New Solitons from Brane Configurations

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Abstract

Recent fertile development in string theory has provided useful tools for investigation of various field theories even non-perturbatively. Among the methods developed, brane configuration technique is an intriguing and promising way to grasp properties of field theories geometrically. Using this technique, various predictions of new phenomena in field theories have been made, and they have been checked by explicit field-theoretical calculations. In this thesis, we concentrate on $\mathcal{N} = 4$ super Yang-Mills theory (SYM) and its non-commutative deformation which are realized as low energy effective theories on parallel D3-branes in type IIB superstring theory. We present two examples which show the powerfulness of the brane technique. The first example is the 1/4 BPS dyon in the SYM. We make predictions concerning this new exotic soliton, using string networks which are stable in string theory. Then we explicitly construct the soliton solutions in the Lagrangian formalism of the SYM. The constructed solutions reproduce the properties of the string networks such as masses, supersymmetries and tension balance. The second example is the non-commutative monopoles. The SYM on some non-commutative spacetime (NCSYM) admits an interpretation of string-theoretically equivalent system which is obtained by introducing background constant Neveu-Schwarz (NS-NS) 2-form field. We construct brane configurations representing solitons in the NCSYM and examine their existence and properties. Then from the Lagrangian formalism of the NCSYM, we explicitly construct the non-commutative monopole solution and check the predicted properties which are proper to the non-commutativity such as non-locality and dipole structure of the non-commutative monopole. These two examples are turned out to be strong evidence of validity and effectiveness of the brane configuration technique.

Contents

Chapter 1

Introduction

Since string theory was found to be a candidate for consistent quantum gravity, vast amount of study has been carried out for the dynamics of string theory. In the 1980's, after the appearance of the notable work of Ref. [34], string theory has become a candidate for the *theory of everything* containing all the physics of the standard model and gravity. However, the most regrettable situation in those days was that we had very little knowledge upon non-perturbative aspects of string theory. This owes to that string theory itself was defined only as a first-quantized theory, not a second-quantized one. Then the theory had thousands of perturbative vacua on which the theory is defined. Resultantly, the dynamics which could be known were only perturbative dynamics, and there was no mean to determine the vacuum of this theory. Therefore this theory had very poor predictive power on phenomenology.

Recent development of this subject in these 5 years was mainly concerning the non-perturbative understanding of string theory. This development was triggered by three intriguing works, Ref. [75] in which full string dualities were analyzed, Ref. [154] in which the existence of M-theory describing the strong coupling region of string theory was conjectured, and Ref. [115] in which extended hypersurfaces called *D-branes* were found to be fundamental objects in string theory. Utilizing the information of the non-perturbative region of string theory given in these works and subsequent studies, string-phenomenology entered a new phase.

Beside the development of the string-phenomenology, the discovery of the importance of the membranes and the D-branes provided a very powerful tool for investigating various field theories. String theory contains an infinite number of particles as excitation of the elementary string, but taking the decoupling limit (in other words, $\alpha' \to 0$ limit), the theory becomes a low energy effective field theory. Using many kinds of D-(and NS-)branes or some compactification of the target space, we can obtain various dimensional field theories with variety of field content.

One interesting example is MQCD which was first proposed in Ref. [153]. This paper gave a geometric realization of Seiberg-Witten curve of some $\mathcal{N} = 2$ field theories, based on the brane realization of these field theories by Ref. [63]. The MQCD is defined with a smooth 5-brane hypersurface embedded in 11-dimensional target space. Since M-theory is expected to describe the non-perturbative effects of string theory, *i.e.* the effects of finite string coupling, the corresponding MQCD describes non-perturbative dynamics of Super QCD. An intriguing result is that the strong-weak duality of $\mathcal{N} = 4$ SYM is realized manifestly and geometrically. (See Refs. [59, 139] and references therein.) The techniques which use configurations of various branes for analyzing physics in field theories are sometimes called "brane configurations" or "brane techniques".

One of the other recent developments with respect to the string-theoretical realization of field theories is AdS/CFT correspondence initiated by Ref. [98]. The spirit of this correspondence is that the large N limit of conformal field theories are equivalent with classical supergravities on AdS space. This AdS space is supplied as a near horizon region of the supergravity solution of the D-branes.

These string-theoretical realizations of field theories have given interesting and intrinsic understanding of dynamics and statics of the field theories. The point is that this realization respects the non-perturbative nature of the field theories. The stringy realization is involved with not only the spectrum of the field theory, solitons living in the theory, and other static properties in the field theory, but also the non-perturbative symmetries including strong-weak and electric-magnetic dualities. However, most of the results from the stringy realization is merely the reproduction of the field theoretical results. Although it was found that the duality conjectured in the field theory side corresponds to some duality in string theory, this correspondence does not give any verification of the duality of the field theory since the S-duality in string theory is still a conjecture.

In a couple of years, many examples confirming the field theory $\leftrightarrow$ string theory correspondence have been turned in, and the credible basis of the correspondence has been gradually constructed. Recently, on the basis of this correspondence, predictions of new interesting phenomena in field theories appeared from the stringy realization. One of the interesting predictions is using the above AdS/CFT correspondence: in Ref. [97], potential between a quark and an anti-quark was provided in the strong ('tHooft) coupling region.

In this thesis, we concentrate on the new solitons in field theories which were first conjectured from the stringy realization. The solitons we treat in this thesis are *1/4 BPS dyons* and *non-commutative monopoles*, both of which are new ingredients from the recent analyses using the brane configuration techniques.

The field theories we know well are 4-dimensional, and the simplest way to realize the 4-dimensional field theories in string theory is to consider D3-branes in the type IIB superstring theory. On this D3-brane whose worldvolume is 4-dimensional, $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) is realized in the low energy region. As we will explain in the following chapter, the rank of the gauge group of the realized field theory is arranged by adopting many parallel D3-branes. The $SU(N)$ theory is provided by N parallel D3-branes. We consider the spectrum of this theory and the exotic deformation of this theory.

Before the appearance of the D-branes, the known spectrum of this $\mathcal{N}=4$ $SU(2)$ SYM consisted of only the W-bosons, monopoles and dyons. These are labeled by electric and magnetic charges denoted as (p,q). These ingredients have their corresponding alternatives in string realization: (p,q)-string suspended between the two D3-branes. This string is a bound state of p fundamental strings and q D-strings. The (p,q)-charge of the dyon is served by the charges of this bound state of strings. Using the correspondences that the eigenvalues of the Higgs fields of the SYM represent the location (and deformation) of the D3-brane surface, and that the deformed part of the D3-brane surface can be interpreted as a string stuck to the D3-brane [32], then the classical configuration of the monopoles and the dyons are directly related to the corresponding brane configurations. In Chap. 2, after briefly reviewing the properties of the low energy effective action of the D-branes, we construct this brane configurations for the conventional solitons such as the monopoles and the dyons.

Using the bound state of the strings, string theory admits as stable states *string networks* which are composed by the strings with various (p,q)-charges and string junctions. If this network is ended on several D3-branes, then this must corresponds to some stable state in the $\mathcal{N}=4$ SYM [20]. In Chap. 3, we give a brief review of the string network and explain this prediction from the brane configurations. Then we explicitly construct the corresponding soliton solutions which are new exotic states preserving 1/4 of the original supersymmetries in the $\mathcal{N}=4$ SYM. These solitons are called 1/4 BPS dyons. The constructed solutions possess beautiful properties reflecting the brane configurations of the string networks.

Another subject in this thesis is the SYM on non-commutative space (NCSYM) which we deal with in Chap. 4. Due to the fact that the non-commutative worldvolume of D3-brane is string-theoretically equivalent with the D3-brane in background NS-NS B-field (See Ref. [131] and references therein), we provide brane configurations corresponding to the monopoles, dyons and 1/4 BPS dyons in NCSYM. From these brane configurations, it is predicted that the monopoles and dyons have the same masses as the ordinary monopoles and dyons, and exhibit certain non-locality and dipole structure. This predictions from the brane configurations are explicitly checked and confirmed by the field theoretical analysis.

This thesis is based on my papers [66, 69, 70, 71, 72, 73] which have been partly done in collaboration with A. Hashimoto, H. Hata, S. Moriyama and N. Sasakura.

Chapter 2

Realization of Conventional Solitons

As explained in the introduction, owing to the fact that on the extended objects called D-branes many kinds of various dimensional field theories are realized, it becomes possible to analyze the field theories in the context of (super)string theories and M-theory. In this chapter, we review the techniques of the string-theoretical realization of the field theories, especially 4-dimensional $\mathcal{N}=4$ super Yang-Mills theory.

2.1 Low energy effective action of D-brane

Since string theory contains infinite number of particle fields as string excitation, if one wants to obtain a field theory with a finite number of fields, some decoupling procedure is necessary. The usual one is the $\alpha' \to 0$ limit which is often called "decoupling limit". This procedure keeps only the fields which are massless (compared to the string scale), and the resultant field theory involves only a finite number of the light fields. This field theory is the low energy effective field theory which reproduces the string scattering amplitudes whose external legs are only the massless excitations.

Although through this decoupling procedure we lose many informations concerning the full spectrum of string theory, surprisingly enough, the important intrinsic properties of string theory are encoded in this low energy effective field theories. For example, the various string dualities [75] were first predicted from the symmetries of these low energy effective field theories.

The D-brane sector of the low energy field theories of the string theories has been also indispensable as an evidence for the string duality. The field theoretical actions for this sector are referred to as "D-brane actions". This D-brane actions exhibit the duality nature.

In this section, we review the structures of the D-brane action. The D-brane is defined as an extended object on which open strings can end. Therefore the dynamics of the D-brane originates in the open string dynamics. The massless excitation of the open superstring is a 1-form $U(1)$ gauge field. The low energy effective action of the open superstrings concerning this gauge field is 10-dimensional Born-Infeld (BI) action [19]. In the case of the D-branes, the low energy sector is described by the following effective action [48, 19, 94, 124]:

$$S = -T_{\mathrm{D}p} \int d^{p+1}x \; e^{-\phi} \sqrt{-\det\left(G_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} + S_{\mathrm{R-R}} + S_{\mathrm{fermion}}. \tag{2.1.1}$$

This action is derived by putting the Dirichlet boundary conditions on the ends of the open strings. Since the open strings are terminated on the D-branes, the fields in the action lives on the brane, *i.e.* the action is $p+1$ dimensional, where p is the spatial dimension of the worldvolume of the D-brane. In the action (2.1.1), $T_{\mathrm{D}p}$ is the tension of the Dp-brane[1], $B_{\mu\nu}$ is the 2-form gauge field from the NS-NS sector, and $F_{\mu\nu}$ is the field strength of the $U(1)$ 1-form gauge field. The metric G is the induced one given by

$$G_{\mu\nu} = \eta_{\mu\nu} + g_{ij}\partial_\mu X^i \partial_\nu X^j, \tag{2.1.2}$$

where g_{ij} is the string metric and X^i denotes the scalar fields on the D-brane (the indices i, j are for the transverse directions, $i, j = p{+}1, \cdots, 9$, and the indices μ, ν are for the longitudinal directions, $\mu, \nu = 0, \cdots, p$). This scalar field is obtained by the dimensional reduction from the gauge field in ten dimension (see Eq. (2.2.2)), since originally both the scalar field and the gauge field came from the open string excitations and the difference between the two is only the boundary condition of the open string.

The action (2.1.1) is derived for the slowly-varying field approximation, $F \gg l_{\mathrm{s}}\partial F$. If taking further approximation $\alpha' F \ll 1$, then from (2.1.1) we obtain the ordinary Maxwell action $F_{\mu\nu}F^{\mu\nu} + \cdots$.

The second term in Eq. (2.1.1) is the contribution from the Ramond-Ramond (R-R) sector. Symbolically, it is given as

$$S_{\mathrm{R-R}} = \int e^{2\pi\alpha' F} \wedge C. \tag{2.1.3}$$

[1]In this thesis, we adopt the following convention for the tensions: $T_{\mathrm{D}p} = 1/(2\pi)^p l_{\mathrm{s}}^{p+1} g$ for Dp-branes and $T_{\mathrm{f}} = 1/2\pi l_{\mathrm{s}}^2$ for fundamental strings. The definition of the slope parameter is given by $\alpha' = l_{\mathrm{s}}^2$, where l_{s} is the string length.

This R-R-coupling is important not only for the duality nature of the D-brane action but also for the soliton realization explained in the following. Let us concentrate on the case of D3-branes in the type IIB superstring theory. In the D3-brane action, there are the following couplings

$$\int d^4x \ \left[B^{\mu\nu}F_{\mu\nu} + C^{\mu\nu}(*F)_{\mu\nu}\right]. \tag{2.1.4}$$

The first term came from the BI part in Eq. (2.1.1), and the second term stemmed from $S_{\mathrm{R-R}}$ (2.1.3). We omitted the numerical factor, and $(*F)$ denotes the Hodge dual of the field strength F. The conjectured S-duality in the type IIB superstring theory exchanges $C_{\mu\nu}$ with $B_{\mu\nu}$, and D3-branes in this theory are invariant under this duality transformation. Thus, for consistency, the D3-brane action must be invariant. Evidently, the couplings (2.1.4) are invariant if we make the following transformation simultaneously:

$$F_{\mu\nu} \leftrightarrow (*F)_{\mu\nu}. \tag{2.1.5}$$

Therefore, the S-duality in the type IIB superstring theory results in the electro-magnetic duality in the D3-brane worldvolume action. Furthermore, under this S-duality (2.1.5), the BI action was shown to be invariant [145]. This is one of the evidence of the conjectured S-duality.

Note that due to the couplings (2.1.4), the D-3brane possibly behave as sources for the fundamental strings and D-strings which possess charges of the 2-form B field and C field respectively [138, 76]. Actually, without the D3-brane couplings (2.1.4), the equation of motion for the B-field in the sense of the 10-dimensional bulk supergravity is

$$d * dB = \delta^{(8)} \tag{2.1.6}$$

where the delta function $\delta^{(8)}$ specifies the locations of the fundamental string. Multiplying d on the both sides of this equation of motion, then we obtain the vanishing of $d\delta^{(8)}$ which means that the string cannot have its end [76]. However, using the above couplings (2.1.4) between the B-field and the gauge field on the D3-brane, we obtain

$$d\delta^{(8)} = d * F \tag{2.1.7}$$

instead. The fundamental string charge, which is specified by the delta function, can flow into the worldvolume of the D3-brane. Thus the fundamental string can end on the D3-brane. This feature is consistent with the fact that originally the D-branes were defined as hypersurfaces on which the fundamental strings can end.

The interesting is that the same analysis can be applied also for the D-strings using the C-field, and resultantly, the D-strings can end on D3-branes. How the ends of the strings can be seen from the D3-brane worldvolume theory? As for the fundamental strings, noting that the right hand side $d * F$ of Eq. (2.1.7) is actually a part of the equation of motion for the gauge fields on the D3-brane, the left hand side of Eq. (2.1.7) is the source for the gauge field. The form of the source $d\delta^{(8)}$ means that the source is placed on the point where the fundamental string ends. On the other hand, concerning the D-string, since the S-duality exchanges the field strength F with its Hodge dual $*F$, the ends of the D-string behaves as a monopole on the worldvolume of the D3-brane. These features of the ends of the open branes are of great importance for the realization of the monopoles and elementary particles by the brane language.

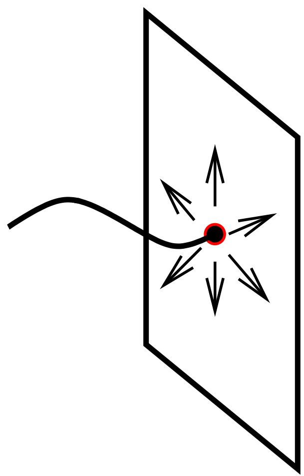

Figure 2.1: Fundamental strings and D-strings can end on D-branes. The arrows represent the flow of the source current.

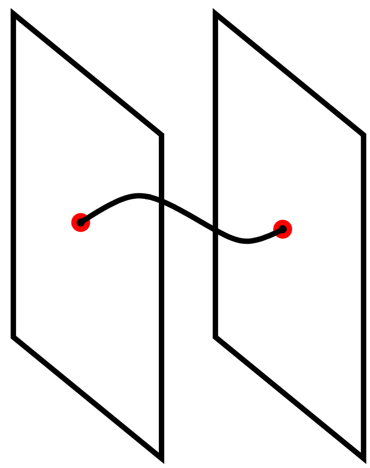

Figure 2.2: The lowest energy state of a fundamental string suspended between two D-branes is the W-boson.

Before ending this section, let us comment on the case with many parallel D-branes [152]. For concreteness, we consider two parallel D3-branes. Since on each brane there exists a $U(1)$ gauge field, the total gauge symmetry is $U(1) \times U(1)$. As explained above, the fundamental string suspended between the two D3-branes have the charge $(1, -1)$ or $(-1, 1)$ depending on its orientation. These stretched strings have masses due to their lengths which are identical with the distance between the two D3-branes, thus according to charges, the lowest energy states of these stretched strings are W-bosons. If the two D3-branes coincide on top of each other, then these stretched strings become massless and the $U(2)$ gauge symmetry is restored.

Taking into account the fact that the diagonal $U(1)$ decouples, the low energy dynamics of the two parallel D-branes is described by the $SU(2)$ non-Abelian gauge theory dimensionally reduced from ten dimensions.

2.2 Born-Infeld dynamics

The Born-Infeld action proposed 60 years ago gets revived as an effective action describing the dynamics of the D-brane, and the properties of the action has contributed to make explicit the non-perturbative nature of string theory such as the string dualities, as explained in the previous section. In this section, we introduce so-called "Born-Infeld dynamics" [32, 56, 93, 64, 47, 142, 36] in which the dynamics of the D-brane is investigated with use of the purely field-theoretical BI action in the approximation of ignoring the gravity. We review the notable paper by C. G. Callan and J. Maldacena [32] in which certain singular classical solutions on the worldvolume of the D-brane were interpreted as strings stuck to the D-brane.

The D-brane action contains the scalar fields X^i in the induced metric on the D-brane as seen in Eq. (2.1.2). These scalar fields are in fact the Nambu-Goldstone bosons with respect to the breaking of the translational invariance in the directions transverse to the D-brane. Therefore, the value of the field X corresponds to the location of the D-brane. Consideration of this scalar field is expected to give the information on the shape of the branes.

As mentioned in the previous section, the dynamics of the Dp-brane is controlled by the dimensionally reduced BI action. Taking the static gauge in the flat spacetime background metric (2.1.2), the action is given by[2]

$$S = -T_{\mathrm{D}p}\int d^{p+1}x\sqrt{-\det(\eta_{\mu\nu}+\partial_\mu X^i\partial_\nu X_i+F_{\mu\nu})}. \tag{2.2.1}$$

This action is derived also by the dimensional reduction from the 10-dimensional BI action

$$S = -T\int d^{10}x\sqrt{-\det(\eta_{MN}+F_{MN})} \tag{2.2.2}$$

with use of the following identity [56]

$$\det\begin{pmatrix} N & A \\ -A^t & M \end{pmatrix} = \det M\cdot\det(N+A^tM^{-1}A), \tag{2.2.3}$$

and by rewriting the gauge field in the transverse directions as $A^i = X^i$. It is possible to derive the relations between the tensions of various p-branes, using the above dimensional reduction and the dualities [125, 126, 127, 128, 78].

[2] In this and the next section, we put $l_s = 1$ and rescale $2\pi F \to F$ for simplicity.

2.2.1 Classical solutions and their interpretation

In this section, we consider how the strings attached to the D-branes can be seen from the viewpoint of the worldvolume theory. Before adopting the fully non-linear treatment of the BI action, let us deal with the linearized action (Maxwell action) which is justified when the field strength is small compared to the string scale. The action is given by expanding the BI action (2.2.1) by the fields as

$$S = T_{\mathrm{D}p}\int d^{p+1}x \left(-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}\partial_\mu X^i(x)\partial^\mu X_i(x)\right). \tag{2.2.4}$$

This action is of course the dimensionally reduced Maxwell action which was originally in 10 dimension.

Let us study a fundamental string stuck to a Dp-brane. Considering the supersymmetries of the configuration in the target space, one finds that a half of the supersymmetries on the worldvolume are left unbroken. The supersymmetry transformation for the gaugino is

$$\delta_\epsilon \lambda = \Gamma^{MN}F_{MN}\epsilon. \tag{2.2.5}$$

Here we write the gauge field in the original 10-dimensional notation, and ϵ is the supersymmetry parameter which is a complex 32-dimensional spinor.

We expect that some BPS state preserving some supersymmetries on the worldvolume represents the above brane configuration. Since the end of the string attached to the D-brane behaves as a point electric charge, a corresponding static solution (in the Coulomb gauge) of the action for $p > 2$ corresponding to the attached string is

$$A_0 = \frac{c_p}{r^{p-2}} \qquad \left(\mathcal{E}_r = -\partial_r A_0 = \frac{c_p(p-2)}{r^{p-1}}\right) \qquad \text{where} \quad r \equiv \sqrt{\sum_{\mu=1}^{p+1} x_\mu x^\mu}. \tag{2.2.6}$$

For this solution, we have

$$\Gamma^{MN}F_{MN} \propto \Gamma^{0\mu}x_\mu. \tag{2.2.7}$$

The solution is found to break all of the supersymmetries since the left hand side of (2.2.7) does not have any zero eigenvalue. So as to construct a BPS solution, we turn on also a single scalar field X^9 which satisfies the equation of motion $\partial^\mu F_{\mu 9} = 0$. Omitting the index 9 for simplicity, the point scalar charge solution is

$$X = \frac{c_p}{\alpha r^{p-2}}, \tag{2.2.8}$$

where α is a parameter specifying the charge. Using (2.2.6) and (2.2.8), the BPS condition reads

$$\left(\Gamma^0 + \alpha\Gamma^9\right)\epsilon = 0. \tag{2.2.9}$$

Taking $\alpha = \pm 1$ preserves half of the supersymmetries, hence the solution with $\alpha = \pm 1$ is BPS.

Callan and Maldacena interpreted this solution as an attached fundamental string elongated along the 9-th direction: the justification of this interpretation is as follows. First, from Eq. (2.2.8), the solution represents deformed Dp-brane surface elongated to the infinity in the 9-th direction. Second, the solution preserves half of the world volume supersymmetries, and the number of the preserved supersymmetries is 8 which is expected from the target space viewpoint. Third, as seen below, the energy of the solution is identical with the energy of the attached string.

The energy of the solution is actually divergent, hence we introduce a cut off δ. Then the energy is given as

$$\begin{aligned} E_{\mathrm{ele}}(\delta) &= T_{\mathrm{D}p}\int d^p\mathbf{r}\ \frac{1}{2}\left(F_{0r}^2 + F_{9r}^2\right) \\ &= \frac{c_p^2(p-2)^2\Omega_p}{(2\pi)^p g}\int_\delta^\infty \frac{dr}{r^{p-1}} \\ &= \frac{|c_p|(p-2)\Omega_p}{(2\pi)^p g}\left|X(\delta)\right|. \end{aligned} \tag{2.2.10}$$

Note that the energy is in proportion to $|X(\delta)|$, thus the energy per unit length in the 9-th direction is given by the coefficient of the right hand side of Eq. (2.2.10). We would like to verify that this coefficient is equal to the tension of the fundamental string. However, now we treat $U(1)$ gauge theory and the electric charge is not quantized but only satisfies the Dirac quantization condition. For this reason, we have to study also the magnetic charge solution. The magnetic field with a point source is (we concentrate on the case $p = 3$)

$$\mathcal{B}_i = \frac{c^{(\mathrm{m})}}{r^2}\hat{\mathbf{r}}_i, \tag{2.2.11}$$

and this is non-BPS by itself in the same manner as before. We excite the scalar field simultaneously as

$$X = \frac{c^{(\mathrm{m})}}{\alpha r}. \tag{2.2.12}$$

Using the solution (2.2.11) and (2.2.12), the factor in the supersymmetry transformation for the gaugino (2.2.5) is found as

$$\Gamma^{MN}F_{MN} = \frac{2c^{(\mathrm{m})}}{r^3}\left[x_1\Gamma_{23} + x_2\Gamma_{31} + x_3\Gamma_{12} + \frac{1}{\alpha}\left(x_1\Gamma_{91} + x_2\Gamma_{92} + x_3\Gamma_{93}\right)\right]. \tag{2.2.13}$$

From this relation, we see that the solution is BPS if the matrix

$$\Gamma_1\Gamma_2\Gamma_3\Gamma_9 - \alpha\mathbf{1} \tag{2.2.14}$$

has a zero eigenvalue. Taking $\alpha = \pm 1$ leaves half of the supersymmetries unbroken, and computing the energy, we obtain

$$E_{\mathrm{mag}}(\delta) = T_{\mathrm{D}p}\int d^p\mathbf{r}\ \frac{1}{2}\left(|\mathcal{B}_i|^2 + F_{9r}^2\right) = \frac{|c^{(\mathrm{m})}|\Omega_3}{(2\pi)^3 g}\,|X(\delta)|\,. \tag{2.2.15}$$

Summarizing the results, for the electric solution and the magnetic solution, the energy per a unit length in the 9-th direction is

$$T_{\mathrm{ele}} = \frac{|c_3|\Omega_3}{(2\pi)^3 g}, \qquad T_{\mathrm{mag}} = \frac{|c^{(\mathrm{m})}|\Omega_3}{(2\pi)^3 g}. \tag{2.2.16}$$

As mentioned above, the coefficients c_3 and $c^{(\mathrm{m})}$ are related with each other by the Dirac quantization condition[3]:

$$c_3 c^{(\mathrm{m})} = \frac{2\pi n}{T_{\mathrm{D3}}(4\pi)^2} \qquad n \in \mathbb{Z}. \tag{2.2.20}$$

For unit electric and magnetic charges, we obtain a relation

$$c_3 c^{(\mathrm{m})} = \pi^2 g. \tag{2.2.21}$$

[3]In the Heaviside unit system, the action and the Coulomb law are

$$S = -\frac{1}{4}\int d^4\sigma \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - e\int_{\mathrm{worldline}} \hat{A}_\mu dx^\mu, \qquad \hat{\mathcal{E}}_i = \frac{e}{4\pi r^2}\hat{\mathbf{r}}_i, \qquad \hat{\mathcal{B}}_i = \frac{m}{4\pi r^2}\hat{\mathbf{r}}_i. \tag{2.2.17}$$

The correspondence can be read from the action (2.2.4) and the solutions (2.2.6) and (2.2.11) as

$$\hat{A}_\mu = \sqrt{T_{\mathrm{D}p}}\,A_\mu, \qquad e = 4\pi\sqrt{T_{\mathrm{D}p}}\,c_3, \qquad m = 4\pi\sqrt{T_{\mathrm{D}p}}\,c^{(\mathrm{m})}. \tag{2.2.18}$$

The phase which is developed by the wave function of an electric charge when rotating twice around the unit magnetic charge is

$$-e\oint \hat{A}_i d\sigma^i = \cdots = -em. \tag{2.2.19}$$

The single-valuedness of the wave function of the electron leads to the condition (2.2.20).

If we identify the energy $T_{\rm ele}$ with the fundamental string tension $1/2\pi$, then using (2.2.16) and (2.2.21), we obtain $T_{\rm mag} = 1/2\pi g$, which is precisely the tension of the D-string. Since the electric-magnetic duality on the D3-brane worldvolume is known to be the S-duality of the type IIB superstring theory, we have seen that the Dirac quantization condition on the worldvolume is consistent with the S-duality of the string theory, under the interpretation by Callan and Maldacena.

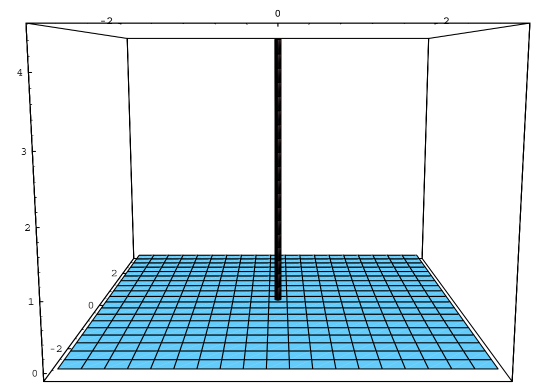

Figure 2.3: A fundamental string stuck to a horizontal Dp-brane.

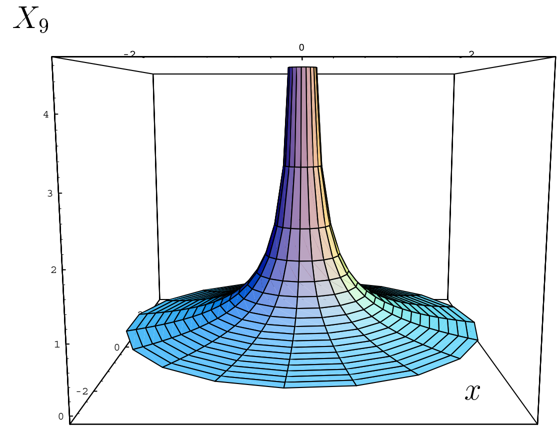

Figure 2.4: The deformed surface of the Dp-brane corresponding to the fundamental string is given by the configuration of the scalar field X^9.

It is straightforward to check the situation in which many fundamental strings are stuck to the D-brane. Since the equations of motion are linear, we can take the superposition of each classical solution. In order to preserve the supersymmetries, we must take the parameterα common for all the strings. The total energy is

$$
\begin{aligned}
E &= T_{{\rm D}p} \int_{M_p - \cup_i D(\mathbf{r}_i,\delta_i)} d^p\mathbf{r} \; \frac{1}{2}\left(\left|\sum_i \mathcal{E}_i\right|^2 + \left|\sum_i \partial_r X_i\right|^2 \right) \\
&= T_{{\rm D}p} \int d^p\mathbf{r} \quad \left|\sum_i \mathcal{E}_i\right|^2 = T_{{\rm D}p} \int d^p\mathbf{r} \quad \left(\sum_i |\mathcal{E}_i|^2 + 2 \sum_{i.j} \mathcal{E}_i \cdot \mathcal{E}_j \right) \\
&= T_{{\rm D}p} \left[\int d^p\mathbf{r} \quad \sum_i |\mathcal{E}_i|^2 + \Omega_p (p-2) \sum_i A_0^{(i)} c_p^{(i)} \right], \qquad (2.2.22)
\end{aligned}
$$

where $A_0^{(i)}$ is the potential which is made by the point charges except for the i-th one $c_p^{(i)}$. Noting that the second term is represented as $\alpha T_{\rm ele} \sum_i X_{(i)}$ where $X_{(i)}$ is the sum of X contributed from the scalar charges except for the i-th one, then the total energy reads

$$E = T_{\rm ele} \sum_i |X(\delta_i)|. \tag{2.2.23}$$

The energy is the sum of all the stuck fundamental strings. This is another consistency check of the interpretation by Callan and Maldacena.

2.2.2 Born-Infeld analysis

In the previous subsection the linearized version of the equations of motion was considered. However, the solution considered above is divergent near the charge, and in that region there is no reason for believing that the linearized Maxwell action is correct. Here in this subsection, we take into account the non-linear nature of the original BI system. Although (2.2.6) and (2.2.8) is a solution of the linear Maxwell-scalar system, in the following we shall see that this solution is also the one for the non-linear BI system.

Restricting our attention to the electric case (for the magnetic case, see Ref. [56]), the BI action is given by

$$S = -T_{{\rm D}p}\int d^{p+1}x \left(\sqrt{(1-|\mathcal{E}_i|^2)(1+|\partial_i X|^2) + (\mathcal{E}_i\partial_i X)^2 - \dot{X}^2} - 1\right) \tag{2.2.24}$$

We turn on a single scalar field X^9 in the same manner as before. The conjugate momenta for $A^\mu (\mu \neq 0)$ and X are

$$\boldsymbol{\pi}_i = T_{{\rm D}p} \frac{\mathcal{E}_i(1+|\partial X|^2) + 2\partial_i X(\mathcal{E}_i\partial_i X)}{\sqrt{(1-|\mathcal{E}_i|^2)(1+|\partial_i X|^2) + (\mathcal{E}_i\partial_i X)^2 - \dot{X}^2}}, \tag{2.2.25}$$

$$P = T_{{\rm D}p} \frac{\dot{X}}{\sqrt{(1-|\mathcal{E}_i|^2)(1+|\partial_i X|^2) + (\mathcal{E}_i\partial_i X)^2 - \dot{X}^2}}. \tag{2.2.26}$$

The equations of motion for the static configuration are derived from this expression as

$$\partial_i \boldsymbol{\pi}_i = 0, \qquad \partial_i \left(\frac{T_{{\rm D}p}^2 \partial_i X + \boldsymbol{\pi}_i(\boldsymbol{\pi}_j \partial_j X)}{\sqrt{T_{{\rm D}p}^2(1+|\partial_i X|^2) + \boldsymbol{\pi}_i^2 + (\boldsymbol{\pi}_i\partial_i X)^2}} \right) = 0. \tag{2.2.27}$$

These complicated equations have the same solution (2.2.6) and (2.2.8), due to the fact that if we put the previous BPS condition $\boldsymbol{\pi}_i = -\partial_i X$ the above non-linear equations reduce to the linear ones. Surprisingly, this solution of the non-linear equations have the same properties: computing the energy of the solution using non-linear Hamiltonian, then the result is the same. The energy is given by the tension times the length of the string. Furthermore, it can be shown that the solution is preserving half of the supersymmetries in the supersymmetric BI system [93]. Thus the configuration is stable.

Though we solved the equations of motion from the BI action, the solution near the origin is still out of the region where the BI action is correct. This is because, the derivative of the field strength is also diverging near the point charge. However, the author of Ref. [142] verified that the solutions (2.2.6) and (2.2.8) are also the solutions of the equations of motion in all order in α' expansion. Due to this verification, we can believe that the solutions (2.2.6) and (2.2.8) give proper background configurations when investigating the D-brane dynamics using these solutions.

