into a single block, using matrix multiplication or matrix addition. Then, the final form has a dimension of $\sum_{l=0}^{n}(l+1) = \frac{1}{2}(n+1)(n+2) = N^{(2)}$. This implies an alternative description of fuzzy S^4 in terms of $N^{(2)} \times N^{(2)}$ block-diagonal matrices, $\overline{X}_A$, which are embedded in $N^{(3)}$-dimensional square matrices and satisfy $\overline{X}_A \overline{X}_A \sim \overline{\mathbf{1}}$, where $\overline{\mathbf{1}} = diag(1, 1, \cdots, 1, 0, 0, \cdots, 0)$ is an $N^{(3)} \times N^{(3)}$ diagonal matrix, with the number of 1's being $N^{(2)}$. Our choice of X_A is, however, convenient in comparison with fuzzy $\mathbf{CP}^3$. The number of 1's in X_A is $(n+1)$. This corresponds to the dimension of an $SU(2)$ subalgebra of $SU(4)$ in the $N^{(3)}(n)$-dimensional matrix representation. (Notice that fuzzy $S^2 = SU(2)/U(1)$ is conventionally described by $(n+1) \times (n+1)$ matrices in this context.) Using the coordinates X_A, we can then confirm the constraint in (54), *i.e.*,

$$
[\mathcal{F}(X), L_\alpha] = 0
\tag{83}
$$

where $\mathcal{F}(X)$ are matrix-functions of X_A's and L_α are the generators of $H = SU(2) \times U(1) \subset SU(4)$, represented by $N^{(3)} \times N^{(3)}$ matrices. If both $\mathcal{F}(X)$ and $\mathcal{G}(X)$ commute with L_α, so does $\mathcal{F}(X)\mathcal{G}(X)$. Thus, there is closure of such "functions" under multiplication. This indicates that fuzzy S^4 follows a closed and associative algebra.

3.4 Construction of fuzzy S^8

We outline construction of fuzzy S^8 in a way of reviewing our construction of fuzzy S^4. As mentioned in section 3.1, $\mathbf{CP}^7$ is a $\mathbf{CP}^3$ bundle over S^8. We expect that we can similarly construct fuzzy S^8 by factoring out fuzzy $\mathbf{CP}^3$ from fuzzy $\mathbf{CP}^7$.

The structure of fuzzy S^4 as a block-diagonal matrix has been derived, based on the following two equations

$$
N^{S^4}(n) \;=\; \sum_{l=0}^{n} \left(N^{(1)}(l) \right)^2 N^{(1)}(n-l),
\tag{84}
$$

$$N^{(3)}(n) = \sum_{l=0}^{n} N^{(1)}(l) \, N^{(1)}(n-l) \tag{85}$$

where $N^{(k)}(l) = \frac{(l+k)!}{k! \, l!}$ as in (2). Fuzzy S^8 analogs of these equations are

$$N^{S^8}(n) = \sum_{l=0}^{n} N^{S^4}(l) \, N^{(3)}(n-l) \,, \tag{86}$$

$$N^{(7)}(n) = \sum_{l=0}^{n} N^{(3)}(l) \, N^{(3)}(n-l) \tag{87}$$

where $N^{S^8}(n)$ is the number of truncated functions on S^8, which can be calculated in terms of the spherical harmonics as in the case of S^4 in (68);

$$
\begin{aligned}
N^{S^8}(n) &= \sum_{a=0}^{n} \sum_{b=0}^{a} \sum_{c=0}^{b} \sum_{d=0}^{c} \sum_{e=0}^{d} \sum_{f=0}^{e} \sum_{g=0}^{f} (2g+1) \\
&= \frac{1}{4 \cdot 7!} (n+1)(n+2)(n+3)(n+4)^2(n+5)(n+6)(n+7) \,.
\end{aligned}
\tag{88}
$$

This number is also calculated by a tensor analysis as in (69);

$$
\begin{aligned}
N^{S^8}(n) &= \sum_{l=0}^{n} \frac{\dim(l,l)}{N^{(6)}(l)} \\
&= \sum_{l=0}^{n} \frac{1}{7!} (2l+7)(l+1)(l+2)(l+3)(l+4)(l+5)(l+6) \\
&= \frac{n+4}{4} \frac{(n+7)!}{7! \, n!}
\end{aligned}
\tag{89}
$$

where $\dim(l,l)$ is the dimension of $SU(8)$ in the (l,l)-representation, *i.e.*, $\dim(l,l) = \frac{1}{7! \, 6!}(2l + 7)\left((l+1)(l+2)\cdots(l+6)\right)^2$. Calculations from (84) to (89) are carried out by use of *Mathematica*.

Equations (86) and (87) indicate that fuzzy S^8 is composed of $N^{(3)}(l)$-dimensional block-diagonal matrices of fuzzy S^4 ($l = 0, 1, \cdots, n$), with the number of these matrices for fixed l being $N^{(3)}(n-l)$. Thus fuzzy S^8 is also described by a block-diagonal matrix whose embedding square matrix has a dimension $N^{(7)}(n)$. Notice that we have a nice matryoshka-like structure for fuzzy S^8, namely, a fuzzy-S^8 box is composed of a number of fuzzy-S^4 blocks and each of those blocks is further composed of a number of fuzzy-S^2 blocks. Fuzzy S^8 is then represented by $N^{(7)} \times N^{(7)}$ block-diagonal matrices X_A which satisfy $X_A X_A \sim \mathbf{1}$ ($A = 1, 2, \cdots, 9$), where $\mathbf{1}$ is the $N^{(7)} \times N^{(7)}$ identity matrix. Similarly to the case of fuzzy S^4, fuzzy S^8 should also obey a closed and associative algebra.

Let us now consider the decomposition

$$SU(8) \to SU(4) \times \underbrace{SU(4) \times U(1)}_{=H^{(4)}} \tag{90}$$

where the two $SU(4)$'s and one $U(1)$ are defined similarly to (53) in terms of the generators of $SU(8)$ in the fundamental representation. Noticing the fact that the number of 1-dimensional blocks in the coordinate X_A of fuzzy S^8 is $N^{(3)}(n)$, we find $[X_A \,, L_\alpha] = 0$ where L_α are now the generators of $H^{(4)}$ represented by $N^{(7)} \times N^{(7)}$ matrices. This is in accordance with the statement that functions on S^8 are functions on $\mathbf{CP}^7 = SU(8)/U(7)$ which are invariant under transformations of $H^{(4)} = SU(4) \times U(1)$. Coming back to the original idea, we can then construct fuzzy S^8 out of fuzzy $\mathbf{CP}^7$ by imposing the particular constraint $[\mathcal{F}, L_\alpha] = 0$, where $\mathcal{F}$ are matrix-functions of coordinates Q_A on fuzzy $\mathbf{CP}^7$, Q_A being defined as in (33) for $k = 7$. This constraint is imposed

on the function $\mathcal{F}(Q_A)$, on top of the fuzzy $\mathbf{CP}^7$ constraints for Q_A, so that it becomes a function on fuzzy S^8, that is, a polynomial of X_A's.

Following the same method, we may construct higher dimensional fuzzy spheres [42, 44, 47]. But we are incapable of doing so as far as we utilize bundle structures analogous to $\mathbf{CP}^3$ or $\mathbf{CP}^7$. This is because, as far as complex number coefficients are used, there are no division algebra allowed beyond octonions. The fact that $\mathbf{CP}^7$ is a $\mathbf{CP}^3$ bundle over S^8 is based on the fact that octonions provide the Hopf map, $S^{15} \to S^8$ with its fiber being S^7. Since this map is the final Hopf map, there are no more bundle structures available to construct fuzzy spheres in a direct analogy with the constructions of fuzzy S^8, S^4 and S^2.

4 Matrix models for gravity

From this chapter on, applications of fuzzy spaces to physical models will be discussed.

4.1 Introduction to NC gravity

As mentioned in section 1.3, noncommutative (NC) spaces can arise as solutions in string and M-theories. Fluctuations of brane solutions are described by gauge theories on such spaces and, with this motivation, there has recently been a large number of papers dealing with gauge theories, and more generally field theories, on noncommutative spaces (see, *e.g.*, [11, 54, 55, 56]). There is also an earlier line of development in close connection with Connes' original idea, using the spectral triple and the so-called 'spectral actions' [58]-[65].

Even apart from their string and M-theory connections, noncommutative spaces are interesting for other reasons. Many of the noncommutative spaces recently discussed have an underlying Heisenberg algebra for different coordinates. Lie algebra structures are more natural from a matrix model point of view; these typically lead to noncommutative analogues of compact spaces and, in particular, fuzzy spaces. Because these spaces are described by finite dimensional matrices, the number of possible modes for fields on such spaces is limited and so one has a natural ultraviolet cutoff. We may think of such field theories as a finite mode approximation to commutative continuum field theories, providing, in some sense, an alternative to lattice gauge theories. Indeed, this point of view has been pursued in some recent work (see, *e.g.*, [16], [21]-[25]). While lattice gauge theories may be most simply described by standard hypercubic lattices, gravity is one case where the noncommutative approach can be significantly better. This can provide a regularized gravity theory preserving the various desirable symmetries, which is hard to do with standard lattice versions. It would be an interesting alternative to the Regge calculus, which is essentially the only finite-mode-truncation of gravity known with the concept of coordinate invariance built in. A finite-mode-truncation is not quantum gravity, but it can give a formulation of standard gravity where questions can be posed and answered in a well defined way.

Partly with this motivation, a version of gravity on noncommutative spaces has been suggested by Nair in [68]. This led to an action for even dimensional, in particular four-dimensional, non-commutative spaces generalizing the Chang-MacDowell-Mansouri approach used for commutative four-dimensional gravity [67]. In this chapter, we shall consider the case of fuzzy S^2 in some detail, setting up the required structures, eventually obtaining an action for gravity in terms of $(N \times N)$-matrices. The large N limit of the action will give the usual action for gravitational fields on S^2. We also construct a finite-dimensional matrix model action for gravity on fuzzy $\mathbf{CP}^2$ and indicate how this may be generalized to fuzzy $\mathbf{CP}^k$ ($k = 1, 2, \cdots$).

4.2 Derivatives, vectors, etc.

We shall primarily be concerned with fuzzy versions of coset spaces of the form G/H for some compact Lie group G, H being a subgroup of G. Most of our discussion will be based on $S^2 = SU(2)/U(1)$. Functions on fuzzy S^2 are given by $(N \times N)$-matrices with elements f_{mn}. As given in (63), the symbol of these fuzzy functions are expressed as $\langle \hat{f} \rangle = \sum_{m,n} f_{mn} \mathcal{D}_{mj}^{*(j)}(g) \mathcal{D}_{nj}^{(j)}(g)$, where $\mathcal{D}_{mk}^{(j)}(g)$ are Wigner $\mathcal{D}$-functions for $SU(2)$ belonging to the spin-j representation. The matrix

dimension N is given by $N = 2j + 1$. In this way of representing functions, derivatives may be realized as the right translation operators R_a on g,

$$R_a \cdot \mathcal{D}^{(j)}_{mk}(g) = \left[\mathcal{D}^{(j)} \left(g \ t_a \right) \right]_{mk} \tag{91}$$

where $t_a = \sigma_a/2$, with σ_a being the Pauli matrices. In order to realize various quantities, particulary an action, purely in terms of matrices, we need to introduce a different but related way of defining derivatives, vectors, tensors, etc., on a fuzzy coset space.

Let g denote an element of the group G and define

$$S_{Aa} \ = \ 2 \ \mathrm{tr}(g^{-1} t_A g t_a) \tag{92}$$

where t_a and t_A are hermitian matrices forming a basis of the Lie algebra of G in the fundamental representation. We normalize these by $\mathrm{tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$, $\mathrm{tr}(t_A t_B) = \frac{1}{2}\delta_{AB}$. The distinction between upper and lower case indices is only for clarity in what follows. For $SU(2)$, $a, A = 1, 2, 3$ and S_{Aa} obey the relations

$$
\begin{aligned}
S_{Aa} \ S_{Ab} &= \ \delta_{ab} \ , \\
S_{Aa} \ S_{Ba} &= \ \delta_{AB} \ , \\
\epsilon_{ABC} \ S_{Aa} \ S_{Bb} &= \ \epsilon_{abc} S_{Cc} \ , \\
\epsilon_{abc} \ S_{Aa} \ S_{Bb} &= \ \epsilon_{ABC} S_{Cc} \ .
\end{aligned}
\tag{93}
$$

Let L_A be the $(N \times N)$-matrix representation of the $SU(2)$ generators, obeying the commutation rules $[L_A, L_B] = i\epsilon_{ABC} L_C$. We then define the operators

$$\mathcal{K}_a \ = \ S_{Aa} L_A - \frac{1}{2} R_a \tag{94}$$

where R_a are the right translation operators, $R_a g = g t_a$. One can think of them as differential operators

$$R_a \ = \ i(E^{-1})^i_a \frac{\partial}{\partial \varphi^i} \tag{95}$$

in terms of the group parameters φ^i and the frame field E^a_i, satisfying

$$g^{-1} dg \ = \ (-i t_a) \ E^a_i d\varphi^i. \tag{96}$$

R_a obey the commutation rules $[R_a, R_b] = i\epsilon_{abc} R_c$. We then find

$$[\mathcal{K}_a, \mathcal{K}_b] \ = \ \frac{i}{4} \epsilon_{abc} \ R_c \, . \tag{97}$$

Identifying the $U(1)$ subgroup generated by t_3 as the H-subgroup, we define derivatives on fuzzy S^2 as $\mathcal{K}_\pm = \mathcal{K}_1 \pm i\mathcal{K}_2$. Notice that this is a hybrid object, being partially a matrix commutator and partially something that depends on the continuous variable g. This is very convenient for our purpose and in the end g will be integrated over anyway.

We now define a matrix-function f on fuzzy S^2 with no g-dependence. The derivative of f is then defined as

$$\mathcal{K}_\mu \cdot f \ \equiv \ [\mathcal{K}_\mu, f] \ = \ S_{a\mu}[L_A, f] \tag{98}$$

where $\mu = \pm$. Since $[\mathcal{K}_+, \mathcal{K}_-] = \frac{1}{2} R_3$ from (97), we find $[\mathcal{K}_+, \mathcal{K}_-] \cdot f = 0$, consistent with the expectation that derivatives commute when acting on a function. Equation (98) also shows that it is natural to define a vector on fuzzy S^2 as

$$V_\mu \ = \ S_{A\mu} \ V_A \tag{99}$$

where V_A are three $(N \times N)$-matrices. On a two-sphere, a vector should only have two independent components, so this is one too many and V_A must obey a constraint. Notice that the quantity $[L_A, f]$ obeys the condition $L_A[L_A, f] + [L_A, f]L_A = 0$, since $L_A L_A$ is proportional to the identity

matrix. This suggests that the correct constraint for a general vector is $L_A V_A + V_A L_A = 0$. In the large N limit, L_A become proportional to x_A, the commutative coordinates of the two-sphere as embedded in $\mathbf{R}^3$ (with $x_A x_A = 1$). So the condition $x \cdot V = 0$ is exactly what we need to restrict the vectors to directions tangential to the sphere. We may thus regard $L_A V_A + V_A L_A = 0$ as the appropriate fuzzy version. As we shall see below this constraint will also emerge naturally when we define integrals on fuzzy S^2. Using

$$[R_a, S_{Ab}] = i\epsilon_{abc} S_{Ac}, \tag{100}$$

we find

$$[\mathcal{K}_+, \mathcal{K}_-] \cdot V_{\pm} = \pm \frac{1}{2} V_{\pm} \tag{101}$$

which is consistent with the Riemann curvature of S^2; $R^+_{+-+} = -R^-_{+--} = \frac{1}{2}$. Higher rank tensors may also be defined in an analogous way with several S_{Aa}'s, $i.e.$, $T_{\mu_1\mu_2\cdots\mu_r} \equiv S_{A_1\mu_1} S_{A_2\mu_2} \cdots S_{A_r\mu_r} T_{A_1 A_2 \cdots A_r}$.