Although the solution is in the same form, non-BPS deformation from the electric BPS configuration exhibits completely different aspects. Let us consider a fluctuation around the BPS solution of the BI system. The equation of motion for the fluctuation coordinate η which denotes the small fluctuation originating in the scalar field other than X^9 is

$$\left(1+\frac{\pi^2 g^2}{r^4}\right)\ddot{\eta}(r,t)-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\eta(r,t)\right)=0. \tag{2.2.28}$$

Here we put $p = 3$ for simplicity (for other p, see Ref. [93]). In the region away from the core, $r \to \infty$, the first term in Eq. (2.2.28) becomes a usual kinetic term for η and then the fluctuation behaves as a free particle moving in the 3-dimensional worldvolume space. On the other hand, in the region near the origin, taking the limit $r \to 0$, we obtain the following equations of motion when changing the variables into $\tilde{x} \equiv \pi g/r$ ($\sim X^9$):

$$\left(1+\frac{\pi^2 g^2}{\tilde{x}^4}\right)\ddot{\eta}(\tilde{x},t)-\frac{\partial^2}{\partial \tilde{x}^2}\eta(\tilde{x},t)=0. \tag{2.2.29}$$

In the $\tilde{x} \to \infty$ limit, this equation describes the free propagation in 1-dimensional space. Thus the fluctuation η corresponds to the fluctuation of the string attached to the D-brane.

If the above description captures the nature of the attached string correctly, the boundary condition for the propagating fluctuation should be Dirichlet-type. Let us calculate the potential between two regions. We introduce a frequency ω as $\ddot{\eta} = -\omega^2\eta$, and changing the variables as

$$y \equiv \omega\tilde{x}, \qquad \kappa \equiv gc\omega^2, \qquad \xi(y) \equiv \int_{\sqrt{\kappa}}^{y} dy\sqrt{1+\kappa^2/y^4}. \tag{2.2.30}$$

Then the equation (2.2.29) is reduced to the 1-dimensional potential problem:

$$\left(\text{potential}-\frac{\partial^2}{\partial \xi^2}\right)\tilde{\eta}=\tilde{\eta}, \qquad \tilde{\eta} \equiv \left(1+\kappa^2/y^4\right)^{1/4}\eta. \tag{2.2.31}$$

where $-\infty < \xi < \infty$. In the weak coupling limit $\kappa \to 0$, this potential approaches to the infinitely high delta function whose singularity is placed at $\xi = 0$. This means that in the weak string coupling (where the perturbative definition of the D-brane is correct), the fluctuation of the fundamental string part feels Dirichlet boundary condition at the point where the string is attached.

The authors of Ref. [93] derived the same type of potential in a completely different manner. They used a string σ model in the gravity background of the Dp-brane solution. The string in that background was found to feel the same potential as obtained above. This is very non-trivial in the sense that in the above analysis we ignored the gravity while in the latter the gravity played an essential role. This sort of gravity $\leftrightarrow$ gauge theory correspondence seems to be universal [120, 27, 81].

Another issue on the BI dynamics is concerning the decay of certain D-brane systems. The D-branes and the fundamental strings stuck to them preserve some of the supersymmetries, and thus they are BPS states. The supersymmetry properties are intimately related to the stability of the state, hence the finite deformation to non-BPS configurations is expected to describe the decay of the D-brane systems [47, 122, 36].

2.3 Generalization to the D-brane bound state

In this section[4], we study the basic properties of the worldvolume BI system in a uniform electric field[5]. A Dp-brane with the uniform electric field can be identified with a bound state of the Dp-brane and fundamental strings (called an (F,Dp) bound state) [152, 4, 136, 95]. This is easily seen from the D-brane action (2.1.1), noting that the constant electric field is equivalent with the constant $B_{0\mu}$ field which indicates the uniform distribution of the fundamental string on the D-brane. In the same manner, the uniform magnetic field is equivalent with the constant $B_{\mu\nu}$ $(\mu, \nu \neq 0)$. This spatially polarized B-field is of importance in the light of the correspondence with the non-commutative geometry (see Chap. 4). Our analysis in this section gives the basis of the analysis in Chap. 4 concerning the monopoles in NCSYM.

We consider a fundamental string ending on this bound state, as a generalization of the previous section and Refs. [32, 56]. Though the supergravity solution representing this (F,Dp) bound state has been constructed recently [96], one of the missing important ingredients in

[4]This section is based on my work [70].
[5]Related issues are found in Refs. [14, 110, 111, 80].

string theory is explicit supergravity solutions representing intersecting branes [147, 55, 53, 79, 65]. Since the gravity $\leftrightarrow$ field theory correspondence mentioned in the previous subsection seems to be universal, the analysis from the field theory side may provide some information in the supergravity side.

In Sec. 2.3.1, we study the BPS property and stability of the configuration. Then in Sec. 2.3.2, we investigate the (Dirichlet and Neumann) boundary conditions for the attached string by means of BI equations of motion, and see that these boundary conditions are consistent with the ones deduced from the viewpoint of string worldsheet à la Polchinski [41]. In Sec. 2.3.3 we perform the gravity analysis using a string σ model in the supergravity background of the (F,Dp) bound state.

2.3.1 Stability of BPS configuration

First, let us see how the uniform electric field and a point charge in it are allowed as stable BPS configurations[6]. We treat the low energy effective theory of a Dp-brane extending along the directions $(012\cdots p)$. The argument on the linearized version of this system given in the previous section and Refs. [32] and [56] tells us that half of the worldvolume supersymmetries are preserved when the fields on the brane satisfy the following BPS condition

$$F_{0\mu} = \alpha F_{9\mu}, \quad \text{with} \quad \alpha = \pm 1, \tag{2.3.1}$$

where $F_{9\mu}$ should be understood with a scalar field $A_9 \equiv X_9$. This BPS equation (2.3.1) is expected to be derived also from the non-linear BI theory [35, 93]. Under the relation (2.3.1), equations of motion of the BI theory agree with the ordinary ones in the Maxwell-scalar system. As a solution[7], it is possible to generalize the solution adopted in the previous section and Ref. [32] so as to include trivially a uniform electric field[8]:

$$X_9 = -A_0 = -\frac{c_p}{r^{p-2}} + \mathcal{E} x_3. \tag{2.3.2}$$

[6] Although we consider here only the BPS system and fluctuations around the BPS configurations, particular non-BPS solutions of the non-linear BI system are fascinating. The first reason is that, they are analogue of the first proposal by Born and Infeld [57, 58, 26, 47, 142] (called "BIon"or "pinched" solution). The second one is that, they are concerned with the brane - anti-brane annihilation [32, 122] (called "throat" solution, or "(charged) catenoidal" solution in Ref. [56]). The brane - anti-brane annihilation can be described also by tachyon condensation [18, 135]. A recent result on the non-perturbative tachyon potential [6] is based on the Matrix theory, in which generally there exists a uniform gauge field strength on the constructed brane. Thus the non-BPS configuration of the (F,D3) bound state is of importance since we can check the calculation of Ref. [6] from another side.

[7] We choose $\alpha = -1$ and $p \geq 3$ in this section.

[8] This background (2.3.2) defines an exact conformal field theory, since the derivation in Ref. [142] depends not on the explicit form of the scalar potential, but only on the BPS relation (2.3.1).

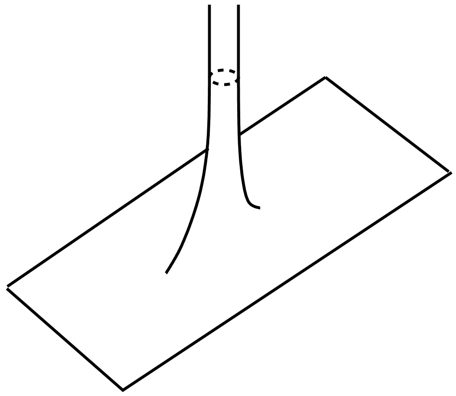

Figure 2.5: "Tube-like" configuration of Dp-brane surface representing a string. Now the Dp-brane is tilted, in order to be a BPS configuration, under the uniform electric field.

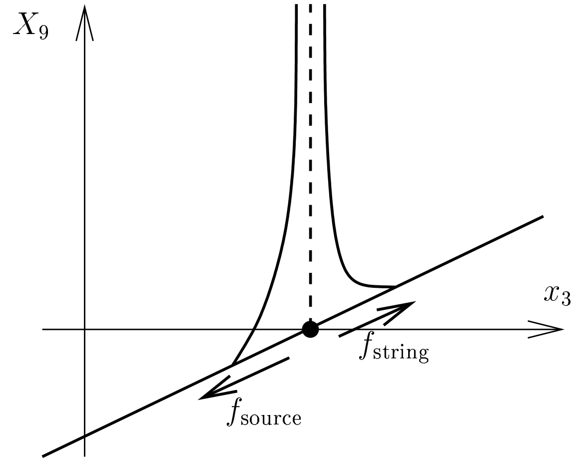

Figure 2.6: Force balance of the configuration.

This configuration represents a charged particle in the background electric field $\vec{\mathcal{E}} = \mathcal{E}\hat{\mathbf{x}}_3$ which is uniform on the Dp-brane[9]. As seen from Eq. (2.3.2), the attached fundamental string is not perpendicular to the Dp-brane, because of the uniform field strength (see Fig. 2.5). In other words, the Dp-brane is now tilted in order to preserve some supersymmetries.

From the viewpoint of target space supersymmetries, it is also possible to see that the configuration given by Eq. (2.3.2) preserves a part of the supersymmetries to be stable. For simplicity, consider the case of $p = 3$. Taking T-dualities in two directions along x_1 and x_2, then the tilted D-string (on which a constant field strength exists) appears. This D-string can be interpreted as a dyonic string carrying both NS-NS and R-R 2-form charges [152]. Following the argument developed in Ref. [133], at the spatial infinity the conserved supercharges are 1/4 of the original ones, under the existence of this dyonic string and a fundamental string perpendicular to the (123)-plane. The T-duality transformation does not change the number of preserved supersymmetries, hence the configuration of Eq. (2.3.2) preserves 1/4 of the original target space supersymmetries (at least at the spatial infinity).

Although this configuration is stable because of its BPS property, naively one might be

[9]A point electric charge in a uniform background magnetic field is not a BPS configuration. A BPS configuration with a magnetic point source in a uniform magnetic field, which corresponds to a D-string ending on the D3-brane - D-string bound state, will be discussed in Chap. 4. We will find that the configuration of this uniform magnetic field is intimately related to the worldvolume non-commutativity, and the tilted configuration provides an essential property of monopoles in NCSYM. An issue concerning the uniform magnetic field is found also in Ref. [92].

afraid of instability due to an electric force on the point charge in the uniform electric field. Actually, the charge at $r=0$ is $(p-2)c_p\Omega_{p-1}$, and the force along the tilted direction can be read as

$$f_{\rm source} = -(p-2)c_p\Omega_{p-1}\cdot T_{{\rm D}p}\,\mathcal{E}\cdot\frac{1}{\sqrt{1+\mathcal{E}^2}}. \tag{2.3.3}$$

Note that the strength of the "physical" electric field[10] is $T_{{\rm D}p}\,\mathcal{E}$, where $T_{{\rm D}p}$ is the tension of the Dp-brane. The final factor in Eq. (2.3.3) is for extracting a component of the force along the tilted direction. Now, what happens is that this force (2.3.3) is exactly cancelled by $T_{\rm f}=1/2\pi$, the tension of the fundamental string emanated from the Dp-brane (see Fig. 2.6). Actually, the contribution of the string tension along the tilted direction is

$$f_{\rm string} = \frac{\mathcal{E}}{\sqrt{1+\mathcal{E}^2}}\frac{1}{2\pi}, \tag{2.3.4}$$

and using the charge quantization condition $(p-2)c_p\Omega_{p-1} = g_{\rm st}(2\pi)^{p-1}$ (see Ref. [32]) and $T_{{\rm D}p}=1/(2\pi)^p g_{\rm st}$, it is easy to see the force balance as

$$f_{\rm source}+f_{\rm string}=0. \tag{2.3.5}$$

This is consistent with the stability expected from the BPS property of the configuration.

2.3.2 Boundary conditions

Worldsheet picture

One of the remarkable properties of the description of the intersecting strings in the BI system is that the boundary conditions for the attached strings can be reproduced, as explained in the previous section [32, 93, 120]. Now, our interest is not the pure D-branes but the (F,Dp) bound state. In this case, the boundary condition in the usual worldsheet picture are known to be modified. We shall study this first, and check the consistency with the BI theory later.

In the worldsheet approach of D-branes by Polchinski et al. [41], the boundary conditions of a fundamental string attached to the D-brane on which a uniform gauge field strength exists

[10] The charge quantization is calculated using the action

$$T_{{\rm D}p}\int d^{p+1}\sigma\sqrt{-\det(h_{\alpha\beta}+F_{\alpha\beta})},$$

therefore the Gauss law derived from this action is $\partial_i(T_{{\rm D}p}\,\mathcal{E}_i)=0$.

are [31, 94, 4]

$$\left(\partial_\sigma X^\mu - F^\mu{}_\nu \partial_\tau X^\nu\right)\Bigg|_{\sigma=0} = 0 \quad (\mu = 0, \cdots, p), \tag{2.3.6}$$

$$\partial_\tau X^i \Bigg|_{\sigma=0} = 0 \quad (i = p+1, \cdots, 9). \tag{2.3.7}$$

For the directions transverse to the Dp-brane (Dirichlet directions), the Dirichlet boundary conditions do not change in spite of the existence of the electric field on the Dp-brane. On the other hand, for the Neumann directions, the uniform electric field $F_{03} \equiv \mathcal{E}$ yields so-called "mixed" boundary conditions, which relate values of two or more scalars at the boundary. How can these mixed boundary conditions be understood in terms of only a single particular scalar, say X^3 ? Defining new coordinate scalars as $X^\pm \equiv (X^0 \pm X^3)/\sqrt{2}$, then the above boundary conditions (2.3.6) along the plane spanned by x^0 and x^3 reads [8]

$$\left(\partial_\sigma X^\pm \pm \mathcal{E} \partial_\tau X^\pm\right)\Bigg|_{\sigma=0} = 0. \tag{2.3.8}$$

A wave solution which respects this phase shift at the boundary $\sigma = 0$ is easily found as

$$X^\pm = A_\pm\Big(\exp\left[i(\tau + \sigma \mp \alpha/2)\right] + \exp\left[i(\tau - \sigma \pm \alpha/2)\right]\Big) \tag{2.3.9}$$

where the phase shift α is defined by a relation $\mathcal{E} = \tan(\alpha/2)$, and $A_\pm$ is the amplitude of the wave, normalized as $|A_\pm| = 1$. Hence the expression for the scalar X^3 is

$$X^3 = \sqrt{2}\left(X^+ - X^-\right) = 2\sqrt{2}\cos(\alpha/2)\Big(\exp[i(\tau+\sigma)] + \exp[i(\tau-\sigma)]\Big). \tag{2.3.10}$$

Here we have chosen $A_+ = -A_- = 1$, since this choice satisfies a requirement that the amplitude of the wave coming in is equal to the one of the wave going out. The expression (2.3.10) indicates that the boundary condition for X^3 is purely Neumann-type, not the mixed type. In the following, we will see that the above boundary conditions are reproduced in the BI analysis.

Transverse (Dirichlet) directions

First, let us investigate the Dirichlet directions. For simplicity, we shall concentrate on the case $p = 3$ hereafter in this section. Denoting the fluctuation in a direction transverse to both the attached string and the D3-brane as η, the equation of motion for this fluctuation is given by [32]

$$\left(1 + |\partial_i X|^2\right)\ddot{\eta} - \Delta\eta = 0. \tag{2.3.11}$$

Substituting the configuration under consideration (2.3.2) and putting the time dependence of the fluctuation $\exp(-i\omega t)$, then Eq. (2.3.11) becomes

$$\left[\left(1+\mathcal{E}^2+\frac{2\pi\mathcal{E}g_{\rm s}}{r^2}\cos\theta+\frac{\pi^2 g_{\rm s}^2}{r^4}\right)\omega^2+\Delta\right]\eta=0, \tag{2.3.12}$$

where $r\equiv\sqrt{x_1^2+x_2^2+x_3^2}$ and $\cos\theta\equiv x_3/r$. Now one can see that the dependence on θ appears in the equation, therefore the solution of this equation cannot be spherically symmetric. We should take into account the θ-dependence of the solution.

Assuming that the solution does not depend on another variable φ in polar coordinates, we can expand η by Legendre functions, a system of orthogonal functions, as

$$\eta(r,\theta)=\sum_{l=0}^{\infty}\eta_l(r)P_l(\cos\theta). \tag{2.3.13}$$

We require that if one takes the limit $\mathcal{E}\to 0$ then the solution should recover the one given in Ref. [32]. It is possible to construct such a solution, and the result is (see App. A.1)

$$\begin{aligned}\eta(r,\theta)=\;&\eta_0^{(0)}(r)\cdot\left(1+\sum_{l=1}^{\infty}P_l(\cos\theta)\left[(2\pi\mathcal{E}g_{\rm s}\omega^2)^l\prod_{i=1}^{l}\frac{1}{(i+1)(2i-1)}\right]\right)\\&+(\text{higher order terms}).\end{aligned} \tag{2.3.14}$$

In the RHS, the "higher order terms" consist of terms which do not contribute to the phase of the total wave flux (the magnitude of those higher order waves dump as taking the limit $r\to 0$. The region $r\sim 0$ corresponds to the tip of the tube in Fig. 2.5, where the initial and final states of the wave are defined on the fundamental string.) The spherically symmetric factor $\eta_0^{(0)}(r)$ is the solution of the equation

$$\left[\left(1+\frac{\kappa^2}{y^4}\right)+\frac{d^2}{dy^2}\right]\eta_0^{(0)}(r)=0, \tag{2.3.15}$$

where we have put $\kappa^2\equiv(1+\mathcal{E}^2)\pi^2 g_{\rm s}^2\omega^4$ and $y\equiv\pi g_{\rm s}\omega/r$. This is exactly the same one obtained in the previous section and Ref. [32], except for the $\mathcal{E}$-dependence of κ. Using the tortoise-like coordinate

$$\xi(y)\equiv\int_{\sqrt{\kappa}}^{y}\sqrt{1+\kappa^2/y^4}, \tag{2.3.16}$$

Eq. (2.3.15) can be rewritten as a Schrödinger-type equation

$$\left(-\frac{d^2}{d\xi^2}+V(\xi)\right)\tilde{\eta}=\tilde{\eta},\quad\text{where}\quad\tilde{\eta}\equiv(1+\kappa^2/y^4)^{1/4}\eta. \tag{2.3.17}$$

The potential $V(\xi)$ approaches to a delta-function with infinite area as one goes to the weak coupling limit $g_{\rm s}\to 0$. Thus the solution (2.3.14) is subject to the Dirichlet boundary condition at the weak coupling limit, as expected by the worldsheet prescription in Sec. 2.3.2.

Longitudinal (Neumann) directions

For the string fluctuation in the longitudinal directions, which is described by the fluctuation of the scalar X^9, we shall follow the argument given in Ref. [123]. Turning on fluctuations of both the gauge field and the scalar field X^9, the authors of Ref. [123] obtained the same equation (2.3.11), for the fluctuation of the gauge field, $\eta = \delta A_i$. The fluctuation of the scalar field, δX^9, is related to δA_i as

$$\partial_i A_i + \partial_t \delta X^9 = 0. \tag{2.3.18}$$

Therefore the boundary conditions for the δX^9 was found to be Neumann-type.

As discussed in Ref. [123], among the various modes in δX^9, the physical mode which precisely corresponds to the fluctuation along the D-brane is the one carrying the angular momentum $l = 1$ in the worldvolume language. The solution (2.3.14) is composed of the modes of all l, hence it may seem to be difficult to extract only the physical mode mentioned above. In order to get the physical fluctuation along x^3, we shall turn on only δA_z. Then the relation (2.3.18) gives

$$-i\omega\delta X^9 \left(= \frac{\partial}{\partial z} A_z\right) = \frac{z}{r}\left(\frac{\partial}{\partial r}\eta_0^{(0)}(r) + \frac{2}{3}(\pi\mathcal{E}g\omega^2)^2\eta_0^{(0)}(r)\right) + \cdots. \tag{2.3.19}$$

The second term in the parenthesis in the RHS stems from the $l = 2$ excited part in the solution (2.3.14), and this mode actually has Dirichlet property. However, taking the weak coupling limit, this term vanishes (and this is also the case for other terms denoted by "$\cdots$" in Eq. (2.3.19), which indicate unphysical $l \neq 1$ modes).

The other longitudinal directions (along x^1 and x^2) can be analyzed in the same way. Summing up all together, we conclude that in the weak coupling limit, for the fluctuations along the longitudinal directions the boundary conditions are Neumann-type, as expected from the worldsheet picture.

Due to the change of the frequency parameter κ in the differential equation, there remains $\mathcal{E}$-dependence at finite coupling g_s. Calculating the transmitting amplitude, the total power emanated from the end of the attached string [123] is now $(1+\mathcal{E}^2)$ times the ordinary flux $\omega^4 g^2$. This result is natural in the sense that in the BI system the speed of light changes under the uniform field strength background. If we turn on the background electric field in the BI-scalar system, the velocity of the fluctuation becomes $1/\sqrt{1+\mathcal{E}^2}$, as seen when we neglect the terms originated from the point source in the differential equation (2.3.12). This change can be viewed also as a change of the frequency ω, therefore, this results in the change of the total energy flux.

2.3.3 Supergravity analysis

In the previous subsections, we have investigated BI dynamics in the uniform electric field, for the BPS configurations and the fluctuations around them. For the BPS configuration, naive force balance ensures the stability of the configuration, and the fluctuations around the configuration satisfy precisely the boundary conditions expected from the worldsheet analysis à la Polchinski.

Here we comment on the relation to the supergravity calculation. In Refs. [93, 120], a test string in the background of supergravity solution of D3-branes are analyzed using the string σ model approach. It was found that fluctuations of the scalar fields on the worldsheet feels the same form of the potential as in the BI analysis. In our case of the D3-brane with the uniform electric field, this property would be able to be confirmed in a similar manner. Recently, the supergravity solution representing the (F,Dp) bound state was constructed [95, 96]. The worldsheet action of the static probe string (which corresponds to the tube part in Fig. 2.5) in this background is

$$S = \int d\tau d\sigma \left[(H')^{1/4} H^{-3/4} \sqrt{(X_3')^2 + H(X_\perp')^2} - X_3' \frac{\mathcal{E}}{\sqrt{1+\mathcal{E}^2}} \frac{1}{H} \right]. \tag{2.3.20}$$

We turn on only the relevant two scalars, X_3 (parameterizing the direction along the electric flux) and $X_\perp$ (the radial coordinate in the transverse directions). In Eq. (2.3.20), H and H' are harmonic functions in the transverse space. The explicit form of these functions is

$$H \equiv 1 + \frac{Q}{(X_\perp)^4}, \qquad H' \equiv 1 + \frac{Q}{(1+\mathcal{E}^2)(X_\perp)^4}. \tag{2.3.21}$$

The second term in Eq. (2.3.20) stems from the coupling of the probe string to the non-trivial NS-NS 2-from background produced by the fundamental strings condensed in the D3-brane.

Near the infinity of the transverse space, two harmonic functions are approximated as $H \sim H' \sim 1$, hence the static solution lead from the equations of motion is satisfying $X_3' = kX_\perp'$, where k is an integration constant. Substituting this relation into the action (2.3.20), we have

$$S \sim \int d\tau d\sigma X_\perp' \left(\sqrt{1+k^2} - k \frac{\mathcal{E}}{\sqrt{1+\mathcal{E}^2}} \right) = \int d\tau X_\perp \Big|_{\text{infinity}} \left(\sqrt{1+k^2} - k \frac{\mathcal{E}}{\sqrt{1+\mathcal{E}^2}} \right). \tag{2.3.22}$$

Noting that $X_\perp$ is positive, the action is minimized at the value $k = \mathcal{E}$. Thus at the spatial infinity we obtain the tilted configuration of the string: $X_3' = \mathcal{E} X_\perp'$, which is exactly matched to the configuration considered in Sec. 2.3.1.

It would be possible to analyze the full structure of the potential which the fluctuation of the string feels, according to the procedure developed in Refs. [93, 120]. Let us define the location of the "core" of the potential by the value of $X_\perp$ where the first term in the harmonic function (2.3.21) (in this case, 1) is comparable to the second term ($\sim 1/X_\perp^4$). Then, the locations of the cores of the two functions H and H' are different from each other. For this reason, the potential which the scalar feels, computed in the supergravity formulation, is expected to be different from the one obtained in Sec. 2.3.2 and Sec. 2.3.2. However, in the weak coupling limit, two cores may overlap with each other and this potential will reproduce the appropriate boundary conditions considered in Sec. 2.3.2.

2.4 SYM ingredients from brane techniques

As well as the supersymmetric $U(1)$ gauge theories analyzed in the previous sections, non-Abelian gauge theories can have their counterparts in the brane description, as explained at the end of Sec. 2.1. As the Dirac monopole is realized as a D-string attached to the D3-brane, the 'tHooft-Polyakov monopole [141, 116] and the Julia-Zee dyon [89] in 4-dimensional SYM are realized in the same manner.

In this section we see how solitons such as the monopoles and dyons are realized by brane configurations. The key which mediates between the conventional SYM picture and the brane configurations is the D-brane action. When many parallel D-branes exist, the low energy effective theory describing this system is a non-Abelian generalization of the BI action [146][11]. Unfortunately, the precise form of the non-Abelian BI action is still beyond the reach of us due to the complexity of the derivation of the action [60, 114, 148, 10, 68, 26]. However, since in the $U(1)$ case the classical BPS solution have the same form for the BI system and the linearized Maxwell system. Thus we expect that this is also the case for the non-Abelian effective theory [25, 64, 73, 69]. In the following, we believe this assumption and adopt the SYM analysis instead of the fully non-linear treatment. We follow mainly Ref. [64]. (See Ref. [25] for the non-Abelian BI analysis.)

[11] The related references before the appearance of the D-brane are Refs. [62, 44, 143, 144, 3].

The action of the bosonic part of the 4-dimensional $\mathcal{N}=4$ SYM is given by[12]

$$S=-\frac{1}{g_{\rm YM}^2}\int d^4x\left(\frac{1}{4}(F^a_{\mu\nu})^2+\frac{1}{2}(D^\mu\Phi^a_i)^2\right), \tag{2.4.1}$$

where we restrict our attention to the case of two parallel D3-branes on which $SU(2)$ SYM is induced. The definition of the field strength and the covariant derivative is

$$F_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+\epsilon^{abc}A^b_\mu A^c_\nu, \tag{2.4.2}$$

$$D_\mu\Phi^a_i=\partial_\mu\Phi^a_i+\epsilon^{abc}A^b_\mu\Phi^c_i, \tag{2.4.3}$$

where the index i runs from 1 to 6, thus specifies the directions transverse to the D3-brane. In the following, we shall turn on only a single scalar field.

One of the candidates of the D-brane action describing low energy dynamics of many parallel D-branes is the following symmetric-trace version of the non-Abelian BI action[13]:

$$S=-T_{\rm D3}{\rm Str}\int d^4x\sqrt{-\det\left(\eta_{\mu\nu}+T_{\rm f}^{-1}F_{\mu\nu}+D_\mu XD_\nu X\right)}. \tag{2.4.4}$$

Expanding the square root and the determinant in the action and picking up the terms quadratic in the field strength and DX, then we obtain

$$S\sim -T_{\rm D3}\,{\rm Tr}\int d^4x\left(\frac{1}{4}T_{\rm f}^{-2}F_{\mu\nu}^2+\frac{1}{2}(D_\mu X)^2\right). \tag{2.4.5}$$

Comparing this action with the above SYM action (2.4.1), we get the correspondence between the ordinary SYM variables and the string-theoretical variables as

$$\Phi(r)=T_{\rm f}X(r),\qquad \frac{1}{g_{\rm YM}^2}=\frac{T_{\rm D3}}{2T_{\rm f}^2}=\frac{1}{4\pi g}. \tag{2.4.6}$$

Here we have identified the gauge field in the SYM action (2.4.1) with the one on the D-brane.

2.4.1 Realization of 'tHooft-Polyakov monopole and Julia-Zee dyon

It is almost evident that a D-string suspended between the two D3-branes corresponds to the 'tHooft-Polyakov monopole. This is easily seen by considering the charges of the configuration as in the case of the W-boson which was explained at the end of Sec. 2.1.

The explicit configuration of the 'tHooft-Polyakov monopole was given in Ref. [118] using the spherically symmetric ansatz

$$A^a_i=\epsilon_{aij}\hat{\mathbf{r}}_j(1-K(r))/r,\qquad A^a_0=\hat{\mathbf{r}}_aJ(r)/r,\qquad \Phi^a=\hat{\mathbf{r}}_aH(r)/r. \tag{2.4.7}$$

[12] We are using the following normalization of the group generators: ${\rm Tr}\,T^aT^b=\delta^{ab}/2$.

[13] In this section, we normalize the factor $T_{\rm f}^{-1}$ in front of X in the square root.

The monopole solution is given by

$$K(r) = Cr/\sinh(Cr), \qquad J(r) = 0, \qquad H(r) = Cr\coth(Cr) - 1. \tag{2.4.8}$$

The eigenvalues of the scalar field Φ represent the location of the surface of the two D3-branes. After diagonalization, the eigenvalues are

$$\Phi(r) = \pm\left(C\coth(Cr) - \frac{1}{r}\right). \tag{2.4.9}$$

Therefore at the infinity of the worldvolume, the eigenvalues approach to some constant values. The effect of the D-string disappears at the infinity, hence these values give the locations of the two D-branes through the relation (2.4.6). The separation Δ of the D3-branes are

$$\Delta = C/T_{\mathrm{f}}. \tag{2.4.10}$$

Reading the asymptotically next-to-leading order terms from the eigenvalues, we have

$$X \sim \frac{\Delta}{2} - \frac{1}{2T_{\mathrm{f}}\, r}. \tag{2.4.11}$$

From the $1/r$ behavior, the magnetic charge placed at the origin is given by $c_{(\mathrm{m})} = \pi$, using $l_{\mathrm{s}} = 1$. This charge is precisely the one given in Sec. 2.2.1. Writing the mass of the monopole in terms of the brane language,

$$\mathrm{Mass} = \frac{4\pi C}{g_{\mathrm{YM}}^2} = \frac{T_{\mathrm{f}}}{g}\Delta = T_{\mathrm{D1}}\Delta. \tag{2.4.12}$$

Hence the mass is correctly reproduced by the naive brane picture. This feature was important in the previous sections for the justification of the interpretation by Callan and Maldacena. Now for the soliton configuration in non-Abelian theories, the field theoretical properties are seen to be reproduced from the brane technique.

The analysis above is easily generalized to the case of the Julia-Zee dyon. In this case, as easily recognized, the suspended string in the brane picture is a (p,q)-string, which is a bound state of p fundamental strings and q D-strings. The explicit field theoretical solution given in Ref. [118] is

$$K(r) = \frac{Cr}{\sinh(Cr)}, \quad J(r) = \sinh\gamma\left[\frac{Cr}{\coth(Cr)} - 1\right], \quad H(r) = \cosh\gamma\left[\frac{Cr}{\coth(Cr)} - 1\right] \tag{2.4.13}$$

This dyon consists of the clouds of both the electric charge $4\pi\sinh\gamma/g_{\mathrm{YM}}$ and the magnetic charge $-4\pi/g_{\mathrm{YM}}$. Quantum mechanically, the electric charge should be quantized as

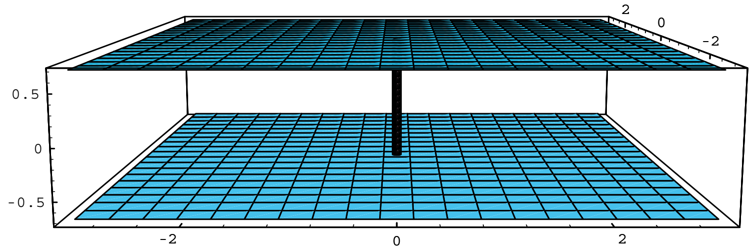

Figure 2.7: A D-string connecting two D3-branes.

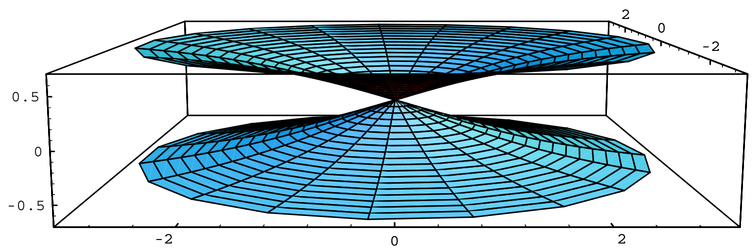

Figure 2.8: The diagonal eigenvalues of the scalar field of the 'tHooft-Polyakov monopole. The corresponding brane configuration is depicted in Fig. 2.7.

$4\pi \sinh\gamma / g_{\rm YM} = p g_{\rm YM}$, where p is an integer. Using the correspondence (2.4.6), we obtain a relation

$$pg = \sinh\gamma. \tag{2.4.14}$$

The brane separation is calculated from the eigenvalues of the solution for the scalar field (2.4.13) in the same manner as

$$\Delta = C \cosh\gamma / T_{\rm f}. \tag{2.4.15}$$

Using these relations (2.4.14) and (2.4.15), the mass of the dyon is rewritten in terms of the brane language as

$${\rm Mass} = (\cosh\gamma)^2 \frac{4\pi C}{g_{\rm YM}^2} = T_{(p,q)}\Delta. \tag{2.4.16}$$

Here we have used the following formula for the (p,q)-string tension

$$T_{(p,q)} = T_{\rm D1}\sqrt{(pg)^2 + q^2} = \cosh\gamma \frac{4\pi}{g_{\rm YM}^2} \tag{2.4.17}$$

with the magnetic charge $q=-1$ of the solution (2.4.13). We have seen again the agreement between the two pictures.

2.4.2 W-boson and S-duality

As for the W-boson, since there is no classical configuration representing it, the direct application is impossible. However, since the S-duality in the type IIB superstring theory exchanges a D-string by a fundamental string, we can use the monopole configuration instead of the W-boson. Let us see the consistency of the above brane picture with the S-duality. The reasoning in the following is similar to the one presented in Sec. 2.2.1.

D3-branes are invariant under the S-duality, because at the supergravity level, the self-dual 4-form gauge field which couples to the D3-brane is left intact under the S-duality transformation. Driven by this fact, Tseytlin showed by an explicit calculation that the D3-brane action is invariant under the electric-magnetic duality [145]. He used the following Abelian BI action

$$S=\int d^4x \left(\sqrt{\det(\delta_{\mu\nu}+F_{\mu\nu}/\sqrt{g})}+\frac{i}{2}\Lambda_{\mu\nu}(F_{\mu\nu}-2\partial_\mu A_\nu)\right). \tag{2.4.18}$$

which is written in Euclidean notation and the field $\Lambda_{\mu\nu}$ is a Lagrange multiplier with anti-symmetric indices. Then integrating out the gauge field A_μ from the action (2.4.18) gives the following action

$$S=\int d^4x \left(\sqrt{\det(\delta_{\mu\nu}+\widetilde{F}_{\mu\nu}\sqrt{g})}\right), \tag{2.4.19}$$

where we have introduced the dual gauge field and its field strength by

$$\Lambda_{\mu\nu}=\epsilon_{\mu\nu\rho\sigma}\partial_\rho\widetilde{A}_\sigma, \qquad \widetilde{F}_{\mu\nu}=\partial_\mu\widetilde{A}_\nu-\partial_\nu\widetilde{A}_\mu. \tag{2.4.20}$$

Therefore, in our notation, the S-duality makes the exchanges

$$T_{\rm D3}=\frac{1}{(2\pi)^3 g l_{\rm s}^4}\to\frac{g}{(2\pi)^3 l_{\rm s}^4}, \qquad g_{\mu\nu}\to\frac{g_{\mu\nu}}{g}. \tag{2.4.21}$$

If we keep only the quadratic terms in the same way as before, the dualized action in terms of the original variables are found as

$$S=-\frac{g}{2\pi}\,{\rm Tr}\int d^4x\left(\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}T_{\rm f}^2 g^{-2}(D_\mu X)^2\right). \tag{2.4.22}$$

Comparing this action with the ordinary SYM action (2.4.1), we obtain the correspondence

$$\Phi(r)=\frac{T_{\rm f}}{g}X(r), \qquad \frac{1}{g_{\rm YM}^2}=\frac{g}{4\pi}. \tag{2.4.23}$$

Using this correspondence, it is straightforward to check the mass and the charge of the monopole . From the asymptotic behavior of the monopole solution provides the brane separation $\Delta = Cg/T_{\rm f}$ and the charge $c^{(3)} = \pi g$ which coincides with the one given in Sec. 2.1. The mass of the monopole in this S-dual theory is equal to $T_{\rm f}\Delta$, which is identical with the energy of the stretched fundamental string.

2.4.3 Nahm construction and D-string realization

In the above realization, monopoles are constructed using the suspended D-strings or (p,q)-strings. Since also on this string there is an effective theory describing the string dynamics, naively one expects that the same brane configuration can be described by this worldsheet gauge theory. In Ref. [43], the worldsheet gauge theory of the D-strings ending on the D3-branes is identified with the $SU(2)$ monopole moduli space of ADHMN construction [106, 107] in a 4-dimensional supersymmetric Yang-Mills-Higgs system. This work [43] provided an intriguing example justifying the fact that the brane techniques give clear understanding of the conventional solitons in a geometrical manner.

The 2-dimensional effective theory on the D-string is obtained by dimensional reduction of the 10-dimensional SYM [152]. Thus the relevant 2-dimensional action is[14]

$$S = T_{\rm D1} \int d^2x \ {\rm Tr} \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} D_\mu \xi^I D^\mu \xi^I + \frac{1}{4}[\xi^I, \xi^J]^2 \right] + S_{\rm f}. \tag{2.4.24}$$

Here $T_{\rm D1}$ is the tension of the D-string. The D-string extends in the direction (01), and the eight scalar fields ξ^I $(I = 2, \cdots, 9)$ contained in the above action represent the transverse fluctuation of the D-string. This system possesses $(8,8)$ supersymmetry (we omit the fermion terms $S_{\rm f}$ for convenience). From the action (2.4.24), we have the following expression for the energy of this system:

$$U = T_{\rm D1} \int dx \, \frac{1}{2} \, {\rm Tr} \left[\mathcal{E}^2 + (D_0\xi^I)^2 + (D_1\xi^I)^2 - \frac{1}{2}[\xi^I, \xi^J]^2 \right]. \tag{2.4.25}$$

The coordinate $x \equiv x^1$ denotes the spatial direction of the worldsheet, and $\mathcal{E}$ is the electric field. With this energy formula, we shall study the BPS nature of the 2-dimensional system.

When one deals with a single D-string, the scalar and the gauge fields are Abelian, and hence there is no potential term. All the scalar fields decouple from each other. First, let us turn on only one scalar field $S \equiv \xi^2$, since we expect that the dyonic string will be realized

[14] In this subsection we put $l_{\rm s} = 1$ and rescale the fields as $2\pi A \to A$ and $2\pi\xi \to \xi$.

by turning on the electric field on the brane and this scalar field is necessary for preserving some supersymmetries, as in Sec. 2.3.1. Then the $(x, S(x))$-plane (which is equivalent to the (ξ^1, ξ^2)-plane) is interpreted as a 2-dimensional plane on which the string lies.

The supersymmetry transformation of the gaugino is

$$\delta\lambda_{\alpha i} = 2\Big[(\sigma^0\bar{\sigma}^1 - \sigma^1\bar{\sigma}^0)_\alpha^{\ \beta}\zeta_{\beta i}\mathcal{E} + (\sigma^0\bar{\sigma}^3 - \sigma^3\bar{\sigma}^0)_\alpha^{\ \beta}\zeta_{\beta i}D_0S + (\sigma^1\bar{\sigma}^3 - \sigma^3\bar{\sigma}^1)_\alpha^{\ \beta}\zeta_{\beta i}D_1S\Big], \tag{2.4.26}$$

where i denotes the index of $SU(4)$ subgroup of R-symmetry, and α is a spinor index. With the BPS condition

$$\mathcal{E} \pm D_1S = 0, \qquad D_0S = 0, \tag{2.4.27}$$

the transformation (2.4.26) vanishes if half of the supersymmetry parameters $\zeta_{\alpha i}$ are set equal to zero[15]:

$$\zeta_{(\pm)i} = 0. \tag{2.4.28}$$

Thus half of the 16 supersymmetries of the D-string effective theory are preserved with the BPS conditions (2.4.27) [16]. Adopting the gauge $A_1 = 0$, a solution of (2.4.27) is found under the requirement of time-independence of the configuration as[42]

$$A_0 = \pm S. \tag{2.4.29}$$

This BPS condition is independent of the worldvolume dimension considered, as seen in Ref. [32]. Due to the Gauss' law, the electric field on the D-string must be a constant. Thus this corresponds to the fundamental string (see Sec. 2.3), and the resultant configuration is the $(p, 1)$ string [152]. Using a proper BI Hamiltonian, the energy of the $(p, 1)$ string is reproduced.

To describe the perpendicular D3-branes in the D-string worldsheet gauge theory, we turn on three more scalar fields $X^i \equiv \xi^{i+2}$ $(i = 1, 2, 3)$, which are interpreted as coordinates parameterizing the D3-brane worldvolume. This means that the D3-branes extend in the direction (0345). Together with the previously considered fields, A_0 and S, the energy formula

[15] The 2-dimensional fermion is denoted as $\zeta = (\zeta_-, \zeta_+)^T$.

[16] The number of the supersymmetries which the (p, q)-string preserves in the target space is 16, not 8. The rest 8 supersymmetries would be supplied from the non-linearly realized supersymmetry [9, 131]. The reason why we impose a condition that the 8 supersymmetries are preserved is for the latter convenience in deriving the BPS conditions for string networks in Sec. 3.1.2.

becomes

$$U = \frac{T_{\rm D1}}{2}\int_{x^{(-)}}^{x^{(+)}} dx \ \ {\rm Tr}\Big[(\mathcal{E} \pm D_1 S)^2 + (D_0 S)^2 \mp 2 D_1 S \mathcal{E} \\ + (D_0 X^i)^2 + (D_1 X^i)^2 - [S, X^i]^2 - \frac{1}{2}[X^i, X^j]^2 \Big]. \quad (2.4.30)$$

The gauge group is chosen to be $SU(2)$ for a reason that will be explained below. The D-string terminates at $x = x^{(\pm)}$, where the D3-branes are located. Thus this theory is defined on the spatial interval $x^{(-)} < x < x^{(+)}$.

In the presence of the scalar fields X^i, the equation of motion for the electric field reads

$$D_1 \mathcal{E} + i[X^i, D_0 X^i] = 0. \quad (2.4.31)$$

With the help of the resultant relation

$${\rm Tr}\, 2 D_1 S \mathcal{E} = {\rm Tr}\, 2\partial_1 (S\mathcal{E}) + i\,{\rm Tr}\, 2[S, X^i] D_0 X^i, \quad (2.4.32)$$

the energy (2.4.30) is bounded in the same way, except for the terms consisting only of the scalar fields X^i, as

$$U = T_{\rm D1}\int dx\, \frac{1}{2}\,{\rm Tr}\Big[(\mathcal{E} \pm D_1 S)^2 + (D_0 S)^2 \mp 2\partial_1(S\mathcal{E}) \\ + \Big(D_0 X^i \mp i[S, X^i]\Big)^2 + (D_1 X^i)^2 - \frac{1}{2}[X^i, X^j]^2 \Big]. \quad (2.4.33)$$

For the last two terms consisting of X^i, we have the identity

$${\rm Tr}\left[(D_1 X^i)^2 - \frac{1}{2}[X^i, X^j]^2\right] \\ = {\rm Tr}\sum_i \left(D_1 X^i \pm \frac{i}{2}\sum_{j,k} \epsilon^{ijk} X^j X^k \right)^2 \mp \frac{2i}{3}\partial_1\, {\rm Tr}(\epsilon^{ijk} X^i X^j X^k). \quad (2.4.34)$$

Hence, finally the energy is bounded as follows:

$$U = T_{\rm D1}\int dx\, \frac{1}{2}\,{\rm Tr}\Big[(\mathcal{E} \pm D_1 S)^2 + (D_0 S)^2 + \Big(D_0 X^i \mp i[S, X^i]\Big)^2 \\ + \Big(D_1 X^i \pm \frac{i}{2}\epsilon^{ijk} X^j X^k\Big)^2 \Big] \\ \mp T_{\rm D1}\,{\rm Tr}\Big[S\mathcal{E}\Big]_{x^{(-)}}^{x^{(+)}} \mp T_{\rm D1}\frac{i}{3}\,{\rm Tr}\Big[\epsilon^{ijk} X^i X^j X^k\Big]_{x^{(-)}}^{x^{(+)}} \\ \geq \mathcal{E}_{\rm boundary} + \mathcal{E}_{\rm D3} \quad (2.4.35)$$

where $\mathcal{E}_{\rm boundary}$ is the energy contributing to the mass of the $(p,1)$-string, and

$$\mathcal{E}_{\rm D3} \equiv T_{\rm D1}\frac{\mp i}{3}\,{\rm Tr}\Big[\epsilon^{ijk} X^i X^j X^k\Big]_{x^{(-)}}^{x^{(+)}}. \quad (2.4.36)$$

Therefore, the conditions for saturating the energy bound are, in addition to (2.4.27),

$$D_0 X^i \mp i[S, X^i] = 0, \tag{2.4.37}$$

$$D_1 X^i \pm \epsilon^{ijk} X^j X^k = 0. \tag{2.4.38}$$

In order to solve these BPS equations, (2.4.27), (2.4.37) and (2.4.38), first note that Eq. (2.4.37) is trivially satisfied with the previous BPS condition (2.4.29), which is a solution of the conditions (2.4.27). Thus we are left with Eq. (2.4.38). If we substitute the gauge fixing condition $A_1 = 0$, this remaining condition (2.4.38) is the Nahm equation for the construction of monopoles in ADHMN method [106, 107].

In Ref. [43], the relation between this Nahm equation (included in Nahm data) and the D-string approach in string theory was discussed. To precisely compare our situation with the Nahm data, we choose the interval defining the worldsheet to satisfy

$$x^{(\pm)} = \pm\frac{\pi}{2a} \tag{2.4.39}$$

without loss of generality. D3-branes are located at these boundaries, $x = x^{(\pm)}$. It is claimed in Ref. [43] (for related discussion, see Ref. [149]) that the boundary conditions which represent k finitely separated D-strings terminating on the D3-branes are

$$\widetilde{X}^i \sim \frac{T_i}{\frac{\pi}{2} \mp z}, \qquad \left(z \sim \pm\frac{\pi}{2}\right) \tag{2.4.40}$$

where new rescaled variables are defined as $\widetilde{X}^i \equiv \mp i X^i/a$, $z \equiv ax$, and the matrices T^i define an irreducible k-dimensional representation of $SU(2)$:

$$[T^i, T^j] = \epsilon^{ijk} T^k. \tag{2.4.41}$$

The asymptotic expression (2.4.40) with (2.4.41) is actually a solution of the Nahm equation (2.4.38) near the boundaries. The scalar X^i diverges at the boundaries, and this implies that there are D3-branes there. Equation (2.4.40) is consistent with the fact that in the D3-brane worldvolume gauge theory,[32] an attached D-string is represented as $|x - x^{(\pm)}| \sim 1/r$ with $r \sim |X^i|$, at $r \sim \infty$.

It should be noted that for $k = 1$ the solution of the Nahm equation is trivially a constant[17]. Extending this trivial solution, one finds that a constant diagonal matrix also satisfies the

[17]With this Nahm data, one can construct a single BPS monopole solution [118, 23] using the ADHMN method.

Nahm equation (2.4.38). Though this vacuum is ordinarily adopted, it belongs to a reducible representation of $SU(2)$ in the Nahm language. The author of Ref. [43] asserts that the reducible representation indicates that D-strings exist infinitely far from each other, and that a configuration of finitely separated D-strings should satisfy an irreducible boundary condition. Furthermore, with a constant diagonal matrix solution, it is impossible to incorporate the effect of the D3-branes. Thus we take the simplest choice $k = 2$ in the following. Therefore we set $T^i = \sigma_i / 2i$.

Let us check the consistency in terms of the energy bound. The term $\mathcal{E}_{\mathrm{D3}}$ in the energy bound (2.4.35) is expected to involve the energy of the D3-branes at the boundaries. In fact, estimation of (2.4.36) by substituting Eq. (2.4.40) leads us to the relation

$$\mathcal{E}_{\mathrm{D3}} = T_{\mathrm{D1}} \frac{1}{2(x^{(+)} - x)^3} \bigg|_{x \to x^{(+)}} + T_{\mathrm{D1}} \frac{1}{2(x - x^{(-)})^3} \bigg|_{x \to x^{(-)}} . \tag{2.4.42}$$

This expression is independent of the length of the interval (the parameter a), and diverges correctly as r^3, which indicates the volume of the D3-brane.

Chapter 3

String Junction and 1/4 BPS Dyon

In Chap. 2, conventional solitons such as monopoles and dyons are realized by the brane configurations. As explained in the introduction, owing to the rapid progress in the string theory side, now the brane technique provides new predictions and results of particular field theories. One of the most interesting one is the prediction and the construction of 1/4 BPS solitons in $\mathcal{N} = 4$ SYM, which was motivated from the discovery of the string networks. The string network is composed by string junctions, and is found to be stable and preserving 1/4 supersymmetries in the target space. A naive extrapolation of the results for the conventional solitons explained in Sec. 2.4 leads to a prediction of the existence of new solitons, corresponding to the configuration of Fig. 3.1.

In this chapter, after reviewing the basic properties of the string junction (network) in Sec. 3.1, we present in Sec. 3.2 an extended review of the notable work [20] by O. Bergmann who predicted the existence of this soliton. Then in Sec. 3.3, based on my works [72, 73] in collaboration with H. Hata and N. Sasakura, we provide an explicit soliton configuration corresponding to this string junction[1]. After explaining our strategy to solve the Bogomol'nyi equations, we investigate some properties of our solutions, and give exact solutions in the case some of the free parameters of the general solutions take certain values. In Sec. 3.4, we show in detail the features of our exact solutions for $SU(3)$, $SU(4)$ and $SU(5)$ as examples. Various interesting properties of the solutions are also discussed there. In Sec. 3.5, we calculate the long-range force between two dyons in the SYM theory, and find the consistency with the idea that a three-pronged string can be produced as a bound state of two strings. Sec. 3.6 contains summary of this chapter and discussions. In App. B, we show that our solutions satisfy the

[1]While the paper [73] was in the final stage of preparation, an article [82] appeared which has an overlap with our discussions on exact solutions for general N.

equations of motion of the non-Abelian BI action, and the other appendices contain several formulas used in the text.

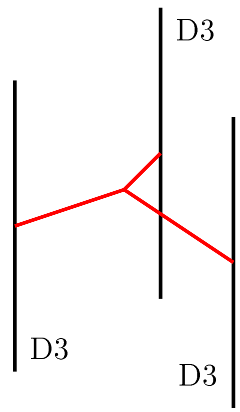

Figure 3.1: A string network is terminated on three D3-branes.

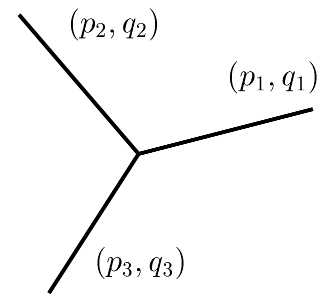

Figure 3.2: Three (p, q)-strings meet at a junction and form a three-pronged string.

3.1 Properties of the string network

The string network is used everywhere now in various situations in string theory. It was conjectured originally in Ref. [125] that a three-pronged string would exist as a stable state possessing the following properties: the forces from the three strings is balanced, and the 2-from charges of the three strings conserve at the junction. The BPS feature of such a pronged string has been shown by several authors in both the string picture [42, 119, 133] and the M-theory picture [88, 103, 83]. The relations to the field theory BPS states have been investigated in several contexts [52, 51, 77, 104, 134, 20, 21, 22].

3.1.1 Pronged string as a 1/4 BPS configuration

A naive expectation is that the three (p, q)-strings can join with forming a junction where the force is balanced and the (p, q)-charge is conserved [125]. These two conditions are written explicitly as

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} q_i = 0, \qquad \sum_{i=1}^{3} T_{(p_i, q_i)} \hat{\mathbf{n}}_i = 0, \tag{3.1.1}$$

where (p_i, q_i) is the NS-NS and R-R charges of the i-th string, and the vector $\hat{\mathbf{n}}_i$ denotes the direction of the i-th string (see Fig. 3.2).

We follow Ref. [133] showing that the three-pronged string satisfying these two conditions (3.1.1) preserve 1/4 of the target space supersymmetries. Let us put the pronged string on the (89)-plane. It is easily checked that the following assignment of the vectors provides the force balance at the junction point:

$$\hat{\mathbf{n}}_i = (\cos\theta_i, \sin\theta_i), \tag{3.1.2}$$

where the angle θ_i is defined as $\theta_i \equiv \mathrm{Arg}(p + iq/g)$. The supersymmetries which the (p, q) string directed along this vector $\hat{\mathbf{n}}$ preserves are [76, 133]

$$\epsilon_{\mathrm{L}} + i\epsilon_{\mathrm{R}} = e^{\theta}\Gamma^0\ \hat{\mathbf{n}}\cdot\mathbf{\Gamma}(\epsilon_{\mathrm{L}} - i\epsilon_{\mathrm{R}}), \tag{3.1.3}$$

where Γ^M are the 10-dimensional gamma matrices. Substituting the definition (3.1.2) for all i into the above supersymmetry constraint, then only for the three-pronged string which satisfies the charge conservation condition, eight target space supersymmetries are unbroken:

$$\epsilon_{\mathrm{L}} = \Gamma^0\Gamma^8\epsilon_{\mathrm{L}}, \qquad \epsilon_{\mathrm{R}} = -\Gamma^0\Gamma^8\epsilon_{\mathrm{R}}, \qquad \epsilon_{\mathrm{L}} = \Gamma^0\Gamma^9\epsilon_{\mathrm{R}}. \tag{3.1.4}$$

Thus at least at the spatial infinity, the three-pronged string is a 1/4 BPS state and hence stable. Connecting many string junctions by component strings, then we obtain a string network. If at all the junction point forces are balanced and the charges are conserved, then it is evident that the whole configuration of the network preserves 1/4 of the original 32 supersymmetries. This is due to that fact that the remained supersymmetries are independent of the charges of the component strings, as seen in Eq. (3.1.4).

3.1.2 Realization of pronged string in 2-dimensional gauge theory

Unfortunately, any supergravity solution representing the pronged string has not been constructed, hence the field theoretical realization of the pronged string is important for the investigation of the pronged strings. First we review the considerations given in Ref. [42][2], where a three-pronged string is realized as a configuration of a 2-dimensional worldsheet effective gauge field theory on a single D-string. The action of this theory is Eq. (2.4.24).

In the picture of the type IIB superstring theory, the configuration of the pronged string preserves eight of the original 32 supercharges. By the existence of the D-string on which

[2] See also Ref. [54].

we carry out computations, the supersymmetry is broken to half of the original supersymmetries, *i.e.*, 16 supercharges in 2 dimensions are preserved at this stage. Therefore the junction configuration observed from the D-string standpoint is expected to preserve half of the 16 supercharges. As seen in Sec. 2.4.3, the BPS condition (2.4.29) left the eight supersymmetries unbroken, thus this condition (2.4.29) is expected to be the one for the pronged string configurations[3].

In the D-brane worldvolume theory, a fundamental string attached to it and extending in the transverse direction is described by a source of the gauge field on the D-brane, since the charge of the string should be conserved at the endpoint of the string as seen in the previous chapter. For the case of a Dp-brane with $p \geq 2$, one can regard a "spike" configuration around the source as a fundamental string [32]. On the other hand, this is not the case when the body in the D-string ($p=1$), because it has only one spatial dimension and the source does not form a spike. The equation of motion for the electric field with a source located at $x = x_0$,

$$D_1 \mathcal{E} = ng\delta(x - x_0), \qquad n \in Z, \tag{3.1.5}$$

has the following BPS solution[4]:

$$A_0 = S = \begin{cases} pg(x - x_0) + q, & (x < x_0) \\ (pg - ng)(x - x_0) + q. & (x > x_0) \end{cases} \tag{3.1.6}$$

Here g is a string coupling constant equal to the fundamental electric charge, and the charges n and p are integral numbers since electric flux on the D-string is quantized [32].

As seen from the solution, the scalar field S is a linear function of x. There is a kink at $x = x_0$ (see Fig. 3.3), and at this point the electric charge is altered by ng. The authors of Ref. [42] interpreted this situation as follows: n invisible fundamental strings[5] are attached to the D-string, and they form a string junction at $x = x_0$. Now the D-strings have electric charges, thus they are interpreted as a $(-p, 1)$ string[6] (for $x < x_0$) and a $(p{-}n, -1)$ string (for $x > x_0$).

This interpretation is justified by the fact that force balance condition is satisfied at a junction point. Naming the regions $x < x_0$ "string 1", $x > x_0$ "string 3", and the invisible

[3]BPS properties of the multi-pronged string are investigated using the worldsheet approach in Ref. [33] in another way.

[4]We adopt the upper sign in Eq. (2.4.29) without loss of generality.

[5]Ref. [42] deals with the case $n = 1$. Though the state with $|n| \geq 2$ is marginal, our treatment in the following sections requires this marginal state.

[6]The charge of a string is defined as the one following a direction oriented toward the junction point.

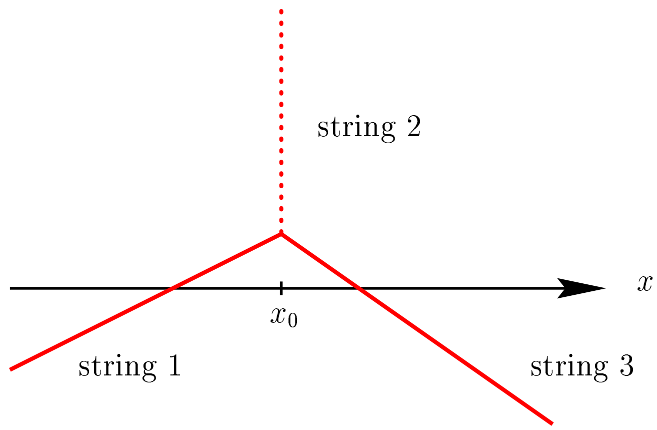

Figure 3.3: The solution (3.1.6) representing a three-pronged string. The dotted line denotes the invisible fundamental string.

fundamental string “string 2”, the tension of each string is read from their charges as

$$T_1 = T\sqrt{p^2 + \frac{1}{g^2}}, \qquad T_2 = nT, \qquad T_3 = T\sqrt{(n-p)^2 + \frac{1}{g^2}}, \tag{3.1.7}$$

where T is the tension of a single fundamental string, $T = gT_{\mathrm{D1}}$. If we assign this tension to each region of the solution, one can confirm that the following force balance conditions are satisfied at the junction point $x = x_0$. One is in the horizontal direction (parallel to x),

$$T_1 \frac{1}{\sqrt{1+(pg)^2}} = T_3 \frac{1}{\sqrt{1+(pg-ng)^2}}, \tag{3.1.8}$$

and the other is in the vertical direction (parallel to the string 2),

$$T_2 = T_1 \frac{pg}{\sqrt{1+(pg)^2}} + T_3 \frac{ng-pg}{\sqrt{1+(pg-ng)^2}}. \tag{3.1.9}$$

A tree web with more prongs can be constructed straightforwardly in the same way. One can see that at every junction point the force balance conditions are satisfied if the charges are conserved there.

In this way of realizing the pronged string, the component strings (strings 1, 2 and 3) are straight and infinitely long. Thus the configuration favors the $SL(2,Z)$ S-duality symmetry as discussed in Ref. [42].

For further justification of the above interpretation (presented in Ref. [42]), let us investigate the energy bound of this configuration. The energy formula (2.4.25) with a single scalar field S is written as

$$\begin{aligned} U &= T_{\rm D1} \int dx \, \frac{1}{2} \, {\rm Tr} \left[\mathcal{E}^2 + (D_0 S)^2 + (D_1 S)^2 \right] \\ &= T_{\rm D1} \int dx \, \frac{1}{2} \, {\rm Tr} \left[(\mathcal{E} \pm D_1 S)^2 + (D_0 S)^2 \mp 2 D_1 S \, \mathcal{E} \right] \\ &= T_{\rm D1} \int dx \, \frac{1}{2} \, {\rm Tr} \left[(\mathcal{E} \pm D_1 S)^2 + (D_0 S)^2 \mp 2 \partial_1 (S \mathcal{E}) \pm 2 S D_1 \mathcal{E} \right]. \end{aligned} \tag{3.1.10}$$

Therefore using Eq. (3.1.5), the energy is bounded by some boundary charges and a source contribution as follows:

$$U \geq | \, \mathcal{E}_{\rm source} + \mathcal{E}_{\rm boundary} \, |, \tag{3.1.11}$$

where

$$\mathcal{E}_{\rm source} = -T_{\rm D1} n g S(x = x_0) \quad \text{and} \quad \mathcal{E}_{\rm boundary} = T_{\rm D1} \Big[{\rm Tr} \, S \mathcal{E} \Big]_{-\infty}^{+\infty}. \tag{3.1.12}$$

The equality in (3.1.11) holds when the BPS conditions (2.4.27) are satisfied.

The term $\mathcal{E}_{\rm source}$ indicates that there actually exists a fundamental string 2, since we can express the source contribution as

$$\mathcal{E}_{\rm source} = nT \left(-S(x{=}x_0) \right). \tag{3.1.13}$$

The value $S(x{=}x_0)$ denotes the coordinate of the endpoint of the fundamental string. Thus its length is $|\infty - S(x = x_0)|$. Equation (3.1.13) appropriately represents the energy of the fundamental string with this length.

On the other hand, one can see that another contribution to the energy, $\mathcal{E}_{\rm boundary}$, represents strings 1 and 3. Since $\mathcal{E}_{\rm boundary}$ is divergent with non-zero n, we restrict the domain of x to the interval $[-L, L]$ for evaluating this term. Then for the solution (3.1.6), we have

$$\begin{aligned} \mathcal{E}_{\rm boundary} &= T_{\rm D1} \Big[{\rm Tr} \, S \mathcal{E} \Big]_{-L}^{+L} \\ &= T_{\rm D1} \Big[(pg - ng)^2 (L - x_0) + (pg)^2 (L + x_0) \Big]. \end{aligned} \tag{3.1.14}$$

Here we have assumed that L is sufficiently large, so that $x_0 \in [-L, L]$. In order to see that the strings attached to the boundaries $x = \pm L$ have tensions T_1 and T_3, we move the junction point by δx_0 (> 0) in the x direction (see Fig. 3.4). The energy changes as a result of this horizontal translation by

$$\begin{aligned} \delta \mathcal{E}_{\rm boundary} &\equiv \delta \mathcal{E}_{\rm boundary}(x_0 + \delta x_0) - \delta \mathcal{E}_{\rm boundary}(x_0) \\ &= T_{\rm D1} \Big[(p - n)^2 g^2 - p^2 g^2 \Big] \delta x_0. \end{aligned} \tag{3.1.15}$$

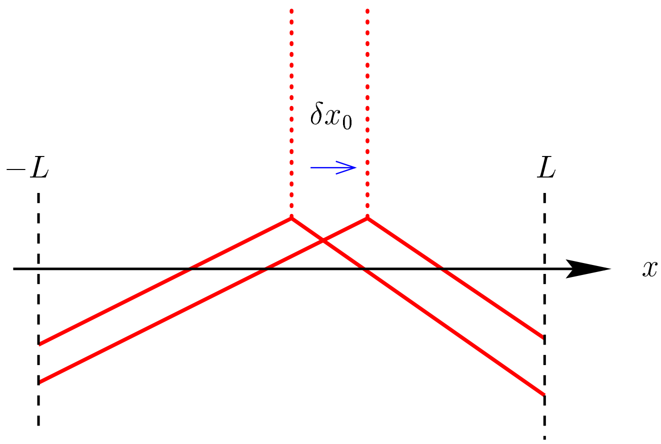

Figure 3.4: Moving the junction point by δx_0 in the x direction.

This expression is consistent with the naive argument from the string picture; in other words, Eq. (3.1.15) can be written in the form

$$|\delta\mathcal{E}_{\text{boundary}}| = T_1 \delta l_1 + T_3 \delta l_3 \tag{3.1.16}$$

with the variations of the length of string 1 and 3

$$\delta l_1 \equiv \delta x_0 \sqrt{1+(pg)^2}, \qquad \delta l_3 \equiv -\delta x_0 \sqrt{1+(ng-pg)^2}. \tag{3.1.17}$$

We have seen that the energy bound (3.1.11) appropriately reflects the energy of a three-pronged string with no endpoint. Now the interpretation presented in Ref. [42] has been justified also from the viewpoint of the BPS energy bound, where necessary information regarding the energy of the three-pronged string is encoded in Eq. (3.1.11). In particular, among three component strings, the existence of the invisible fundamental string (string 2) is guaranteed by the source contribution of the energy bound.

3.2 1/4 BPS states in 4-dimensional SYM from multi-pronged strings

3.2.1 Prediction from string theory

As explained in Chap. 2, particular sorts of field theories have their counterparts in string theory, as a low energy effective theories on D-branes. A typical example of such interplay is given by the 4-dimensional $\mathcal{N}=4$ $SU(N)$ SYM broken spontaneously to $U(1)^{N-1}$. As we have seen in the previous chapter, this theory can be studied as an effective field theory on N parallel D3-branes [152, 145, 61]. The invariance of D3-branes under the $SL(2,Z)$ duality transformation of the type IIB superstring theory implies the $SL(2,Z)$ duality symmetry of the $\mathcal{N}=4$ SYM [105, 132]. The $(\pm 1, 0)$ strings stretching between different D3-branes preserve 1/2 supersymmetry of the D3-brane world volume and appear as the massive W-bosons of the broken gauge symmetries in the field theory. The $SL(2,Z)$ duality symmetry implies the existence of other BPS states of the field theory corresponding to the general (p,q) strings with relatively prime integers p and q. The states with $(p,\pm 1)$ are monopoles and dyons, the field configurations of which are explicitly known in the form of the Prasad-Sommerfield solution [118]. The existence of the states with $q=2$ was shown by the quantization of the collective modes of the two monopole solutions [132], while the existence of the states with $q>2$ was discussed in [130, 117].

On the other hand, as we have seen in Sec. 3.1, these suspended strings are possibly generalized to the multi-pronged strings. This configuration preserves 1/4 of the 32 supersymmetries in the type IIB superstring theory (3.1.4), and thus a BPS object. Therefore, it is natural to expect a configuration depicted in Fig. 3.1, in which legs of the three-pronged string are terminated on the D3-branes. This configuration is possible since strings with any (p,q)-charges can end on D-branes due to the invariance of the D3-branes under the type IIB S-duality. Furthermore, the configuration of Fig. 3.1 preserves 1/8 of the original supersymmetries: the existence of the parallel D3-branes along the (0123) directions put the following constraint for unbroken supersymmetries

$$\epsilon_{\mathrm{L}}=\Gamma^0\Gamma^1\Gamma^2\Gamma^3\epsilon_{\mathrm{R}}. \tag{3.2.1}$$

This constraint preserves a half of the supersymmetries originally preserved by the multi-pronged strings (3.1.4). From the viewpoint of the D3-brane worldvolume theory, the number of the preserved supersymmetries is 1/4 of the $\mathcal{N}=4$ SYM.

How can this pronged-string be seen from the field theory side? Bergman presented the resolution of this question [20]. He obtained an evidence for the existence of the corresponding 1/4 BPS states in the $\mathcal{N}=4$ $SU(N)$ SYM theory with $N>2$, showing that the mass of the three-pronged string agrees with the mass of the corresponding field theory state under the assumption of its BPS saturation. In the following, we give an extended review of his work [20].

In that paper, however, the configuration itself in the field theory was not obtained. Such a 1/4 BPS state was shown to have non-parallel electric and magnetic charges [20, 49], and this fact makes it a non-trivial issue to solve the equations of BPS saturation. The first explicit construction was given by me and collaborators [72, 73][7], and this will be explained from the next sections.

3.2.2 BPS bound and Bogomol'nyi equations

Although first predicted in Ref. [20] for the specific example of the $SU(3)$ case, we treat here general $SU(N)$ case. This enables us to consider the BPS saturated solutions which correspond to the string theory BPS states of multi-pronged strings connecting N D3-branes[8].