We now turn to a definition of 'integration' on fuzzy S^2. We will only need, and will only define, integration of the fuzzy analogue of an antisymmetric rank-2 tensor or a two-form. Such a quantity has components of the form $W_{+-} = (S_{A+}S_{B-} - S_{A-}S_{B+})W_{AB}$. From the properties of S_{Aa}, we find $S_{A+}S_{B-} - S_{A-}S_{B+} = -2i\epsilon_{ABC}S_{C3}$. Integration of W_{+-} over g (with the trace of the matrix W_{AB}) will give zero. To get a nonzero integral we must introduce a density factor ρ. Such a factor must commute with R_3 to be properly defined on $SU(2)/U(1)$ and must give nonzero upon g-integration with S_{C3}. The only choice is $\rho = \frac{1}{3} S_{K3} L_K$, up to normalization, which can be determined by $\int_g S_{K3}S_{C3} = 3\delta_{KC}$ where factor 3 corresponds to $\dim SU(2)$. The appearance of such a density factor is actually very natural. If we consider a commutative S^2 embedded in $\mathbf{R}^3$ with coordinates x_A, then $x_A = S_{A3}$ in a suitable parametrization. The usual volume element is oriented along $x_A = S_{A3}$ and so we can expect a factor $\rho = \frac{1}{3} S_{K3} L_K$ in the fuzzy case. With the introduction of the factor ρ, we can consider an 'integral' of the form $\int_g \text{Tr}(\rho W)$. However, if we consider $\int_g \text{Tr}(\rho W f)$ where f is a function, we do not have the expected cyclicity property since $[\rho, f] \neq 0$ in general. Cyclicity property can be obtained if we symmetrize the factors inside the trace except the density factor ρ. Gathering these points, we now define an 'integral' over fuzzy S^2, denoted by $\rceil\!\!\lceil$, as follows:

$$\rceil\!\!\lceil A_1 A_2 \cdots A_l = \int_g \text{Tr}\left[\rho \frac{1}{l} \sum_{cycl.} (A_1 A_2 \cdots A_l)\right] \tag{102}$$

where $A_1, A_2, \cdots, A_l$ are functions, vectors, tensors, etc., such that the product is an antisymmetric rank-2 tensor (of the form W_{+-}), $i.e.$, a fuzzy analogue of a two-form on S^2. The summation in (102) is taken over cyclic permutations of the arguments. Note that we can express such a two-form as $A_1 A_2 \cdots A_l = (-2i)\epsilon_{ABC}S_{C3}(A_1 A_2 \cdots A_l)_{AB}$. So the integral is further written as

$$
\begin{aligned}
\rceil\!\!\lceil A_1 A_2 \cdots A_l &= (-2i)\epsilon_{ABC}\,\text{Tr}\left[L_C \frac{1}{l}\sum_{cycl.}(A_1 A_2 \cdots A_l)_{AB}\right] \\
&= (-2i)\epsilon_{ABC}\,\text{STr}\,[L_C A_1 A_2 \cdots A_l]_{AB} \tag{103}
\end{aligned}
$$

where STr is the symmetrized trace over the $(N \times N)$ matrices inside the bracket, the lower indices A, B being assigned to some of the matrices in A_1, A_2, etc.

In a similar fashion, we now consider a fuzzy analogue of an exterior derivative, in particular, the analogue of a two-form corresponding to the curl of a vector $V_\mu = S_{A\mu}V_A$, $\mu = \pm$. Since we have defined $\mathcal{K}_\pm$ as derivatives on fuzzy S^2, a fuzzy analogue of such a term can be given by

$$
\begin{aligned}
dV &\equiv [\mathcal{K}_+, V_-] - [\mathcal{K}_-, V_+] \\
&= (S_{A+}S_{B-} - S_{A-}S_{B+})[L_A, V_B] - 2S_{C3}V_C \\
&= (-2i)\,S_{C3}\,(\epsilon_{ABC}[L_A, V_B] - iV_C)\,. \tag{104}
\end{aligned}
$$

If h is a function on fuzzy S^2, we also have

$$\begin{aligned}
V\ dh &\equiv V_+[\mathcal{K}_-, h] - V_-[\mathcal{K}_+, h] \\
&= (-2i)\ \epsilon_{ABC}\ S_{C3}V_A[L_B, h]\,.
\end{aligned} \tag{105}$$

Using the definition of the integral (102) we find

$$\begin{aligned}
\int dV\ h &= (-2i)\frac{1}{2}\ \mathrm{Tr}\left[L_K\left\{\epsilon_{ABC}[L_A, V_B] - iV_C\right\}\ h \right.\\
&\qquad\qquad \left. + \left\{\epsilon_{ABC}[L_A, V_B] - iV_C\right\} L_K\ h \right]\int_g \frac{1}{3}S_{K3}S_{C3} \\
&= (-2i)\frac{1}{2}\ \mathrm{Tr}\left[L_C\left\{\epsilon_{ABC}[L_A, V_B] - iV_C\right\}\ h \right.\\
&\qquad\qquad \left. + \left\{\epsilon_{ABC}[L_A, V_B] - iV_C\right\} L_C\ h \right]
\end{aligned} \tag{106}$$

where we used $\int_g S_{K3}S_{C3} = 3\delta_{KC}$. Similarly we have

$$\int V\ dh = (-2i)\frac{1}{2}\ \mathrm{Tr}\left[\epsilon_{ABC}(L_C V_A + V_A L_C)[L_B, h]\right]\,. \tag{107}$$

By using cyclicity of the trace for the finite dimensional matrices L_A, V_B, h, etc., we find that the desired partial integration property

$$\int dV\ h = \int V\ dh \tag{108}$$

holds if V_A obey the constraint

$$L_A V_A + V_A L_A = 0\,. \tag{109}$$

This relation has been introduced earlier based on geometric properties of S^2. We have now justified this relation as a correct constraint for vectors on fuzzy S^2, based on integration properties. When V_A are gauge fields, this constraint will have to be slightly modified for reasons of gauge invariance. The relevant constraint is shown in (113).

4.3 Action for gravity on fuzzy S^2

We are now in a position to discuss actions for gravity on fuzzy S^2. As mentioned in chapter 1, we follow the proposal of [68] for the action of gravity on noncommutative G/H space, where the gravitational fields (*i.e.*, frame fields e_μ and spin connections Ω_μ) are described by $U(k)$ gauge fields, with $U(k)$ being specified by $G \subseteq U(k)$. In our case, the gauge group is then chosen as $U(2)$ and the gauge fields are written as

$$\begin{aligned}
\mathcal{A}_\mu &= \mathcal{A}_\mu^a I^a \\
&= e_\mu^+ I^+ + e_\mu^- I^- + \Omega_\mu^3 I^3 + \Omega_\mu^0 I^0.
\end{aligned} \tag{110}$$

The components $(\Omega_\mu^0, \Omega_\mu^3, e_\mu^\pm)$ are vectors on fuzzy S^2 as defined in the previous section. The upper indices of these vectors correspond to components for the Lie algebra of $U(2)$, $(I^0, I^3, I^\pm)$, form the (2×2)-representation of $U(2)$. Specifically, in terms of the Pauli matrices σ_i, $I^0 = \frac{1}{2}\mathbf{1}$, $I^3 = \frac{1}{2}\sigma_3$, $I^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. $\mathcal{A}_\mu$ is thus a vector on fuzzy S^2 which also takes values in the Lie algebra of $U(2)$. This $U(2)$ is the group acting on the upper indices of $\mathcal{A}_\mu$ or the tangent frame indices. Notice that, with L_A, R_a and the I's, we have three different actions for $SU(2)$. In terms of $\mathcal{A}_\mu$ we now define a field strength $F_{\mu\nu}$ as

$$[\mathcal{K}_\mu + \mathcal{A}_\mu, \mathcal{K}_\nu + \mathcal{A}_\nu] = \frac{i}{4}\epsilon_{\mu\nu\alpha}R_\alpha + F_{\mu\nu}\,. \tag{111}$$

In our description, gravity is parametrized in terms of deviations from S^2. The vectors $e^{\pm}_{\mu}$ are the frame fields for this and Ω^{α}_{μ} ($\alpha = 0, 3$) are the spin connections. As opposed to the commutative case, there can in general be a connection for the I^0 component, since we need the full $U(2)$ to form noncommutative gauge fields. One can expand $F_{\mu\nu}$ as

$$F_{\mu\nu} = F^0_{\mu\nu} \, I^0 \; + \; \mathcal{R}^3_{\mu\nu} \, I^3 \; + \; \mathcal{T}^a_{\mu\nu} \, I^a \tag{112}$$

where $\mathcal{T}^a_{\mu\nu}$ is the torsion tensor and $\mathcal{R}^3_{\mu\nu}$ is of the form $R_{\mu\nu}(\Omega) + 2(e^+_{\mu} e^-_{\nu} - e^-_{\mu} e^+_{\nu})$ where $R_{\mu\nu}(\Omega)$ is the Riemann tensor on commutative S^2. The expression for $\mathcal{R}^3_{\mu\nu}$ is thus a little more involved for fuzzy S^2.

In defining an action, we shall use our prescription for the integral. The gauging of $\mathcal{K}_{\mu}$ is equivalent to the gauging $L_A \rightarrow L_A + \mathcal{A}_A$. Thus we must also change our definition of ρ to $\rho = \frac{1}{3} S_{K3}(L_K + \mathcal{A}_K)$. The constraint (109) is now replaced by

$$(L_A + \mathcal{A}_A)(L_A + \mathcal{A}_A) = L_A L_A \,. \tag{113}$$

Note that $\mathcal{A}_A$ is expanded in terms of the I^a as in (110). This constraint was first proposed in [51] as the correct condition to be used for gauge fields on fuzzy S^2.

The data for gravity is presented in the form of the gauge field $\mathcal{A}_A$. Following the action suggested in [68], as a generalization McDowell-Mansouri approach for commutative gravity, we can express an action for gravity on fuzzy S^2 as

$$\mathcal{S} \; = \; \alpha \; \int \; \text{tr} \, (QF) \tag{114}$$

where tr denotes the trace over the I's regarded as (2×2)-matrices. F denotes a two-form on fuzzy S^2 corresponding to the field strength; it is in general expressed by $F = F^a I^a$, being in the algebra of $U(N) \otimes U(2)$. For higher even-dimensional G/H-spaces, the actions are given in the following form [68]:

$$\mathcal{S} \; \sim \; \int \; \text{tr} \, (QFF...F) \tag{115}$$

where Q is a combination of the I's which commutes with the H-subgroup of G. For the present case, we can choose $Q = I^3$. However, unlike the case of four and higher dimensions, the term involving F^0 in $\int \text{tr}(I^3 F)$ vanishes, which would be the fuzzy analogue of the statement that the two-dimensional Einstein-Hilbert action $\int R\sqrt{g}$ is a topological invariant. As in the commutative context, we may need to use a Lagrange multiplier scalar field η to obtain nontrivial actions. In the present case, the analogous action is given by

$$\mathcal{S} \; = \; \alpha \; \int \; \text{tr} \, (I^3 \eta F) \tag{116}$$

where $\eta = \eta^0 I^0 + \eta^3 I^3 + \eta^+ I^+ + \eta^- I^-$, $(\eta^0, \eta^3, \eta^{\pm})$ being scalar functions on fuzzy S^2. Using the decomposition (112) for the field strength, we can simplify this expression as

$$\mathcal{S} \; = \; -i \, \frac{\alpha}{2} \, \text{Tr} \, \left[I^3 \eta \, [\, (L_C + \mathcal{A}_C) F_C + F_C(L_C + \mathcal{A}_C) \,] \right] \tag{117}$$

where F_C is defined as follows:

$$F_C \; \equiv \; F^0_C I^0 + F^3_C I^3 + F^+_C I^+ + F^-_C I^- \,, \tag{118}$$

$$F^0_C \; = \; \frac{1}{2} \left\{ [L_A, \Omega^0_B] + \frac{1}{2}(\Omega^0_A \Omega^0_B + \Omega^3_A \Omega^3_B) + (e^+_A e^-_B + e^-_A e^+_B) \right\} \epsilon_{ABC}$$

$$-\frac{i}{2} \Omega^0_C \,, \tag{119}$$

$$F^3_C \; = \; \frac{1}{2} \left\{ [L_A, \Omega^3_B] + \frac{1}{2}(\Omega^0_A \Omega^3_B + \Omega^3_A \Omega^0_B) + (e^+_A e^-_B - e^-_A e^+_B) \right\} \epsilon_{ABC}$$

$$-\frac{i}{2}\Omega_C^3 \ , \qquad (120)$$

$$F_C^- \ = \ \frac{1}{2}\left\{[L_A, e_B^-] + \frac{1}{2}e_A^-(\Omega_B^0 + \Omega_B^3) - \frac{1}{2}(\Omega_B^0 - \Omega_B^3)e_A^-\right\}\epsilon_{ABC}$$

$$-\frac{i}{2}e_C^- \ , \qquad (121)$$

$$F_C^+ \ = \ \frac{1}{2}\left\{[L_A, e_B^+] + \frac{1}{2}e_A^+(\Omega_B^0 - \Omega_B^3) - \frac{1}{2}(\Omega_B^0 + \Omega_B^3)e_A^+\right\}\epsilon_{ABC}$$

$$-\frac{i}{2}e_C^+ \ . \qquad (122)$$

Equation (117), with (118)-(122), is the action for gravity on fuzzy S^2. They are expressed entirely in terms of finite dimensional ($N \times N$)-matrices, $L_A, e_A^\pm$, and Ω_A^α ($\alpha = 0, 3$). As mentioned earlier, this action provides a new regularization scheme for gravity, and, in principle, we can calculate many interesting physical quantities, correlation functions in particular, from (117)-(122) by analyzing it as a matrix model.

In what follows, we shall analyze the action (117) a bit further and discuss its commutative limit. Variations of the action with respect to η's provide four equations of motion, *i.e.*,

$$\mathcal{F}^a \ \equiv \ [(L_C + \mathcal{A}_C)F_C + F_C(L_C + \mathcal{A}_C)]^a = 0 \qquad (123)$$

for $a = 0, \ 3, \pm$. The components $a = \pm$ correspond to the vanishing of torsion. $\mathcal{F}^3$ is not quite the Riemann tensor associated with Ω^3, due to the e^+e^--term. The vanishing of $\mathcal{F}^3$ shows that the Riemann tensor is proportional to the e^+e^--term.

There are also equations of motion associated with the variation of the $e^\pm$, Ω^3, Ω^0, which are equations coupled to η's. We do not write them out here, they can be easily worked out from the expressions (119)-(122) for the F_C's. Notice however that one solution of such equations of motion is easy to find. The variation of the action with respect to the $e^\pm$, Ω^3, Ω^0 is of the form

$$\delta\mathcal{S} \ = \ -i\frac{\alpha}{2}\,\mathrm{Tr}\left[\, I^3 \ \eta \ \delta \ [(L_C + \mathcal{A}_C)F_C + F_C(L_C + \mathcal{A}_C)] \ \right]. \qquad (124)$$

This evidently shows that $\eta = 0$ is a solution.