The properties of supersymmetry-preserving states are constrained by the supersymmetry algebra including the Hamiltonian which determines the mass of the state. We consider 4-dimensional $\mathcal{N}=4$ $SU(N)$ SYM. Omitting the fermionic terms, the energy of this system reads,

$$U=\int d^3x\frac{1}{2}\,\mathrm{Tr}\left\{(\mathcal{B}_i)^2+(\mathcal{E}_i)^2+(D_iX)^2+(D_iY)^2+(D_0X)^2+(D_0Y)^2-[X,Y]^2\right\}, \quad (3.2.2)$$

where $\mathcal{E}_i=F_{0i}$ and $\mathcal{B}_i=1/2\epsilon_{ijk}F_{jk}$ are the electric and magnetic fields, and the covariant derivative is defined by $D_\mu X=\partial_\mu X-i\,[A_\mu,X]$. We have put the Yang-Mills coupling constant equal to one, and shall consider the case of vanishing vacuum theta angle. In Eq. (3.2.2), we have kept only two adjoint scalars, X and Y, among the original six which should describe the transverse coordinates of the D3-branes. This is because we are interested in the multi-pronged BPS string states preserving 1/4 of the supersymmetry, and such string networks must necessarily lie on a two-dimensional plane.

In this chapter we assume the D-flatness[9] $[X,Y]=0$. To derive the BPS saturation condition, we rewrite the energy (3.2.2) as [49]

$$\begin{aligned}U=\int d^3x\frac{1}{2}\,\mathrm{Tr}\Big\{&(\mathcal{E}_i\cos\theta-\mathcal{B}_i\sin\theta-D_iX)^2+(\mathcal{B}_i\cos\theta+\mathcal{E}_i\sin\theta-D_iY)^2\\&+(D_0X)^2+(D_0Y)^2\Big\}+(Q_X+M_Y)\cos\theta+(Q_Y-M_X)\sin\theta,\end{aligned} \quad (3.2.3)$$

[7]See also Ref. [91] for the construction using the ADHMN formalism.

[8]The following analysis is base on the paper [73], and the case with specific $SU(3)$ case was discussed in Ref. [72].

[9]Due to the D-flatness, we can diagonalize X and Y simultaneously and hence consider definite D3-brane surfaces described by the eigenvalues of X and Y as we shall do in later subsections. The Bogomol'nyi equations without imposing the D-flatness are given in Refs. [82, 91]

where θ is an arbitrary angle, and $Q_{X,Y}$ and $M_{X,Y}$ are defined by

$$Q_X = \int d^3x \,\mathrm{Tr}\left(\mathcal{E}_i D_i X\right) = \int_{r\to\infty} dS_i \,\mathrm{Tr}\left(\mathcal{E}_i X\right), \quad M_X = \int d^3x \,\mathrm{Tr}\left(\mathcal{B}_i D_i X\right) = \int_{r\to\infty} dS_i \,\mathrm{Tr}\left(\mathcal{B}_i X\right). \tag{3.2.4}$$

Since each term in the integrand of (3.2.3) is non-negative and θ is arbitrary, we obtain the BPS bound:

$$U \geq E_{\mathrm{BPS}} \equiv \sqrt{\left(Q_X + M_Y\right)^2 + \left(Q_Y - M_X\right)^2}. \tag{3.2.5}$$

The lower bound in (3.2.5) is saturated when the following conditions hold:

$$D_i X = \mathcal{E}_i \cos\theta - \mathcal{B}_i \sin\theta, \tag{3.2.6}$$

$$D_i Y = \mathcal{B}_i \cos\theta + \mathcal{E}_i \sin\theta, \tag{3.2.7}$$

$$D_0 X = D_0 Y = 0, \tag{3.2.8}$$

$$[X, Y] = 0, \tag{3.2.9}$$

with the angle θ given by

$$\sin\theta = \frac{Q_Y - M_X}{E_{\mathrm{BPS}}}, \quad \cos\theta = \frac{Q_X + M_Y}{E_{\mathrm{BPS}}}. \tag{3.2.10}$$

In addition to the four equations (3.2.6)–(3.2.9), we have to impose the Gauss law,

$$D_i \mathcal{E}_i = 0, \tag{3.2.11}$$

since we used it in converting the volume integration of $Q_{X,Y}$ (3.2.4) into the surface one. Note that Eq. (3.2.10) is an automatic consequence of the two equations (3.2.6) and (3.2.7) and need not be imposed independently. In the next subsection, we show that the SYM configurations satisfying Eqs. (3.2.6)–(3.2.9) preserve 1/4 supersymmetry.

3.2.3 Supersymmetric aspects of the solution

It was argued in [49, 20] that the solution with non-parallel electric and magnetic charges has 1/4 supersymmetry. Here we shall examine this property in detail.

The Lagrangian of the 4-dimensional $\mathcal{N} = 4$ SYM is given by [113]

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{i}{2}\overline{\lambda}^a_K\gamma^\mu D_\mu\lambda^a_K + \frac{1}{2}(D_\mu A^a_i)^2 + \frac{1}{2}(D_\mu B^a_i)^2 \\ &-\frac{i}{2}g f_{abc}\overline{\lambda}^a_K(\alpha^i_{KL}A^b_i + i\beta^i_{KL}\gamma_5 B^b_i)\lambda^c_L \\ &-\frac{1}{4}g^2\left(\left(f_{abc}A^b_iA^c_j\right)^2 + \left(f_{abc}B^b_iB^c_j\right)^2 + 2\left(f_{abc}A^b_iB^c_j\right)^2\right),\end{aligned} \tag{3.2.12}$$

where K and L are the indices of the $\mathbf{4}$ representation of the $SU(4)$ R-symmetry. The matrices α and β span the antisymmetric part of $\mathbf{4}\times\mathbf{4}$, which transform in $\mathbf{6}$:

$$\alpha_1 = \begin{pmatrix} & \sigma_1 \\ -\sigma_1 & \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} & \sigma_3 \\ -\sigma_3 & \end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix} i\sigma_2 & \\ & i\sigma_2 \end{pmatrix},$$
$$\beta_1 = \begin{pmatrix} & i\sigma_2 \\ i\sigma_2 & \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} & \mathbf{1} \\ -\mathbf{1} & \end{pmatrix}, \qquad \beta_3 = \begin{pmatrix} -i\sigma_2 & \\ & i\sigma_2 \end{pmatrix}. \tag{3.2.13}$$

Here σ_i denote the Pauli matrices. The Lagrangian is invariant under the $\mathcal{N}=4$ supersymmetry transformation. The supersymmetry transformation of the gaugino is given by

$$\delta\lambda_K = \Big[\frac{-i}{2}\sigma^{\mu\nu}F_{\mu\nu} - i\gamma^\mu D_\mu\left(\alpha^i A_i + i\beta^i\gamma_5 B_i\right)$$
$$+\frac{i}{2}g\epsilon_{ijk}\alpha^k[A^i,A^j] + \frac{i}{2}g\epsilon_{ijk}\beta^k[B^i,B^j] + g\alpha^i\beta^j[A^i,B^j]\gamma_5\Big]_{KL}\epsilon_L, \tag{3.2.14}$$

where the parameter ϵ_L denotes a Majorana spinor.

In investigating the supersymmetric aspects of our solutions, we identify (A_3, B_3) with (X,Y) and put $A_{1,2}$ and $B_{1,2}$ equal to zero due to the $SO(6)$ ($\sim SU(4)$) R-symmetry of the action. Plugging the Bogomol'nyi equations (3.2.6)–(3.2.9) into the above supersymmetry transformation (3.2.14), we obtain

$$\delta\lambda_K^a = \left[-i(\sigma^{0i}\delta_{KL} + \gamma^i\alpha^3_{KL})\mathcal{E}^a_i - \left(\frac{i}{2}\sigma^{ij}\epsilon_{ijk}\delta_{KL} - \gamma_k\gamma_5\beta^3_{KL}\right)\mathcal{B}^{ka}\right]\epsilon_L. \tag{3.2.15}$$

Since the angle θ can be absorbed by a rotation in the (X,Y) plane, we have put $\theta=0$. In our solutions the electric and magnetic fields behaves differently from each other, and thus we obtain two constraints[10] on ϵ_L:

$$\left[\delta_{KL}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} - i\begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}_{KL}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\right]\epsilon_L = 0,$$
$$\left[\delta_{KL}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} + i\begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}_{KL}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}\right]\epsilon_L = 0. \tag{3.2.16}$$

Representing the Majorana spinor ϵ_L by a Weyl spinor η_L

$$\epsilon_L = \begin{pmatrix} i\sigma_2\eta_L^* \\ \eta_L \end{pmatrix}, \tag{3.2.17}$$

[10]Our convention of the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu].$$

and substituting (3.2.17) into (3.2.16), we find the conditions

$$\begin{pmatrix} 1 & i & & \\ -i & 1 & & \\ & & 1 & -i \\ & & i & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \begin{pmatrix} 1 & -i & & \\ i & 1 & & \\ & & 1 & -i \\ & & i & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = 0. \tag{3.2.18}$$

The solution is obtained as

$$\eta_L = a_L \eta, \qquad (a_1, a_2, a_3, a_4) = e^{i\theta_R} \left(0, 0, i, 1\right), \tag{3.2.19}$$

where η is a constant Weyl spinor. Thus the BPS states discussed in this chapter preserve one supersymmetry (3.2.19) out of the original four, and the phase factor in Eq. (3.2.19) indicates the remaining $U(1)$ R-symmetry.

In this $\mathcal{N} = 4$ system, we have two independent central charges Q_X and M_Y after an appropriate $SO(6)$ R-transformation (a subgroup of this transformation is used to put θ to zero. The formula for the lower bound of the energy is given by

$$\begin{aligned} U &= \int d^3x \, \frac{1}{2} \operatorname{Tr} \left\{ (\mathcal{E}_i \mp D_i X)^2 + (\mathcal{B}_i \mp D_i Y)^2 + (D_0 X)^2 + (D_0 Y)^2 - [X, Y]^2 \right\} \pm Q_X \pm M_Y \\ &\geq \max \Big\{ \, |Q_X + M_Y| \, , \, |Q_X - M_Y| \, \Big\}, \end{aligned} \tag{3.2.20}$$

which depends on the relative sign of the charges Q_X and M_Y. Though we solve only one of the two cases in this chapter, this is not essential because the Bogomol'nyi equations of another case are just obtained by the substitution $X \to -X$. Since our solutions have non-zero Q_X and M_Y, their masses saturate either $|\, Q_X + M_Y|$ or $|\, Q_X - M_Y|$, and not both. Generally, states saturating k bounds are generated by $2(\mathcal{N} - k)$ fermionic creation operators made of the supercharges. Hence our BPS configuration of SYM belongs to a supermultiplet with $n \times 2^6$ (n: integer) components. This implies that this multiplet contains states with spins higher than or equal to 3/2.

3.2.4 SYM solutions and the IIB picture of string networks

In this subsection, we shall discuss the asymptotic ($r \to \infty$) behavior of the solutions and its relation to the IIB picture of the string networks. Suppose that we have a static solution $(A_\mu(\hat{\mathbf{r}}), X(\hat{\mathbf{r}}), Y(\hat{\mathbf{r}}))$ to the equations (3.2.6)–(3.2.9) and (3.2.11), and that their asymptotic forms are, after a suitable gauge transformation, given (locally) as follows:

$$X \sim \operatorname{diag}(x_a) + \frac{1}{2r} \operatorname{diag}(u_a), \tag{3.2.21}$$

$$Y \sim \mathrm{diag}(y_a) + \frac{1}{2r}\,\mathrm{diag}(v_a), \tag{3.2.22}$$

$$\mathcal{E}_i \sim \frac{\hat{\hat{\mathbf{r}}}_i}{r^2}\,\frac{1}{2}\,\mathrm{diag}(e_a), \tag{3.2.23}$$

$$\mathcal{B}_i \sim \frac{\hat{\hat{\mathbf{r}}}_i}{r^2}\,\frac{1}{2}\,\mathrm{diag}(g_a), \tag{3.2.24}$$

with $\hat{\hat{\mathbf{r}}} \equiv \hat{\mathbf{r}}/r$. For these asymptotic behaviors, we have

$$Q_X = 2\pi \sum_{a=1}^{N} e_a x_a, \qquad M_X = 2\pi \sum_{a=1}^{N} g_a x_a, \tag{3.2.25}$$

and similarly for Q_Y and M_Y.

Now, this solution is interpreted as representing a configuration of N D3-branes $a = 1, 2, \ldots, N$ at transverse coordinates (x_a, y_a) from which a string with two-form (NS-NS and R-R) charges (e_a, g_a) are emerging in the direction (u_a, v_a). This is because the eigenvalues

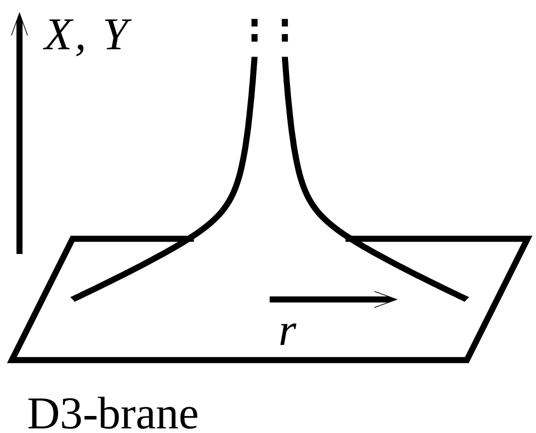

Figure 3.5: “Tube-like” configuration of D3-brane surface representing a string.

of the scalars (X, Y) are interpreted as the transverse coordinates of the D3-branes and the “tube-like” part of the D3-brane surface (corresponding to smaller $\hat{\mathbf{r}}$) can be regarded as a string [32] (see Fig. 3.5). The string directions (u_a, v_a) are not arbitrary but are related to (e_a, g_a) by [133]

$$\begin{pmatrix} u_a \\ v_a \end{pmatrix} = -\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_a \\ g_a \end{pmatrix}, \tag{3.2.26}$$

where θ is the angle given by (3.2.10). Leaving the derivation of Eq. (3.2.26) for our SYM solutions in Sec. 3.3.1, we shall discuss the IIB picture of string networks deduced from (3.2.26).

In the analysis we shall consider tree[11] string networks lying on a flat plane. We can show that, given (e_a, g_a), (x_a, y_a) and (u_a, v_a) $(a = 1, 2, \ldots, N)$ satisfying Eqs. (3.2.10) and (3.2.26), it is possible to draw "generalized" tree string networks on a plane made of 3-string junctions satisfying the following two properties:

(A) The positions of the 3-string junctions in the network is consistently given in such a way that the direction of each component string A is parallel or anti-parallel[12] to the vector (u_A, v_A) which is related to the charge vector (e_A, g_A) by Eq. (3.2.26). Here, the component strings include the N strings emerging directly from the D3-branes as well as the internal strings connecting 3-string junctions, and the charges (e_A, g_A) are uniquely determined by the conservation condition at the junctions.

(B) The sum of the masses of the component strings constituting the network coincides with the BPS bound E_{BPS} (3.2.5) of the energy:

$$E_{\text{BPS}} = \sum_A T_A \ell_A. \tag{3.2.27}$$

Here, $T_A = \pm 2\pi\sqrt{e_A^2 + g_A^2}$ and ℓ_A are the tension and the length of the string A, respectively, and the sign factor of the tension T_A is positive (negative) if the string A is parallel (anti-parallel) to the vector (u_A, v_A).

In appendix A.2, we present a proof of the properties (A) and (B) valid for general N using a technique which reduces the problem to the simple $N = 3$ case. The string networks satisfying (A) and (B) are "generalized" ones in the sense that we are allowing strings with *negative* tension (see Fig. 3.6). As seen from the proof given in appendix A.2, there are in general more than one possible string networks for a given set of (e_a, g_a) and (x_a, g_a). In order for it to be possible to identify a *physical* network consisting only of strings with positive tension, (x_a, y_a) cannot be arbitrary but have to satisfy conditions determined by the charges (e_a, g_a). Such a condition is given by Eq. (3.5.1) for the $N = 3$ case. In Sec. 3.4.2, we analyze our exact SYM solutions satisfying the BPS saturation condition for the $N = 4$ case to find that for every parameter value of the solution there are corresponding physical networks. Though

[11]Here we allow a tree network to contain crossings of the strings. Such string networks with crossings may be regarded as containing loops if we identify the crossing points as four-string junctions. We shall discuss this point later in Sec. 3.4.2.

[12]A string A connecting two 3-string junctions (or a D3-brane and a junction) α and β with coordinates (x_α, y_α) and (x_β, y_β) is parallel (anti-parallel) to the vector (u_A, v_A) defined to be directed from α to β if t_A satisfying $(x_\alpha, y_\alpha) + t_A(u_A, v_A) = (x_\beta, y_\beta)$ is positive (negative).

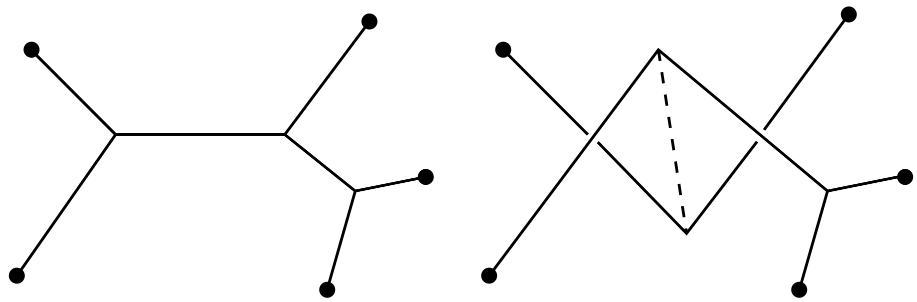

Figure 3.6: String networks on a plane corresponding to the same set of (e_a, g_a) and (x_a, y_a). The blobs represent D3-branes. The network (a) is physical, while (b) is unphysical since it contains a string with negative tension (dashed line).

it is generally expected that, for (x_a, y_a) and (e_a, g_a) associated with a BPS saturated SYM configuration, we can always identify physical networks, we do not yet have a proof of it.

We mentioned at the beginning of this subsection the interpretation of the eigenvalues of the scalars (X, Y) as the coordinates of the strings emerging from the D3-branes. However, as we shall find later, such interpretation does not match the string network picture for finite r since the lines corresponding to the eigenvalues of the scalars are generally *curved*. Moreover, multiple 3-string junctions in the networks with $N \geq 4$ is impossible in this interpretation due to the analyticity of the scalars as functions of the four dimensional coordinates of the SYM.

3.3 Explicit construction of the 1/4 BPS dyon

3.3.1 Spherical symmetry and monopole solutions

Let us prepare a framework for constructing the BPS saturated SYM configurations. Putting $\theta = 0$ as explained before, the equations to be solved are

$$D_i X = \mathcal{E}_i, \tag{3.3.1}$$

$$D_i Y = \mathcal{B}_i, \tag{3.3.2}$$

as well as (3.2.8), (3.2.9) and (3.2.11). Our strategy for the construction of the solutions is the same as in Ref. [72] for the $SU(3)$ case. First, we prepare an $SU(N)$ monopole solution $(A_i(\hat{\mathbf{r}}), Y(\hat{\mathbf{r}}))$ to Eq. (3.3.2). Then, Eq. (3.3.1) is automatically satisfied by putting $A_0(\hat{\mathbf{r}}) =$

$-X(\hat{\mathbf{r}})$, while Eq. (3.2.8) holds due to Eq. (3.2.9) and the time-independence of our solution. Therefore, we have only to solve Eq. (3.2.11), i.e.,

$$D_i D_i X = 0, \tag{3.3.3}$$

under the D-flatness condition (3.2.9). As we did in Ref. [72], we adopt as the monopole solutions to Eq. (3.3.2) those given in Ref. [150] constructed using a general formalism for spherically symmetric solutions of Ref. [151]. In the following, we shall present our problem in the formalism of Ref. [151] and explain the monopole solutions of Ref. [150] (see also Ref. [17]).

Our solutions (A_i, X, Y) are assumed to be spherically symmetric with respect to an angular momentum operator $\boldsymbol{J}$, i.e., they satisfy

$$[J_i, A_j] = i\epsilon_{ijk} A_k, \qquad [J_i, X] = [J_i, Y] = 0, \tag{3.3.4}$$

The present $\boldsymbol{J}$ is the sum of the space and the gauge group rotations:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{T}, \tag{3.3.5}$$

where $\boldsymbol{L} = -i\hat{\mathbf{r}} \times \boldsymbol{\nabla}$ is the generator of the space rotation and $\boldsymbol{T}$ is the maximal $SU(2)$ embedding in $SU(N)$ with $T_3 = \frac{1}{2}\,\mathrm{diag}\,(N-1, N-3, \cdots, -N+1)$. The monopole solution of [150] assumes the following form for the vector potential:

$$\boldsymbol{A}(\hat{\mathbf{r}}) = \Big(\boldsymbol{M}(r, \hat{\mathbf{r}}) - \boldsymbol{T}\Big) \times \hat{\mathbf{r}}/r, \tag{3.3.6}$$

where the Lie algebra valued function M_i should satisfy the spherical symmetry condition,

$$[J_i, M_j] = i\epsilon_{ijk} M_k. \tag{3.3.7}$$

Various formulas are derived by using the expression $\boldsymbol{\nabla} = \hat{\mathbf{r}}\partial/\partial r - (i/r)\hat{\mathbf{r}} \times \boldsymbol{L}$ for the space derivative as well as the spherical symmetry properties, Eqs. (3.3.4) and (3.3.7). We need in particular the following three:

$$\boldsymbol{D}Y = \hat{\mathbf{r}}Y' + \frac{i}{r}\hat{\mathbf{r}} \times [\boldsymbol{M}, Y], \tag{3.3.8}$$

$$\mathcal{B}_i = -\frac{i}{r^2}\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\left(\frac{1}{2}\epsilon_{jk\ell}[M_k, M_\ell] - iT_j\right) - \frac{1}{r}\left(\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \boldsymbol{M}'\right)\right)_i, \tag{3.3.9}$$

$$D_i D_i X = X'' + \frac{2}{r}X' - \frac{1}{r^2}\left(\delta_{ij} - \hat{\mathbf{r}}_i\hat{\mathbf{r}}_j\right)[M_i, [M_j, X]], \tag{3.3.10}$$

where the prime denotes the differentiation $\partial/\partial r$.

Due to the spherical symmetry it is sufficient to construct solutions on the positive z-axis. The monopole equation (3.3.2) and the Gauss law constraint (3.3.3) on the z-axis are reduced to

$$r^2 Y' = \frac{1}{2}\left[M_+, M_-\right] - T_3, \tag{3.3.11}$$

$$M'_\pm = \mp \left[M_\pm, Y\right], \tag{3.3.12}$$

$$X'' + \frac{2}{r}X' - \frac{1}{2r^2}\Big([M_+,[M_-,X]] + [M_-,[M_+,X]]\Big) = 0, \tag{3.3.13}$$

with $M_\pm \equiv M_1 \pm iM_2$. As seen from Eqs. (3.3.1), (3.3.2) and (3.3.8), the z-component of the electric and magnetic fields on the positive z-axis are equal to the derivatives of the scalars:

$$\mathcal{E}_z = X', \qquad \mathcal{B}_z = Y'. \tag{3.3.14}$$

The matrix-valued functions $M_\pm(r)$ and $Y(r)$ of Ref. [150] are given on the positive z-axis as

$$(M_+)_{mn} = \delta_{m,n-1}\, a_m(r), \qquad M_- = M_+^T, \tag{3.3.15}$$

$$Y = \mathrm{diag}\left(Y_m(r)\right) = \frac{1}{2r}\,\mathrm{diag}\left(\Psi_m(r) - \Psi_{m-1}(r)\right), \quad \Psi_0 = \Psi_N = 0, \tag{3.3.16}$$

using a_m and Ψ_m, which are further expressed in terms of $Q_m(r)$ as

$$a_m(r) = \frac{r}{Q_m}\left(m\overline{m}Q_{m-1}Q_{m+1}\right)^{1/2}, \tag{3.3.17}$$

$$\Psi_m(r) = -r\left(\ln Q_m\right)' + m\overline{m} \qquad (\overline{m} \equiv N - m), \tag{3.3.18}$$

with $Q_0 = Q_N = 1$. The functions a_m and $M_\pm$ satisfy Eqs. (3.3.11) and (3.3.12) if $Q_m(r)$ is a solution to

$$(Q'_m)^2 - Q_m Q''_m = m\overline{m}\, Q_{m+1}Q_{m-1}, \tag{3.3.19}$$

for $m = 1, 2, \ldots, N-1$. In Ref. [150] they found the following Q_m satisfying (3.3.19):

$$Q_m = \gamma_m \sum_{D_m} \prod_{a\in D_m} e^{-2y_a r} \prod_{b\in \overline{D}_m} \left(2y_b - 2y_a\right)^{-1}, \tag{3.3.20}$$

where γ_m is given by

$$\gamma_m = \frac{\prod_{n=1}^{N-1} n!}{\prod_{k=1}^{m-1} k! \cdot \prod_{l=1}^{\overline{m}-1} l!}, \tag{3.3.21}$$

and the sum in (3.3.20) is over the $\binom{N}{m}$ distinct ways of dividing the integers $\{1, 2, \ldots, N\}$ into two groups, D_m with m elements and $\overline{D}_m$ with $\overline{m}$ elements.

In (3.3.20), y_a $(a = 1, 2, \ldots, N)$ are arbitrary parameters satisfying $\sum_{a=1}^{N} y_a = 0$. They are nothing but y_a appearing in the asymptotic expression (3.2.22) of the scalar Y if the condition $y_1 < y_2 < \cdots < y_N$ is satisfied. This is seen from the asymptotic behavior of $\ln Q_m$,

$$\ln Q_m = -2 \sum_{a=1}^{m} y_a \, r + \text{const.} + O\left(e^{-2(y_m - y_{m-1})r}\right) \quad (r \to \infty), \tag{3.3.22}$$

and Eq. (3.3.18). Using (3.3.14), we find that the asymptotic form of the magnetic field is $\mathcal{B}_z \sim -T_3/r^2$ and hence the magnetic charges g_a of the present solution are given by

$$(g_a) = (-N+1, -N+3, \ldots, N-1)\,. \tag{3.3.23}$$

On the other hand, the behavior of $Q_m(r)$ of (3.3.20) near the origin $r = 0$ is

$$Q_m = r^{m\overline{m}} \left(1 + q_m r^2 + O(r^3)\right) \quad (r \sim 0), \tag{3.3.24}$$

where the coefficient q_m of the sub-leading term is given by[13]

$$q_m = \frac{2m\overline{m}}{N(N^2-1)} \sum_{a=1}^{N} y_a^2. \tag{3.3.25}$$

The leading behavior $r^{m\overline{m}}$ of Q_m is consistent with the regularity of the solution at $r = 0$.

Since we have adopted the diagonal form (3.3.16) for the scalar Y, the D-flatness condition (3.2.9) implies that the other scalar X is also diagonal. The diagonal form of X is also a consequence of the spherical symmetry $[J_3, X] = 0$ (3.3.4) implying $[T_3, X] = 0$ on the z-axis. Therefore, we express X in terms of $N-1$ functions Φ_m $(m = 1, 2, \ldots, N-1)$ as

$$X = \operatorname{diag}(X_m) \equiv \frac{1}{2r} \operatorname{diag}(\Phi_m - \Phi_{m-1})\,, \qquad \Phi_0 = \Phi_N = 0. \tag{3.3.26}$$

Then, Eq. (3.3.13) for X is reduced to

$$\Phi_m'' - \frac{a_m^2}{r^2}(2\Phi_m - \Phi_{m+1} - \Phi_{m-1}) = 0, \qquad m = 1, \ldots, N-1. \tag{3.3.27}$$

This differential equation can be solved numerically and for a certain special values of (y_a) analytically. In the next subsection, we shall present an exact solution.

Before closing this subsection, we derive the relation (3.2.26) for our SYM solutions. This is an immediate consequence of (3.3.14) for $\mathcal{E}_z$ and $\mathcal{B}_z$ and the asymptotic expressions (3.2.21)

[13] One way to derive Eq. (3.3.25) is as follows. Eq. (3.3.19) implies that q_m satisfies the recursion relation $(2m\overline{m} - 2)q_m = m\overline{m}(q_{m+1} - q_{m-1})$, whose solution is given by $q_m \propto m\overline{m}$ (incidentally, this recursion equation is the $p = 1$ case of Eq. (3.3.31), which is solved in App. A.3). Taking into account the initial condition $q_1 = \frac{2}{N(N+1)} \sum_{a=1}^{N} y_a^2$ obtained from (3.3.20), we get (3.3.25).

– (3.2.24). They lead to Eq. (3.2.26) with $\theta = 0$, $(u_a, v_a) = -(e_a, g_a)$. (Note that, since $M_\pm(r)$ decays exponentially as $r \to \infty$ as seen from Eqs. (3.3.17) and (3.3.22), so do the other components $\mathcal{E}_{x,y}$ and $\mathcal{B}_{x,y}$.) Eq. (3.2.26) in the case of non-vanishing θ is obtained by a rotation of (X, Y) and hence (u_a, v_a).

3.3.2 Exact solutions

It is possible to construct an exact solution to the equations (3.3.27) for the following special values of (y_a):

$$(y_1, y_2, \ldots, y_N) = C\,(-N+1, -N+3, \ldots, N-1)\,, \tag{3.3.28}$$

where C is a positive constant. For this (y_a), Q_m is given explicitly by[14]

$$Q_m = \left(\frac{\sinh Cr}{C}\right)^{m\overline{m}}, \tag{3.3.29}$$

and Eq. (3.3.27) becomes

$$\Phi_m'' - \frac{m\overline{m}}{\sinh^2 r}\left(2\Phi_m - \Phi_{m+1} - \Phi_{m-1}\right) = 0, \qquad m = 1, \cdots, N-1, \tag{3.3.30}$$

where we have put $C = 1$ since C can be absorbed into the rescaling of r.

To rewrite the differential equation (3.3.30) into a diagonal form, let us consider the eigenvalue problem for the non-differential part of (3.3.30):

$$m\overline{m}\left(2v_m^{(p)} - v_{m+1}^{(p)} - v_{m-1}^{(p)}\right) = p(p+1)v_m^{(p)}, \qquad m = 1, \cdots, N-1, \tag{3.3.31}$$

where the eigenvalue is expressed as $p(p+1)$, and we have $v_0^{(p)} = v_N^{(p)} = 0$. The solution to Eq. (3.3.31) is given in App. A.3. There we show that p characterizing the eigenvalue takes the integer values; $p = 1, 2, \ldots, N-1$. The explicit form of the eigenvectors $v_m^{(p)}$ is also given there. Then, by expressing Φ_m as

$$\Phi_m(r) = \sum_{p=1}^{N-1} v_m^{(p)} \varphi^{(p)}(r), \tag{3.3.32}$$

the differential equations (3.3.30) are reduced to diagonal ones for $\varphi^{(p)}$:

$$\varphi^{(p)}(r)'' - \frac{p(p+1)}{\sinh^2 r}\varphi^{(p)}(r) = 0. \qquad p = 1, \cdots, N-1, \tag{3.3.33}$$

[14] The easiest way to derive (3.3.29) is to first obtain Q_1 from Eq. (3.3.20) and then use the recursion relation (3.3.19) to get Q_m with higher m.

In order to solve (3.3.33), we make a change of variables from r to $y \equiv e^{2r}$, and consider $\widetilde{\varphi}^{(p)}(y) \equiv (y-1)^p \varphi^{(p)}$ instead of $\varphi^{(p)}$. Eq. (3.3.33) is then transformed into a Gauss hypergeometric differential equation:

$$y(1-y)\frac{d^2}{dy^2}\widetilde{\varphi}^{(p)} + [1-(1-2p)y]\,\frac{d}{dy}\widetilde{\varphi}^{(p)} - p^2\widetilde{\varphi}^{(p)} = 0. \tag{3.3.34}$$

The general solution to (3.3.33) is therefore given as a linear combination of two independent ones:

$$\varphi^{(p)} = \beta_p\,(y-1)^{-p}\Big(F(-p,-p,1;y) + c_p\,[F(-p,-p,1;y)\ln y + F^*(-p,-p,1;y)]\Big) \tag{3.3.35}$$

where[15]

$$F(-p,-p,1;y) = \sum_{k=0}^{p}\binom{p}{k}^2 y^k, \tag{3.3.36}$$

$$F^*(-p,-p,1;y) = -2\sum_{k=1}^{p}\binom{p}{k}^2\sum_{l=1}^{k}\left(\frac{1}{p-l+1}+\frac{1}{l}\right)y^k, \tag{3.3.37}$$

and β_p and c_p are constants. For our present purpose, c_p must be chosen in such a way that $\varphi^{(p)}$ of (3.3.35) is non-singular at $y=1$ ($r=0$). Such c_p is given by

$$c_p = \left(2\sum_{l=1}^{p}\frac{1}{l}\right)^{-1}. \tag{3.3.38}$$

The derivation of (3.3.38) is given in appendix A.4. There we obtain the solution using the analytic continuation of the hypergeometric functions. For example, $\varphi^{(p)}$ with $p=1,2$ and 3 are[16]

$$\begin{aligned}
\varphi^{(1)} &= \beta_1\,(r\coth r - 1)\,,\\
\varphi^{(2)} &= -\beta_2\left(\coth r - r\frac{2\cosh^2 r + 1}{3\sinh^2 r}\right),\\
\varphi^{(3)} &= \beta_3\left(-1 - \frac{15}{11\sinh^2 r} + r\frac{3\cosh r(2\cosh^2 r + 3)}{11\sinh^3 r}\right).
\end{aligned} \tag{3.3.39}$$

As for the other scalar Y for the present (y_a) of (3.3.28), Ψ_m of (3.3.16) is given using (3.3.18) and (3.3.29) by

$$\Psi_m = -m\overline{m}\,(r\coth r - 1)\,. \tag{3.3.40}$$

Since $v_m^{(1)} \propto m\overline{m}$, this is the $p=1$ term of (3.3.32).

[15] $F^*(\alpha,\beta,\gamma;y) \equiv (\partial/\partial\alpha + \partial/\partial\beta + 2\partial/\partial\gamma)\,F(\alpha,\beta,\gamma;y)$.

[16] $\varphi^{(1)}$ and $\varphi^{(2)}$ were written as φ_+ and φ_- for the $N=3$ case in our paper [72].

3.4 Behavior of the solutions

In this section, we shall discuss various aspects of the solutions: explicit analysis of the exact solutions for the cases $N = 3$ (Sec. 3.4.1) and $N = 4, 5$ (Sec. 3.4.2), the behavior of the solutions near the origin for general N (Sec. 3.4.3), the concept of the effective charges (Sec. 3.4.4), and new solutions for the cases where some of y_a are degenerate (Sec. 3.4.5).

3.4.1 Exact solutions for $SU(3)$

Following the procedure of obtaining exact solutions in the previous subsection, let us analyse the properties of the exact solutions for $SU(3)$. According to Eq. (3.3.28), we choose the special values of (y_a):

$$(y_1, y_2, y_3) = (-C, 0, C). \tag{3.4.1}$$

Then using eigenvectors $v^{(1)} = (2, 2)$ and $v^{(2)} = (-2, 2)$ and the explicit solutions $\varphi^{(1)}$ and $\varphi^{(2)}$ in Eq. (3.3.39), we obtain an exact solution. From the solution, we can read off the following values for the locations of the D3-branes and the electric and magnetic charges of the strings:

$$(x_a, y_a) = \left\{ \left(\beta_1 - \frac{2}{3}\beta_2,\ -1\right), \quad \left(\frac{4}{3}\beta_2,\ 0\right), \quad \left(-\beta_1 \frac{2}{3}\beta_2,\ 1\right) \right\}, \tag{3.4.2}$$

$$(e_a, g_a) = \left\{ (2\beta_1 - 2\beta_2,\ -2), \quad \left(4\beta_2,\ 0\right), \quad (-2\beta_1 - 2\beta_2,\ 2) \right\}. \tag{3.4.3}$$

In particular, for $(\beta_1, \beta_2) = (-1/4, 1/4)$, the three charges are $(-1, -2)$, $(1, 0)$ and $(0, 2)$. In Fig. 3.7, we plot the trajectories of the D3-brane coordinates in course of changing r by the solid lines. In course of decreasing r, the D3-branes approach the origin of (X, Y) and meet there at $r = 0$, where the gauge symmetry is restored. The trajectory of the brane D_2 is just a straight line. This is a general feature of our exact solutions and comes from the boundary condition $y_2 = 0$. We obtain a bending trajectory of D_2 for a general case of $y_2 \neq 0$, in which we solved the equations only numerically.

The branes D_1 and D_3 connect smoothly to each other at the origin. The behavior of the solutions near the origin will be discussed in Sec. 3.4.3. Noticing the fact that the brane D_2 has no magnetic charge[17] while the branes D_1 and D_3 have non-zero magnetic charges, this might come from our technical preference that we describe the BPS states by the classical treatment of the SYM theory, *i.e.* electrically. Then the interpretation of the trajectories

[17] We mean the charge (e_a, g_a) by the electric and magnetic charges of the brane D_a.

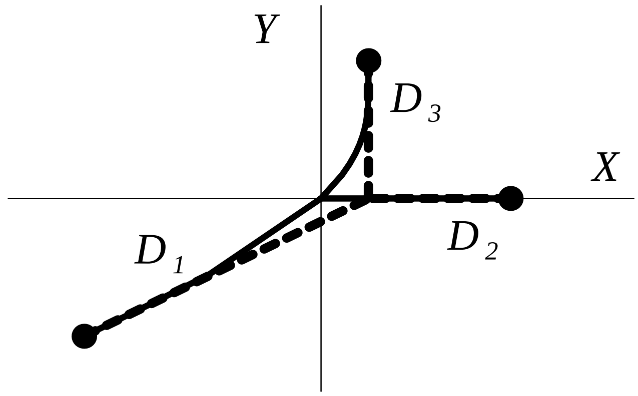

Figure 3.7: The D3-brane configurations (solid lines) and the three-pronged strings from the IIB picture (dashed lines).

might be that the branes D_1 and D_3 are the two parts of one very heavy smooth magnetic object pulled and bent by the light electric brane D_2.

In Fig. 3.7, we also draw three dashed straight lines tangent to the D3-brane trajectories at $r = \infty$. They meet at one point and their forces balance. This would be a non-trivial consistency check of our approach. As discussed in Sec. 3.2.4, this configuration agrees with the three-pronged strings in the IIB picture. On the other hand, the behavior of our solution at finite r is quite different from the above IIB picture. The bending of the trajectories of the D3-branes will be analyzed in Sec. 3.4.3.

The obtained BPS saturated spherically symmetric regular configurations in $\mathcal{N} = 4$ $SU(3)$ SYM theory are carrying non-parallel electric and magnetic charges $(a, 2)$, $(b, 0)$ and $(-a - b, -2)$, where a and b take arbitrary real values. Assuming the quantization of the electric charges a and b and the $SL(2, Z)$ duality symmetry, our solutions imply, in general, the existence of the junctions of the three IIB strings carrying the two-form charges (p, q), (lr, ls) and $(-p - lr, -q - ls)$, respectively, where l, p, q, r, s are integers satisfying $ps - qr = 2$.

In addition to the bending of the solutions, another difference between our solutions and the IIB picture is the number of degrees of freedom. Now let us count the number of degrees of freedom of a 3-string junction in the string picture. As for the charges, the three magnetic charges are fixed to $(-2, 0, 2)$ from the beginning, but we have the freedom of two electric charges (the other electric charge is determined by the charge conservation). Then, since the string tensions are determined by the charges, the relative directions of the three strings are

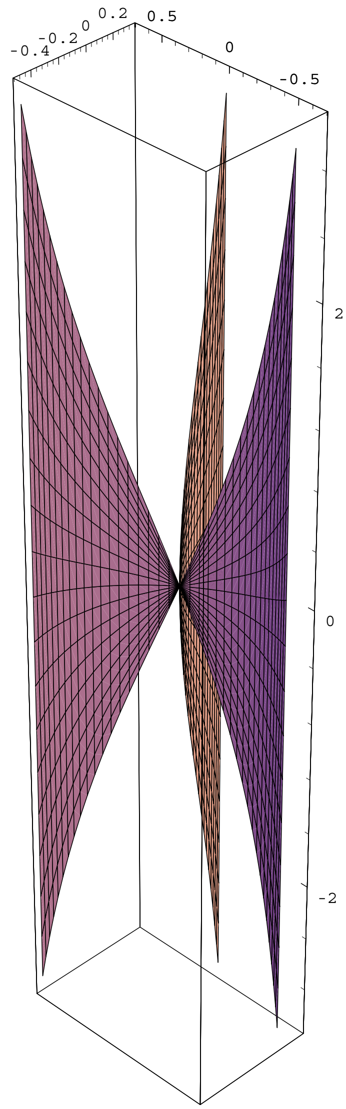

Figure 3.8: The deformation of the D3-brane surface drawn by the eigenvalues of the scalar fields. The vertical direction is along the D3-brane worldvolume, and the horizontal directions are spanned by X and Y.

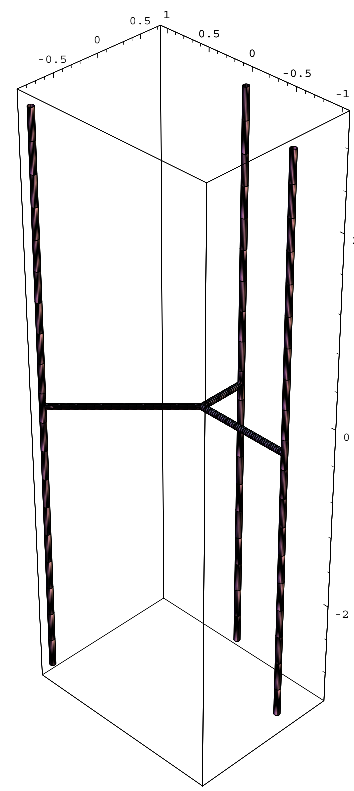

Figure 3.9: Corresponding brane configuration in string theory.

determined by the force balance condition. We have the freedom to choose the lengths of each string. We should not count the freedom of rotating or shifting parallelly the 3-string junction in the two-dimensional plane, because we fixed these degrees of freedom in solving the equations. We absorbed the angle θ in Sec. 3.3.1, and the center of the three D3-branes are fixed by the tracelessness of the adjoint scalar fields. Thus we have in total five degrees of freedom in the string picture. However, the number of degrees of freedom of our solutions is four, consisting of $y_1, y_2, \beta_1, \beta_2$. This does not agree with five[18].

This difference might again come from the limitation of the electric description of the 3-string junction. Since electrically charged objects are the fundamental degrees of freedom themselves in the electric description, the introduction of a bare electric charge would necessarily cause the problem of singularities in the solutions. Nevertheless, in our solutions, we have one D3-brane (D_2 in Fig. 3.7) which is charged only electrically. However, we may have the possibility that, if we had the freedom to introduce a bare electric charge, the number of the degrees of freedom would increase from four to five.

In fact, the difficulty of a bare electric charge appears in a different way in our construction of the solutions. We took the maximal embedding of $SU(2)$ to $SU(3)$ in the analysis. Because of this, the magnetic charge is 2 and not 1. Taking the minimal embedding of $SU(2)$ does not work. It turns out that, to obtain non-parallel electric and magnetic charges, we have a singularity at the origin, which is the bare source of the electric field.

We tried another way to introduce a unit magnetic charge. In the degenerate case of $y_1 = y_2$, the magnetic charges of the D3-branes become $(1, 1, -2)$ [17, 150]. But in this case the asymptotic behavior at $r \sim \infty$ of the differential equation for X changes from the non-degenerate cases, and the solutions regular at $r = 0$ diverge at $r = \infty$ ($\lim_{r\to\infty} \phi_1(r) = \infty$) except the case of parallel electric and magnetic charges. Thus we could not introduce one unit of magnetic charge in our solutions[19]. The degenerate case for $N > 3$ will be studied in Sec. 3.4.5.

3.4.2 Exact solutions for $SU(4)$ and $SU(5)$

In this subsection, we shall study the exact solutions presented in the Sec. 3.3.2 in more detail for the cases $N = 4$ and 5. First, let us consider the $SU(4)$ case. Since the eigenvectors $v_m^{(1,2,3)}$

[18] See Sec. 3.6 for the discussion for general $SU(N)$.

[19] Without the spherical symmetry ansatz, it is possible to obtain the unit magnetic charge [91].

(with appropriate normalization) read (see App. A.3)

$$\vec{v}^{(1)} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}, \qquad \vec{v}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{v}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \tag{3.4.4}$$

the X-coordinates of the four D3-branes are given by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \frac{1}{2r} \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & -3 \\ -1 & -1 & 3 \\ -3 & 1 & -1 \end{pmatrix} \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \\ \varphi^{(3)} \end{pmatrix}, \tag{3.4.5}$$

in terms of $\varphi^{(1,2,3)}$ of Eq. (3.3.39). The charges (e_a, g_a) and the D3-brane coordinates at infinity (x_a, y_a) determined from the asymptotic form of the present exact solution are

$$\begin{aligned} (e_1, g_1) &= (3\beta_1 + \beta_2 + \beta_3\ ,\ -3), \quad (e_2, g_2) = (\beta_1 - \beta_2 - 3\beta_3\ ,\ -1), \\ (e_3, g_3) &= (-\beta_1 - \beta_2 + 3\beta_3\ ,\ 1), \quad (e_4, g_4) = (-3\beta_1 + \beta_2 - \beta_3\ ,\ 3), \end{aligned} \tag{3.4.6}$$

and

$$\begin{aligned} (x_1, y_1) &= \left(\frac{3}{2}\beta_1 + \frac{1}{3}\beta_2 + \frac{3}{11}\beta_3, \frac{-3}{2}\right), \quad (x_2, y_2) = \left(\frac{1}{2}\beta_1 - \frac{1}{3}\beta_2 - \frac{9}{11}\beta_3, \frac{-1}{2}\right), \\ (x_3, y_3) &= \left(-\frac{1}{2}\beta_1 - \frac{1}{3}\beta_2 + \frac{9}{11}\beta_3, \frac{1}{2}\right), \quad (x_4, y_4) = \left(-\frac{3}{2}\beta_1 + \frac{1}{3}\beta_2 - \frac{3}{11}\beta_3, \frac{3}{2}\right). \end{aligned} \tag{3.4.7}$$

For $(\beta_1, \beta_2, \beta_3) = (-\beta, 0, 0)$, we have $X_a = \beta Y_a$. In this special case, all the D3-branes and strings are located on a straight line which passes the origin. The charges (e_a, g_a) are parallel to each other and the configuration preserves 1/2 supersymmetries.

In Fig. 3.10, an example of the typical configurations is presented with solid curves. Notice that the four D3-brane surfaces bend and have a common tangent at the origin. We have chosen the parameters β_p in order for the electric charges to take integer values.

As explained in Sec. 3.2.4, we can find the configuration of a four-pronged string in the string picture from the asymptotic behavior of the solution. In the case of four D3-branes, there are three different ways to connect the D3-branes by a four-pronged string (see Fig. 3.11). Among them, the physical configurations are chosen by the condition that the tensions of all the strings should be positive. The four-pronged string in Fig. 3.10 (dotted lines) corresponds to the case (a) of Fig. 3.11. In general, the correspondence between the parameters β_p and the type of the four-pronged string is as follows:

$$\text{case (a)} \quad \Rightarrow \quad |\beta_2| > 5\sqrt{\frac{3}{11}}|\beta_3| \quad \text{or} \quad |\beta_2| < \frac{25}{11}|\beta_3|,$$

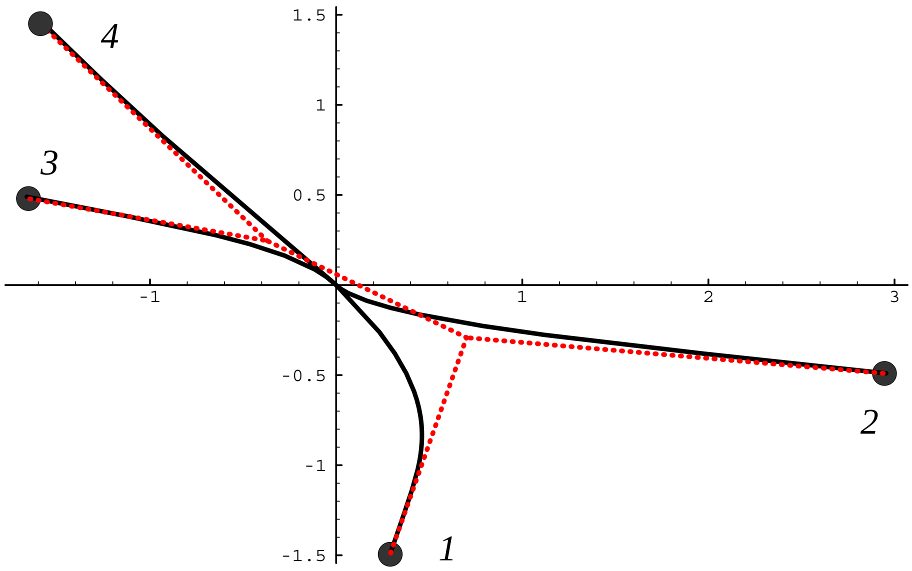

Figure 3.10: The trajectories of (X_a, Y_a) of the $N = 4$ exact solution (solid lines). The parameters are chosen as $(\beta_1, \beta_2, \beta_3) = (1, -2, -2)$, which give the electric and magnetic charges $(e_a, g_a) = (-1, -3), (9, -1), (-5, 1), (-3, 3)$. The dotted lines describe the corresponding four-pronged string in the string picture.

$$\begin{aligned} \text{case (b)} \quad &\Rightarrow \quad 5\sqrt{\frac{3}{11}}|\beta_3| < |\beta_2| < \frac{45}{11}|\beta_3|, \\ \text{case (c)} \quad &\Rightarrow \quad \frac{15}{22}|\beta_3| < |\beta_2| < 5\sqrt{\frac{3}{11}}|\beta_3|. \end{aligned} \tag{3.4.8}$$

Note that the whole parameter space of β_p is covered, namely, we can identify at least one physical four-pronged string configurations for every value of β_p. When $|\beta_2| = 5\sqrt{\frac{3}{11}}|\beta_3|$ holds, the three cases degenerate. There is no internal string, and the four strings meet at a point, which is not generally at the origin.

An interesting fact is that in the regions

$$5\sqrt{\frac{3}{11}}|\beta_3| < |\beta_2| < \frac{45}{11}|\beta_3|, \qquad \frac{15}{22}|\beta_3| < |\beta_2| < \frac{25}{11}|\beta_3|, \tag{3.4.9}$$

two cases satisfy the positive tension condition. In the former region, the condition is satisfied by (a) and (b), and in the latter by (a) and (c). In fact, for the trajectories of (X_a, Y_a) described in Fig. 3.10 (where the four-pronged strings configuration corresponding to the case (a) is drawn by dotted lines), there is another string configuration corresponding to the case (c) shown in Fig. 3.12. The string configuration of Fig. 3.12 has a crossing of strings and hence

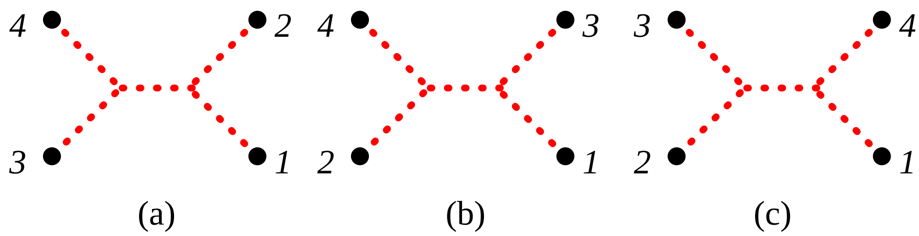

Figure 3.11: Three different ways of connecting the four D3-branes.

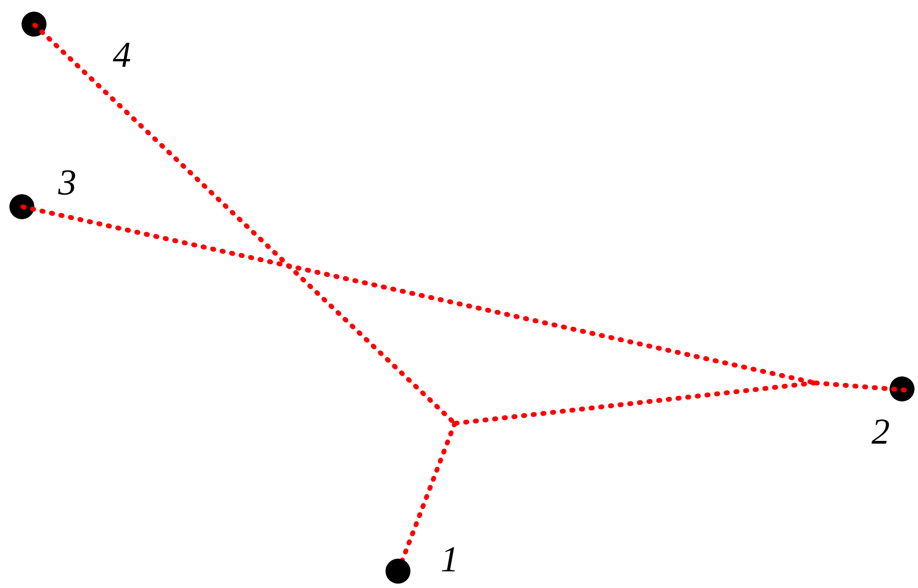

Figure 3.12: Another four-pronged string configuration corresponding to the trajectories (solid lines) of Fig. 3.10.

contains a loop in it if we regard the crossing point as a four-string junction. It is generally the case that, in the regions of (3.4.9), one of the two string configurations contains a string crossing and the other does not. Therefore, if we discard the string configuration with a crossing by classifying it as a loop configuration, then we recover the one-to-one correspondence between the BPS states in the string theory and those in the field theory.

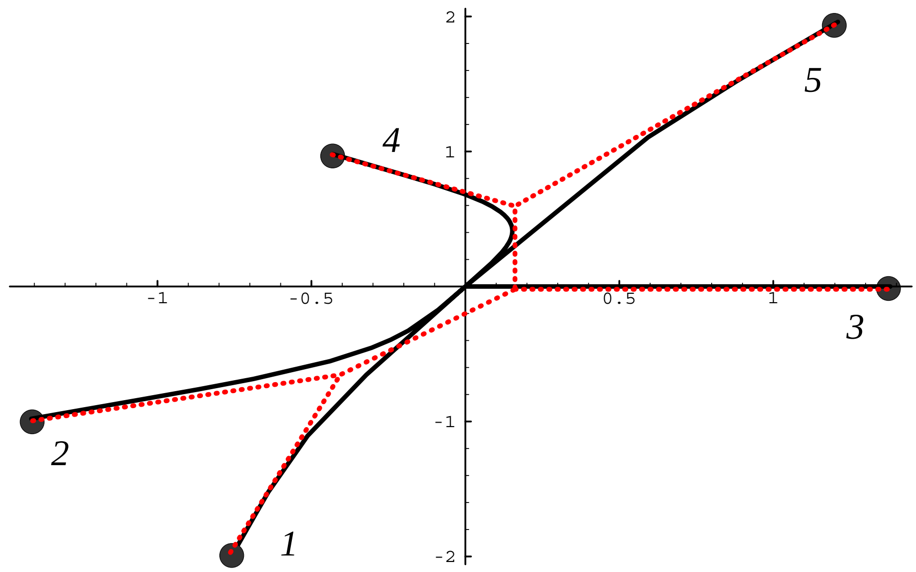

Figure 3.13: The trajectories of the five D3-brane coordinates (X_a, Y_a) of the exact solution (solid lines), with charges $(e_a, g_a) = (-1, -4)$, $(-5, -2)$, $(6, 0)$, $(-3, 2)$, and $(3, 4)$. The corresponding five-pronged string in the string picture is represented by dotted lines.

The analysis of the exact solution for $SU(5)$ can be carried out in the same manner. In Fig. 3.13, we present the trajectories of the five D3-brane surfaces (X_a, Y_a) and the corresponding five-pronged string configuration for a certain β_p. Notice that the D3-brane surfaces 1, 2, 4 and 5 have a common tangent at the origin $(X, Y) = (0, 0)$, while the surface 3 sticks to the others.

3.4.3 D3-brane surfaces near the origin

Since the two scalars X and Y must vanish simultaneously at $r = 0$ due to the regularity of the solution, all the N D3-brane surfaces, which are described by the eigenvalues of the scalars, meet at the origin $(X, Y) = (0, 0)$ of the transverse plane. In this subsection we shall study how the D3-brane surfaces meet at the origin for general N.

For this purpose, let us consider the behaviors near the origin of the scalars X and Y in our exact solution of Sec. 3.3.2. Rather than expanding the expression (3.3.35) with respect to r, it is easier to return to the differential equation (3.3.33) to know the behavior of $\varphi^{(p)}$ near $r = 0$. In fact, Eq. (3.3.33) is approximated near $r = 0$ as

$$\varphi^{(p)}(r)'' - \frac{p(p+1)}{r^2}\varphi^{(p)}(r) = 0 \qquad (r \sim 0), \tag{3.4.10}$$

and the solution regular at $r = 0$ should behave as $\varphi^{(p)} \sim r^{p+1}$. Therefore, the leading contribution to Φ_m (3.3.32) comes from the $p = 1$ term. This fact, together with the expression (A.3.6) for $v_m^{(1)}$ implies that $\Phi_m(r) \sim 2\alpha_X m\overline{m}r^2$ (α_X is an m-independent constant) and hence that the leading behavior of the eigenvalue of the scalar X is given by

$$X_m(r) \sim \alpha_X(N - 2m + 1)r. \tag{3.4.11}$$

The behavior of the other scalar Y should be the same as (3.4.11) except the constant α_X since Y is also a solution to the differential equation (3.3.27). Therefore, we get

$$(X_m, Y_m) \sim (N - 2m + 1)\,(\alpha_X, \alpha_Y)\,r \qquad (r \sim 0). \tag{3.4.12}$$

The concrete expression of the constant α_Y is obtained from Eqs. (3.3.18) and the q_m term of (3.3.24):

$$\alpha_Y = -\frac{2}{N(N^2-1)}\sum_a y_a^2. \tag{3.4.13}$$

From Eq. (3.4.12) we can deduce the followings. The shape of the junction of N D3-brane surfaces at the origin $(X, Y) = (0, 0)$ of the transverse plane differs depending on whether N is even or odd. When N is even, all the N D3-brane surfaces have a common tangent at the origin: half of the D3-brane surfaces X_m ($1 \le m \le N/2$) are "smoothly" connected to the other half ($N/2+1 \le m \le N$) of the surfaces at the origin $(X, Y) = (0, 0)$. On the other hand, when N is odd, the above picture is true except for the D3-brane surface with $m = (N+1)/2$ since, for this particular m, (3.4.12) vanishes and the leading behavior of $\left(X_{(N+1)/2}, Y_{(N+1)/2}\right)$ is given by the next $O(r^2)$ term which is generically not parallel to (α_X, α_Y). Therefore, among N D3-brane surfaces, $N-1$ have a common tangent at the origin, while the remaining one meets with the others with an angle. We saw in Sec. 3.4.2 that the above behaviors are actually realized in the cases with $N = 3$, 4 and 5 (see Figs. 3.7, 3.10 and 3.13).

The above analysis about the behavior of the scalars near $r = 0$ was concerning the exact solution of Sec. 3.3.2 for X. To obtain the behavior of a general solution corresponding to (y_a)

other than (3.3.28), we have to analyze the differential equation (3.3.27) near $r = 0$. Naive substitution of the leading expression of a_m^2, $a_{\overline{m}}^2 \sim m\overline{m}$, derived from (3.3.24) into (3.3.27) gives Eq. (3.3.30) with $\sinh^2 r$ replaced by r^2. Therefore, the same expansion as (3.3.32) leads to (3.4.10) and one would conclude that the above behaviors of the scalars near the origin are not restricted to the exact solutions. Though we have confirmed that this naive analysis is valid for $N = 3$, 4 and 5, it must be checked that the other $\varphi^{(q)}$ with $q \neq p$ enters in a harmless way in the apparently higher order terms of Eq. (3.4.10).

Finally in this subsection, we shall give two comments. Our first comment is concerning the behaviors of the scalars near the origin when N is odd: one D3-brane (X_m, Y_m) with $m = (N+1)/2$ stick to the rest $N-1$ branes with an angle when N is odd. Noticing that the R-R charge g_m vanishes only for this particular m (c.f. (3.3.23)), this phenomenon would be ascribed to the fact that we are describing the BPS states by the classical treatment of the SYM theory (i.e. electrically) as we mentioned in Sec. 3.4.1.

Secondly, the fact that all the D3-brane surfaces have a common tangent at the origin when N is even means that all of them cannot be globally straight in the transverse plane. Otherwise, the charges (e_a, g_a) are all parallel. This is the case also when N is odd as is seen from the force balance condition.

3.4.4 Effective charges

From the discussions in the previous subsection, the solutions with non-parallel electric and magnetic charges are necessarily described by curved lines. As discussed in Sec. 3.2.4 and appendix A.2, the asymptotic behavior ($r \to \infty$) of the curved lines reproduces the multi-pronged string configuration in the string picture. In this subsection, we shall point out that this feature can be extended to any finite r.

The basic equations used in the proof of Appendix A.2 are[20]

$$\sum_{a=1}^{N} (u_a y_a - v_a x_a) = 0, \tag{3.4.14}$$

$$(u_a, v_a) = -(e_a, g_a), \tag{3.4.15}$$

$$U = |Q_X + M_Y| = 2\pi \left| \sum_{a=1}^{N} e_a x_a + \sum_{a=1}^{N} g_a y_a \right| . \tag{3.4.16}$$

[20] We take $\theta = 0$ for simplicity without losing generality.

Thus if we can define quantities at finite r which satisfy the equations similar to (3.4.14), (3.4.15) and (3.4.16), then one can conclude the same results as in appendix A.2, that is, we can form a multi-pronged string in the string picture using the tangent vectors at r, and the configuration has the satisfactory properties concerning the tensions of the strings, the BPS bound, the formation of a planer tree junction diagram, etc.

The natural generalization of x_a and y_a would be just given by the diagonal elements of the Higgs fields $X_a(r)$ and $Y_a(r)$, where we assume a gauge in which $X(r)$ and $Y(r)$ are diagonal. Viewing the asymptotic behaviors (3.2.21) and (3.2.22), the generalization of (u_a, v_a) would be $-2r^2(X_a'(r), Y_a'(r))$, where the prime denotes the derivative in the radial direction. Then the generalization of (3.4.14) would be expected as

$$W(r) \equiv r^2 \,\mathrm{Tr}\,(X'(r)Y(r) - X(r)Y'(r)) = 0. \tag{3.4.17}$$

It is enough to show (3.4.17) on the positive z-axis because of the spherical symmetry. Both X_a and Y_a satisfy the same differential equations (3.3.13), which is in the form $(rX_a)'' + \sum_{b=1}^{N} A_a{}^b(r)(rX_b) = 0$ with a symmetric matrix $A_a{}^b(r)$. Using this fact and taking a derivative of (3.4.17), one finds $W' = 0$. Since $W = 0$ at $r = \infty$ from (3.4.14), the equation (3.4.17) holds in the whole space.

Now let us define the r-dependent charges $(e_a(r), g_a(r))$ by the electric and magnetic fields in the radial direction:

$$\mathrm{diag}\,(e_a(r)) = 2r^2\mathcal{E}_r(r), \qquad \mathrm{diag}\,(g_a(r)) = 2r^2\mathcal{B}_r(r), \tag{3.4.18}$$

where the diagonal form of the $\mathcal{E}_r(r)$ and $\mathcal{B}_r(r)$ is guaranteed by (3.3.8) coming from the spherical symmetry ansatz. Then (3.4.15) trivially holds from the Bogomol'nyi equations (3.3.1) and (3.3.2).

As for the last equation (3.4.16), we first define the energy in the spherical region $|x| < r$ in the D3-brane world volume by

$$U_r = \int_{|x|<r} d^3x\, \mathcal{U}, \tag{3.4.19}$$

where $\mathcal{U}$ is the energy density appearing in (3.2.3). Because of BPS saturation, the equality in (3.2.5) holds for the energy U_r and the charges $Q_{X,Y}(r)$ and $M_{X,Y}(r)$, the definitions of which follow (3.2.4) for the spherical region $|x| < r$. From (3.3.1) and (3.3.2), $Q_Y(r) - M_X(r) = 0$ holds, and hence the two charges do not contribute to the BPS bound of U_r. Plugging (3.3.8)

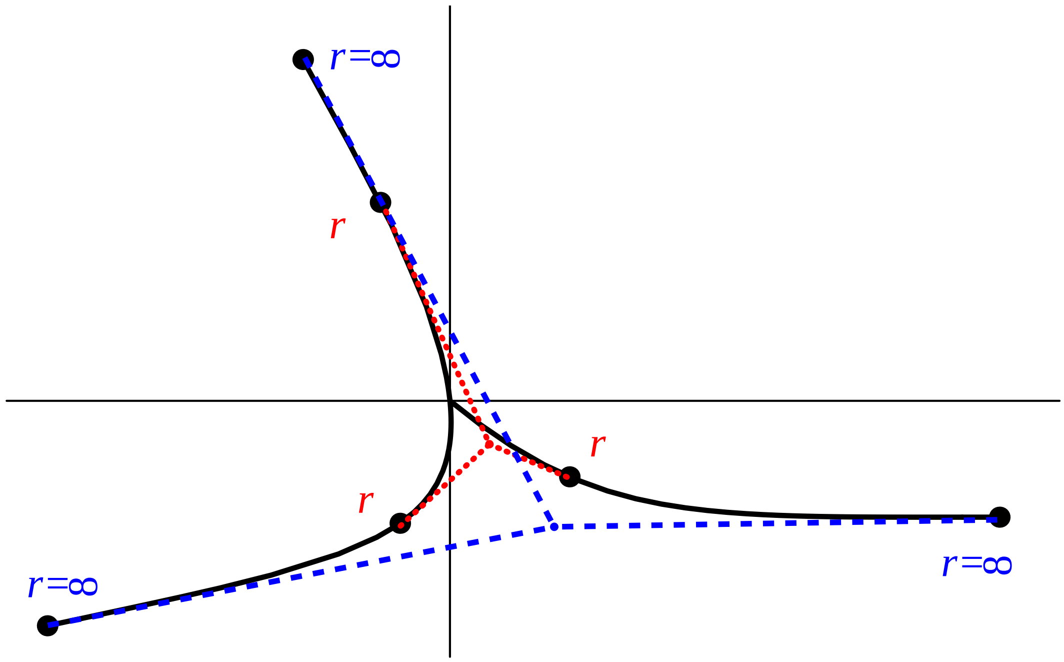

Figure 3.14: The tangent lines to the D3-brane surfaces (solid curves) at $r = \infty$ (dashed lines) and at a finite r (dotted lines).

and the same equation for X into the definitions of the charges $Q_X(r)$ and $M_Y(r)$ and noting the fact that the second term of (3.3.8) does not contribute, they become

$$Q_X(r) = 4\pi r^2 \mathrm{Tr}(X(r)X'(r)), \quad M_Y(r) = 4\pi r^2 \mathrm{Tr}(Y(r)Y'(r)). \tag{3.4.20}$$

Thus from the definition of the effective charges (3.4.18) and the Bogomol'nyi equations (3.3.1) and (3.3.2), the same equation as (3.4.16) holds for the quantities we have defined.

Because of the regularity of the solutions at $r = 0$, it is a general property that the effective charges $(e_a(r), g_a(r))$ vary as functions of r and tend to zero as $r \to 0$. If the ratio $e_a(r) : g_a(r)$ depends on r, the D3-brane trajectories bend. We show the $SU(3)$ case in Fig. 3.14 as an example. The three straight lines starting at $(X_a(r), Y_a(r))$ in the direction $-(e_a(r), g_a(r))$ meets at a common point. Namely, the trajectories of the D3-branes on the transverse plane is developed in such a way that for every r the tangent lines to the trajectories form a string junction carrying the effective charges.

3.4.5 Solutions with degenerate y_a

Up to now we have considered the cases with distinct y_a. When some of the y_a degenerate, we can construct solutions with magnetic charges different from (3.3.23) [150]. In this subsection, we shall discuss solutions with degenerate y_a for the $SU(4)$ case.