The equations for the connections $e^\pm$, Ω^3, Ω^0 in (123) are also solved by setting all $F_{\mu\nu}$ to zero. This corresponds to the pure gauge solutions, *i.e.*, the choice of $\mathcal{A}_\mu = S_{B\mu}\mathcal{A}_B$, $\mathcal{A}_B = iU^{-1}[\hat{L}_B, U]$ where U is a matrix which is an element of $U(N) \otimes U(2)$, and $\hat{L}_B$ is viewed as $L_B \otimes \mathbf{1}$. In other words, it is an element of $U(2)$ with parameters which are ($N \times N$)-matrices. This solution corresponds to the fuzzy S^2 itself.

4.4 Commutative limit

We now consider the commutative limit of the action (117) by taking the large N limit. The matrices L_A's are matrix representations of the generators of $SU(2)$ in the spin $n/2$-representation. The matrix dimension N is then given by $N = n + 1$. We introduce the states of fuzzy S^2, $|\alpha\rangle$ ($\alpha = 0, 1, \cdots, n$), characterized by $\langle z|\alpha\rangle = 1, z, \cdots, z^n$ for each α. The operators $\mathcal{L}_A$ acting on such states can be expressed as [106]:

$$\begin{aligned}\mathcal{L}_+ &= \frac{n+2}{2}\phi_+ + z^2\frac{\partial}{\partial z} \ , \quad \phi_+ \ = \ \frac{2z}{1+z\bar{z}} \ ,\\ \mathcal{L}_- &= \frac{n+2}{2}\phi_- - \frac{\partial}{\partial z} \ , \quad \phi_- \ = \ \frac{2\bar{z}}{1+z\bar{z}} \ ,\\ \mathcal{L}_3 &= \frac{n+2}{2}\phi_3 + z\frac{\partial}{\partial z} \ , \quad \phi_3 \ = \ \frac{1-z\bar{z}}{1+z\bar{z}}\end{aligned} \qquad (125)$$

where ϕ's are the coordinates on S^2, obtained by usual stereographic projection on a complex plane. Note that $\mathcal{L}_A$'s correspond to those obtained in (31) for $k = 1$. Using such a Hilbert space,

we can consdier the vectors (Ω's and e's) as functions of $z, \bar{z}$. Large n limits of the matrix operator L_A and the commutator $[L_A, \Omega_B]$ can then be given by the following replacements:

$$L_A \longrightarrow \frac{n+2}{2}\phi_A,\tag{126}$$

$$
\begin{aligned}
[L_A, \Omega_B] &\longrightarrow \frac{1}{n+1}\left\{\frac{n+2}{2}\phi_A, \Omega_B\right\} \\
&= \frac{n+2}{2}\frac{1}{n+1}(1 + z\bar{z})^2(\partial\phi_A\bar{\partial}\Omega_B - \bar{\partial}\phi_A\partial\Omega_B) \\
&= \frac{n+2}{2}\frac{1}{n+1}\, k_A\, \Omega_B
\end{aligned}
\tag{127}
$$

where $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial\bar{z}}$ and the operators k_A are defined in terms of a Poisson bracket $k_A\Omega_B \equiv \{\phi_A, \Omega_B\} = (1 + z\bar{z})^2(\partial\phi_A\bar{\partial}\Omega_B - \bar{\partial}\phi_A\partial\Omega_B)$ with

$$k_+ = 2(z^2\partial + \bar{\partial}),\quad k_- = -2(\partial + \bar{z}^2\bar{\partial}),\quad k_3 = 2(z\partial - \bar{z}\bar{\partial}).\tag{128}$$

Notice that $\frac{1}{2}k_A$ satisfy the $SU(2)$ algebra;

$$\left[\frac{k_+}{2}, \frac{k_-}{2}\right] = 2\frac{k_3}{2},\quad \left[\frac{k_3}{2}, \frac{k_+}{2}\right] = \frac{k_+}{2},\quad \left[\frac{k_3}{2}, \frac{k_-}{2}\right] = -\frac{k_-}{2}.\tag{129}$$

Actions of k_A's on ϕ_B's can be calculated as

$$
\begin{aligned}
&k_+\phi_+ = 0,\quad k_+\phi_- = 4\phi_3,\quad k_+\phi_3 = -2\phi_+,\\
&k_-\phi_+ = -4\phi_3,\quad k_-\phi_- = 0,\quad k_-\phi_3 = 2\phi_-,\\
&k_3\phi_+ = 2\phi_+,\quad k_3\phi_- = -2\phi_-,\quad k_3\phi_3 = 0.
\end{aligned}
\tag{130}
$$

The replacement of commutator with Poisson bracket in (127) is analogous to the passage from the quantum theory to the classical theory, $1/(n+1)$ serving as the analogue of Planck's constant. As we have discussed, this correspondence can be best seen by geometric quantization of S^2. Notice also that, because of (126), the term L_A dominates in the expression of $L_A + \mathcal{A}_A$ for large values of n.

It is instructive to consider the large n limit of one of the terms in the action, say the term involving η^0, in some detail. Denoting this term as $\mathcal{S}[\eta^0]$ and using (126), (127), we find

$$
\begin{aligned}
\mathcal{S}[\eta^0] &= -i\frac{\alpha}{2}\left(\frac{n+2}{2}\right)\,\epsilon_{ABC}\\
&\quad\text{Tr}\left[\eta^0\left(\frac{n+2}{2(n+1)}\phi_C k_A\Omega_B^3 + \phi_C(e_A^+ e_B^- - e_A^- e_B^+)\right) + \mathcal{O}\left(\frac{1}{n}\right)\right]\\
&\approx -i\frac{\beta}{n+1}\,\epsilon_{ABC}\\
&\quad\text{Tr}\left[\eta^0\phi_C\left(\frac{1}{2}\,k_A\Omega_B^3 + (e_A^+ e_B^- - e_A^- e_B^+)\right) + \mathcal{O}\left(\frac{1}{n}\right)\right]
\end{aligned}
\tag{131}
$$

where $\beta = \frac{\alpha}{2}\left(\frac{n+2}{2}\right)(n+1)$. This will be taken as an n-independent constant. In carrying out these simplifications, it is useful to keep in mind that the Ω_A obey the constraint

$$\phi_A\,\Omega_A + \Omega_A\,\phi_A \approx 2\,\phi_A\,\Omega_A \approx 0\tag{132}$$

which is a natural reduction of the constraint for vectors on fuzzy S^2 as shown in (109). Since $\frac{1}{2}k_A$ can serve as derivative operators on S^2, we can define $k_A\Omega_B$ as

$$k_A\Omega_B = 2\frac{\partial}{\partial\phi^A}\Omega_B \equiv 2\,\partial_A\Omega_B.\tag{133}$$

As in the general case given in (25), the trace over $(N \times N)$-matrices can be replaced by the integral over z and $\bar{z}$;

$$\frac{1}{n+1}\,\text{Tr} \longrightarrow \int\frac{dzd\bar{z}}{\pi(1 + z\bar{z})^2} \equiv \int_{z,\bar{z}}.\tag{134}$$

We can now rewrite (131) as

$$\mathcal{S}[\eta^0] \approx -i\beta\epsilon_{ABC} \int_{z,\bar{z}} \eta^0\phi_C \left[\partial_A\Omega_B^3 + (e_A^+e_B^- - e_A^-e_B^+)\right] . \tag{135}$$

Similar results can be obtained for the rest of η's. With a simple arrangement of notation, (135) and the analogous formulae for the other η's, we recover the commutative action

$$\mathcal{S} \sim \epsilon_{AB} \int_{z,\bar{z}} \eta F_{AB} \tag{136}$$

where $\eta(z,\bar{z})$ is the Lagrange multiplier and $F_{AB}(z,\bar{z})$ is the Riemann tensor on S^2. This action is known as the two-dimensional Jackiw-Teitelboim action on S^2 [66]. We have therefore checked that, in the large N limit, the matrix action (117) for gravity reduces to a corresponding commutative action.

4.5   Generalizations

Even though we have derived the matrix action (117) via our definitions of $\mathcal{K}_\mu$, the final result is simple and can be interpreted more directly. The key quantity that enters in the action is the combination $L_A + \mathcal{A}_A^a I^a$. We can write this as

$$\begin{aligned} L_A + \mathcal{A}_A^a I^a &= D_A^a I^a \equiv D_A , \\ D_A^0 &= L_A + \mathcal{A}_A^0 , \\ D_A^a &= \mathcal{A}_A^a \qquad (a \neq 0) \end{aligned} \tag{137}$$

where a denotes the full $U(2)$ indices $(\pm, 0, 3)$. The key ingredient is thus a set of $(N \times N)$ hermitian matrices D_A^a. The definition of the curvatures is seen to be

$$\begin{aligned} \left[D_A, D_B\right] &= \left[D_A^a I^a, D_B^b I^b\right] = i\epsilon_{ABC}D_C^c I^c + F_{AB}^c I^c \\ &= i\epsilon_{ABC}D_C + F_{AB}. \end{aligned} \tag{138}$$

The action (117) is then given by

$$\begin{aligned} \mathcal{S} &= -i\frac{\alpha}{2} \, \mathrm{Tr} \left[I^3\eta \, \epsilon_{ABC} \left(D_C F_{AB} + F_{AB}D_C\right)\right] \\ &= -2i\alpha \, \mathrm{Tr} \left[I^3\eta \, \left(\epsilon_{ABC}D_A D_B D_C - iD^2\right)\right]. \end{aligned} \tag{139}$$

The constraint on the the D's is $D_A D_A = L_A L_A$. It is only in this constraint that the restriction to the sphere arises. Notice that for this particular case, we could absorb the factor of I^3 inside the trace into the field η.

   The general structure is as thus follows. We start with an irreducible finite dimensional representation of the Lie algebra of $SU(2) \times U(1)$ given by the I^a with the commutation relation $[I^a, I^b] = if^{abc}I^c$. Specifically, here we have $f^{abc} = \epsilon^{abc}$ for $a, b, c = 3, \pm$ and zero otherwise. We then construct the combinations $D_A = D_A^a I^a$ where the D_A^a are arbitrary hermitian matrices of some given dimension $N = n + 1$. Using the same $SU(2)$ structure constants we define the curvatures by $F_{AB} = [D_A, D_B] - if_{ABC}D_C$. This does not make any reference to the sphere yet. We restrict to the sphere by imposing the constraint $D_A D_A = L_A L_A$. The action is then constructed in terms of F_{AB} as in (139).

   We can use this structure to generalize to $SU(3)$, which will apply to the case of gravity on fuzzy $\mathbf{CP}^2$. Let I^a, $a = 1, 2, ..., 8$ be a set of (3×3)- matrices forming a basis of the Lie algebra of $SU(3)$, with the commutation rules $[I^a, I^b] = if^{abc}I^c$. We include $I^0 = \frac{1}{\sqrt{6}}\mathbf{1}$ to make up the algebra of $U(3)$. Let L_A denote an irreducible representation of the $SU(3)$ algebra in terms of $(N \times N)$-matrices, with $[L_A, L_B] = if_{ABC}L_C$. Note that N is restricted by $N \equiv N^{(2)} = (n+1)(n+2)/2$ $(n = 1, 2, \cdots)$ as in (2). The dynamical variables are then given by D_A^a which are a set of arbitrary $(N \times N)$-matrices. (There are 72 matrices since $A = 1, 2, ..., 8$ and $a = 0, 1, ..., 8$.) The curvatures

are defined by $F_{AB} = [D_A, D_B] - if_{ABC}D_C$, $D_A = D_A^a I^a$. As the constraints to be obeyed by the D's, we choose

$$D_A D_A = L_A L_A,$$

(140)

$$d_{ABC} D_B D_C = \left(\frac{n}{3} + \frac{1}{2}\right) D_A$$

(141)

where the constant in (141) is given by the relation (42) for $k = 2$. The continuum limit of these conditions gives $\mathbf{CP}^2$ as an algebraic surface in $\mathbf{R}^8$ and they have been used to construct noncommutative, and particularly fuzzy, versions of $\mathbf{CP}^2$ [83, 107, 38]. Following the construction of the action given in [68] and our general discussion in section 4.3, we can write the action for gravity on fuzzy $\mathbf{CP}^2$ as

$$
\begin{aligned}
\mathcal{S} &= \alpha \, \mathrm{Tr} \left[I^8 \left(D_A F_{KL} F_{MN} + F_{KL} F_{MN} D_A \right) \right] f_{KLB} f_{MNC} d_{ABC} \\
&= \alpha \, \mathrm{Tr} \left[I^8 \left(D_A \{ [D_K, D_L] - if_{KLR} D_R \} \{ [D_M, D_N] - if_{MNS} D_S \} \right. \right. \\
&\qquad \left. \left. + \{ [D_K, D_L] - if_{KLR} D_R \} \{ [D_M, D_N] - if_{MNS} D_S \} D_A \right) \right] \\
&\qquad\qquad\qquad\qquad\qquad \times f_{KLB} f_{MNC} d_{ABC} \, .
\end{aligned}
$$

(142)

This action, along with the constraints (140) and (141), gives gravity on fuzzy $\mathbf{CP}^2$ as a matrix model. One can also check directly that the large N limit of this will reduce to the MacDowell-Mansouri version of the action for gravity on commutative $\mathbf{CP}^2$.

It is clear that similar actions can be constructed for all $\mathbf{CP}^k$ ($k = 1, 2, \cdots$). Notice that the quantity $f_{KLB} f_{MNC} d_{ABC}$ is the fifth rank invariant tensor of $SU(3)$. For $\mathbf{CP}^k$ we can use k factors of F's and one factor of D and then contract indices with $\omega^{A_1 \cdots A_{2k+1}}$, the invariant tensor of $SU(k+1)$ with rank $(2k+1)$. For an explicit form of such a tensor, see (174). Actions for gravity on fuzzy $\mathbf{CP}^k$ are then written in a generalized form as

$$\mathcal{S} = \alpha \mathrm{Tr} \left[I^{((k+1)^2 - 1)} \left(D_{A_1} F_{A_2 A_3} F_{A_4 A_5} \cdots + F_{A_2 A_3} F_{A_4 A_5} \cdots D_{A_1} \right) \right] \omega^{A_1 \ldots A_{2k+1}} \, .$$

(143)

In the large N limit, such an action will contain the Einstein term (in the MacDowell-Mansouri form), but will also have terms with higher powers of the curvature. The action (143) has to be supplemented by suitable constraints on the D's, which may also be taken as the algebraic constraints for fuzzy $\mathbf{CP}^k$ shown in (34) and (35).