Since we have $y_1 \leq y_2 \leq y_3 \leq y_4$, the patterns of the degeneracy are classified as

$$(y_1, y_2, y_3, y_4) = \begin{cases} \text{(i)} & (-s, -s, s, s), \\ \text{(ii)} & (-s, -s, -s, 3s) \text{ or } (-3s, s, s, s), \\ \text{(iii)} & \text{Only one pair in } (y_a) \text{ is degenerate,} \end{cases} \tag{3.4.21}$$

with $s > 0$. As stated in Sec. 3.4.3, the analysis of the solutions of the differential equations (3.3.27) at $r \sim 0$ does not depend on the values of y_a. On the other hand, as we shall show below, the asymptotic behaviors of the solutions change drastically.

Let us first examine the case (i). Since Eq. (3.3.20) is ill-defined for degenerate y_a, we introduce an infinitesimally small parameter δ as follows:

$$(y_1, y_2, y_3, y_4) = (-s + \alpha_1\delta, -s + \alpha_2\delta, s + \alpha_3\delta, s + \alpha_4\delta), \tag{3.4.22}$$

where $\sum_{a=1}^{4} \alpha_a = 0$. Substituting this into Eq. (3.3.20) and taking the limit $\delta \to 0$, we obtain the asymptotic behaviors

$$Q_1(r) \sim \frac{3}{8s^2} r e^{2sr}, \quad Q_2(r) \sim \frac{3}{64s^4} e^{4sr}, \quad Q_3(r) \sim \frac{-3}{8s^2} r e^{2sr}, \quad (r \to \infty). \tag{3.4.23}$$

It follows from these equations that the magnetic charges of this solution are given by $(g_a) = (-2, -2, 2, 2)$. Then, substituting (3.4.23) into (3.3.17), the differential equation (3.3.27) in the asymptotic region is approximated as

$$\mathbf{\Phi}'' - \frac{1}{r^2} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix} \mathbf{\Phi} + \ldots = 0, \tag{3.4.24}$$

where $\mathbf{\Phi}$ is a column vector $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)^T$, and the dotted part is multiplied by the powers of the exponentially decaying factor e^{-2sr}. Since the eigenvalues of the 3×3 matrix in (3.4.24) are 0 and 2 (doubly degenerate), there are two independent divergent modes which behave as r^2 for large r. In order to obtain a physically sensible Φ_m, we have to adjust two of the parameters of the solution to eliminate the divergent modes. As we mentioned in Sec. 3.4.3, the number of free parameters in the $SU(4)$ solution X regular at $r = 0$ is three. Therefore, we are left with only one free parameter. Recalling that the other scalar Y is already a solution to the same differential equation as for X with regular behavior (3.2.22) at $r \to \infty$, we find that X must be proportional to Y, implying that the solution is an uninteresting one with parallel charge vectors. Analysis for the case (ii) leads to the same situation, namely, two divergent modes exist at infinity. Hence we conclude that for (i) and (ii) there is no non-trivial regular solution.

Next, let us consider the case (iii). In this case, the magnetic charges of the solutions are $(-2,-2,1,3)$, $(-3,0,0,3)$ and $(-3,-1,2,2)$ corresponding to three different degeneracy patterns of (y_a), $y_1 = y_2$, $y_2 = y_3$ and $y_3 = y_4$, respectively. We shall take the case $y_1 = y_2$. Then, similarly to Eq. (3.4.24), the differential equation (3.3.27) is approximated for large r as

$$\boldsymbol{\Phi}'' - \frac{1}{r^2}\begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{\Phi} + \ldots = 0. \tag{3.4.25}$$

In this case, the eigenvalues of the 3×3 matrix are 0 (doubly degenerate) and 2, and there is only one harmful divergent mode. Therefore, contrary to the cases (i) and (ii), there can be non-trivial solutions since we still have two free parameters in the solution after eliminating one divergent mode.

For these solutions regular at $r \to \infty$ we have

$$(X_1, Y_1) - (X_2, Y_2) = O\left(\frac{1}{r^2}\right). \tag{3.4.26}$$

This is because Eq. (3.4.25) implies

$$(2\Phi_1 - \Phi_2)'' - \frac{2}{r^2}(2\Phi_1 - \Phi_2) + \ldots = 0 \tag{3.4.27}$$

for $2\Phi_1 - \Phi_2 = 2r(X_1 - X_2)$, and the divergent r^2 term in the solution at $r \to \infty$ has been eliminated by fine-tuning to leave the $1/r$ term. From Eq. (3.4.26) and the asymptotic expressions (3.2.21) and (3.2.22), we find that $(x_1, y_1) = (x_2, y_2)$ and $(e_1, g_1) = (e_2, g_2)$ hold for the present solution. In the IIB picture, two strings 1 and 2 degenerate completely, and the situation is reduced to the three-pronged string.

An example of the solutions for $(y_a) = (-1/2, -1/2, 0, 1)$ obtained by a numerical method is presented in Fig. 3.15. We see that two D3-brane surfaces 1 and 2 approaches to each other as $r \to \infty$ as mentioned above, though at the origin they behave in the way explained in Secs. 3.4.2 and 3.4.3.

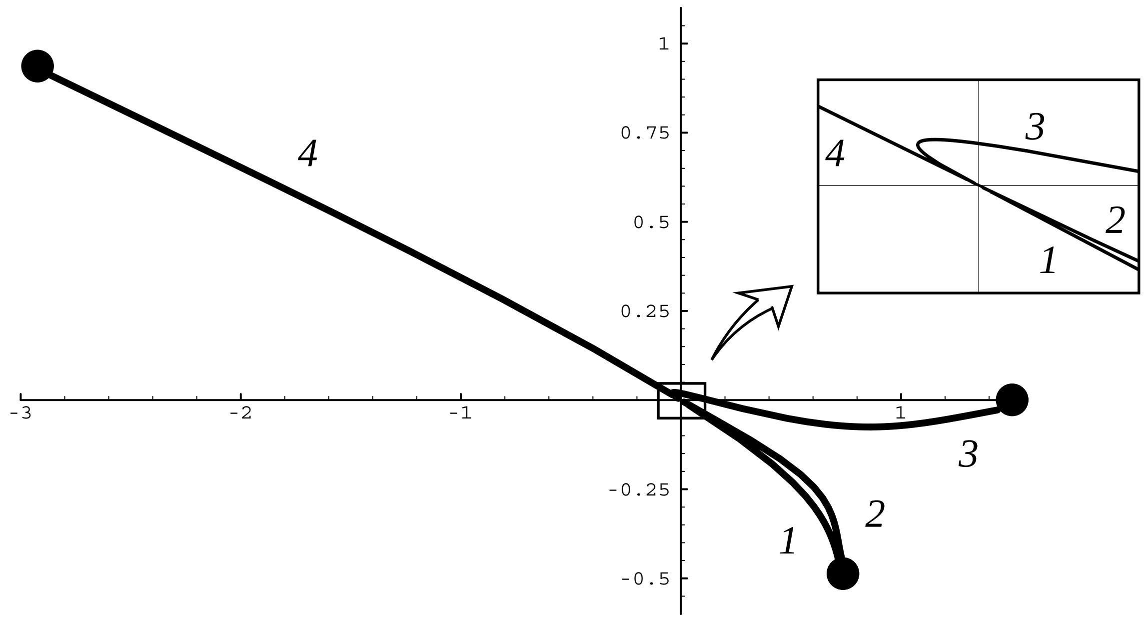

Figure 3.15: A configuration with $(y_a) = (-1/2, -1/2, 0, 1)$. The locations of two D3-brane surfaces 1 and 2 at $r = \infty$ coincide. The behavior near $r = 0$ is magnified in the small window.

3.5 Existence condition of three-pronged string

In the string picture, the existence of a three-pronged string connecting three parallel D3-branes depends on its two-form charges and the relative locations of the D3-branes [20]. Let us consider three D3-branes denoted by D_1, D_2 and D_3, and a three-pronged string connecting them (Fig. 3.16). We denote the two-form charges of the strings ending on the D_1, D_2 and D_3 by (e_1, g_1), (e_2, g_2) and (e_3, g_3), respectively, where the charges are conserved at the junction: $\sum_{i=1}^3 e_i = \sum_{i=1}^3 g_i = 0$. Since the force balance condition determines the relative directions of the three strings meeting at the junction, the shape of the triangle formed by the three D3-branes is constrained. One can show easily that the necessary and sufficient conditions for the existence of a three-pronged string connecting the D3-branes are given by the following conditions for each angle θ_i at the vertices D_i of the triangle:

$$\theta_i \leq \mathrm{Angle}\left[(e_j, g_j), (e_k, g_k)\right] \quad \text{for different } i, j, k. \tag{3.5.1}$$

Here Angle$[\boldsymbol{v}_1, \boldsymbol{v}_2]$ denotes the angle between the two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ having a common initial point. When the length of one of the strings vanishes, one of the equalities holds in (3.5.1). In such cases the configuration is a stable BPS state with two strings located at the same point in the D3-brane world volume. When the D3-brane locations are changed further out of (3.5.1), the stability may be lost. Thus the process of changing D3-brane locations from (3.5.1) to the outside may be observed as a decay process of a field theory BPS state of a three-pronged string to two dyon states [20, 21]. On the other hand, when two strings

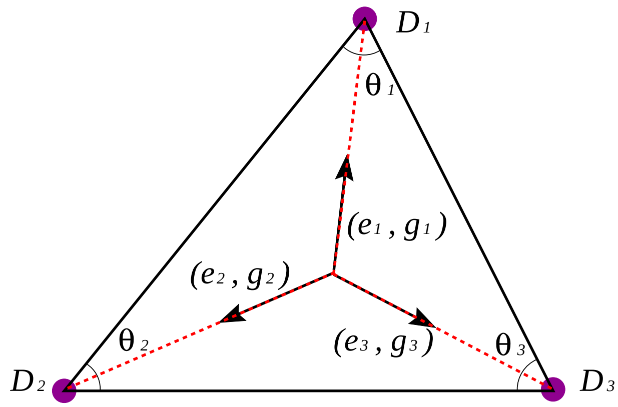

Figure 3.16: The triangle formed by three parallel D3-branes (solid lines). The charge vectors are described by the lines with an arrow. The dotted lines denote a three-pronged string connecting the D3-branes.

with some two-form charges are located at a finite distance in the D3-brane world volume, and the configuration of the D3-branes satisfies (3.5.1), the strings may attract each other and decay to the stable BPS state of a three-pronged string. Thus the evaluation of the force between two dyons would be another way to see the conditions (3.5.1). In this section, we will estimate the force between well-separated two dyons following essentially the approach of Refs. [49, 100, 101, 90], and will actually reproduce (3.5.1).

Let us consider the following configurations of D3-branes and strings. Three parallel D3-branes D_1, D_2 and D_3 are located at (x_1, y_1), (x_2, y_2) and (x_3, y_3) in the transverse two-dimensional plane, respectively (Fig. 3.17). An (e_1, g_1) string connects D_1 and D_3, and an (e_2, g_2) string connects D_2 and D_3. The two strings are separated by a distance R in the D3-brane world volume. The two strings appear as two dyons with the electric and magnetic charges (e_1, g_1) and (e_2, g_2) under the different $U(1)$ subgroups of $SU(3)$, respectively, in the D3-brane world volume theory. The dyon corresponding to the (e_1, g_1) string is denoted by d_1 and the other one by d_2. We assume the distance R is large enough to allow treating the dyons as point-like particles.

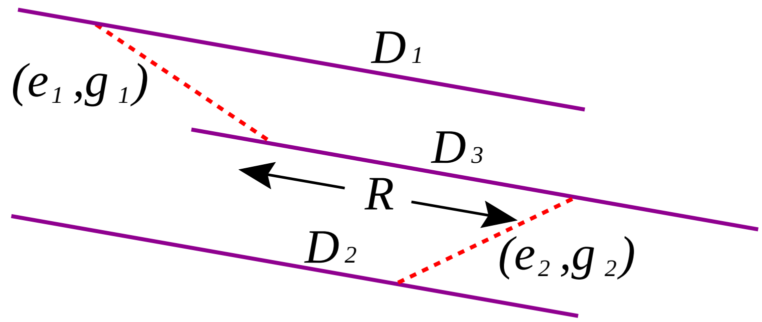

Figure 3.17: Two strings with charges (e_1, g_1) and (e_2, g_2) (dotted lines) connect one common and two different parallel D3-branes (solid lines). The two strings are separated by R in the D3-brane world volume.

There act a Coulomb force between the two dyons. The Coulomb potential is obtained as

$$V_{\rm Coulomb}(R) = \frac{\pi}{R}(e_1 e_2 + g_1 g_2), \tag{3.5.2}$$

by simply adding the electric and magnetic contributions.

Another force is generated by the long-range Higgs field. Let us first evaluate the mass of the dyon d_1 by neglecting the effect of the other dyon d_2. By calculating $Q_{X,Y}$ and $M_{X,Y}$ from the asymptotic behaviors of the electric and magnetic fields (3.2.21)-(3.2.24), and plugging them into (3.2.5), the mass of the dyon d_1 is obtained as

$$m_1 = 2\pi\sqrt{e_1^2 + g_1^2}\; l_{13}, \tag{3.5.3}$$

where $l_{13} = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}$ is the distance between D_1 and D_3. This mass agrees with the corresponding string mass by identifying $2\pi\sqrt{e_1^2 + g_1^2}$ as the string tension. We shall evaluate the potential energy between the dyons by taking into account the R-dependence of the distance l_{13} caused by the long-range effect of the other dyon d_2.

From (3.2.21) and (3.2.22), the long-range behaviors of the Higgs field generated by the dyon d_2 are given by

$$\begin{aligned} X &\sim {\rm diag}(x_1, x_2, x_3) + \frac{1}{2R}\,{\rm diag}(0, u_2, u_3), \\ Y &\sim {\rm diag}(y_1, y_2, y_3) + \frac{1}{2R}\,{\rm diag}(0, v_2, v_3), \end{aligned} \tag{3.5.4}$$

where the u_a and v_a are related to the charges by (3.2.26). The R-dependent terms of the third components cause the R-dependence of the distance l_{13}. The factors are expressed explicitly as

$$(u_3, v_3) = \sqrt{e_2^2 + g_2^2}\, \hat{\boldsymbol{l}}_{23}, \tag{3.5.5}$$

where $\hat{\boldsymbol{l}}_{23}$ is a unit vector parallel to the string connecting D_2 and D_3:

$$\hat{\boldsymbol{l}}_{23} = \frac{(x_2 - x_3, y_2 - y_3)}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}}. \tag{3.5.6}$$

Plugging these asymptotic behaviors into (3.5.3) and taking the first order in $1/R$, we obtain the potential energy from the long-range Higgs field as

$$V_{\text{Higgs}}(R) = -\frac{\pi}{R}\sqrt{e_1^2 + g_1^2}\sqrt{e_2^2 + g_2^2}\, \hat{\boldsymbol{l}}_{13} \cdot \hat{\boldsymbol{l}}_{23}. \tag{3.5.7}$$

Adding this to the Coulomb potential (3.5.2), the total potential energy is obtained as

$$V(R) = -\frac{\pi}{R}\left(\sqrt{e_1^2 + g_1^2}\sqrt{e_2^2 + g_2^2}\, \hat{\boldsymbol{l}}_{13} \cdot \hat{\boldsymbol{l}}_{23} - e_1 e_2 - g_1 g_2\right). \tag{3.5.8}$$

The cancellation between the Coulomb and the Higgs contributions occur when the angle between $\hat{\boldsymbol{l}}_{13}$ and $\hat{\boldsymbol{l}}_{23}$ equals that between the two vectors (e_1, g_1) and (e_2, g_2). In case the former angle is smaller than the latter, the force between the two dyons is attractive, and vice versa. Thus we obtain the inequality for θ_3 in (3.5.1) from a field theoretical viewpoint. Repeating similar discussions for the other combinations of two strings, we obtain all the inequalities in (3.5.1).

3.6 Summary and discussions

In this chapter, according to the predictions from string theory, we have constructed 1/4 BPS states in $\mathcal{N} = 4$ $SU(N)$ SYM theory by solving explicitly the Bogomol'nyi equations under the assumption of spherical symmetry. The solutions correspond to the string theory BPS states of a multi-pronged string connecting N different parallel D3-branes. Here each string ending on D3-branes carries the two-form charges $(e_1, N-1), (e_2, N-3), \cdots, (e_N, -N+1)$, where e_a take real values satisfying $\sum_a e_a = 0$. The NS-NS charges e_a (or electric charges in the SYM theory) may be quantized if the quantization of the collective modes of our solutions is performed. We have also shown that, by fine-tuning some of the parameters, we can construct solutions with magnetic charges different from above.

As we have studied in Secs. 3.2.4, 3.4 and appendix A.2, the trajectories of D3-branes have interesting behaviors. We have shown that, from the asymptotic behaviors of our solutions, we can generate the BPS saturated configuration of the corresponding multi-pronged string, that is, straight strings compose a tree diagram with junctions and the forces balance at each junction. On the other hand, the behaviors at finite r are quite different from the above IIB picture. The trajectories bend non-trivially and all the D3-brane surfaces meet at the origin $r = 0$. Nonetheless, an interesting feature of the trajectories is that we can actually generate the string picture configuration at every r by defining appropriately the effective electric and magnetic charges. At the origin $r = 0$ the directions of all the charge vectors with non-vanishing magnetic charges degenerate, and hence all the D3-brane trajectories with non-vanishing R-R two-form string charges meet there with a common tangent. This feature apparently violates the $SL(2, Z)$ duality symmetry, and may come from our technical preference that we treat the SYM theory merely classically, i.e. electrically.

We have constructed exact solutions for general $SU(N)$ in the case the vacuum expectation values of one of the Higgs fields (Y) are parallel to the magnetic charges. The number of the free parameters of our exact solutions are given by N, one from the overall factor of the Higgs field and $N-1$ from the vacuum expectation values of another Higgs field (X). In the general cases of our (non-exact) solutions, the vacuum expectation values of the first Higgs field can be varied continuously, and hence the total number of the free parameters is expected to be given by $2N - 2$. The solutions should be regular at the whole space including $r = 0$ and should converge to finite values at $r \to \infty$. We have checked the number of the free parameters up to $SU(6)$ by expanding explicitly the second-order differential equations (3.3.27) at $r \sim 0$ and checking numerically the safe convergence of the solutions at $r \to \infty$. A general proof of this fact has not been given.

This number of the free parameters of our solutions is interesting because it does not agree with the expectation in the IIB picture, as discussed in Sec. 3.4.1. Let us consider a tree diagram composed of straight string segments and three-string junctions in a two-dimensional plane. Here we assume the number of external strings ending on D3-branes is N and they have the same fixed R-R charges as the magnetic charges of our solutions. We have the freedom to take $N-1$ NS-NS charges (the other one is determined by the charge conservation). Then the relative directions of the strings in the diagram are determined by the force balance condition. We have the freedom to take the $2N - 3$ lengths of each string segment in the diagram. We should not count the freedom of the parallel shift and the rotation of the whole diagram, because we fixed these degrees of freedom in obtaining our solutions (see Sec. 3.3.1). Thus the total number of the degrees of freedom is given by $3N - 4$, which exceeds the above degrees of freedom of our solutions by $N - 2$. We do not have any reliable explanation of this discrepancy, but may point out that the freedom of choosing the two-form (or electric and magnetic) charges does not seem to be fully incorporated as can be seen from the $r \sim 0$ behaviors of our solutions discussed in the second paragraph of this section. Thus we hope the number of free parameters of such solutions may agree with the string picture if we find a better formulation in field theory respecting the $SL(2, Z)$ duality symmetry to describe the 1/4 BPS states.

In certain parameter regions of our solutions for $SU(4)$, we found two distinct multi-pronged string networks which can correspond to the same field theory BPS states, as discussed in Sec. 3.4.2. However, regarding a crossing of strings as a four-string junction, one of the networks always contains a loop [133, 84, 1], while the other one is genuinely tree-like in such cases. Thus, if we choose the tree one as the corresponding string configuration, we recover the one-to-one correspondence between the BPS states in the string theory and those in the field theory. More precisely, we can continuously deform the diagram of Fig. 3.12 to Fig. 3.10 by shrinking the internal loop. As seen in Sec. 3.2.4, from the soliton solution we only observe the locations of the D3-branes and (p, q)-charges of the *external* strings, thus we have no information concerning the internal structure of the network including loops. The information on the loop deformation might be obtained by studying the zero modes of the solution with the parameters in the region (3.4.9). The zero modes of the 1/4 BPS dyon was investigated in Ref. [16].

In Sec. 3.5, we discussed the force between two strings connecting one common and two different parallel D3-branes by regarding the strings as dyons and calculating the long-range force between them in the SYM theory. In a certain parameter region of the D3-brane coordinates, the force is attractive, and a three-pronged string can be formed as a bound state of the two strings. This result is another support to the existence of a three-pronged string connecting D3-branes.

The supermultiplet of a 1/4 BPS state should contain states with spins higher than one (see Sec. 3.2.3). The appearance of a field with such a high spin is unusual in a field theory without gravity. It is intriguing to note that a 1/4 BPS state is a non-local object and cannot appear as an elementary local field. To see this, let us consider a $(0, 1)$ string connecting two parallel D3-branes separated by a distance l. This string state corresponds to a monopole in $SU(2)$ SYM theory. One sees that the energy of a monopole is given by $E_{(0,1)} = Tl/g$, while its width is $\delta r_{(0,1)} = (Tl)^{-1}$ by viewing the explicit expression of the monopole configuration, where T and g are the string tension and the string coupling constant, respectively (see Sec.

2.4). By performing the $SL(2,Z)$ duality transformation on the equation $E_{(0,1)}\delta r_{(0,1)} = 1/g$, we obtain $E_{(p,q)}\delta r_{(p,q)} = |p+q\tau|^2/\mathrm{Im}\tau$ for a (p,q) string in general. Thus one cannot take $\delta r \to 0$ limit simultaneously for two strings with non-parallel NS-NS and R-R charges by taking an appropriate limit of τ. This fact implies that a 1/4 BPS state cannot appear as an elementary local field in any $SL(2,Z)$ equivalent description, because it has non-parallel electric and magnetic charges.

In appendix B, we have shown that our solutions are also solutions of the non-Abelian BI action proposed in [146]. Among the developments of the BI physics explained in Secs. 2.2 and 2.3 and found in Refs. [32, 64, 56, 93, 57, 58, 26, 47, 142, 36], a remarkable fact is the coincidence between the brane worldvolume approach and the target space approach [93]. Since any solution of supergravity corresponding to string junctions has not been constructed yet, we hope our brane realization of string junctions will give a certain insight to a supergravity realization of string junctions.

It is clear that there are some unsatisfactory points to be overcome in our formulation to describe the general 1/4 BPS states. We need more freedom to choose the magnetic charges. An $SL(2,Z)$ invariant formulation is also needed. It is known that Nahm equations [107, 106, 40] describe monopoles on D3-branes by 1-brane dynamics (see Sec. 2.4.3 or Ref. [43]) rather than the D3-brane dynamics as we did. It might shed light on the above problems if the formulation is extended to multi-pronged strings considering appropriate boundary conditions of 1-brane fields at junctions like those given in [33]. See Ref. [69] for the progress in this direction.

Since we use only two adjoint Higgs fields in our strategy of solving the Bogomol'nyi equations, our results may be also applicable to $\mathcal{N}=2$ SQCD. Thus the states discussed in this chapter might be related to the exotic states with non-parallel electric and magnetic charges observed recently in [139] in $\mathcal{N}=2$ MQCD [153] (See also [49, 2] for some discussions on such exotic states).

After this work [72, 73] was carried out, much progress has been made concerning the properties of the 1/4 BPS dyons [12, 13, 15, 16]. In these references, the low energy dynamics of this exotic dyon was discussed, and unified description of the creation of this state as a bound state of many dyons was provided.

Chapter 4

Non-Commutative Monopoles and Tilted Brane Configurations

The development in string theory in this half a decade has enabled us to understand various perturbative and non-perturbative phenomena in field theories by geometrical and intuitive pictures. This progress in string theory is now beyond the province of reproduction of the result of the field theories, that is, now the string theory has predictive power in various field theories.

As we have seen in the previous chapter, one of the most intriguing examples in the above progress is the 1/4 BPS dyon solution in 4-dimensional $\mathcal{N}=4$ SYM. The study of the 1/4 BPS states in the SYM was triggered by the discovery of the stable string network in the type IIB superstring theory [129, 42]. When this string network has its ends on D3-branes, these states preserve 1/4 supersymmetries of the original D3-brane worldvolume theory [20]. After the study from the string theory side, there appeared some works [72, 73, 82, 91] in which explicit field theoretical solutions for the corresponding solitons were constructed. The properties of the solution favor the string theory interpretation with respect to their (p, q)-charges, masses and supersymmetries.

Recently, quantum field theories on non-commutative geometries have received renewed attention following the observation that they arise naturally as a decoupled limit of open string dynamics on D-branes [38]. In the formalism of Ref. [38], SYM on non-commutative spacetime (NCSYM) arises from Fourier transforming the winding modes of D-branes living in a transverse torus in the presence of NS-NS 2-form background [45]. To be concrete, consider a D-string oriented along the 01-plane and localized on a square torus in the 23-plane in the background of B_{23}. In the absence of B_{23}, the Fourier transform is equivalent to acting by

T-duality in the 23-directions. In the presence of the B_{23}, however, the Fourier transform **(I)** and T-duality **(II)** acts differently. On one hand, **(I)** gives rise to the NCSYM with non-commutativity scale

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}. \tag{4.0.1}$$

On the other hand, **(II)** gives rise to D3-branes in the NS-NS 2-form background. The precise map of degrees of freedom between **(I)** and **(II)** is highly non-local and was described in a recent paper [131] as a perturbative series in the non-commutativity parameter θ. The physics of **(I)** at large 'tHooft coupling can further be related to **(II)** in the near horizon region [67, 99] in the spirit of the AdS/CFT correspondence [98]. Yet, these equivalences have contributed very little to the understanding of the localized observables in the NCSYM. The difficulty stems largely from the fact that we do not yet understand the encoding of the observables in one formulation in terms of the other with sufficient detail.

To study the localized structures, it is natural to introduce localized probes. Topologically stable solution such as a magnetic monopole seems particularly suited for such a task. Instantons on non-commutative space-times have also been studied [109, 50, 140, 102] along this line.

In this chapter, we will study the static properties of magnetic monopoles, dyons, and other related structures in the NCSYM with $\mathcal{N} = 4$ supersymmetry[1]. Since the non-commutativity modifies the equation of motion for the gauge fields, one must first establish the fact that these solutions exist in the first place. To this end, the equivalence between **(I)** and **(II)** will prove to be extremely useful; magnetic monopoles and dyons can be understood in **(II)** in the language of brane configurations. Masses, charges, and supersymmetries of these objects can be analyzed in the language of **(II)**. The fact that these objects stay in the spectrum of the theory in the decoupling limit provides a strong evidence that objects with the corresponding mass, charge, and supersymmetry exist in the NCSYM. In the language of **(II)**, it is also straightforward to argue for the existence and stability of exotic dyons which arise from three-pronged strings [1, 42, 133, 20] and other complicated brane configurations.

The analysis in **(II)** provides new feature of the monopole in the NCSYM: it is predicted that the NCSYM monopoles exhibit a certain non-locality and dipole structure. This prediction should be examined directly from the field theoretical analysis of **(I)**. We carry out the explicit construction of the NCSYM monopoles, and show that this feature actually emerges.

[1]Related 1/2-BPS and 1/4-BPS constant field-strength solutions on tori were discussed in Refs. [85, 86, 87].

This chapter is based on Refs. [66, 71]. We will begin in Sec. 4.1 by briefly reviewing some basic facts about the NCSYM **(I)** and how they arise as a decoupling limit. Then we will explain the basic notions of the monopoles in the NCSYM in Sec. 4.2. In Sec. 4.3, we provide brane configurations in **(II)** which correspond to the monopoles, dyons and string 1/4 BPS dyons in **(I)**. Following the prediction concerning the NCSYM monopole in Sec. 4.3, we verify in Sec. 4.4 this prediction by constructing the explicit configuration in the Lagrangian formalism of the NCSYM in **(I)**.

4.1 Non-commutative Yang-Mills from string theory

In this section, we will review the string theory origin of the NCSYM. To be specific, let us take our spacetime to have 3+1 dimensions. We will not consider the effect of making time non-commutative. Then, without loss of generality, we can restrict our attention to the case where the only non-vanishing component of the non-commutativity parameter is $\theta_{23} = -\theta_{32} = 2\pi\Delta^2$. ($\Delta$ has the dimension of length.) The NCSYM with coupling $\hat{g}_{\rm YM}$ and non-commutativity $\theta_{\mu\nu}$ is defined by the action

$$S = -\frac{1}{\hat{g}_{\rm YM}^2}\,{\rm Tr}\int d^4x\,\left(\frac{1}{2}\hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} + D_\mu\hat{\Phi}_i * D^\mu\hat{\Phi}_i + \ldots\right) \tag{4.1.1}$$

where the index a runs from 4 to 9 specifying 6 scalar fields. In Eq. (4.1.1), "..." corresponds to the fermion terms, $\hat{F}$ is the covariant field strength

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i\hat{A}_\mu * \hat{A}_\nu + i\hat{A}_\nu * \hat{A}_\mu, \tag{4.1.2}$$

and $D_\mu\hat{\Phi}$ is given with the covariant derivative as

$$D_\mu\hat{\Phi} = \partial_\mu\hat{\Phi} - i\hat{A}_\mu * \hat{\Phi} + i\hat{\Phi} * \hat{A}_\mu. \tag{4.1.3}$$

The $*$-product is defined with the non-commutativity parameter θ_{ij} by

$$(f * g)(x) \equiv f(x)\exp\left(\frac{i}{2}\theta_{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right)g(x) = f(x)g(x) + \frac{i}{2}\{f,g\}(x) + O(\theta^2), \tag{4.1.4}$$

where $\{f,g\}$ is the Poisson bracket,

$$\{f,g\}(x) \equiv \theta_{ij}\partial_i f(x)\,\partial_j g(x). \tag{4.1.5}$$

Relevant details about non-commutative geometry and the NCSYM are reviewed in Refs. [38, 131].

According to the construction of Ref. [38], this theory is equivalent to D3-branes in the background NS-NS 2-form in the $\alpha' \to 0$ limit while scaling

$$g_{\rm s} = \frac{1}{4\pi}\hat{g}_{\rm YM}^2\sqrt{\frac{\alpha'^2}{\alpha'^2+\Delta^4}}, \qquad V_{23} = \Sigma_B^2 = \frac{\alpha'^2}{\alpha'^2+\Delta^4}\Sigma^2, \qquad B_{23} = \frac{\Delta^2}{\alpha'}, \tag{4.1.6}$$

and keeping Δ, Σ and $\hat{g}_{\rm YM}$ fixed. In the presence of D-branes, longitudinally polarized constant NS-NS 2-form is not a pure gauge and has the effect of inducing a magnetic flux on the world volume. The magnetic fluxes in this context can be interpreted as the non-threshold bound state of D-strings oriented along the 1-direction. When multiple parallel D3-branes are present, the same number of D-strings get induced on each of the D3-branes. When the 23-directions is compactified on a torus of size $\Sigma_B = \alpha'\Sigma/\Delta^2$, the ratio of the number of induced D-strings and the number of D3-branes is precisely $n_1/n_3 = \Sigma^2/\Delta^2$.

The map between gauge fields $\hat{A}_\mu$ of the NCSYM **(I)** and the gauge fields A_μ living on the D1-D3 bound state **(II)** was constructed in Ref. [131] to leading non-trivial order in θ, and takes the form

$$\hat{A}_i = A_i - \frac{1}{4}\theta^{kl}\left[A_k\left(\partial_l A_i + F_{li}\right) + \left(\partial_l A_i + F_{li}\right)A_k\right] + \mathcal{O}(\theta^2) \tag{4.1.7}$$

The resummation of this series is not well understood at the present time[2].

4.2 Basic notions of NCSYM monopoles

We are interested in studying the properties of the monopole-like objects in the NCSYM **(I)**. To simplify our discussions, we shall take as the gauge group the simplest one $U(2)$. Note that the group $SU(2)$ is not allowed here since the algebra of any special unitary group is not closed when the multiplication is defined by the $*$-product. Some basic properties of the NCSYM action is already manifest. First, the $*$-product acts like an ordinary product for the constant fields in the Cartan subalgebra of the gauge group. Therefore, NCSYM can be Higgsed just like the ordinary SYM. This is important since BPS monopoles exist as a stable state in the Higgsed SYM. Second, if we assume that only the magnetic field and one component of the scalar (say $\hat{\Phi}_9$) is non-zero, the energy of the system without the electric field is given by

$$E = \frac{1}{\hat{g}_{\rm YM}^2}\,{\rm Tr}\int d^3x\left[\frac{1}{2}\hat{F}_{ij} * \hat{F}_{ij} + D_i\hat{\Phi} * D_i\hat{\Phi}\right], \tag{4.2.1}$$

[2] The higher order corrections to (4.1.7) were studied recently in [5]. See Ref. [112] for related issues.

The energy (4.2.1) is bounded below by a surface integral as follows[3]:

$$\begin{aligned} E &= \frac{1}{2\hat{g}_{\rm YM}^2}\,{\rm Tr}\int d^3x \Big[\mp\epsilon_{ijk}\left(\hat{F}_{ij} * D_k\hat{\Phi} + D_k\hat{\Phi} * \hat{F}_{ij}\right) + (\hat{F}_{ij} \pm \epsilon_{ijk}D_k\hat{\Phi}) * (\hat{F}_{ij} \pm \epsilon_{ijl}D_l\hat{\Phi})\Big] \\ &\geq \frac{1}{\hat{g}_{\rm YM}^2}\,{\rm Tr}\int d^3x\partial_k\Big[\mp\epsilon_{ijk}\left(\hat{F}_{ij} * \hat{\Phi}\right)\Big]. \end{aligned} \qquad (4.2.2)$$

Thus the notion of the BPS bound exists also in the non-commutative theory.

Now, by definition, a magnetic monopole solution should have the property that

$$\hat{\Phi} \to \frac{U}{2}\sigma^3 \qquad (4.2.3)$$

at large r, so the bound on the energy can be made to take the form

$$E = \frac{U}{2\hat{g}_{\rm YM}^2}{\rm Tr}\int_{S_2} dS_k\,\epsilon^{ijk}\hat{F}_{ij}\sigma^3. \qquad (4.2.4)$$

Furthermore, in order for the action to be finite, F_{ij} should decay according to

$$\hat{\mathcal{B}}^k = \frac{1}{2}\epsilon^{ijk}\hat{F}_{ij} = \frac{x^k\sigma^3}{2r^3}Q \qquad (4.2.5)$$

at sufficiently large r where the system looks spherically symmetric. Therefore, the bounded energy (4.2.4) is evaluated as

$$E = \frac{4\pi Q}{\hat{g}_{\rm YM}^2}U. \qquad (4.2.6)$$

In commutative theories, Q takes on integer values due to the Dirac quantization condition. It is an important question whether there are corrections to Q in powers of (ΔU) for the non-commutative theory. Even in the non-commutative theory, however, the fields are slowly varying for large enough r, so we expect the standard commutative gauge invariance argument to hold. Therefore, we are lead to conclude that the magnetic monopoles of NCSYM have the same masses and charges as their commutative counterparts.

Here we have argued in general terms that a self-dual magnetic monopole solution will saturate the BPS bound and has the same mass and the charge as in the commutative theory, provided that they exist. Unfortunately, the field equations of the non-commutative theory contain an infinite series of higher derivative interactions, making the task of proving the existence, as well as studying the detailed structure of these solutions, a serious challenge. However, even without the detailed understanding of magnetic monopole solutions in NCSYM,

[3]In deriving the inequality (4.2.2), we assumed that $\hat{F}_{ij} \pm \epsilon_{ijk}D_k\hat{\Phi}$ decay sufficiently fast at the infinity so that we can apply the formula $\int d^3x\,(f * g)(x) = \int d^3x f(x)g(x)$ for these quantities.

the equivalence between **(I)** and **(II)** can be exploited to establish some basic properties of these objects. For example, the existence, the stability, the mass, and the supersymmetry of these states can be understood in the language of brane construction in **(II)**. In this formalism, it is also easy to establish similar properties of (p,q)-dyons and string junctions. These brane constructions provide a strong evidence that the corresponding objects exist in **(I)**.

4.3 Brane construction of NCSYM solitons

In this section, we will study a variety of dyonic states in the NCSYM. It will however be convenient to first study the case of the BPS monopole as a prototype. The analysis for other cases will follow a similar pattern.

4.3.1 Monopoles in NCSYM

In the formalism of the field theory brane constructions, magnetic monopoles in Higgsed SYM have a natural realization as D-strings suspended between a pair of parallel but separated D3-branes as seen in Sec. 2.4. Similar configuration exists in **(II)** and is a natural candidate for a state which gets mapped to the magnetic monopole of **(I)** under the relation (4.1.7). One important difference between **(II)** and the usual situation is the fact that the background NS-NS 2-form B_{23} also induces a background R-R 2-form $A_{01} = \frac{1}{g}\sqrt{\frac{B_{23}^2}{1+B_{23}^2}}$ which couples to the world volume of the suspended D-string [99, 121]. This effect can also be interpreted as the force felt by the magnetic charge at the endpoint of the suspended D-string in the background of constant magnetic field in the 1-direction. The overall effect is to tilt the suspended D-string in the 1-direction and to change the overall energy of the configuration (see Fig. 4.1). The extent of the tilt and the change in the energy can be found by obtaining the minimal energy configuration of the D-string DBI action in the R-R 2-form background at weak string coupling

$$S = \frac{1}{2\pi\alpha'}\int_0^{2\pi\alpha' U} dx_9 \left(\frac{1}{g_{\rm s}}\sqrt{1+\left(\frac{dx_1}{dx_9}\right)^2} + A_{01}\frac{dx_1}{dx_9}\right). \tag{4.3.1}$$

It is an elementary exercise to show that this expression is minimized for $dx_1/dx_9 = B$, and that the minimum mass is

$$m = \frac{U}{g_{\rm s}}\left(\sqrt{1+B^2} - \frac{B^2}{\sqrt{1+B^2}}\right) = \frac{4\pi}{\hat{g}^2_{\rm YM}}U \tag{4.3.2}$$

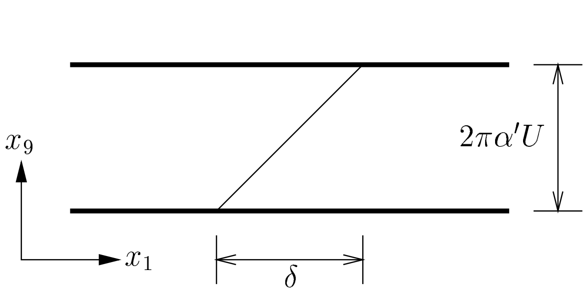

Figure 4.1: Brane configuration of a D-string suspended between a pair of parallel D3-branes in the background of the constant NS-NS 2-form **(II)**. The induced magnetic field on the D3-brane world volume gives rise to a tilt in the D-string orientation.

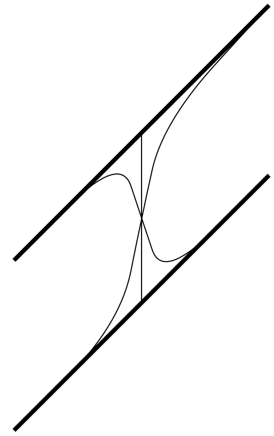

Figure 4.2: Magnetic monopole solution in the tilted D3-brane picture **(III)** where the world volume fields are single-valued.

where we used Eq. (4.1.6) to express the result in terms of the parameters of the NCSYM **(I)**. Despite the fact that the suspended D-string was tilted in the 1-direction in response to the background fields, the mass remained exactly the same as in the ordinary SYM.

It is also interesting to compute the "non-locality" of the suspended D-string indicated by "δ" in Fig. 4.1:

$$\delta = \frac{dx_1}{dx_9} 2\pi\alpha' U = 2\pi\Delta^2 U. \tag{4.3.3}$$

This length therefore remains constant in the decoupling limit $\alpha' \to 0$ in spite of the fact that the slope dx_1/dx_9 diverges in this limit.

It is straightforward to count the number of supersymmetries preserved by this configuration. Let us denote the spinors representing 32 supercharges of the type IIB superstring theory by

$$\epsilon_- = \epsilon_L - \epsilon_R, \qquad \epsilon_+ = \epsilon_L + \epsilon_R. \tag{4.3.4}$$

As we mentioned earlier, D3-branes in the background of B_{23} can be thought of as a bound state of n_1 D-strings and n_3 D3-branes. Such a configuration places a constraint

$$\epsilon_- = \Gamma^0\Gamma^1(\sin(\phi)\epsilon_- + \Gamma^2\Gamma^3\cos(\phi)\epsilon_+) \tag{4.3.5}$$

on the supercharges, where $\tan(\phi) = B$. This result can be easily obtained by following the supersymmetry of $(p, q) = (n_1, n_3)$ string through a chain of duality transformations. On the other hand, a D-string tilted in the 19-plane by the angle $\phi = \tan^{-1}(B)$ preserves

$$\epsilon_- = \Gamma^0\Gamma^\phi\epsilon_-, \qquad \epsilon_+ = -\Gamma^0\Gamma^\phi\epsilon_+ \tag{4.3.6}$$

where

$$\Gamma^\phi = \Gamma^1\sin(\phi) + \Gamma^9\cos(\phi). \tag{4.3.7}$$

The two constraints in Eq. (4.3.6) reduces the number of preserved supersymmetries from 32 to 16. It turns out that Eq. (4.3.5) closes among spinors satisfying Eq. (4.3.6), and reduce the number of independent supersymmetries from 16 to 8. Therefore, this brane configuration preserves the same number of supersymmetries as the magnetic monopole of $\mathcal{N} = 4$ SYM.

We are interested in the supersymmetry of these states in the field theory limit where we scale $B = \Delta^2/\alpha' \to \infty$ keeping Δ fixed. In this limit linear combinations of (4.3.5) and (4.3.6) can be assembled into the following independent set of conditions

$$\epsilon_- = \Gamma^0\Gamma^1\epsilon_-, \qquad \epsilon_+ = -\Gamma^0\Gamma^1\epsilon_+, \tag{4.3.8}$$

$$\epsilon_- = \Gamma^9\Gamma^1\Gamma^2\Gamma^3\epsilon_+. \tag{4.3.9}$$

These conditions are satisfied by 8 spinor components, indicating that the magnetic monopole preserves 8 out of 16 supercharges in the field theory limit.

The brane configuration described in this section is precisely the S-dual of the configuration considered in Ref. [70], except for the fact that in Ref. [70], it was the D3-brane that was tilted instead of the D-string. The two description can be mapped from one to the other by simply rotating the entire system. Although rotating the branes seem like a trivial operation, it amounts to changing the static gauge condition in the language of DBI action. The fact that this makes implicit reference to the gravitational sector of the theory means that this is not a symmetry in the field theory limit. It is more like a duality transformation mapping equivalent physical system between two descriptions. Let us therefore refer to the tilted D3-brane description as **(III)**.

One particular advantage of **(III)** is the fact that the field configuration corresponding to this brane configuration is easily understood. Thinking of the pair of D3-branes as giving rise to $U(2) = U(1)\times SU(2)$ gauge theory, the configuration of Fig. 4.2 is simply the $F_{23} = \partial\Phi_9 = B$ embedded into the $U(1)$ sector and an ordinary Prasad-Sommerfield monopole embedded into the $SU(2)$ sector [64].

The equivalence between **(II)** and **(III)** also sheds light on the nature of **(II)** when expanded in θ. When **(III)** is interpreted as a BIon, the fields are well defined as a single valued function. When **(III)** is rotated to **(II)**, this single-valuedness is lost. The field configuration must now contain branch cuts to account for multi-valuedness in some region of the D3-brane world volume. Since such a field configuration is non-analytic, expansion in θ is likely not to yield a uniformly converging series, and this may have profound implication for the map between **(I)** and **(II)**. Especially in light of the fact that **(II)** seem pathological from many points of view, having a more conventional alternative description **(III)** may prove to be extremely useful in future investigations.

Before concluding this section, let us pause for a moment and briefly describe what happens to the magnetic monopoles in the NCSYM with large 'tHooft coupling and large N. Consider $SU(N+1)$ broken to $SU(N) \times U(1)$. At large coupling, this $SU(N)$ sector is described by the supergravity background [67, 99] and the $U(1)$ sector appears as a D3-brane probe in this background. The supergravity background describing the near horizon of the N D3-branes in the background of B_{23} is given by

$$\begin{aligned}
ds^2 &= \alpha' \left\{ \left(\frac{U^2}{\sqrt{\lambda}}\right)(-dt^2 + dx_1^2) + \left(\frac{\sqrt{\lambda}U^2}{\lambda + \Delta^4 U^4}\right)(dx_2^2 + dx_3^2) + \frac{\sqrt{\lambda}}{U^2} dU^2 + \sqrt{\lambda} d\Omega^2 \right\}, \\
e^{\phi} &= \frac{\hat{g}_{\rm YM}^2}{2\pi}\sqrt{\frac{\lambda}{\lambda + \Delta^4 U^4}}, \qquad A_{01} = \frac{2\pi}{\hat{g}_{\rm YM}^2}\frac{\alpha' \Delta^2 U^4}{\lambda}, \qquad B_{23} = \frac{\alpha' \Delta^2 U^4}{\lambda + \Delta^4 U^4},
\end{aligned} \tag{4.3.10}$$

where $\lambda = 4\pi \hat{g}_{\rm YM} N$. We wish to find the minimal configuration for the probe D-string action

$$S = \frac{1}{2\pi\alpha'} \int dx_1 \left(e^{-\phi} \sqrt{-G_{00}(G_{11} + G_{UU}(\partial U(x_1))^2)} - A_{01} \right). \tag{4.3.11}$$

Near the probe D3-brane, magnetic charge of the D-string will feel the same force as in the case of the flat space, so we impose the boundary condition that $\alpha' \partial U = \alpha'/\Delta^2$ at U where we place the probe D3-brane. Rather remarkably, the configuration

$$U(x_1) = \frac{1}{\Delta^2} x_1, \tag{4.3.12}$$

i. e. a tilted straight line, is a solution to this problem, and when the solution and the background is substituted into Eq. (4.3.11) we find

$$S = \int dx\, \frac{2\pi}{\hat{g}_{\rm YM}^2}\frac{1}{\Delta^2} = \frac{2\pi}{\hat{g}_{\rm YM}^2} U \tag{4.3.13}$$

which, as expected for a BPS state, is the same mass that we found in the weakly coupled limit.

4.3.2 (p,q)-Dyons in NCSYM

In the previous section, we described the interpretation of magnetic monopoles of the NCSYM in the language of **(II)** and found that they have the same mass as the ordinary SYM. It is extremely straightforward to repeat the analysis of the previous section to the case of (p,q)-dyons. There will be some qualitative difference in the pattern of supersymmetry breaking which we will discuss below. Once the basic properties of the (p,q)-dyons are understood, it is natural to consider the possibility of forming a state corresponding to a pronged string [1, 42, 133, 20]. We will examine the existence, the stability, and the supersymmetry of these junction states.

It is extremely straightforward to generalize the discussion of the previous section to the (p,q)-dyon. The expression for the action (4.3.1) is generalized to

$$S = \frac{1}{2\pi\alpha'}\int_0^{2\pi\alpha' U} dx_9 \left(\sqrt{p^2+\frac{q^2}{g_s^2}}\sqrt{1+\left(\frac{dx_1}{dx_9}\right)^2} + qA_{01}\frac{dx_1}{dx_9}\right). \tag{4.3.14}$$

which is minimized by setting

$$\frac{dx_1}{dx_9} = \frac{qB}{\sqrt{(1+B^2)g_s^2 p^2 + q^2}}. \tag{4.3.15}$$

The minimum mass is

$$m = \sqrt{\frac{(1+B^2)g_s^2 p^2 + q^2}{(1+B^2)g_s^2}}U = \sqrt{p^2+\frac{16\pi^2 q^2}{\hat{g}_{\rm YM}^4}}U \tag{4.3.16}$$

which is precisely identical to the result one would expect from the ordinary SYM.

Let us now investigate the number of preserved supersymmetries for these dyons. For the sake of concreteness, we will first consider $(p,q)=(1,0)$, which is a W-boson. As in the previous section, the D3-brane puts the constraint (4.3.5). The $(1,0)$-string, on the other hand, preserves

$$\epsilon_- = \Gamma^0\Gamma^9\epsilon_+. \tag{4.3.17}$$

For spinors satisfying Eq. (4.3.17), the supersymmetry constraint (4.3.5) simplifies to

$$\left(1-\Gamma^0\Gamma^1\sin(\phi)-\Gamma^1\Gamma^2\Gamma^3\Gamma^9\cos(\phi)\right)\epsilon_- = 0. \tag{4.3.18}$$

Conditions (4.3.17) and (4.3.18) are satisfied by 8 independent spinor components for arbitrary values[4] of ϕ. Therefore, we learn that the W-boson in the field theory limit $\tan(\phi)=B=\Delta^2/\alpha'\to\infty$ also preserves 8 supercharges.

[4] We thank M. Krogh for pointing out an error regarding this point in the earlier version of our paper [66].

It is straight forward to extend this analysis to the case of (p,q) dyons. Taking the (p,q)-string to be oriented in the direction given by (4.3.15), it is easy to obtain a set of independent constraints in a manner similar to the monopole in the last section. The number of the unbroken supersymmetries is 8, and in the decoupling limit, the surviving supersymmetries are specified by Eq. (4.3.8) in addition to the constraint

$$\sqrt{p^2 + \left(\frac{4\pi q}{\hat{g}_{\rm YM}^2}\right)^2}\,\Gamma^9\epsilon_- = \Gamma^1\left(p + \frac{4\pi q}{\hat{g}_{\rm YM}^2}\Gamma^2\Gamma^3\right)\epsilon_+, \tag{4.3.19}$$

which reduces to (4.3.9) when $(p,q)=(0,1)$. We conclude that in the field theory limit, the (p,q)-dyons are 1/2 BPS objects, precisely analogous to the situation in the ordinary $\mathcal{N}=4$ SYM.

Just as in the magnetic monopole case, one can consider the analogue of **(III)** where one tilts the D3-brane in such a way to make the (p,q)-string point upward. This will simply correspond to embedding the Julia-Zee dyon in the $SU(2)$ sector and turning on the $U(1)$ part independently. From this standpoint, it is easy to see that the number of preserved supersymmetries is 8.

The large N and large 'tHooft coupling limit of the (p,q)-dyon is also straightforward to analyze. One simply generalizes Eq. (4.3.11) to

$$S = \frac{1}{2\pi\alpha'}\int dx_1\left(\sqrt{p^2+q^2e^{-2\phi}}\sqrt{-G_{00}(G_{11}+G_{UU}(\partial U(x_1))^2)} - qA_{01}\right). \tag{4.3.20}$$

The minimal action configuration satisfying the appropriate boundary condition is simply

$$U(x) = \frac{x}{\Delta^2}\sqrt{1+\frac{\hat{g}_{\rm YM}^4 p^2}{16\pi^2 q^2}}, \tag{4.3.21}$$

and we find the mass of the (p,q)-dyon to be

$$m = U\sqrt{p^2+\frac{16\pi^2 q^2}{\hat{g}_{\rm YM}^4}}, \tag{4.3.22}$$

in agreement with the earlier result from weak coupling (4.3.16).

4.3.3 1/4 BPS dyons in NCSYM

Having established the existence and some basic properties of (p,q)-dyons, it is natural to consider the status of the 1/4 BPS dyons discussed in Chap. 3. In the absence of the background NS-NS 2-form, the existence of pronged string relied on the property of (p,q)-strings, that their tension can be balanced

$$\sum_i \vec{T}_{p_i,q_i} = 0 \tag{4.3.23}$$

where

$$\vec{T}_{p,q} = \left(p, \frac{q}{g_s}\right) \tag{4.3.24}$$

for $\sum p_i = \sum q_i = 0$. The components of $\vec{T}$ can be, say, in the 8 and the 9 directions.

When the effect of the B-field is taken in to account, these vectors are rotated out of the 89-plane into the 1-direction. Now one needs to make sure that the tension balance condition is satisfied in the 1, 8, and 9 directions simultaneously. It turns out, however, that the entire effect of the B-field can be accounted for by rotating the tension vector in the 19-plane so that the (1,8,9) components read

$$\vec{T}_{p,q} = \left(\frac{q}{g_s}\sin(\phi),\ p,\ \frac{q}{g_s}\cos(\phi)\right), \qquad \tan(\phi) = B. \tag{4.3.25}$$

It is straightforward to verify that this vector is oriented relative to the D3-brane world volume with the appropriate slope (4.3.15) by rotating $T_{p,q}$ in the 89-plane to point in the 19-directions.

Since we can just as easily tilt the D3-branes instead of tilting the (p,q)-strings, there is a version of **(III)** for the string junction. The fact that the field configuration for such a state is known (as seen in Chap. 3 and Refs. [72, 73, 82, 91]) might prove useful in the same way that the Prasad-Sommerfield solution in **(III)** is related to the magnetic monopole in the NCSYM **(I)**.

Clearly, the condition for sum of $\vec{T}_{p,q}$ to vanish for conserved (p,q)-charges in a pronged string is still valid, so the pronged string exists as a stable state in the presence of the B field. Though different supersymmetries are preserved by the respective component (p,q)-strings in the string network, in view of this stability the whole configuration is expected to preserve some of the supersymmetries. Let us therefore investigate the field theory limit of these configurations more closely.

Consider a junction of strings (p_i, q_i), $i = 1, 2, 3$, supported by D3-branes localized in the 89-plane with strings meeting at the origin. In order to take the field theory limit of such a configuration, we should scale the distance of the D3-brane to the origin as $\alpha' U_i$ with $\alpha' \to 0$ and oriented in the $(p, \frac{q}{g_s\sqrt{1+B^2}})$ direction in the 89-plane. In other words, the Higgs expectation value of the (Φ_8, Φ_9) field should be chosen to scale according to

$$\vec{U}_i = (\Phi_8, \Phi_9)_i = \frac{U_i}{\sqrt{p_i^2 + \frac{q_i^2}{(1+B^2)g_s^2}}}\left(p_i, \frac{q_i}{g_s\sqrt{(1+B^2)}}\right). \tag{4.3.26}$$

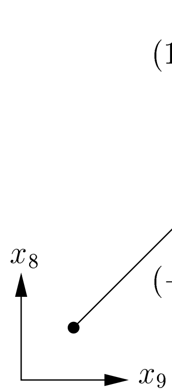

Figure 4.3: Configuration of three string junction in a NS-NS 2-form background. The dots denote the D3-branes perpendicular to the 89-plane. The orientation of the branes resembles the conventional junction in the 89-plane. The components of the pronged string is tilted in the 19-plane in response to the NS-NS 2-form background.

To take the field theory limit, we scale g_s and B according to Eq. (4.1.6). Expressed in terms of $\hat{g}_{\rm YM}$ and Δ, (4.3.26) reads

$$\vec{U}_i = (\Phi_8, \Phi_9)_i = \frac{U_i}{\sqrt{p_i^2 + \frac{16\pi^2}{\hat{g}_{\rm YM}^4} q_i^2}} \left(p_i, \frac{4\pi}{\hat{g}_{\rm YM}^2} q_i \right) \tag{4.3.27}$$

and has a trivial $\alpha' \to 0$ limit. These 1/4 BPS states therefore appear to exist in the field theory limit and orient itself in the usual way in the 89-plane as we illustrate in Fig. 4.3. Fig. 4.3 does not represent the orientation of the strings outside the 89-plane but it should be remembered that they are tilted in the 19-plane. The mass of the 1/4 BPS dyon takes the same form as in the commutative case

$$m = \sum_{i=1,2,3} \sqrt{p_i^2 + \frac{16\pi^2}{\hat{g}_{\rm YM}^4} q_i^2} |\vec{U}_{p_i,q_i}|. \tag{4.3.28}$$

The unbroken supersymmetries of the junction in the field theory limit corresponds to the spinor components of the supercharges satisfying the constraints of both the monopoles and W-bosons, Eqs. (4.3.8), (4.3.9), and (4.3.17). This can be seen easily from the fact that, since the (p_i, q_i)-string is now oriented in the direction (4.3.27) in the 89-plane in the decoupling limit, the constraint for the component (p_i, q_i)-string becomes

$$\left(p_i \Gamma^8 + \frac{4\pi q_i}{\hat{g}_{\rm YM}^2} \Gamma^9 \right) \epsilon_- = \Gamma^1 \left(p_i + \frac{4\pi q_i}{\hat{g}_{\rm YM}^2} \Gamma^2 \Gamma^3 \right) \epsilon_+, \tag{4.3.29}$$

as a generalization of Eq. (4.3.19). We conclude, therefore, that objects in the NCSYM corresponding to the field theory limit of the pronged strings preserves 4 supercharges, just like their commutative counterparts.

4.4 Explicit configuration of NCSYM monopole

Having established some basic properties of the monopoles, dyons and 1/4 BPS dyons in the NCSYM in the language of brane construction **(II)**, it is natural to wonder how much of this can be understood strictly in the framework of the Lagrangian formalism of **(I)**. One fascinating prediction from this brane technique **(II)** is that the monopole in the NCSYM has non-locality δ (4.3.3) due to the tilt of the D-string suspended between two D3-branes (see Fig. 4.1). Believing that the brane configuration of this figure precisely captures the field-theoretical properties, the configuration of the monopole in the NCSYM should reproduce the tilted line, as the eigenvalues of the Higgs field. The ends of the D-string appear to be magnetic charges, hence the field theoretical solution should contain dipole structure.

In this section, we explicitly solve the BPS equation for the monopole of the NCSYM to the first non-trivial order in θ_{ij} which specifies the non-commutativity. This 1/2 BPS solution has the same mass as the ordinary SYM monopole, in agreement with the prediction in the previous section. Solving the non-commutative eigenvalue equation in Sec. 4.4.2, we show in Sec. 4.4.3 that the solution actually reproduces the tilt of the suspended D-string. Examining the magnetic field, the dipole structure is also found.

4.4.1 Non-commutative BPS equation and its solution

The Bogomol'nyi-Prasad-Sommerfield (BPS) monopole solution [23, 118] of the ordinary SYM is saturating a particular energy bound which is usually called the BPS bound. Since this bound is topologically sensible, the state saturating the bound is stable. Now in the case of the NCSYM, unfortunately the topological argument seems not to be valid due to the high complexity of the $*$-product. However, even with this complexity, we can argue a similar mass bound as in Sec. 4.2. If the solution of the non-commutative BPS (NCBPS) equation[5],

$$D_i\Phi + \frac{1}{2}\epsilon_{ijk}F_{jk} = 0, \tag{4.4.1}$$

[5] For simplicity, we have put the Yang-Mills coupling constant $\hat{g}_{\rm YM}$ equal to unity, and abbreviate $\hat{A}$ to A in the following.

has the same asymptotic behavior as the ordinary BPS solution, then the energy remains the same. This fact will be confirmed to the first non-trivial order in θ by explicitly solving the NCBPS equation (4.4.1).

We shall solve the NCBPS equation (4.4.1) to $O(\theta)$ in the small θ expansion. Let us express the gauge field as

$$A_i \equiv (A_i^a + a_i^a)\,\frac{1}{2}\sigma_a + \left(A_i^0 + a_i^0\right)\frac{1}{2}\mathbb{1}, \tag{4.4.2}$$

where the upper (lower) case component fields are of $O(\theta^0)$ ($O(\theta^1)$). The scalar Φ is expressed in a similar manner. First, the $O(\theta^0)$ part of the NCBPS equation (4.4.1) is nothing but the ordinary BPS equation, and we adopt the well-known BPS monopole solution as the $O(\theta^0)$ part of the solution:

$$\Phi^a = \frac{x^a}{r}F(r), \quad A_i^a = \epsilon_{aij}\frac{x_j}{r}W(r), \quad \Phi^0 = A_i^0 = 0, \tag{4.4.3}$$

where the the functions appearing in the solution are

$$F(r) \equiv C\coth(Cr) - \frac{1}{r}, \qquad W(r) \equiv \frac{1}{r} - \frac{C}{\sinh(Cr)}. \tag{4.4.4}$$

The dimensionful parameter C determines the mass scale of the monopole. For later convenience, we present the asymptotic behavior of these functions:

$$F(r) = C - \frac{1}{r} + O(e^{-Cr}), \qquad W(r) = \frac{1}{r} + O(e^{-Cr}). \tag{4.4.5}$$

Next let us proceed to the $O(\theta)$ part of the NCBPS equation (4.4.1). Plugging (4.4.2) into the NCBPS equation (4.4.1) and taking the $O(\theta)$ part, the $U(1)$ component reads

$$\partial_i\varphi^0 + \frac{1}{2}\{A_i^a, \Phi^a\} + \epsilon_{ijk}\left(\partial_j a_k^0 + \frac{1}{4}\{A_j^a, A_k^a\}\right) = 0, \tag{4.4.6}$$

and does not contain the $SU(2)$ fields (a_i^a, φ^a). On the other hand, the $SU(2)$ component of the $O(\theta)$ part of the NCBPS equation decouples from the $U(1)$ fields, and is in fact the linearized equation for the fluctuation (a_i^a, φ^a) obtained from the ordinary BPS equation. Since any solution for (a_i^a, φ^a) corresponds to a θ-dependent gauge transformation on the BPS solution (4.4.3), we take $a_i^a = \varphi^a = 0$ hereafter.

Now our task is to solve the equation (4.4.6). The ansatz for the BPS monopole solution (4.4.3) was the covariance under the rotation of the diagonal $SO(3)$ subgroup of $SO(3)_{\text{gauge}} \times SO(3)_{\text{space}}$. In order to solve Eq. (4.4.6), we put the following ansatz for the $U(1)$ fields

(a_i^0, φ^0) respecting the covariance under the generalized rotation[6], in which we rotate also the parameter θ_{ij}:

$$a_i^0 = \theta_{ij} x_j A(r), \qquad \varphi^0 = \theta_{ij} \epsilon_{ijk} x_k B(r), \tag{4.4.7}$$

where $A(r)$ and $B(r)$ are functions of r to be determined. Putting these ansatz (4.4.7) and the explicit forms of the BPS solution (4.4.3) into the differential equation (4.4.6), we obtain the following two equations as the coefficients of different tensor structures:

$$-A + B + rB' + \frac{1}{4r^2} W(W + 2F) = 0, \tag{4.4.8}$$

$$A' + 2B' - \frac{d}{dr}\left[\frac{1}{4r^2} W(W + 2F)\right] = 0. \tag{4.4.9}$$

The solution to Eqs. (4.4.8) and (4.4.9) is given by

$$A(r) = \frac{1}{4r^2} W(W + 2F) - 2\frac{c_1}{r^3} + c_2, \qquad B(r) = \frac{c_1}{r^3} + c_2, \tag{4.4.10}$$

with two arbitrary constant parameters, c_1 and c_2. The parts in (4.4.10) containing these constant parameters are actually solution to the homogeneous part of Eq. (4.4.6):

$$\partial_i \varphi^0 + \epsilon_{ijk} \partial_j a_k^0 = 0. \tag{4.4.11}$$

Since the c_2 part of the scalar φ^0 diverges at the infinity, we put $c_2 = 0$. As for the c_1 part, a careful substitution into the left hand side of Eq. (4.4.11) gives in fact a term proportional to $\partial_i \partial_i (1/r) = -4\pi\delta^3(\boldsymbol{r})$. Hence the c_1 part is not a solution at the origin, and we shall also put $c_1 = 0$. Finally the desired solution of the equation (4.4.6) is

$$a_i^0 = \theta_{ij} x_j \frac{1}{4r^2} W(W + 2F), \qquad \varphi^0 = 0. \tag{4.4.12}$$

Note in particular that the scalar field receives no correction to this order. Since the whole solution has the same leading asymptotic behavior as the BPS solution (4.4.3), we find that the non-commutativity does not change the mass of the monopole.

4.4.2 Non-commutative eigenvalue equation

The configuration of the D-string suspended between two parallel D3-branes is described by the deformation of the surface of the D3-branes in the spirit of the BIon (BI soliton) physics

[6] The generalized rotational covariance for a_i^0 allows two other terms with different structures, $\epsilon_{ijk}\theta_{jk}$ and $x_i \epsilon_{jkl} \theta_{jk} x_l$. However, Eq. (4.4.6) implies the vanishing of these two terms.

studied in Sec. 2.2.1. The extent of this deformation of the D3-brane surface is given by the eigenvalues of the scalar field on the D3-branes. We saw in the previous section that there is no additional contribution of $O(\theta)$ in the scalar field configuration. However, since we are now dealing with the theory with the $*$-product, the eigenvalue problem should be different from that in the usual commutative case. In this section, we see that the $O(\theta)$ terms are actually generated in the eigenvalues of the scalar field.

We propose that the eigenvalue equation for a hermitian matrix valued function M to be considered in the non-commutative case is

$$M * \boldsymbol{v} = \lambda * \boldsymbol{v}, \tag{4.4.13}$$

where $\boldsymbol{v}$ and λ are the eigenvector and the eigenvalue, respectively. Though there are other candidates for the non-commutative eigenvalue equation, the present one (4.4.13) has advantages over the others in various respects as we shall see in this and the final sections.

For solving (4.4.13) to $O(\theta)$, let us make the expansion

$$M = M_0 + M_1, \quad \boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1, \quad \lambda = \lambda_0 + \lambda_1, \tag{4.4.14}$$

where the subscript number specifies the order of θ. Then, the $O(\theta^0)$ part of (4.4.13) is $M_0\boldsymbol{v}_0 = \lambda_0\boldsymbol{v}_0$, and the $O(\theta)$ part reads

$$M_0\boldsymbol{v}_1 + M_1\boldsymbol{v}_0 + \frac{i}{2}\{M_0, \boldsymbol{v}_0\} = \lambda_0\boldsymbol{v}_1 + \lambda_1\boldsymbol{v}_0 + \frac{i}{2}\{\lambda_0, \boldsymbol{v}_0\}. \tag{4.4.15}$$

Multiplying $\boldsymbol{v}_0^\dagger$ from the left, we obtain the formula which gives the $O(\theta)$ part of the eigenvalue:

$$\lambda_1 = \frac{1}{\boldsymbol{v}_0^\dagger\boldsymbol{v}_0}\left(\frac{i}{2}\boldsymbol{v}_0^\dagger\{M_0 - \lambda_0\mathbb{1}, \boldsymbol{v}_0\} + \boldsymbol{v}_0^\dagger M_1\boldsymbol{v}_0\right). \tag{4.4.16}$$

In view of the application to the present NCBPS solution, let us consider the particular case with

$$M_0 = m_0(\boldsymbol{r})\,\widehat{x}_a\sigma_a \quad (\widehat{x}_i \equiv x_i/r), \qquad M_1 = 0, \tag{4.4.17}$$

and hence $\lambda_0 = \pm m_0(\boldsymbol{r})$ and $\widehat{x}_a\sigma_a\boldsymbol{v}_0 = \pm\boldsymbol{v}_0$. Then, Eq. (4.4.16) is calculated to give

$$\lambda_1 = \frac{i}{2\boldsymbol{v}_0^\dagger\boldsymbol{v}_0}\frac{m_0(\boldsymbol{r})}{r}\theta_{ij}\left(\boldsymbol{v}_0^\dagger\sigma_i\partial_j\boldsymbol{v}_0 \mp \widehat{x}_i\boldsymbol{v}_0^\dagger\partial_j\boldsymbol{v}_0\right) = -\frac{m_0(\boldsymbol{r})}{2r^2}\theta_i\widehat{x}_i, \tag{4.4.18}$$

with $\theta_i \equiv (1/2)\epsilon_{ijk}\theta_{jk}$. Note that λ_1 (4.4.18) is independent of the sign of λ_0. We obtained the last expression of (4.4.18) using the explicit form $\boldsymbol{v}_0 = (x_1 - ix_2, \pm r - x_3)^{\mathrm{T}}$. However, the general formula for λ_1, Eq. (4.4.16), is in fact invariant under the local phase and scale transformation of $\boldsymbol{v}_0$ due to the identity $\boldsymbol{v}_0^\dagger\{M_0-\lambda_0\mathbb{1}, f\mathbb{1}\}\boldsymbol{v}_0 = 0$ valid for any $f(\boldsymbol{r})$. This corresponds to the fact that the eigenvalue λ of (4.4.13) is invariant under the right multiplication $\boldsymbol{v} \to \boldsymbol{v} * h$ for an arbitrary $h(\boldsymbol{r})$. We shall discuss the gauge transformation property of the eigenvalues in the final section.