5 Fuzzy spaces as brane solutions in M(atrix) theory

5.1 Introduction to M(atrix) theory

There has been extensive interest in the matrix model of M-theory or the M(atrix) theory since its proposal by Banks, Fischler, Shenker and Susskind (BFSS) [78]. As mentioned in section 1.3, in M(atrix) theory nine dimensions out of eleven are described by $(N \times N)$-matrices, while the other dimensions correspond to light-front coordinates. This structure arises as a natural extension of matrix regularization of bosonic membranes in light-front gauge. The ordinary time component and the extra spatial direction, the so-called longitudinal one, emerge from the light-front coordinates in M(atrix) theory. The longitudinal coordinate is considered to be toroidally compactified with a radius R. In this way, the theory can be understood in 10 dimensions. This is in accordance with one of the features of M-theory, i.e., as a strongly coupled limit of type IIA string theory, since the radius R can be related to the string coupling constant g by $R = gl_s$ where l_s is the string length scale. From 11-dimensional points of view, one can consider certain objects which contain a longitudinal momentum N/R as a Kaluza-Klein mode. Partly from these observations

it has been conjectured that the large N limit of M(atrix) theory should describe M-theory in the large longitudinal momentum limit or in the so-called infinite momentum frame (IMF). This BFSS conjecture has been confirmed in various calculations, especially in regard to perturbative calculations of graviton interactions (see, *e.g.*, [108, 109]), capturing another feature of M-theory, *i.e.*, emergence of 11-dimensional supergravity in the low energy limit. There also exits a related matrix model by Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT) [89] which corresponds to type IIB string theory. This IKKT model has been investigated with a lot of attention as well. (For a review of this model, one may refer to [90].)

Besides gravitons, M(atrix) theory does contain extended and charged objects, namely, memberanes and 5-branes. The membrane in matrix context appeared originally in the quantization of the supermembrane a number of years ago by de Wit, Hoppe and Nicolai [110]. Membranes of spherical symmetry in M(atrix) theory have been obtained in [80, 81]. As regards 5-branes, they were obtained as longitudinal 5-branes or L5-branes [84, 85, 86]. The L5-branes are named after the property that one of their five dimensions coincides with the longitudinal direction in M(atrix) theory. One may think of the existence of transverse 5-branes as opposed to L5-branes, but it turns out that there is no classically conserved charges corresponding to the transverse 5-branes. Thus it is generally believed that the L5-branes are the only relevant 5-branes in M(atrix) theory at least in the classical level. (In a modified M(atrix) theory, *i.e.*, the so-called plane wave matrix theory [111], the existence of transverse 5-branes is discussed at a quantum level [112].) L5-branes with spherical symmetry in the transverse directions have also been obtained in [82]. Although this spherical L5-brane captures many properties of M-theory, it is as yet unclear how to include matrix fluctuations contrary to the case of spherical membranes. The only other L5-brane that is known so far is an L5-brane with $\mathbf{CP}^2$ geometry in the transverse directions [83]. Matrix configuration of this L5-brane is relevant to that of the fuzzy $\mathbf{CP}^2$.

Fuzzy spaces are one of the realizations of noncommutative geometry in terms of $(N \times N)$-matrices, hence, those extended objects in M(atrix) theory are possibly described by the fuzzy spaces as far as the transverse directions are concerned. Following this idea, in the present chapter we shall consider fuzzy complex projective spaces $\mathbf{CP}^k$ $(k = 1, 2, \cdots)$ as ansätze to the extended objects or the brane solutions in M(atrix) theory. This approach towards a solution to M(atrix) theory was originally pursued by Nair and Randjbar-Daemi in [83] which, among the other known brane solutions, revealed the existence of the L5-brane of $\mathbf{CP}^2 \times S^1$ geometry. At this stage, we are familiar to the fact that fuzzy $\mathbf{CP}^k$ are constructed in terms of matrix representations of the algebra of $SU(k+1)$ in the $(n, 0)$-representation under a certain set of algebraic constraints. This fact makes it relatively straightforward to include transverse fluctuations of branes with $\mathbf{CP}^2$ (or $\mathbf{CP}^k$) geometry in comparison with the case of the spherical L5-brane. This point is one of the advantages to consider fuzzy $\mathbf{CP}^k$ as ansätze for the brane solutions. Note that fluctuations of branes are described by gauge fields on noncommutative geometry. This means that the dynamics of the extended objects in M(atrix) theory can be governed by gauge theories on fuzzy spaces.

From a perspective of type IIA string theory, the gravitons, membranes and L5-branes of M-theory are respectively relevant to D0, D2 and D4 brane solutions. Type IIA string theory also contains a D6 brane. The D6 brane is known to be a Kaluza-Klein magnetic monopole of 11-dimentional supergravity compactified on a circle and is considered to be irrelevant as a brane solution in M(atrix) theory. Naively, however, since D6 branes are Hodge dual to D0 branes in the same sense that D2 and D4 branes are dual to each other, we would expect the existence of L7-branes in M(atrix) theory. It is important to remind that fuzzy spaces can be constructed only for compact spaces. If we parametrize branes by fuzzy spaces, the transverse directions are also all compactified in the large N limit. As far as the capture of a Kaluza-Klein mode in the scale of N/R is concerned, one can not distinguish the longitudinal direction from the transverse ones. The gravitons or the corresponding D0 branes of M-theory would possibly live on the transverse directions in this case. Thus we may expect the existence of L7-branes as a Hodge dual description of such gravitons in an M-theory perspective. Construction of L7-branes (or transverse D6-branes) has been suggested in [86, 113], however, such extended objects have not been obtained in the matrix model. Besides the fact that no L7-brane charges appear in the supersymmetry algebra of M(atrix) theory, there is a crucial obstruction to the construction of L7-brane, that is, as shown by Banks, Seiberg and Shenker [86], the L7-brane states have an infinite energy in the large N

limit, where the energy of the state is interpreted as an energy density in the transverse directions. Indeed, as we shall discuss in the next section, an L7-brane of $\mathbf{CP}^3 \times S^1$ geometry leads to an infinite energy in the large N limit and, hence, one can not make sense of the theory with such an L7-brane.

In order to obtain an L7-brane as a solution to M(atrix) theory, it would be necessary to introduce extra potentials or fluxes to the M(atrix) theory Lagrangian such that the brane system has a finite energy as $N \to \infty$. Since M(atrix) theory is defined on a flat space background, such an additional term suggests the description of the theory in a nontrivial background. The most notable modification of the M(atrix) theory Lagrangian would be the one given by Berenstein, Maldacena and Nastase (BMN) to describe the theory in the maximally supersymmetric parallel-plane (pp) wave background [111]. There has been a significant amount of papers on this BMN matrix model of M-theory. (For some of the earlier papers, see [114]-[119].) Another important approach to the modification of BFSS M(atrix) theory is to introduce a Ramond-Ramond (RR) field strength as a background such that it couples to brane solutions. Specifically, one may have a RR 4-form as an extra potential from a IIA string theory viewpoint. As shown by Myers [120], the matrix equation of motion with this RR flux allows fuzzy S^2 as a static solution, meaning that the corresponding IIA theory has a spherical D2-brane solution. The RR field strength is associated with a charge of this D2 brane. The modified equation of motion also allows a diagonal matrix configuration as a solution which corresponds to N D0-branes, with N being the dimension of matrices. One may interpret these solutions as bound states of a spherical D2-brane and N D0-branes. From a D0-brane perspective, the RR field strength is also associated with a D0-brane charge. So the extra RR flux gives rise to a D-brane analog of a dielectic effect, known as Myers effect. A different type of flux, *i.e.*, a RR 5-form which produces bound states of N D1-branes and a D5-brane with $\mathbf{CP}^2$ geometry has been proposed by Alexanian, Balachandran and Silva [88] to describe a generalized version of Myers effect from a viewpoint of IIB string theory. From a M(atrix) theory perspective, the D5 brane corresponds to the L5-brane of $\mathbf{CP}^2 \times S^1$ geometry. In this chapter, we consider further generalization along this line of development, namely, we consider a general form for all possible extra potentials that allows fuzzy $\mathbf{CP}^k$ as brane solutions or solutions of modified matrix equations of motion. We find several such potentials for $k \leq 3$.

The extra potentials we shall introduce in the consideration of a possible L7 brane solution to M(atrix) theory are relevant to fluxes on a curved space of $(\mathbf{CP}^3 \times S^1) \times \mathcal{M}_4$ where $\mathcal{M}_4$ is an arbitrary four-dimensional manifold. We shall show that one of the potentials can be interpreted as a 7-form flux in M(atrix) theory. According to Freund and Rubin [121], existence of a 7-form in 11 dimensional (bosonic) theories implies compactification of 7 or 4 space-like dimensions. The existence of the 7-form in M(atrix) theory is interesting in a sense that it would lead to a matrix version of Freund-Rubin type compactification. As we shall discuss later, fluctuations from a stabilized L7-brane solution will turn on an infinite potential in the large N limit. So the fluctuations are suppressed and we have a reasonable picture of matrix compactification, provided that the $\mathcal{M}_4$ does not affect the energy of the L7-brane state.

The plan of the rest of this chapter is as follows. In the next section, following [83], we show that fuzzy $\mathbf{CP}^k$ ($k \leq 4$) provide solutions to bosonic matrix configurations in M(atrix) theory. Along the way we briefly review definitions and properties of fuzzy $\mathbf{CP}^k$. We further discuss the energy scales of the solutions and see that the energy becomes finite in the large N limit only in the cases of $k = 1, 2$, corresponding to the membrane and the L5-brane solutions in M(atrix) theory. In section 5.3, we examine supersymmetry of the brane solutions for $k \leq 3$. We make a group theoretic analysis to show that those brane solutions break the supersymmetries in M(atrix) theory. Our discussion is closely related to the previous analysis [83] in the case of $k = 2$. In section 5.4, we introduce extra potentials to the M(atrix) theory Lagrangian which are suitable for the fuzzy $\mathbf{CP}^k$ brane solutions. We consider the effects of two particular potentials to the theory. These effects can be considered as generalized Myers effects. We find a suitable form of potentials for the emergence of static L7-brane solutions, such that the potentials lead to finite L7-brane energies in the large N limit. Section 5.5 is devoted to the discussion on possible compactification models in non-supersymmetric M(atrix) theory. We show that one of the extra potentials introduced for the presence of L7-branes can be interpreted as a matrix-valued or 'fuzzy' 7-form in M(atrix) theory. This suggests compactification down to 7 dimensions. We also consider compactification down to

4 dimensions by use of fuzzy S^4 which, as discussed in chapter 3, is defined in terms of fuzzy $\mathbf{CP}^3$.

5.2 Fuzzy $\mathbf{CP}^k$ as brane solutions

The M(atrix) theory Lagrangian can be expressed as

$$\mathcal{L} = \mathrm{Tr}\left(\frac{1}{2R}\dot{X}_I^2 + \frac{R}{4}[X_I, X_J]^2 + \theta^T\dot{\theta} + iR\theta^T\Gamma_I[X_I, \theta]\right) \tag{144}$$

where X_I $(I = 1, 2, \cdots, 9)$ are hermitian $N \times N$ matrices, θ denotes a 16-component spinor of $SO(9)$ represented by $N \times N$ Grassmann-valued matrices, and Γ_I are the $SO(9)$ gamma matrices in the 16-dimensional representation. The Hamiltonian of the theory is given by

$$\mathcal{H} = \mathrm{Tr}\left(\frac{R}{2}P_I P_I - \frac{R}{4}[X_I, X_J]^2 - iR\theta^T\Gamma_I[X_I, \theta]\right) \tag{145}$$

where P_I is the canonical conjugate to X_I; $P_I = \frac{\partial\mathcal{L}}{\partial\dot{X}_I}$. As discussed in the introduction, we will be only interested in those energy states that have finite energy in the limit of the large longitudinal momentum N/R. Since the Hamiltonian (145) leads to an infinite energy state in the limit of $R \to \infty$, we will consider the large N limit with a large, but fixed value for R. With this limit understood, the theory is defined by (144) or (145) with a subsidiary Gauss law constraint

$$[X_I, \dot{X}_I] - [\theta, \theta^T] = 0. \tag{146}$$

In this section, we shall consider the bosonic part of the theory, setting the θ's to be zero. The relevant equations of motion for X_I are given by

$$\frac{1}{R}\ddot{X}_I - R[X_J, [X_I, X_J]] = 0 \tag{147}$$

with a subsidiary constraint

$$[X_I, \dot{X}_I] = 0. \tag{148}$$

We shall look for solutions to these equations, taking the following ansätze

$$X_I = \begin{cases} r(t)Q_i & \text{for } I = i = 1, 2, \cdots, 2k \\ 0 & \text{for } I = 2k+1, \cdots, 9 \end{cases} \tag{149}$$

where Q_i denote the local coordinates of fuzzy $\mathbf{CP}^k = SU(k+1)/U(k)$ $(k = 1, 2, \cdots)$. Since X_I are defined for $I = 1, 2, \cdots, 9$, the ansätze are only valid for $k \leq 4$.

As discussed in section 2.4, fuzzy $\mathbf{CP}^k$ can be constructed in terms of certain matrix generators of $SU(k+1)$ as embedded in $\mathbf{R}^{k^2+2k}$ under a set of algebraic constraints. Here we shall briefly review such a construction. Let L_A $(A = 1, 2, \cdots, k^2 + 2k = \dim SU(k+1)$ be $N^{(k)} \times N^{(k)}$-matrix representations of the generators of $SU(k+1)$ in the $(n, 0)$-representation. The coordinates of fuzzy $\mathbf{CP}^k$ as embedded in $\mathbf{R}^{k^2+2k}$ are parametrized by $Q_A = L_A/\sqrt{C_2^{(k)}}$ where $C_2^{(k)}$ is the quadratic Casimir of $SU(k+1)$ in the $(n, 0)$-representation

$$C_2^{(k)} = \frac{nk(n+k+1)}{2(k+1)}. \tag{150}$$

The matrix dimension is given by

$$N^{(k)} = \frac{(n+k)!}{k!\,n!} \sim n^k. \tag{151}$$

The fuzzy $\mathbf{CP}^k$ coordinates, as embedded in $\mathbf{R}^{k^2+2k}$, are then defined by the following two constraints on Q_A:

$$Q_A\,Q_A = \mathbf{1}, \tag{152}$$
$$d_{ABC}\,Q_A\,Q_B = c_{k,n}\,Q_C \tag{153}$$

where $\mathbf{1}$ is the $N^{(k)} \times N^{(k)}$ identity matrix, d_{ABC} is the totally symmetric symbol of $SU(k+1)$ and the coefficient $c_{k,n}$ is given by

$$c_{k,n} = \frac{(k-1)}{\sqrt{C_2^{(k)}}} \left(\frac{n}{k+1} + \frac{1}{2} \right). \tag{154}$$

The first constraint (152) is trivial due to the definition of Q_A. The second constraint (153) is what is essential for the global definition of fuzzy $\mathbf{CP}^k$ as embedded in $\mathbf{R}^{k^2+2k}$. For $k \ll n$, the coefficient $c_{k,n}$ becomes $c_{k,n} \rightarrow c_k = \sqrt{\frac{2}{k(k+1)}}(k-1)$ and this leads to the constraints for the coordinates q_A of commutative $\mathbf{CP}^k$, i.e., $q_A\, q_A = 1$ and $d_{ABC}q_A q_B = c_k q_C$. As discussed earlier in section 2.4, the latter constraint restricts the number of coordinates to be $2k$ out of $k^2 + 2k$. Similarly, under the constraint (153), the coordinates of fuzzy $\mathbf{CP}^k$ are effectively expressed by the local coordinates Q_i $(i = 1, 2, \cdots, 2k)$ rather than the global ones Q_A $(A = 1, 2, \cdots, k^2 + 2k)$.

Let us consider the commutation relations of Q_i's. By construction they are embedded in the $SU(k+1)$ algebra. We split the generators L_A of $SU(k+1)$ into $L_i \in \underline{SU(k+1)} - \underline{U(k)}$ and $L_\alpha \in \underline{U(k)}$, where $\underline{G}$ denotes the Lie algebra of group G. The index $i = 1, 2, \cdots, 2k$ is relevant to the $\mathbf{CP}^k$ of our interest, while the index $\alpha = 1, 2, \cdots, k^2$ denotes the $U(k)$ subgroup of $SU(k+1)$. We shall continue to distinguish these indices in what follows. The $SU(k+1)$ algebra, $[L_A, L_B] = i f_{ABC} L_C$ with the structure constant f_{ABC}, is then expressed by the following set of commutation relations

$$[Q_i, Q_j] = i\, \frac{c_{ij\alpha}}{\sqrt{C_2^{(k)}}}\, Q_\alpha, \tag{155}$$

$$[Q_\alpha, Q_\beta] = i\, \frac{f_{\alpha\beta\gamma}}{\sqrt{C_2^{(k)}}}\, Q_\gamma, \tag{156}$$

$$[Q_\alpha, Q_i] = i\, \frac{f_{\alpha ij}}{\sqrt{C_2^{(k)}}}\, Q_j \tag{157}$$

where we use $Q_A = L_A / \sqrt{C_2^{(k)}}$ and denote $f_{ij\alpha}$ by $c_{ij\alpha}$ to indicate that it is relevant to the commutators of Q_i's. $f_{\alpha\beta\gamma}$ is essentially the structure constant of $SU(k)$ since the $U(1)$ part of the $U(k)$ algebra can be chosen such that it commutes with the rest of the algebra. We can calculate $c_{\alpha ij} c_{\beta ij}$ as

$$c_{\alpha ij}\, c_{\beta ij} = f_{\alpha AB} f_{\beta AB} - f_{\alpha\gamma\delta} f_{\beta\gamma\delta} = \delta_{\alpha\beta} \tag{158}$$

by use of the relations $f_{\alpha AB} f_{\beta AB} = (k+1)\delta_{\alpha\beta}$ and $f_{\alpha\gamma\delta} f_{\beta\gamma\delta} = k\delta_{\alpha\beta}$. Notice that the result (158) restricts possible choices of the $\mathbf{CP}^k$ indices (i, j). For example, in the case of $k = 2$ we have $(i, j) = (4, 5), (6, 7)$ with the conventional choice of the structure constant f_{ABC} of $SU(3)$. Similarly, in the case of $k = 3$ we have $(i, j) = (9, 10), (11, 12), (13, 14)$. Under such restrictions, we can also calculate $c_{ij\alpha} f_{j\alpha k}$ as

$$c_{ij\alpha}\, f_{j\alpha k} = c_{ij\alpha}\, c_{kj\alpha} = \delta_{ik}. \tag{159}$$

Using (155)-(159), we can easily find that $[Q_j, [Q_i, Q_j]] = -Q_i/C_2^{(k)}$. Thus, with the ansätze (149), we can express the equation of motion (147) solely in terms of the matrix Q_i, which reduces the equation to an ordinary differential equation of $r(t)$. Note that the subsidiary constraint (148) is also satisfied with (149). So our ansätze are consistent and we can now proceed to the equations of motion for $r(t)$.

With the ansätze, the bosonic part of the Lagrangian (144) becomes

$$\begin{aligned}
\mathcal{L} &= \mathrm{Tr} \left(\frac{\dot{r}^2}{2R} Q_i Q_i - \frac{R}{4} \frac{r^4}{C_2^{(k)}} Q_\alpha Q_\alpha \right) \\
&= \frac{N^{(k)}}{k^2 + 2k} \left(\frac{\dot{r}^2}{2R} - \frac{Rr^4}{4C_2^{(k)}} \right)
\end{aligned} \tag{160}$$

where we use the relation

$$\mathrm{Tr}(Q_A Q_B) = \frac{N^{(k)}}{k^2 + 2k} \delta_{AB}. \tag{161}$$

Note that we choose a particular index i for the kinetic term and that the factor $Q_\alpha Q_\alpha$ in the potential term should be understood with the restricted choices of (i,j) in the use of (158). For example, in the case of $k = 2$ and $(i,j) = (4,5)$, $Q_\alpha Q_\alpha$ should be expressed as $Q_\alpha Q_\alpha = \frac{1}{4} Q_3 Q_3 + \frac{3}{4} Q_8 Q_8$, while we have $Q_\alpha Q_\alpha = \frac{1}{4} Q_3 Q_3 + \frac{1}{12} Q_8 Q_8 + \frac{2}{3} Q_{15} Q_{15}$ for $k = 3$, $(i,j) = (9,10)$. The apparent illness of the expression $Q_\alpha Q_\alpha$ is caused by the fact that the fuzzy $\mathbf{CP}^k$ can, as we have seen, not be defined locally in terms of Q_A's. The trace of $Q_\alpha Q_\alpha$, however, always leads to the factor $\mathrm{Tr} Q_\alpha Q_\alpha = \frac{N^{(k)}}{k^2 + 2k}$, which is in consistent with the relation (158). In what follows, we shall use the symbol $c_{ij\alpha}$ when it is necessary to make a local analysis on the matrix Lagrangian.

The equation of motion corresponding to (160) is given by

$$\ddot{r} + \frac{R^2}{C_2^{(k)}} r^3 = 0. \tag{162}$$

A general solution to this equation is written as

$$r(t) = A \operatorname{cn}\left(\alpha(t - t_0); \kappa^2 = \frac{1}{2}\right) \tag{163}$$

where $\alpha = \sqrt{R^2/C_2^{(k)}}$ and $\operatorname{cn}(u; \kappa) = \operatorname{cn}(u)$ is one of the Jacobi elliptic functions, with κ ($0 \leq \kappa \leq 1$) being the elliptic modulus. A and t_0 are the constants determined by the initial conditions. Using the formula

$$\frac{d}{du} \operatorname{cn}(u; \kappa) = -\operatorname{sn}(u; \kappa) \operatorname{dn}(u; \kappa)$$

$$= -u + \frac{1 + 4\kappa^2}{3!} u^3 - \cdots, \tag{164}$$

we can express $\dot{r}$ as

$$\dot{r} = -A\alpha \operatorname{sn}(\alpha(t - t_0)) \operatorname{dn}(\alpha(t - t_0)). \tag{165}$$

In the limit of large N (or n), $\dot{r}$ is suppressed by $\dot{r} \sim 1/n^2$. So the solution (163) corresponds to a static solution in the large N limit.

The potential energy is given by

$$V = \frac{N^{(k)}}{k^2 + 2k} \left(\frac{R}{4 C_2^{(k)}} r^4 \right) \sim n^{k-2} R r^4. \tag{166}$$

From this result we can easily tell that for $k = 1, 2$ we have finite energy states in the large N limit. These states respectively correspond to the spherical membrane and the L5-brane of $\mathbf{CP}^2$ geometry in M(atrix) theory. By contrast, for $k = 3, 4$ we have infinite energy states. So, although these may possibly correspond to $L7$ and $L9$ brane solutions, they are ill-defined and we usually do not consider such solutions in M(atrix) theory.

5.2.1 Global analysis

Since the fuzzy $\mathbf{CP}^k$ is algebraically defined in terms of Q_A ($A = 1, 2, \cdots, k^2 + 2k$), we need to replace Q_i by Q_A in order to make a global analysis on the matrix Lagrangian. With our ansätze (149), the M(atrix) theory Lagrangian is then written as

$$\begin{aligned}
\mathcal{L} &= \mathrm{Tr}\left(\frac{\dot{r}^2}{2R} Q_A Q_A + \frac{R r^4}{4} [Q_A, Q_B]^2 \right) \\
&= N^{(k)} \left(\frac{\dot{r}^2}{2R} - (k+1) \frac{R r^4}{4 C_2^{(k)}} \right) \\
&= N^{(k)} \sqrt{k+1} \left(\frac{\dot{r}^2}{2\tilde{R}} - \frac{\tilde{R} r^4}{4 C_2^{(k)}} \right)
\end{aligned} \tag{167}$$

where we introduce $\tilde{R} = \sqrt{k+1}R$ by which we can reproduce the same equation of motion as in (162). The potential energy is now given by $N^{(k)}(k+1)Rr^4/4C_2^{(k)} \sim n^{k-2}Rr^4$. Thus the large n (and R) behavior of the potential does not change from (166). Up to the overall scaling factor $1/(k^2+2k)$, the global Lagrangian (167) is therefore equivalent to the local Lagrangian (160), with the scaling of R to $\tilde{R}$ being understood.

5.3   Supersymmetry breaking

We have set the ferimionic matrix variables θ to be zero. In this section, we now consider the supersymmetry transformations of the brane solutions in M(atrix) theory. The supersymmetric variation of θ is given by

$$\delta\theta_r = \frac{1}{2}\left(\dot{X}_I(\Gamma_I)_{rs} + [X_I, X_J](\Gamma_{IJ})_{rs}\right)\epsilon_s + \delta_{rs}\xi_s \tag{168}$$

where ϵ and ξ are 16-component spinors of $SO(9)$ represented by $N \times N$ matrices ($r, s = 1, 2, \cdots, 16$) and Γ_I's are the corresponding gamma matrices as before. Γ_{IJ} are defined by $\Gamma_{IJ} = \frac{1}{2}[\Gamma_I, \Gamma_J]$. With our ansätze, the equation (168) reduces to

$$\delta\theta_r = \frac{1}{2}\left(\dot{r}Q_i(\gamma_i)_{rs} + r^2\frac{ic_{ij\alpha}}{\sqrt{C_2^{(k)}}}Q_\alpha(\gamma_{ij})_{rs}\right)\epsilon_s + \delta_{rs}\xi_s \tag{169}$$

where γ_i's are the gamma matrices of $SO(2k)$ under the decomposition of $SO(9) \to SO(2k) \times SO(9-2k)$. Accordingly, we here set $i = 1, 2, \cdots, 2k$ and $r, s = 1, 2, \cdots, 2^k$. For the static solution we make $\dot{r} \sim n^{-2}$ vanish. Indeed, if $\delta\theta \sim n^{-2}$, we have $\mathrm{Tr}(\delta\theta^T\delta\dot{\theta}) \sim N^{(k)}n^{-4} \sim n^{k-4}$ and, for $k = 1, 2$ and 3, this term vanishes in the large N limit. The other term $\mathrm{Tr}(iR\delta\theta^T\Gamma_I[X_I, \delta\theta])$ in the Lagrangian vanishes similarly. So, for static solutions, the condition $\delta\theta = 0$ is satisfied when $c_{ij\alpha}Q_\alpha\gamma_{ij}$ becomes a **c**-number in the $SO(2k)$ subspace of $SO(9)$ such that the ϵ-term can be cancelled by ξ in (169). In what follows, we examine this BPS-like condition for $k = 1, 2, 3$. Note that we shall not consider the case of $k = 4$ or a 9-brane solution to M(atrix) theory. (The 9-branes are supposed to correspond to "ends of the world" which describe gauge dynamics of the 9-dimensional boundary of M-theory and they are considered to be irrelevant as brane solutions of the theory.)

It is known that the spherical membrane solution breaks all supersymmetries. Let us rephrase this fact by examining the BPS condition ($\delta\theta = 0$) for $k = 1$. The 2-dimensional gamma matrices are given by $\gamma_1 = \sigma_1$ and $\gamma_2 = \sigma_2$, where σ_i is the (2×2)-Pauli matrices. The factor $c_{ij\alpha}Q_\alpha\gamma_{ij}$ becomes proportional to $Q_3\sigma_3$ where Q_3 is an $N^{(1)} \times N^{(1)}$ matrix representing the $U(1)$ part of the $SU(2)$ generators in the spin-$n/2$ representation. Now the factor σ_3 is not obviously proportional to identity in the $SO(2)$ subspace of $SO(9)$, so we can conclude that the BPS condition is broken.

For $k = 2$, we can apply the same analysis to the factor of $c_{ij\alpha}Q_\alpha\gamma_{ij}$. We use the conventional choice for the structure constant of $SU(3)$ where the group elements are defined by $g = \exp(i\theta^a\frac{\lambda^a}{2})$ with the Gell-Mann matrices λ^a ($a = 1, 2, \cdots, 8$). As discussed earlier, with this convention the set of (i, j) is restricted to $(i, j) = (4, 5)$ or $(6, 7)$. The relevant $c_{ij\alpha}$'s are given by $c_{453} = 1/2$, $c_{458} = \sqrt{3}/2$, $c_{673} = -1/2$ and $c_{678} = \sqrt{3}/2$. Introducing the usual 4-dimensional gamma matrices γ_i ($i = 4, 5, 6, 7$)

$$\gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix},$$
$$\gamma_6 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_7 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}, \tag{170}$$

we can calculate the factor of interest as

$$c_{45\alpha}Q_\alpha\gamma_{45} \sim \left(Q_3 + \sqrt{3}Q_8\right)\begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix},$$
$$c_{67\alpha}Q_\alpha\gamma_{67} \sim \left(-Q_3 + \sqrt{3}Q_8\right)\begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix} \tag{171}$$

where $\gamma_{ij} = \frac{1}{2}[\gamma_i, \gamma_j]$, and Q_3, Q_8 are $N^{(2)} \times N^{(2)}$ matrices representing diagonal parts of the algebra of $SU(3)$ in the $(n, 0)$-representation. In either case, it is impossible to make the factor $c_{ij\alpha}Q_\alpha\gamma_{ij}$ be proportional to identity or zero in terms of the (4×4)-matrix which corresponds to γ_i's. This indicates that the brane solution corresponding to $k = 2$ breaks the supersymmetries of M(atrix) theory as originally analyzed in [83].

The same analysis is applicable to the case of $k = 3$ and we can show that the brane solution corresponding to $k = 3$ also breaks the supersymmetries. For the completion of discussion, we present the factors $c_{ij\alpha}Q_\alpha\gamma_{ij}$ for $(i, j) = (9, 10), (11, 12), (13, 14)$ in suitable choices of $c_{ij\alpha}$ and 6-dimensional gamma matrices:

$$c_{9\,10\,\alpha}\, Q_\alpha\, \gamma_{9\,10} \sim \left(\sqrt{3}Q_3 + Q_8 + 2\sqrt{2}Q_{15}\right) \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & -\sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & -\sigma_1 \end{pmatrix},$$

$$c_{11\,12\,\alpha}\, Q_\alpha\, \gamma_{11\,12} \sim \left(-\sqrt{3}Q_3 + Q_8 + 2\sqrt{2}Q_{15}\right) \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix},$$

$$c_{13\,14\,\alpha}\, Q_\alpha\, \gamma_{13\,14} \sim \left(-2Q_8 + 2\sqrt{2}Q_{15}\right) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{172}$$

where Q_3, Q_8 and Q_{15} are the $N^{(3)} \times N^{(3)}$ matrices representing diagonal parts of $SU(4)$ algebra in the $(n, 0)$-representation. In the last line, $\mathbf{1}$ denotes the 4×4 identity matrix.

5.4 L7-branes and extra potentials

As we have seen in (166), the potential energy of a prospective L7-brane with $\mathbf{CP}^3 \times S^1$ geometry is proportional to n, leading to an infinite energy in the large N limit. In this section, we introduce extra potentials to the bosonic part of the M(atrix) theory Lagrangian so that the total potential energy of the L7-brane becomes finite in the large N limit. The kinetic energy of brane states with $\mathbf{CP}^k \times S^1$ geometry is proportional to n^{k-4}/R. This can be seen by (160) or (167) with $\dot{r} \sim n^{-2}$. Since the kinetic energy is suppressed by n^{k-4}, one can consider the brane solution for any k ($k = 1, 2, 3$) as a static solution. Consideration of potential energies will suffice for the stability analysis of brane solutions. In what follows, we first consider a general form of the extra potentials which is appropriate for the fuzzy $\mathbf{CP}^k$ brane solutions. We then consider two particular cases in some details, eventually obtaining a suitable form of the extra potential for the emergence of L7-branes.

We have been interested in the brane solutions of the fuzzy $\mathbf{CP}^k$ ansätze (149). The matrix coordinates are to be parametrized in terms of the fuzzy $\mathbf{CP}^k$ coordinates, $X_A = r(t)Q_A$, which are globally defined on $\mathbf{R}^{k^2+2k}$. Extra potentials are given by polynomials of X_A's, with the indices being contracted by some invariant tensor on $\mathbf{R}^{k^2+2k}$. A possible form of these extra potentials is then expressed as

$$F_{2r+1}(X) = \mathrm{STr}(F_{A_1 A_2 \cdots A_{2r+1}} X_{A_1} X_{A_2} \cdots X_{A_{2r+1}}), \tag{173}$$

$$F_{A_1 A_2 \cdots A_{2r+1}} = (-i)^{2r+1}\mathrm{tr}(t_{[A_1} t_{A_2} \cdots t_{A_{2r+1}]}) \tag{174}$$

where $F_{A_1 A_2 \cdots A_{2r+1}}$ is the rank-$(2r+1)$ invariant tensors of $SU(k+1)$, with r being $r = 1, 2, \cdots, k$. t_A's are the generators of $SU(k+1)$ in the fundamental representation. STr means the symmetrized trace over X_A's and the square bracket in (174) means the antisymmetrization of the arguments. A trace over t_A's is denoted by tr. Note that there are no rank-$(2s)$ invariant tensors ($s = 1, 2, \cdots$) due to cyclicity of the trace. The invariant tensors $F_{A_1 A_2 \cdots A_{2r+1}}$ are associated with differential $(2r+1)$-forms of $SU(k+1)$. The potentials $F_{2r+1}(X)$ are closely related to these differential forms. We shall show this relation in the next section.

For $r = 1$, we have

$$F_3(X) = -\frac{1}{12} f_{IJK} \mathrm{Tr}(X_I X_J X_K) \tag{175}$$

where we use $t_{[I}t_{J]} = \frac{1}{2}[t_I, t_J]$ and $f_{IJK} = -i2\mathrm{tr}([t_I, t_J]t_K)$. Since $f_{IJK} \sim \epsilon_{IJK}$, the addition of F_3 to the M(atrix) theory Lagrangian essentially leads to Myers effect from a viewpoint of IIA string theory [120]. For $r = 2$, we can calculate F_5 as

$$
\begin{aligned}
F_5(X) &= \frac{(-i)^5}{5}\mathrm{STr}(\ \mathrm{tr}(t_K t_L t_M t_N t_O) X_{[K} X_L X_M X_{N]} X_O\ ) \\
&= \frac{1}{5} \cdot \frac{i}{4} f_{KLA} f_{MNB} \mathrm{tr}(t_A t_B t_O) \frac{1}{8} \mathrm{Tr}(\{[X_K, X_L], [X_M, X_N]\} X_O) \\
&= \frac{i}{160} f_{KLA} f_{MNB} \mathrm{tr}(t_A t_B t_O) \mathrm{Tr}(X_K[X_L, \{[X_M, X_N], X_O\}])
\end{aligned}
\tag{176}
$$

where indices A, B are symmetric and we have used

$$
\begin{aligned}
X_{[K} X_L X_M X_{N]} &= \frac{1}{4!}(\{[X_K, X_L], [X_M, X_N]\} + \{[X_K, X_M], [X_N, X_L]\} \\
&\quad + \{[X_K, X_N], [X_L, X_M]\})\,.
\end{aligned}
\tag{177}
$$

Note that F_{KLMNO} exists for any $SU(k+1)$ with $k \geq 2$. F_5 is a natural generalization of Myers term in a higher rank. The variation of F_5 with respect to X_K is expressed as

$$
\frac{\delta}{\delta X_K} F_5 = \frac{i}{32} f_{KLA} f_{MNB} \mathrm{tr}(t_A t_B t_O)[X_L, \{[X_M, X_N], X_O\}]\,.
\tag{178}
$$

5.4.1 Modification with F_5

The M(atrix) theory Lagrangian with the extra potential F_5 is given by

$$
\begin{aligned}
\mathcal{L}^{(5)} &= \mathcal{L} + \lambda_5 F_5(X)\,,
\tag{179} \\
\mathcal{L} &= \mathrm{Tr}\left(\frac{\dot{X}_I^{\,2}}{2R} + \frac{R}{4}[X_I, X_J]^2\right)
\tag{180}
\end{aligned}
$$

where $\mathcal{L}$ is the bosonic part of the original M(atrix) theory Lagrangian (144) and λ_5 is a coefficient of the rank-5 invariant tensor F_{KLMNO}. The matrix equations of motion are expressed as

$$
\frac{1}{R}\ddot{X}_I - R[X_J, [X_I, X_J]] - \lambda_5 \frac{\delta}{\delta X_I} F_5 = 0
\tag{181}
$$

where the variation of F_5 with respect to X_I is given by (178). We now evaluate the variation with the fuzzy $\mathbf{CP}^k$ ansätze (149). The variation (178) becomes

$$
\begin{aligned}
\left.\frac{\delta}{\delta X_K} F_5\right|_{X=rQ} &= \frac{i}{32} r^4 \left(i \frac{k+1}{\sqrt{C_2^{(k)}}}\right) f_{KLA} \mathrm{tr}(t_A t_B t_O)[Q_L, \{Q_B, Q_O\}] \\
&= -\frac{i}{64} r^4 \left(\frac{k+1}{\sqrt{C_2^{(k)}}}\right)^2 c_{k,n} Q_K
\end{aligned}
\tag{182}
$$

where we use the relations

$$
\begin{aligned}
[Q_K, Q_L] &= i f_{KLA} Q_A / \sqrt{C_2^{(k)}}\,, \\
f_{KLA} f_{KLB} &= (k+1)\delta_{AB}
\end{aligned}
\tag{183}
$$

and the fuzzy $\mathbf{CP}^k$ constraints, i.e., $d_{KLM} Q_K Q_L = c_{k,n} Q_M$ and $Q_K Q_K = \mathbf{1}$ with $d_{ABC} = 2\mathrm{tr}(\{t_A, t_B\}t_C)$. Notice that, because of closure of the algebra, we can regard Q_A and Q_B as the fuzzy $\mathbf{CP}^k$ coordinates so that they also satisfy the constraints. Explicit values of $C_2^{(k)} \sim n^2$ and $c_{k,n} \sim 1$ are given in (150) and (154), respectively.

The matrix equations of motion (181) becomes

$$
\begin{aligned}
0 &= \frac{\ddot{r}}{R}Q_I + r^3 R \frac{k+1}{C_2^{(k)}} Q_I + \frac{i\lambda_5}{64} r^4 \left(\frac{k+1}{\sqrt{C_2^{(k)}}} \right)^2 c_{k,n} Q_I \\
&= \sqrt{k+1} \left[\frac{\ddot{r}}{\tilde{R}} + \frac{\tilde{R}}{C_2^{(k)}} r^3 \left(1 + \frac{i\lambda_5 r}{64R}(k+1)c_{k,n} \right) \right] Q_I
\end{aligned}
\tag{184}
$$

where we define $\tilde{R}$ by $\tilde{R} = \sqrt{k+1}R$ as in (167). Notice that the equation is linear in Q_I. So we can reduce the matrix equation to an equation of $r(t)$ as in the case without the extra potential. In this sense, the form of F_5 is suitable for our fuzzy $\mathbf{CP}^k$ ansätze, namely, the fuzzy $\mathbf{CP}^k$ ansätze provide solutions to the modified theory as well. Evaluation of F_5 on these ansätze can be easily done as

$$
\begin{aligned}
F_5(Q) &= \frac{i}{160} r^5 \left(i \frac{k+1}{\sqrt{C_2^{(k)}}} \right)^2 \mathrm{tr}(t_A t_B t_O) \mathrm{Tr}(\{Q_A, Q_B\} Q_O) \\
&= -\frac{i}{320} r^5 \left(\frac{k+1}{\sqrt{C_2^{(k)}}} \right)^2 c_{k,n} N^{(k)} \sim n^{k-2} r^5
\end{aligned}
\tag{185}
$$

where we use $\mathrm{Tr}(Q_A Q_A) = \mathrm{Tr}\mathbf{1} = N^{(k)}$. Note that the n and R dependence of (185) is the same as the M(atrix) theory potential (166). The presence of $F_5(Q)$ is arrowed for $k \geq 2$.

5.4.2 Modification with F_7

The modified Lagrangian is given by

$$
\mathcal{L}^{(7)} = \mathrm{Tr}\left(\frac{\dot{X}_I^{\,2}}{2R} + \frac{R}{4}[X_I, X_J]^2 \right) + \lambda_7 \, F_7(X)
\tag{186}
$$

where λ_7 is a coefficient of the rank-7 invariant tensor. The matrix equations of motion are expressed as

$$
\frac{1}{R}\ddot{X}_I - R[X_J, [X_I, X_J]] - \lambda_7 \frac{\delta}{\delta X_I} F_7 = 0 \,.
\tag{187}
$$

From (173) we can express F_7 as

$$
\begin{aligned}
F_7(X) &= \frac{(-i)^7}{7} \mathrm{STr}\left(\mathrm{tr}(t_I t_J t_K t_L t_M t_N t_O) X_{[I} X_J X_K X_L X_M X_{N]} X_O \right) \\
&= \frac{1}{56} f_{IJA} f_{KLB} f_{MNC} \mathrm{tr}(t_A t_B t_C t_O) \\
&\quad \times \frac{1}{48} \mathrm{Tr}\Big[\, \big(\, \{[X_I, X_J], [X_K, X_L]\}[X_M, X_N] \\
&\qquad\qquad + [X_M, X_N]\{[X_I, X_J], [X_K, X_L]\} \\
&\qquad\qquad + [X_I, X_J][X_M, X_N][X_K, X_L] \\
&\qquad\qquad + [X_K, X_L][X_M, X_N][X_I, X_J] \, \big) \, X_O \, \Big]
\end{aligned}
\tag{188}
$$

where indices A, B, C are symmetric and we have used the antisymmetrization of six X_I's in the following form

$$
\begin{aligned}
X_{[1} X_2 X_3 X_4 X_5 X_{6]} &= \frac{1}{6!} \big(\, [(1234)56] + [(1235)46] + [(1245)63] \\
&\qquad\quad + [(1345)26] + [(2345)16] \, \big) \,.
\end{aligned}
\tag{189}
$$

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Here [(1234)56] and so on are given by

$$
\begin{aligned}
[(1234)56] \;=\;& \{[12],[34]\}[56] + [56]\{[12],[34]\} + [12][56][34] + [34][56][12] \\
& + (1234) \leftrightarrow (1342) \; + \; (1234) \leftrightarrow (1423) \,, \quad\quad\quad (190) \\
[IJ] \;\equiv\;& [X_I, X_J] \quad \text{for } I, J = 1, 2, \cdots, 6 \,.
\end{aligned}
$$

In (190), the replacement $(1234) \leftrightarrow (1342)$ means the transpositions of $[12]$, $[34]$ with $[13]$, $[42]$. The same applies to the second replacement $(1234) \leftrightarrow (1423)$. The variation of F_7 with respect to X_I is given by

$$
\begin{aligned}
\frac{\delta}{\delta X_I} F_7 \;=\;& \frac{1}{8} \cdot \frac{1}{48} f_{IJA} f_{KLB} f_{MNC} \mathrm{tr}(t_A t_B t_C t_O) \\
& \times \big[\; X_J \,, \; \{[X_K, X_L], [X_M, X_N]\} X_O \\
& \quad\quad\quad + X_O \{[X_K, X_L], [X_M, X_N]\} \\
& \quad\quad\quad + [X_K, X_L] X_O [X_M, X_N] \\
& \quad\quad\quad + [X_M, X_N] X_O [X_K, X_L] \; \big] \,. \quad\quad (191)
\end{aligned}
$$

We now enforce the fuzzy $\mathbf{CP}^k$ ansätze on (191):

$$
\frac{\delta}{\delta X_I} F_7 \bigg|_{X=rQ} = \frac{1}{384} r^6 \left(i \frac{k+1}{\sqrt{C_2^{(k)}}} \right)^2 f_{IJA} \mathrm{tr}(t_A t_B t_C t_O) [Q_J, 3! Q_{(B} Q_C Q_{O)}] \quad\quad (192)
$$

where $Q_{(B} Q_C Q_{O)}$ means the symmetrized product of Q's. Noticing that the indices A, B, C are symmetric, we can calculate the factor involving $\mathrm{tr}(t_A t_B t_C t_O)$ as

$$
\begin{aligned}
f_{IJA} \mathrm{tr}(t_A t_B t_C t_O) Q_{(B} Q_C Q_{O)} \;=\;& \mathrm{tr}(t_A t_B t_C t_O) 4 f_{IJ(A} Q_B Q_C Q_{O)} \\
=\;& \mathrm{tr}(\{t_A, t_B\}\{t_C, t_O\}) f_{IJA} Q_B Q_C Q_O \\
=\;& \left(\frac{1}{k+1} \delta_{AB} \delta_{CO} + \frac{1}{2} d_{ABD} d_{DCO} \right) f_{IJA} Q_B Q_C Q_O \\
=\;& f_{IJA} \left(\frac{1}{k+1} + \frac{c_{k,n}^2}{2} \right) Q_A \quad\quad (193)
\end{aligned}
$$

where we have used

$$
\{t_A, t_B\} = \frac{1}{k+1} \delta_{AB} \mathbf{1} + d_{ABC} t_C \,. \quad\quad (194)
$$

From (192) and (193), we find

$$
\frac{\delta}{\delta X_I} F_7 \bigg|_{X=rQ} = -\frac{i}{64} r^6 \left(\frac{k+1}{\sqrt{C_2^{(k)}}} \right)^3 \left(\frac{1}{k+1} + \frac{c_{k,n}^2}{2} \right) Q_I \,. \quad\quad (195)
$$

The equations of motion (187) becomes

$$
\sqrt{k+1} \left[\frac{\ddot{r}}{\tilde{R}} + \frac{\tilde{R}}{C_2^{(k)}} r^3 \left(1 + \frac{i\lambda_7 r^3}{64R} \frac{(k+1)^2}{\sqrt{C_2^{(k)}}} \left(\frac{1}{k+1} + \frac{c_{k,n}^2}{2} \right) \right) \right] Q_I \;=\; 0 \,. \quad\quad (196)
$$

Since these matrix equations are linear in Q_I, the fuzzy $\mathbf{CP}^k$ ansätze ($k \geq 3$) provide solutions to the modified M(atrix) theory with F_7. Evaluating F_7 on such solutions, we have

$$
F_7(Q) = -\frac{i}{448} r^7 \left(\frac{k+1}{\sqrt{C_2^{(k)}}} \right)^3 \left(\frac{1}{k+1} + \frac{c_{k,n}^2}{2} \right) N^{(k)} \;\sim\; n^{k-3} r^7 \,. \quad\quad (197)
$$

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5.4.3 Emergence of L7-branes

To recapitulate, we are allowed to include the extra potentials of the form $F_{2r+1}(X)$ ($r \leq k$, $k = 1, 2, 3$) in the M(atrix) theory Lagrangian as far as the brane solutions of $\mathbf{CP}^k$ geometry in the transverse directions are concerned. Evaluated on the globally defined fuzzy $\mathbf{CP}^k$ ansätze, these extra potentials are expressed as

$$F_3 = \frac{-i}{24} N^{(k)} \left(\frac{k+1}{\sqrt{C_2^{(k)}}} \right) r^3 \sim n^{k-1} r^3 \,, \tag{198}$$

$$F_5 = \frac{-i}{320} N^{(k)} \left(\frac{k+1}{\sqrt{C_2^{(k)}}} \right)^2 c_{k,n} r^5 \sim n^{k-2} r^5 \,, \tag{199}$$

$$F_7 = \frac{-i}{448} N^{(k)} \left(\frac{k+1}{\sqrt{C_2^{(k)}}} \right)^3 \left(\frac{1}{k+1} + \frac{c_{k,n}^2}{2} \right) r^7 \sim n^{k-3} r^7 \,, \tag{200}$$

$$V = N^{(k)}(k+1) \frac{R}{4 C_2^{(k)}} r^4 \sim n^{k-2} R r^4 \tag{201}$$

where we include the M(atrix) theory potential V. As mentioned earlier, we consider static solutions. So the effective Lagrangian is given by

$$\mathcal{L}_{eff} = -V_{tot} = -V + \lambda_3 F_3 + \lambda_5 F_5 + \lambda_7 F_7 \,. \tag{202}$$

From (198)-(201), we can express V_{tot} as

$$V_{tot}(r) = N^{(k)}(k+1) \frac{R}{C_2^{(k)}} v(r) \sim n^{k-2} R \,, \tag{203}$$

$$v(r) = \frac{r^4}{4} - \mu_3 r^3 + \mu_5 r^5 + \mu_7 r^7 \tag{204}$$

where μ's are given by

$$\mu_3 = \frac{i\lambda_3 \sqrt{C_2^{(k)}}}{24R} \,,$$
$$\mu_5 = -\frac{i\lambda_5}{320R}(k+1) c_{k,n} \,,$$
$$\mu_7 = -\frac{i\lambda_7}{448R} \frac{(k+1)^2}{\sqrt{C_2^{(k)}}} \left(\frac{1}{k+1} + \frac{c_{k,n}^2}{2} \right) \,. \tag{205}$$

In the case of $k = 1$, there is no F_5 or F_7; only F_3 exists and the potential $v(r)$ becomes $v_3(r) \equiv \frac{r^4}{4} - \mu_3 r^3$. This potential is relevant to Myers effect. In Myers' analysis [120], the coefficient λ_3 is determined such that it satisfies the equations of motion $\frac{\partial v_3}{\partial r} = r^3 - 3\mu_3 r^2 = 0$. So $\mu_3 \sim r/3 \sim 1$ ($r > 0$), or $\lambda_3 \sim R/n$. Analogously, we may require $\lambda_5 \sim R$, $\lambda_7 \sim nR$ such that $v(r) \sim 1$. Note that we demand $\mu_5, \mu_7 > 0$ so that the potential $v(r)$ is bounded below. We also assume $\mu_3 > 0$ so that $v(r)$ always has a minimum at $r > 0$, provided that $\lambda_3 F_3$ never vanishes in (202).

Regardless the value of $v(r)$ or r, the total potential $V_{tot}(r) \sim n^{k-2} R$ becomes finite for $k = 1, 2$ in the large n limit. The brane solutions corresponding to $k = 1, 2$ therefore exist no matter how they are unstable or not. For $k = 3$, however, $V_{tot}(r)$ diverges in the large n limit *unless* $v(r) = 0$. Let us consider a potential without F_5; $v_7(r) \equiv \frac{r^4}{4} - \mu_3 r^3 + \mu_7 r^7$. The equation of motion is given by

$$\frac{\partial v_7}{\partial r} = 7\mu_7 r^2 \left(r^4 + \frac{r}{7\mu_7} - \frac{3\mu_3}{7\mu_7} \right) = 0 \,. \tag{206}$$

Denoting the nonzero solution by $r = r_*$, we now substitute this back to $v_7(r)$; $v_7(r_*) = \frac{r_*^3}{7}(\frac{3r_*}{4} - 4\mu_3)$. If we fix μ_3 as $\mu_3 = \frac{3}{16}r_*$, $v_7(r_*)$ vanishes. In this case, $V_{tot}(r_*)$ becomes finite in the large n limit and the corresponding L7-branes are allowed to present as a stable solution at the minimum $r = r_*$. The L7-branes exist for a particular value of μ_3. In this sense, the strength of F_3 flux can be considered as a controlling parameter for the emergence of L7-branes. The same analysis applies to a potential without F_7; $v_5(r) \equiv \frac{r^4}{4} - \mu_3 r^3 + \mu_5 r^5$. In the case of the potential $v(r)$ with nonzero μ_{2r+1}'s, the existence of L7-branes can similarly be shown at a minimum of $v(r)$, with two of the three μ_{2r+1}'s serving as the controlling parameters.

If we introduce fluctuations from the minimum, the potential $v(r)$ becomes nonzero and consequently the total potential $V_{tot}(r)$ diverges in the large n limit. In other words, fluctuations from the stabilized L7-branes are suppressed.

The extra potentials are expressed as $F_{2r+1}(Q) \sim \text{Tr}\mathbf{1}$ where $\mathbf{1}$ is the $N^{(3)} \times N^{(3)}$ identity matrix. They can be regarded as a constant matrix-valued potentials. This suggests that the analysis in the previous section also holds with $F_{2r+1}(Q)$, preserving the L7-brane solutions non-supersymmetric.

In parametrizing the transverse part of the L7-brane with the fuzzy $\mathbf{CP}^3$ ansatz in (149), we squash the three irrelevant directions. We could however include contributions of the three directions to the total potential in (203) such that they do not affect the existence condition for the L7-branes, namely, the finiteness of $V_{tot}(r)$ at the minimum in the large n limit. Note that an addition of n-independent constants to $v(r)$ does not affect the existence condition with suitable shifts of controlling parameters. Obviously, an addition of constants to $V_{tot}(r)$ is also possible. In terms of M(atrix) theory as a 11-dimensional theory, the emergence of L7-branes and the suppression of their fluctuations suggest a compactification of the theory down to 7 dimensions. We shall discuss this point further in the next section.

5.5   Compactification scenarios

As mentioned in section 5.1, the existence of a 7-form suggests a compactification of the M(atrix) theory down to 7 or 4 dimensions. In this section, we first show that the extra potential $F_7(X)$ can be considered as a 7-form in M(atrix) theory. We then discuss that the effective Lagrangian (202) with $k = 3$ may be used as a 7-dimensional matrix model of M-theory compactification. We also consider a M(atrix) compactification scenario down to 4 dimensions by use of fuzzy S^4.

5.5.1 F_7 as a fuzzy 7-form

The general expression of $F_{2r+1}(X)$ in (173) is closely related to differential $(2r + 1)$-forms of $SU(k+1)$ $(r = 1, 2, \cdots, k)$. The differential forms of $SU(k+1)$ are constructed by the Lie algebra valued one-form

$$g^{-1}dg = -it_A E_A^a d\theta^a = -it_A E_A \tag{207}$$

where $g = \exp(-it^a \theta^a)$ is an element of $SU(k + 1)$, θ^a's are the continuous group parameters, t_A's are the generators of $SU(k + 1)$ in the fundamental representation with $\text{tr}(t_A t_B) = \frac{1}{2}\delta_{AB}$, and $E_A = E_A^a(\theta)d\theta^a$ are the one-form frame fields on $SU(k + 1)$ $(a, A = 1, 2, \cdots, k^2 + 2k)$. The differential $(2r + 1)$-forms $\Omega^{(2r+1)}$ of $SU(k + 1)$ are then defined by

$$
\begin{aligned}
\Omega^{(2r+1)} &= \text{tr}(g^{-1}dg)^{2r+1} \\
&= \frac{1}{(2r+1)!}(-i)^{2r+1}\text{tr}(t_{A_1}t_{A_2}\cdots t_{A_{2r+1}})E_{A_1} \wedge E_{A_2} \wedge \cdots \wedge E_{A_{2r+1}} \\
&= F_{A_1 A_2 \cdots A_{2r+1}} E_{A_1} E_{A_2} \cdots E_{A_{2r+1}}
\end{aligned} \tag{208}
$$

where we use $E_{A_1} \wedge E_{A_2} \wedge \cdots \wedge E_{A_{2r+1}} = (2r+1)! \, E_{[A_1}E_{A_2}\cdots E_{A_{2r+1}]}$. The rank-$(2r+1)$ invariant tensors $F_{A_1 A_2 \cdots A_{2r+1}}$ are defined in (174). Note that, due to cyclicity of the trace, the differential $(2s)$-forms $(s = 1, 2, \cdots)$ vanishes; $\Omega^{(2s)} = \text{tr}(g^{-1}dg)^{2s} = 0$. $\Omega^{(2r+1)}$ are nonzero in general. Using $d(g^{-1}dg) = dg^{-1}dg = -g^{-1}dgg^{-1}dg = -(g^{-1}dg)^2$, we find $d\Omega^{(2r+1)} = 0$ since $\Omega^{(2r+2)} = 0$. Thus $\Omega^{(2r+1)}$ are closed differential forms. One can show that $\Omega^{(2r+1)}$ are not exact. In fact, it is known that $\Omega^{(2r+1)}$ are elements of $\mathcal{H}^{2r+1}(SU(k + 1), \mathbf{R})$, the $(2r + 1)$-th cohomology group of $SU(k+1)$ $(r = 1, 2, \cdots, k)$ over the real numbers. The rank-$(2r+1)$ invariant tensors $F_{A_1 A_2 \cdots A_{2r+1}}$,

or Casimir invariants, are in one-to-one correspondence with cohomology classes for the Lie group $SU(k+1)$. This correspondence is related to the so-called Weil homomorphism between Chern classes and Casimir invariants. For descriptions of these mathematical aspects, one may refer to [34] (see pp.315-319).

Replacing E_A by X_A in (208), we now define

$$\Omega^{(2r+1)}(X) \equiv F_{A_1 A_2 \cdots A_{2r+1}} X_{A_1} X_{A_2} \cdots X_{A_{2r+1}} \,. \tag{209}$$

Then $F_{2r+1}(X)$ of (173) is written as

$$F_{2r+1}(X) = \mathrm{STr}\, \Omega^{(2r+1)}(X) \,. \tag{210}$$

So we can naturally interpret $F_{2r+1}(X)$ as matrix-valued differential forms. In section 5.4, we have evaluated $F_4(X)$ and $F_7(X)$ on the ansätze $X_I = r(t)Q_I$ where Q_I are the $N^{(k)} \times N^{(k)}$ matrix representations of $SU(k+1)$, satisfying the fuzzy $\mathbf{CP}^k$ constraints. As shown in (182) and (195), the variations of both $\overline{F_5(X)}$ and $F_7(X)$ with respect to X_I are linear in Q_I when they are evaluated on fuzzy $\mathbf{CP}^k$. Since Q_I are traceless matrices, (182) and (195) correspond to the fact that $\Omega^{(5)}(Q)$ and $\Omega^{(7)}(Q)$ are matrix-valued *closed* differential forms.

In (185) and (197), it is shown that both $F_5(Q)$ and $F_7(Q)$ are nonzero constants. This is related to the fact that $\Omega^{(2r+1)}(Q)$ are matrix-valued *non-exact* differential forms. Note that the non-exactness of a differential form, say $\Omega^{(3)}$, can be shown by $\int_{S^3} \Omega^{(3)} \neq 0$, where the integration is taken over $SU(2) = S^3$. (If $\Omega^{(3)}$ is exact, *i.e.*, $\Omega^{(3)} = d\alpha$, Stokes' theorem says $\int_{S^3} \Omega^{(3)} = \int_{\partial S^3} \alpha$ where ∂S^3 is the boundary of S^3. Since S^3 is a compact manifold, $\int_{\partial S^3} \alpha = 0$. Thus $\Omega^{(3)}$ can not be exact. One can similarly show the non-exactness of $\Omega^{(2r+1)}$ in general, using the fact that the volume element of $SU(k+1)$ can be constructed in terms of the wedge products of $\Omega^{(2r+1)}$'s.) $F_3(Q)$ is a fuzzy analogue of $\int_{S^3} \Omega^{(3)}$. So the value of $F_3(Q)$ in (198) corresponds to the nonzero volume element of a fuzzy version of S^3. Locally, we may parametrize S^3 as $S^3 \approx \mathbf{CP}^1 \times S^1$. So $F_3(Q)$ can also be seen as the volume element of a fuzzy version of $\mathbf{CP}^1 \times S^1$. Analogously, we can make a local argument to show that $F_{2k+1}(Q)$ ($k = 2, 3$) correspond to the volume elements of fuzzy versions of $S^{2k+1} \approx \mathbf{CP}^k \times S^1$. Note that since $\mathbf{CP}^k = S^{2k+1}/S^1$, we can locally express S^{2k+1} as $\mathbf{CP}^k \times S^1$ in general. Globally, we can not distinguish the $\mathbf{CP}^k$ coordinates from the S^1. This is in accordance with the fact that the $(2k + 1)$ matrices in F_{2k+1} are democratically defined on $\mathbf{R}^{k^2+2k}$. From a local point of view, one may interpret $\Omega^{(2k+1)}(Q)$ ($k = 2, 3$) as longitudinal $(2k + 1)$-forms in M(atrix) theory. But there is no notion of matrix for the longitudinal direction. So it is not appropriate to make such a local interpretation. We would rather consider $\Omega^{(2r+1)}(Q)$ ($r = 1, 2, \cdots, k$) as matrix-valued or 'fuzzy' $(2r + 1)$-forms in a global sense.

The fact that we can interpret $\Omega^{(7)}(Q)$ (with $k = 3$) as a 7-form in M(atrix) theory is interesting in search for compactification models of the theory. As mentioned in section 5.1, according to Freund and Rubin [121], existence of a differential s-form in d-dimensional theories suggests compactification of $(d - s)$ or s space-like dimensions ($s < d$). Usually the Freund-Rubin type compactification is considered in 11-dimensional supergravity which contains a 4-form. Although this compactification has a problem in regard to the existence of chiral fermions, the Freund-Rubin compactification of M-theory has been shown to avoid such a problem and presumably provides a realistic model of M-theory in lower dimensions [122]. The presence of the above-mentioned 7-form then supports a possibility of the Freund-Rubin type compactification in M(atrix) theory. It is not clear at this point how the effective Lagrangian (202) relates to compactified 7-dimensional supergravity in the low energy limit. However, as discussed before, the Lagrangian (202) with $k = 3$ does capture a desirable physical property for the compactification of M(atrix) theory down to 7 dimensions.

In terms of the 11-dimensional M-theory, the potential $F_7(Q)$ corresponds to a flux on a curved space of $(\mathbf{CP}^3 \times S^1) \times \mathcal{M}_4$ geometry where $\mathcal{M}_4$ is some four-dimensional manifold. In our ansätze for the brane solutions, we neglect contributions from $\mathcal{M}_4$. The Freund-Rubin type compactification however requires the manifold $\mathcal{M}_4$ to be a positively curved Einstein manifold. So we may describe $\mathcal{M}_4$ either by fuzzy $\mathbf{CP}^2$ or fuzzy S^4 in M(atrix) theory. As we have seen in chapter 3, fuzzy S^4 can be represented by $N^{(3)} \times N^{(3)}$ block-diagonal matrices. Thus, in order to include $\mathcal{M}_4$ contributions to the Lagrangian (202) with $k = 3$, it would be natural to parametrize $\mathcal{M}_4$ by fuzzy S^4 such that

it does not affect the existence condition for L7-branes, *i.e.*, the finiteness of L7-brane energies in the large N limit. As analyzed before, this is possible if the diagonal components of fuzzy S^4 contain a large number of zeros. Note that one of the four dimensions in $\mathcal{M}_4$ represents the time component in M(atrix) theory. So the use of fuzzy spaces for $\mathcal{M}_4$ does not exactly fit the framework of M(atrix) theory. However, as in the case of the IKKT model [89], one can consider the time component in terms of matrices. So we may describe $\mathcal{M}_4$ by fuzzy S^4 as far as a matrix model building of M-theory in the large N limit is concerned. The effective L7-brane Lagrangian (202) with $k = 3$ therefore provides a compactification model of M(atrix) theory.

5.5.2 Emergence of fuzzy S^4

Compactification of M(atrix) theory down to 4 dimensions is also possible for the Freund-Rubin compactification in the presence of the 7-form. We shall discuss this possibility by use of fuzzy S^4. As shown in (54), functions on fuzzy S^4 can be constructed from functions on fuzzy $\mathbf{CP}^3$ by imposing the following constraint:

$$[\mathcal{F}(Q_A), Q_{\tilde{\alpha}}] = 0 \qquad (211)$$

where $\mathcal{F}(Q_A)$ are arbitrary polynomial functions of the fuzzy $\mathbf{CP}^3$ coordinates Q_A ($A = 1, 2, \cdots, 15$). In what follows, any Q_A denote the fuzzy $\mathbf{CP}^3$ coordinates unless otherwise mentioned. The indices $\tilde{\alpha}$ in $Q_{\tilde{\alpha}}$ corresponds to the algebra of $\tilde{H} = SU(2) \times U(1)$ in terms of the decomposition, $SU(4) \rightarrow SU(2) \times SU(2) \times U(1)$, as considered in (53). With an imposition of (211), the functions on fuzzy $\mathbf{CP}^3$ become the functions on fuzzy S^4. Notice that fuzzy $\mathbf{CP}^3$ is defined globally in terms of Q_A with algebraic constraints given in (152) and (153). So the condition (211) is a further constraint on top of these fuzzy $\mathbf{CP}^3$ constraints for Q_A.

As analyzed in section 3.3, upon the imposition of (211) the fuzzy $\mathbf{CP}^3$ coordinates Q_A become fuzzy S^4 coordinates which are no more represented by full $N^{(3)} \times N^{(3)}$ matrices but by $N^{(3)} \times N^{(3)}$ block-diagonal matrices. The block-diagonal matrix is composed of $(n + 2 - m)$ blocks of dimension m, where we take account of all values of $m = 1, 2, \cdots, n + 1$. Let Y_μ ($\mu = 1, 2, 3, 4$) be local coordinates of fuzzy S^4. We can express Y_μ as

$$Y_\mu = \text{block-diag}(\underbrace{1, 1, \cdots, 1}_{n+1}, \underbrace{\square_2, \square_2, \cdots, \square_2}_{n}, \cdots, \square_n, \square_n, \square_{n+1}) \qquad (212)$$

where $\square_m$ denotes an $(m \times m)$ matrix ($m = 2, 3, \cdots, n + 1$). Note that the dimension of Y_μ can be counted by $\sum_{m=1}^{n+1}(n + 2 - m)m = \frac{1}{6}(n + 1)(n + 2)(n + 3) = N^{(3)}$, while the number of nonzero matrix elements becomes $\sum_{m=1}^{n+1}(n + 2 - m)m^2 = \frac{1}{12}(n + 1)(n + 2)^2(n + 3) \equiv N^{S^4}$, which gives the number of coefficients in mode expansion of truncated functions on S^4. Notice that Y_μ should satisfy $Y_\mu Y_\mu + Y_0 Y_0 = \mathbf{1}_{N^{(3)}}$, where Y_0 denotes a radial component of fuzzy S^4 and $\mathbf{1}_{N^{(3)}}$ is the $N^{(3)} \times N^{(3)}$ identity matrix. It is easily seen that Y_μ commute with $N^{(1)} \times N^{(1)}$ block matrices where $N^{(1)} = n + 1$ is the number of 1's in (212). Since $Q_{\tilde{\alpha}}$ is essentially expressed by $N^{(1)} \times N^{(1)}$ matrix representations of $\underline{SU(2)}$, Y_μ commute with $Q_{\tilde{\alpha}}$ and really satisfy the condition (211). Although the matrix configuration (212) is the most natural one in comparison with fuzzy $\mathbf{CP}^3$, it is not the only one that describes fuzzy S^4. As mentioned earlier, one can also locate the same-size blocks in a single block, using matrix multiplication (or matrix addition). The dimension of this matrix configuration becomes $\sum_{m=1}^{n+1} m = \frac{1}{2}(n + 1)(n + 2) = N^{(2)}$. So fuzzy S^4 is also described by $N^{(2)} \times N^{(2)}$ block-diagonal matrices $\tilde{Y}_\mu$, satisfying $\tilde{Y}_\mu \tilde{Y}_\mu + \tilde{Y}_0 \tilde{Y}_0 = \mathbf{1}_{N^{(2)}}$.

Let us now impose the constraint (211) on the effective Lagrangian (202) with $k = 3$. The corresponding equation of motion becomes linear in Y_μ. So the fuzzy S^4 also provides static brane solutions to the modified M(atrix) theory. Since $\Omega^{(2r+1)}(Q)$ are proportional to the identity matrix, they remain the same after the imposition of (211). The local coordinates of fuzzy $\mathbf{CP}^3$ (which we earlier denote by Q_i) are simply replaced by Y_μ after the imposition of (211). So, as in the case of the L7-branes, we can similarly show the emergence of L5-branes with fuzzy-S^4 parametrization. On the other hand, if we represent fuzzy S^4 by $N^{(2)} \times N^{(2)}$ block-diagonal matrices $\tilde{Y}_\mu$, the M(atrix) theory potential would scale to 1 as in the case of the L5-branes with $\mathbf{CP}^2 \times S^1$ geometry. So, in this case, we may avoid the problem of infinite energy for the brane solution, and it is not necessary to include the extra potentials in order to show the existence of the brane solutions.

The brane solution we construct here is an L5-brane of $S^4 \times S^1$ geometry. The transverse directions of this L5-brane are purely spherical, so it is different from the previously obtained 'spherical' L5-brane [82]. Note that the latter was constructed under the condition $\epsilon_{ijklm} X_i X_j X_k X_l \sim X_m$ where the local matrix coordinates in the transverse directions are described by four out of the five X_i's ($i = 1, 2, \cdots, 5$) and the index m labels any fixed value between 1 and 5. Strictly speaking, X_i's do not describe S^4 geometry, but rather $\mathbf{CP}^3$. This can be seen if we express the above condition as

$$f_{IJA} f_{KLB} d_{ABM} Q_I Q_J Q_K Q_L \sim d_{ABM} Q_A Q_B \sim X_M \tag{213}$$

where we replace $X_i \to Q_I$ and $\epsilon_{ijklm} \to f_{IJA} f_{KLB} d_{ABM}$. (As we have seen in (176), the rank-five invariant tensor of $SU(4)$ can be given by $f_{IJA} f_{KLB} d_{ABM}$, if Q_I's are symmetrized.) As mentioned earlier, there has been a difficulty to include fluctuations in the spherical L5-branes. Our version of spherical L5-brane apparently avoids this difficulty; since the fuzzy S^4 is defined by Q_A's along with a set of constraints on them, the fluctuations can naturally be described by $Q_A \to Q_A + A_A$.

The spherical L5-brane solution lies on a 11-dimensional manifold $(S^4 \times S^1) \times \mathcal{M}_6$. In the contex of Freund-Rubin type M(atrix) coompactification, $\mathcal{M}_6$ is to be described by fuzzy $\mathbf{CP}^3$. We may include $\mathcal{M}_6$ contributions to the M(atrix) theory Lagrangian such that the contributions are irrelevant to the existence condition of the L5-brane. As before, the $\mathcal{M}_6$ can be constructed by parametrizing the diagonal components of fuzzy $\mathbf{CP}^3$ with a large number of zeros.

Fuzzy S^4 is most naturally defined by $N^{(3)} \times N^{(3)}$ block-diagonal matrices Y_μ in the presence of fuzzy $\mathbf{CP}^3$. If we adhere to such a definition, it would be necessary to introduce extra potentials as before so that the energy of the spherical L5-branes at a minimum of the total potential becomes finite. In this sense, the effective Lagrangian (202), with the imposition of the fuzzy S^4 condition (211), can also be used as a compactification model of M(atrix) theory in 4 dimensions.

In terms of the local coordinates of fuzzy $\mathbf{CP}^3$ Q_i, the M(atrix) theory potential is calculated by $\mathrm{Tr}\frac{Rr^4}{4}[Q_i, Q_j]^2 = -\frac{N^{(3)}}{15}\frac{Rr^4}{4C_2^{(3)}}$ as in (160). The sum of the extra potentials for the emergence of L7-branes has been given by $\frac{N^{(3)}}{15}\frac{Rr_*^4}{4C_2^{(3)}}$, with r_* being at the minimum of $v(r)$ in (204). In terms of the local coordinates of fuzzy S^4 in (212), a matrix Lagrangian for the emergence of the spherical L5-branes is then expressed as

$$\mathcal{L}_{S^4 \times S^1} = \mathrm{Tr}\left(\frac{\dot{r}^2 Y_\mu^2}{2R} + \frac{Rr^4}{4}[Y_\mu, Y_\nu]^2 + \frac{Rr_*^4}{60C_2^{(3)}}\mathbf{1}_{N^{(3)}}\right) \tag{214}$$

where we include the kinetic term. Since the extra potentials are constant, we expect $\dot{r}$ remains to be $\dot{r} \sim n^{-2}$. So the kinetic energy $\mathrm{Tr}(\dot{r} Y_\mu^2/2R) \sim 1/nR$ vanishes in the large n limit, and we can consider (214) as an effective Lagrangian for the static spherical L5-branes. The value of r_* is determined by the controlling parameters for the emergence of the spherical L5-branes. For example, consider the potential $v(r)$ of the form $v_5(r) = \frac{r^4}{4} - \mu_3 r^3 + \mu_5 r^5$ where μ_3, μ_5 are given by (205) with $k = 3$. In this case, the controlling parameter is given by μ_3 as discussed before. From $\frac{\partial v_5}{\partial r}\big|_{r_*} = 0$ and $v_5(r_*) = 0$, we can easily find $r_* = 8\mu_3$. Note that r_* is independent of n or R since μ_3 scales to 1.

In order to obtain compactification of M(atrix) theory down to 4 dimensions, we simply eliminate the longitudinal direction in the spherical L5-branes. The relevant brane solution would be a transverse 4-brane of spherical geometry. Apparently, this brane solution does not have a time component in the M(atrix) theory framework but, as mentioned earlier, it is possible to express the time component by matrices as far as a matrix model building of M-theory in the large N limit is concerned. Bearing this possibility in mind, we can conjecture the action for such a spherical 4-brane or fuzzy S^4 solution as

$$\mathcal{S}_4 = \frac{r^4 R}{4}\mathrm{Tr}\left([Y_\mu, Y_\nu]^2 + \frac{\beta}{C_2^{(3)}}\mathbf{1}_{N^{(3)}}\right), \tag{215}$$

$$\beta = \frac{1}{15}\left(\frac{r_*}{r}\right)^4 \sim 1. \tag{216}$$

There are basically two fundamental parameters, R and $N = N^{(3)} \sim n^3$. We consider that in the large N/R limit, or in the IMF limit, the matrix action (215) describes compactification of

M-theory in 4 dimensions. R is essentially the 11-dimensional Planck length l_p. (Remember that R is given by $R = gl_s = g^{2/3}l_p$ where g is the string coupling constant and l_s is the string length scale.) Since the time component is supposed to be embedded in matrix configurations, we can assume that r is some constant to be absorbed into the definition of R. The fuzzy S^4 solutions are also non-supersymmetric since they are constructed from the L7-branes of $\mathbf{CP}^3 \times S^1$ geometry. Finally we should note that the above action may be used as a realistic 4-dimensional matrix model of M-theory compactification.

6 Conclusions

Mathematical and physical aspects of fuzzy spaces have been explored in this dissertation. As for mathematical part, we consider construction of fuzzy spaces of certain types. In chapter 2, we review construction of fuzzy complex projective spaces $\mathbf{CP}^k$ ($k = 1, 2, \cdots$), following a scheme of geometric quantization. This construction has particular advantages in defining symbols and star products for fuzzy $\mathbf{CP}^k$. Algebraic construction of fuzzy $\mathbf{CP}^k$ has also been included in this chapter. In chapter 3, we have presented construction of fuzzy S^4, utilizing the fact that $\mathbf{CP}^3$ is an S^2 bundle over S^4. Fuzzy S^4 is obtained by imposing an additional constraint on fuzzy $\mathbf{CP}^3$. We find the constraint is appropriate by considering commutative limits of functions on fuzzy S^4 in terms of homogeneous coordinates of $\mathbf{CP}^3$. We propose that coordinates on fuzzy S^4 are described by block-diagonal matrices whose embedding square matrix represents the fuzzy $\mathbf{CP}^3$. Along the way, we have shown a precise matrix-function correspondence for fuzzy S^4, providing different ways of counting the number of truncated functions on S^4. Because of its structure, the fuzzy S^4 should follow a closed and associative algebra. Analogously, we also obtain fuzzy S^8, using the fact that $\mathbf{CP}^7$ is a $\mathbf{CP}^3$ bundle over S^8.

In the second part of this dissertation, we have considered physical applications of fuzzy spaces. Fuzzy spaces are particulary suitable for the studies of matrix models. In chapter 4, we consider matrix models for gravity on fuzzy spaces. Such models can give a finite mode truncation of ordinary commutative gravity. We obtain the actions for gravity on fuzzy S^2 and on fuzzy $\mathbf{CP}^2$ in terms of finite dimensional matrices. The commutative large N limit is also discussed. Lastly, in chapter 5, we have discussed application of fuzzy spaces to M(atrix) theory. Some of the previously known brane solutions in M(atrix) theory are reviewed by use of fuzzy $\mathbf{CP}^k$ as ansätze. We show that, with an inclusion of extra potential terms, the M(atrix) theory also has brane solutions whose transverse directions are described by fuzzy S^4 and fuzzy $\mathbf{CP}^3$. The extra potentials can be considered as matrix-valued or 'fuzzy' differential $(2r + 1)$-forms or fluxes in M(atrix) theory ($r = 1, 2, \cdots, k$). Compactification of M(atrix) theory is discussed by use of these potentials. In particular, we have conjectured a compactification model of M(atrix) theory in four dimensions. The resultant action (215) is expressed in terms of the local coordinates of fuzzy S^4 (212) and can be used as a realistic matrix model of M-theory in four dimensions.

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