Let us evaluate various eigenvalues of the system using the formula (4.4.18). First, the scalar eigenvalues are obtained by substituting $m_0(\boldsymbol{r}) = F(r)/2$ as

$$\lambda_\Phi = \pm\frac{1}{2}F(r) - \frac{\theta_i\widehat{x}_i}{4r^2}F(r) + O(\theta^2). \tag{4.4.19}$$

Next, we shall consider the eigenvalues of the magnetic field $B_i \equiv (1/2)\epsilon_{ijk}F_{jk}$ near the infinity $r \to \infty$. The magnetic field itself is given from the solution (4.4.12) as[7]

$$B_i = -\frac{\widehat{x}_i}{2r^2}\widehat{x}_a\sigma_a + \frac{C}{2r^3}\left(\delta_{ij} - 3\widehat{x}_i\widehat{x}_j\right)\theta_j\mathbb{1} + O\left(\frac{1}{r^4}\right). \tag{4.4.20}$$

We would like to evaluate the $O(\theta)$ contribution to the eigenvalues by putting $m_0(\boldsymbol{r}) = -\widehat{x}_i/2r^2$ and $M_1(\boldsymbol{r},\theta) = C\left(\delta_{ij} - 3\widehat{x}_i\widehat{x}_j\right)\theta_j\mathbb{1}/2r^3$. Since $m_0(\boldsymbol{r})$ in this case is $O(1/r^2)$, using the formula (4.4.18), the order of the correction to the eigenvalues from this part is found as $O(1/r^4)$. Thus near the infinity, the $O(1/r^3)$ part of the eigenvalues of the magnetic field is saturated by m_1.

4.4.3 Interpretation of the eigenvalues

In this section, we shall see how the eigenvalues (4.4.19) and (4.4.20) reproduce the brane configuration depicted in Fig. 4.1. In the brane picture, the end of the D-string is seen as a magnetic charge in a single D3-brane worldvolume theory. The prediction from the brane configuration is that the magnetic charge on each end of the D-string is actually moved in different directions between the upper and lower D3-branes, as shown in Fig. 4.1. This shift is specified by the spatial vector δ_i.

Now, the eigenvalues of the magnetic field (4.4.20) indicate that the $U(1)$ part of the magnetic field exhibits a dipole structure. This structure is exactly the one expected from the brane picture above. Noting that the zero-th order solution (4.4.3) represents $-1/2$ charge on the upper D3-brane and $1/2$ charge on the lower, it is easy to derive the non-locality δ_i as

$$\delta_i = C\theta_i. \tag{4.4.21}$$

[7] Since the definition of the magnetic field contains the $*$-product in itself, we should calculate also the Poisson bracket term. However, this term contributes only to the $O(1/r^4)$ part in (4.4.20).

This result verifies the prediction of the previous section with the identification $C = U$.

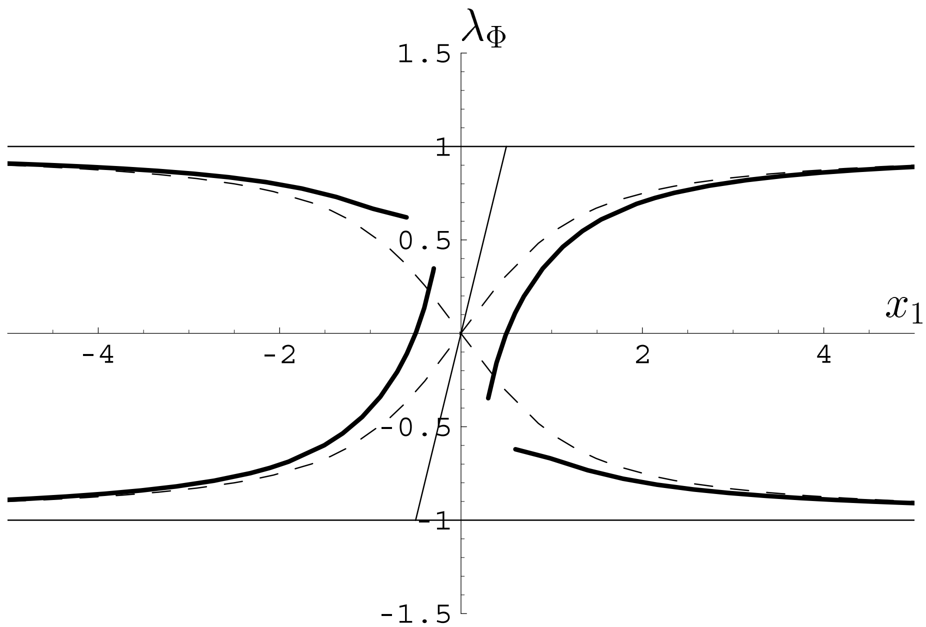

Figure 4.4: The eigenvalues of the solution for the scalar field. We choose $C = 2$ and $\theta_{23} = 1/2$, and other θ's are set to zero.

Although the magnetic charges are expected to indicate only the locations of the ends of the D-string, the eigenvalues of the scalar field must reproduce not only the locations of the ends but also the slope of the D-string. In fact, the asymptotic behavior of the eigenvalues (4.4.19) is given using (4.4.5) as

$$\begin{aligned} \lambda_\Phi &= \pm\frac{C}{2} \mp \frac{1}{2r} + \left(-\frac{C}{4r^2} + \frac{1}{4r^3}\right)\theta_i \hat{x}_i + O(e^{-Cr}) \\ &= \pm\frac{C}{2} \mp \frac{1}{2}\left|x_i \mp \left(\frac{C}{2} - \frac{1}{2r}\right)\theta_i\right|^{-1} + O(e^{-Cr}). \end{aligned} \tag{4.4.22}$$

Eq. (4.4.22) implies first that in the upper and the lower D3-brane the end of the D-string sits at $x_i = C\theta_i/2$ and $x_i = -C\theta_i/2$, respectively. Hence the non-locality is precisely given by $\delta_i = C\theta_i$, in agreement with the result (4.4.21) from the magnetic field. Secondly, in order to read off the slope of the D-string from (4.4.22), we rewrite it as

$$\lambda_\Phi = \pm\frac{C}{2} \mp \frac{1}{2}\left|x_i - \lambda_\Phi\theta_i\right|^{-1} + O(e^{-Cr}). \tag{4.4.23}$$

This equation means that, for a given value of λ_Φ, the corresponding worldvolume coordinate is located on a sphere with its center at $x_i = \lambda_\Phi \theta_i$. Interpreting the trajectory of the center as the D-string, our analysis reproduces precisely the tilt of the suspended D-string.

In Fig. 4.4 we present the curves of the eigenvalues of the scalar field. The thin straight lines represent the brane configuration of Fig. 4.1. The dashed curves denote the eigenvalues of the scalar field with $\theta = 0$. Comparing these with the bold curves representing the eigenvalues (4.4.19) with $\theta \neq 0$, we can read off the simple brane configuration of the previous section. (The reason why the bold curves are cutoff for small $|x_1|$ in Fig. 4.4 is that the $O(\theta)$ term of the scalar eigenvalues (4.4.19) are actually divergent at the origin $r = 0$. We shall discuss this problem in the next section.)

4.4.4 Generalization to dyon

It is straightforward to apply the analysis carried out above to the dyons. In this section, we see the non-commutative dyon exhibits the similar non-locality which was predicted by the brane configuration technique in the previous section.

The NCBPS equations for the dyons are given by

$$B_i + D_i\Phi \cos\alpha = 0, \qquad E_i + D_i\Phi \sin\alpha = 0, \tag{4.4.24}$$

where the parameter α specifies the electric charge of the soliton. Note that the form of the non-commutative BPS equation is the same as the ordinary one, except for the fact that the multiplication of fields are now defined with the $*$-product. For static solutions, the second equation in Eq. (4.4.24) can be easily solved by putting

$$A_0 = \Phi \sin\alpha. \tag{4.4.25}$$

Hence the non-commutative dyon solution can be easily obtained from the non-commutative monopole solution.

Let us evaluate the solution to the $O(\theta)$. Explicitly, the zero-th order solution is [23, 118]

$$A_0^a = \frac{x^a}{r} F(r) \tan\alpha, \qquad \Phi^a = \frac{x^a}{r} F(r) \frac{1}{\cos\alpha}, \tag{4.4.26}$$

and A_i^a is given by Eq. (4.4.3). However, without explicit use of this zero-th order solution, we can deduce the properties of the non-commutative dyons. One has only to multiply the resultant quantities of the non-commutative monopole by the factor $1/\cos\alpha$. For example, the two D3-branes are separated by the length $C/\cos\alpha$. The non-locality derived from the eigenvalues of the non-commutative dyon solution is $\delta_i = C\theta_i$. This non-locality is unchanged since the multiplication is only for the direction perpendicular to the D3-branes.

On the other hand, the prediction from the brane configuration technique is given by Eq. (4.3.15) as

$$\delta = \left| 2\pi\alpha' U \frac{qB}{\sqrt{(1+B^2)(pg)^2 + q^2}} \right|. \tag{4.4.27}$$

Using the following identification between the two pictures

$$U = \frac{C}{\cos\alpha}, \qquad pg = \tan\alpha, \qquad 2\pi\alpha' B_{23} = \theta_{23}, \qquad (1+B^2)g^2 = \left(\frac{g_{\mathrm{YM}}^2}{4\pi}\right)^2, \tag{4.4.28}$$

we obtain $\delta = C\theta$ in precise agreement with our field theoretical analysis.

The brane configuration has predicted that the locations of the electric charges are also shifted as the magnetic charges. From the NCBPS equation (4.4.24), we have

$$E_i = B_i \tan\alpha. \tag{4.4.29}$$

Then the non-commutative analysis for the magnetic charge of the monopole is also applicable to the electric part. The electric charges of the non-commutative dyon are located at the point where the magnetic charges sit, and form dyon charges. The ratio $\tan\alpha$ describes the correct electric charges.

It would be interesting to apply our formulation to the case of the non-commutative 1/4 BPS dyons. The solution in **(I)** is expected to exhibit the non-locality studied in Sec. 4.3.3.

4.5 Summary and discussion

Our analysis presented in this chapter is one of the intriguing examples of the predictive power of brane techniques: first, naive brane configurations are provided in which some supersymmetries are preserved and forces balance, then those configurations have the counterparts in field theories which inherited the properties of the brane configurations.

In Sec. 4.3, instead of working with the Lagrangian formulation of NCSYM **(I)**, we took advantage of the equivalence between NCSYM **(I)** and the decoupling limit of D3-branes in a background NS-NS 2-form potential **(II)** to study the stable brane configurations corresponding to these states. Using this approach, it is easy to show that there are stable brane configurations corresponding to magnetic monopoles, (p,q)-dyons, and 1/4 BPS dyons in the NCSYM, and that they survive in the field theory limit. From this analysis, the non-commutative version of the monopoles, dyons and 1/4 BPS dyons were shown to have the same masses and supersymmetry properties as the ordinary SYM counterparts.

In Sec. 4.4, we solved the BPS equation for the non-commutative monopole to the first non-trivial order in θ. Evaluating the eigenvalues of the solution, we explicitly showed that the solution exhibits the brane configuration of the tilted D-string given in Sec. 4.3. Magnetic field has the dipole structure, and the scalar field is also shifted to reproduce the tilted trajectory of the D-string. For the non-commutative dyons, similar properties are verified in the same manner.

Some comments are in order. Our first comment is on the gauge transformation property of the eigenvalue λ in our non-commutative eigenvalue equation (4.4.13). Of course, the eigenvalue λ in Eq. (4.4.13) is never strictly invariant under the local gauge transformation of M,

$$M \to U^{-1} * M * U, \tag{4.5.1}$$

where U^{-1} is the inverse of U with respect to the $*$-product, $U * U^{-1} = U^{-1} * U = \mathbb{1}$. However, we can show that the eigenvalue has a fairly nice property under (4.5.1). Consider an infinitesimal gauge transformation δ_ϵ on M with $U = \mathbb{1} + i\epsilon$ ($\epsilon^\dagger = \epsilon$),

$$\delta_\epsilon M = i(M * \epsilon - \epsilon * M). \tag{4.5.2}$$

Letting δ_ϵ act on (4.4.13) and $*$-multiplying the resultant equation by $\boldsymbol{v}^\dagger$ from the left, we obtain

$$\boldsymbol{v}^\dagger * \delta_\epsilon \lambda * \boldsymbol{v} = i\boldsymbol{v}^\dagger * (\lambda * \epsilon - \epsilon * \lambda) * \boldsymbol{v}. \tag{4.5.3}$$

Taking the $O(\theta)$ part of (4.5.3) and using $\delta_\epsilon \lambda_0 = 0$, $\delta_\epsilon \lambda_1$ is given as

$$\delta_\epsilon \lambda_1 = \boldsymbol{v}_0^\dagger \{\epsilon, \lambda_0\} \boldsymbol{v}_0, \tag{4.5.4}$$

for a normalized $\boldsymbol{v}_0$. Eq. (4.5.4) implies that, at least to $O(\theta)$, the gauge transformation corresponds to a coordinate transformation on the eigenvalue $\lambda(\boldsymbol{r})$. In the $U(2)$ case with $\epsilon(\boldsymbol{r}) = \epsilon_a(\boldsymbol{r})\sigma_a + \epsilon_0(\boldsymbol{r})\mathbb{1}$, the form of the coordinate transformation is $\delta_\epsilon x_i = -\theta_{ij}\left(\partial_j \epsilon_0 + \boldsymbol{v}_0^\dagger \sigma_a \boldsymbol{v}_0\, \partial_j \epsilon_a\right)$.

Therefore, we have shown that the eigenvalue of Eq. (4.4.13) for M and the one for $U^{-1} * M * U$ are related by a coordinate transformation on the D3-branes and hence are physically equivalent, at least to the first non-trivial order in θ. Recalling also that the eigenvalue of (4.4.13) is independent of the choice of the eigenvector and that the scalar eigenvalue (4.4.19) has an invariance under the simultaneous rotation of x_i and θ_{ij}, the eigenvalue equation (4.4.13) we have adopted is a satisfactory one. These good properties, in particular, the gauge transformation property of λ, are expected to persist to higher orders in θ.

Our second comment is on another type of non-commutative eigenvalue equation,

$$M * \boldsymbol{v} = \boldsymbol{v} * \lambda, \tag{4.5.5}$$

where, compared with Eq. (4.4.13), $\boldsymbol{v}$ and λ are interchanged on the right hand side. Eq. (4.5.5) has a property that the eigenvalue λ is invariant under the gauge transformation (4.5.1) of M. However, for a given M the eigenvalues are not unique and do depend on the choice of the eigenvectors $\boldsymbol{v}$. What is worse, in the analysis similar to those in Sec. 3 by adopting (4.5.5), we can show that it is impossible to obtain the the scalar eigenvalues possessing an invariance under the simultaneous rotation of x_i and θ_{ij}. These are the reasons why we did not adopt (4.5.5).

Our third comment is concerning the singular nature of the scalar eigenvalues (4.4.19) at the origin $r = 0$. Since we have adopted the eigenvalue equation (4.4.13), the matrix whose diagonal entries are these eigenvalues is no longer a solution of the BPS equation. Hence although the eigenvalues (4.4.22) diverge at the origin, the energy of the configuration is still finite. We would need a technique beyond the expansion in powers of θ to dissolve the singularity at the origin.

Our final comment is for the analysis of all order in θ. In fact, the straightforward generalization of our analysis of Sec. 4.4 to the higher order in θ seems to be difficult. However, it is very encouraging that the construction of instanton solutions via the ADHM method admits a natural non-commutative generalization [109]. Indeed, Nahm's construction of the magnetic monopole [106, 107, 40] also admits a simple non-commutative generalization. One simply solves for the normalized zero modes of the operator

$$0 = \hat{\boldsymbol{\Delta}}^\dagger * \hat{\mathbf{v}} = i\frac{d}{dz}\hat{\mathbf{v}}(z,x) - \mathbf{x} * \hat{\mathbf{v}}(z,x) - \mathbf{T}\hat{\mathbf{v}}(z,x), \tag{4.5.6}$$

and computes

$$\hat{A}_i = \int_{-U/2}^{U/2} dz\, \mathbf{v}^\dagger(z,x) * \partial_i \mathbf{v}(z,x), \qquad \hat{\Phi} = \int_{-U/2}^{U/2} dz\, z\mathbf{v}^\dagger(z,x) * \mathbf{v}(z,x). \tag{4.5.7}$$

The non-commutativity is reflected in the $*$-product in (4.5.6), and as long as $\mathbf{\Delta}^\dagger * \mathbf{\Delta}$ satisfies the usual requirement that it be invertible and that it commutes with the quarternions, all the steps in the argument leading to the self-duality of Eq. (4.5.7) follow immediately from the same argument in the commutative case [39, 108]. Note that in Eq. (4.5.6) the $*$-product appears only in the product $\mathbf{x} * \hat{\mathbf{v}}$, and thus the expansion in θ ends up with no higher order terms. The $O(\theta)$ approximation of Eq. (4.5.6) is exact. Hence it might be possible to obtain the result without small θ perturbation in this ADHMN formalism. Since our result in **(I)** in Sec. 4.4 verifies perfectly the prediction of **(II)**, hence there must not be any higher corrections concerning the dipole structure. This will be checked in the future study[8].

[8]While writing the paper [71], we became aware of the paper [11] which has an overlap concerning the solution of Sec. 4.4. The author of Ref. [11] used non-commutative ADHMN construction presented in Ref. [66].

Chapter 5

Conclusion

In this thesis, we argued the brane realization of solitonic objects in field theories. Taking the 1/4 BPS dyons and non-commutative monopoles as examples, we have proved that the brane technique is so powerful that we can make even predictions on the existence of new solitons and their properties without constructing explicit configurations of the solitons in the field theories.

In Chap. 2, we presented a review of the brane realization of the conventional solitons such as the monopoles and dyons in the 4-dimensional $SU(2)$ SYM. The correspondence between the soliton configurations in the field theory and the corresponding brane configurations is based on the the following facts:

- The eigenvalues of the scalar fields on the D-brane describes the location of the D-brane. When the scalar field depends on the worldvolume coordinates, then it tells that the surface of the D-brane is deformed.
- So as to preserve some of the supersymmetries of the worldvolume theory, we must turn on also the electro-magnetic fields on the brane if one deforms the D-brane by turning on the scalar fields. The scalar charge and electro-magnetic charge on the D-brane indicates that there is a string stuck to the D-brane. This is lead from the charge conservation of the stuck string. Thus the deformed part of the D-brane corresponds to the string stuck to it.

Using these, we have shown that the monopoles and (p,q)-dyons in $\mathcal{N}=4$ $SU(2)$ SYM is realized by (p,q)-strings suspended between two D3-branes in the type IIB superstring theory.

The Nahm equation for the construction of the monopoles has been shown to be the BPS equations of the 2-dimensional gauge theory on the D-string worldsheet.

In Chap. 3, we have constructed explicit configurations of the 1/4 BPS dyons, by following the prediction from the brane configuration in Ref. [20]. This exotic dyon corresponds to the string network suspended among many D3-branes. The constructed configurations of the dyons in $\mathcal{N}=4$ $SU(N)$ SYM reproduce the brane configurations of the string network at the infinity of the worldvolume. We found, however, that the behavior near the core of the soliton does not resemble to the string network in the type IIB picture.

In Chap. 4, using the equivalence between the worldvolume non-commutativity and the background B-field in string theory, we investigate solitons in NCSYM. From the analysis of the brane technique, it was predicted that the monopoles and such have the same masses and preserve the same supersymmetries as in the ordinary SYM. Furthermore, the prediction of the new phenomena in the NCSYM from the brane technique was that the solitons such as monopoles possess a certain non-locality. We have examined these predictions explicitly in the Lagrangian formalism of the NCSYM. We have performed the analysis up to the first non-trivial order in the perturbation of the non-commutativity parameter θ, and the result confirms all of the predictions concerning the monopoles in the NCSYM.

The main theme of this thesis is to show the powerfulness of the brane techniques. As seen in the analysis presented in this thesis, it is obvious that the brane technique especially against the solitons in SYM is really an useful method. Therefore there is no doubt that the brane technique will contribute to study of solitons in various field theories.

Let us comment on the future directions. What should be done and can be done first is to check the non-locality of the 1/4 BPS dyon of the NCSYM by explicitly constructing the configuration. This is expected to give further confirmation of the validity of the brane techniques. Then, it would be possible to apply the brane technique to solitons in other field theories. For example, in lower dimensional field theories, the vortex solutions may have its generalizations to the non-commutative case. The properties of this vortex solutions would be predicted simply from the brane techniques.

Investigation of the solitons gives information of the spectrum of the theory. It would be interesting to study the symmetry of the theory in terms of the brane configurations or in general string theory. For example, from the brane configurations the NCSYM monopoles have the same mass as in ordinary SYM, thus there may be S-duality symmetry in the spectrum of the NCSYM. This may be examined from the string theory realization of the NCSYM.

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Appendix A

Verification of the formulas

In this appendix, we verify the formulas used in this thesis.

A.1 Solution for the fluctuation of the Born-Infeld system

In this section, we present a way to solve the equation (2.3.12) using the expansion (2.3.13). Making an abbreviation

$$\mathcal{O} \equiv \left(1 + E^2 + \frac{\pi^2 g_s^2}{r^4}\right)\omega^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right), \tag{A.1.1}$$

the equation (2.3.12) is decomposed due to the orthogonality of the Legendre functions as follows:

$$\begin{aligned}
&\mathcal{O}\eta_0 + \frac{2\epsilon}{r^2}\eta_1 = 0,\\
&\left(\mathcal{O} - \frac{2}{r^2}\right)\eta_1 + \frac{2\epsilon}{r^2}\left(\eta_0 + \frac{2}{5}\eta_2\right) = 0,\\
&\left(\mathcal{O} - \frac{6}{r^2}\right)\eta_2 + \frac{2\epsilon}{r^2}\left(\frac{2}{3}\eta_1 + \frac{3}{7}\eta_3\right) = 0, \quad \cdots
\end{aligned} \tag{A.1.2}$$

where we put $\epsilon \equiv \pi E g_s \omega^2$. Due to the structure of these equations, it is possible to expand η_l further as

$$\eta_l = \epsilon^l \eta_l^{(l)} + \epsilon^{l+2}\eta_l^{(l+2)} + \cdots, \tag{A.1.3}$$

where, in particular, $\eta_0^{(0)}$ is the zero mode of the operator $\mathcal{O}$. Then easily one can deduce that the leading terms of η_l in each angular momentum l satisfy relations

$$\eta_1^{(1)} = \eta_0^{(0)}, \quad \eta_2^{(2)} = \frac{2}{9}\eta_1^{(1)}, \quad \cdots, \tag{A.1.4}$$

therefore intrinsically all $\eta_l^{(l)}$ are identical with $\eta_0^{(0)}$. The next-to-leading terms $\eta_l^{(l+2)}$ are determined by evaluating the next-to-leading terms in the decomposed equations (A.1.2). As an example, let us consider the first one:

$$\mathcal{O}\eta_0^{(2)} + \frac{2}{r^2}\eta_1^{(1)} = 0. \tag{A.1.5}$$

At $r \sim 0$ ($y \sim \infty$), the region which represents the place where the effect of the end point of the attached string is not expected to appear, the operator $\mathcal{O}$ is approximated by

$$\mathcal{O} \sim \frac{1}{\pi^2 g_{\mathrm{s}}^2 \omega^2} y^4 \left(1 + \frac{d^2}{dy^2}\right). \tag{A.1.6}$$

Therefore, using the asymptotic ($r \sim 0$) behavior of the solution $\eta_1^{(1)} = \eta_0^{(0)} \sim \exp(\pm iy)$ at the weak coupling limit, we obtain a solution of Eq. (A.1.5) as

$$\eta_0^{(2)} \sim \frac{1}{y}\exp(\pm iy). \tag{A.1.7}$$

This does not contribute when we discuss the phase shift of the boundary, since the magnitude of this mode is dumping fast enough at $r \sim 0$. Owing to similar argument, the following relations can be derived:

$$\eta_l^{(l+2k)} \sim y^{-k}\exp(\pm iy) \quad \text{at } r \sim 0. \tag{A.1.8}$$

These modes ($k \geq 1$) are collectively represented in Eq. (2.3.14) as "higher order terms".

A.2 String networks in the IIB picture

In this section, we present a proof of the properties (A) and (B) of Sec. 3.2.4 concerning the presence of the "generalized" string networks for arbitrary N. The following proof is based on a substitution rule which allows us to reduce the number N of the strings emerging from the D3-branes (we call these strings "external string" hereafter) by one.

For any two among the N external strings, say the strings 1 and 2, let us consider a new string 12 carrying the charge $(e_{12}, g_{12}) = (e_1 + e_2, g_1 + g_2)$. The transverse coordinate (x_{12}, y_{12}) from which the new string 12 emerges is determined by the conditions that the BPS bound E_{BPS} (3.2.5) of the energy and the angle θ of Eq. (3.2.10) are common in both the original system of N external strings $(1, 2, 3, \ldots, N)$ and the new system of $N-1$ external strings $(12, 3, \ldots, N)$. Then, this new string 12 satisfies the two properties:

(I) The junction point of the strings 1 and 2 lies on the string 12.

(II) Taking this junction point as the common end of the three strings 1, 2 and 12, the mass of the string 12 is equal to the sum of the masses of the strings 1 and 2.

To show the properties (I) and (II), note first that the condition determining (x_{12}, y_{12}) is expressed as

$$C_{12}\begin{pmatrix} x_{12} \\ y_{12} \end{pmatrix} = C_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + C_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \tag{A.2.1}$$

where C_a $(a = 1, 2, 12)$ is the 2×2 matrix given by

$$C_a = \begin{pmatrix} g_a & -e_a \\ e_a & g_a \end{pmatrix}. \tag{A.2.2}$$

Then, the property (I) claims that, for t_1 and t_2 determined by the first equality of

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t_1\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + t_2\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_{12} \\ y_{12} \end{pmatrix} + t_{12}\begin{pmatrix} u_{12} \\ v_{12} \end{pmatrix}, \tag{A.2.3}$$

there exits t_{12} which satisfies the second equality. Such t_{12} is easily found by eliminating (x_a, y_a) $(a = 1, 2)$ from Eq. (A.2.1) using Eq. (A.2.3) to get

$$t_{12}C_{12}\begin{pmatrix} u_{12} \\ v_{12} \end{pmatrix} = t_1C_1\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + t_2C_2\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \tag{A.2.4}$$

which is equivalent to

$$t_{12}\left(e_{12}^2 + g_{12}^2\right) = t_1\left(e_1^2 + g_1^2\right) + t_2\left(e_2^2 + g_2^2\right). \tag{A.2.5}$$

This equation (A.2.5) also implies the property (II) since the tension and the length of the strings are given by $T_a = 2\pi\,\mathrm{sgn}(t_a)\sqrt{e_a^2 + g_a^2}$ and $\ell_a = |t_a|\sqrt{e_a^2 + g_a^2}$, respectively:

$$T_{12}\ell_{12} = T_1\ell_1 + T_2\ell_2. \tag{A.2.6}$$

Now, the properties (I) and (II) implies that we can reduce the problem of proving (A) and (B) for N external strings to that for $N - 1$ external strings. Since we know that (A) and (B) hold for $N = 3$, we conclude that they are valid for an arbitrary N.

A.3 Solving the eigenvalue problem (3.3.31)

In this section, we shall solve the eigenvalue problem (3.3.31), namely, we obtain the eigenvalues $p(p + 1)$ and the corresponding eigenvector $v_m^{(p)}$. For this purpose let us introduce a polynomial $V^{(p)}(z)$ of a variable z as

$$V^{(p)}(z) \equiv \sum_{m=1}^{N-1} v_m^{(p)} z^m. \tag{A.3.1}$$

Then, Eq. (3.3.31) is expressed as a second order differential equation for $V^{(p)}(z)$:

$$\begin{aligned} z(z-1)^2 V^{(p)\prime\prime} + (z-1)\Big[N+1-(N-3)z\Big]V^{(p)\prime} \\ + \left[\frac{N+1}{z} - (N-1)z - p(p+1)\right] V^{(p)} = 0, \end{aligned} \tag{A.3.2}$$

where the prime denotes the differentiation with respect to z. In order to transform the differential equation (A.3.2) into a more familiar one, we rewrite it in terms of $\widetilde{V}^{(p)}(z)$ defined by

$$V^{(p)}(z) = z(1-z)^{p-1}\widetilde{V}^{(p)}(z). \tag{A.3.3}$$

The differential equation for $\widetilde{V}^{(p)}$ reads

$$\begin{aligned} z(1-z)\widetilde{V}^{(p)\prime\prime} + \Big[-N+1-(2p-N+3)z\Big]\widetilde{V}^{(p)\prime} \\ -(p-N+1)(p+1)\widetilde{V}^{(p)} = 0. \end{aligned} \tag{A.3.4}$$

This hypergeometric differential equation has a polynomial solution when p takes integer values $p = 1, 2, \ldots, N-1$, and we have

$$V^{(p)}(z) = z(1-z)^{p-1} \sum_{k=0}^{N-p-1} \frac{\binom{N-p-1}{k}\binom{k+p}{k}}{\binom{N-1}{k}} z^k. \tag{A.3.5}$$

The eigenvector $v_m^{(p)}$ can be read off from (A.3.1) and (A.3.5). For example, $v_m^{(1)}$ and $v_m^{(2)}$ are given (up to the normalization constant) by

$$v_m^{(1)} = m\overline{m}, \tag{A.3.6}$$

$$v_m^{(2)} = m\overline{m}\left(m - \overline{m}\right), \tag{A.3.7}$$

with $\overline{m} \equiv N - m$.

A.4 The coefficient c_p

In this section we derive the solution of Eq. (3.3.34) in an infinite Taylor series at $y = 1$ and obtain the coefficient c_p in (3.3.38) by an analytic continuation to the expression (3.3.35).

We should find the solutions $\tilde{\varphi}^{(p)}$ of Eq. (3.3.34) which damps faster than $(1-y)^p$ at $y \sim 1$. Although the solution of the hypergeometric differential equation (3.3.34) which is regular at $y=1$ may be expressed formally by the hypergeometric series $F(-p,-p,-2p;1-y)$, this does not make sense because of a zero in the denominators. Thus the real solution may be obtained by taking an appropriate limit of $\epsilon_i \to 0$ in the expression $F(-p+\epsilon_1,-p+\epsilon_2,-2p+\epsilon_3;1-y)$. The higher terms from order $(1-y)^{2p+1}$ of this series have a $1/\epsilon_3$ singularity at $\epsilon_3 = 0$, and the series of these singular terms[1] can be shown to satisfy by itself a hypergeometric differential equation which converges to Eq. (3.3.34) in the limit $\epsilon_i \to 0$. Since this partial series $\lim_{\epsilon_3\to 0} \epsilon_3 F(-p+\epsilon_1,-p+\epsilon_2,-2p+\epsilon_3;1-y)$ behaves like $\epsilon_1\epsilon_2$ at $\epsilon_{1,2}\to 0$, we cancel this factor and obtain a solution

$$\tilde{\varphi}^{(p)} = \lim_{\epsilon_{1,2}\to 0}\lim_{\epsilon_3\to 0}\frac{\epsilon_3}{\epsilon_1\epsilon_2}F(-p+\epsilon_1,-p+\epsilon_2,-2p+\epsilon_3;1-y), \tag{A.4.1}$$

where we wrote explicitly that the limit $\epsilon_3 \to 0$ should be taken first.

A well-known formula for an analytic continuation of the hypergeometric function reads

$$\begin{aligned} F(\alpha,\beta,\gamma;1-y) &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta-\gamma+1;y) \\ &\quad + y^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;y). \end{aligned} \tag{A.4.2}$$

We also use a well-known expansion formula for the gamma function:

$$\Gamma(\epsilon-n) = \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon}+\sum_{l=1}^{n}\frac{1}{l}+\psi(1)+O(\epsilon)\right), \tag{A.4.3}$$

where $\psi(1)$ is the Euler constant. Taking the limit in Eq. (A.4.1) by using Eqs. (A.4.2) and (A.4.3), we obtain

$$\tilde{\varphi}^{(p)} = -\frac{(p!)^2}{(2p)!}\left(2\left(\sum_{l=1}^{p}\frac{1}{l}\right)F(-p,-p,1;y)+F^*(-p,-p,1;y)+\ln(y)F(-p,-p,1;y)\right), \tag{A.4.4}$$

where we have used the definition $F^*(\alpha,\beta,\gamma;z) \equiv \left(\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}+2\frac{\partial}{\partial\gamma}\right)F(\alpha,\beta,\gamma;z)$. This is the solution (3.3.35) with $c_p = 1/(2\sum_{l=1}^{p}\frac{1}{l})$.

[1] Hence this series damps faster than $(1-y)^p$ as desired.

Appendix B

Non-Abelian Born-Infeld action

We shall show in this appendix that the BPS saturated solutions obtained in Sec. 3.2 and Sec. 2.4.3 are also solutions of the non-Abelian BI action proposed in Ref. [146]. This property for the conventional solitons are summarized in Ref. [25], and here we will present solutions of the BPS equations other than the ones considered in that reference: 1/4 BPS dyon in 4-dimensional SYM (Sec. 3.2) and 1/4 BPS state in 2-dimensional SYM (Sec. 2.4.3). We will follow the discussions given in Ref. [64], where the usual monopole and dyon solutions in $SU(2)$ SYM theory were shown to satisfy the equations of motion from the non-Abelian BI action.

The ordinary D3-brane effective action is given by the Abelian BI action

$$S = \int d^4x \, \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})}, \tag{B.0.1}$$

where we have ignored the overall numerical factor, which is irrelevant in the discussions here, and have appropriately normalized the gauge fields to be consistent with Eq. (3.2.2). In our solutions, the D3-branes fluctuate in the transverse two dimensional plane with coordinates X and Y. Hence the induced metric $g_{\mu\nu}$ on the D3-branes is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu X \partial_\nu X + \partial_\mu Y \partial_\nu Y, \tag{B.0.2}$$

where $\eta_{\mu\nu}$ denotes the flat Minkowski metric. The action (B.0.1) with the metric (B.0.2) can be simply deduced [56] by the dimensional reduction from the 6-dimensional BI action with a flat metric:

$$S = \int d^6x \, \sqrt{-\det(\eta_{ab} + F_{ab})}. \tag{B.0.3}$$

Here the fluctuations of the four-dimensional induced metric (B.0.2) are incorporated in the six-dimensional gauge fields in (B.0.3) by identifying X and Y with some components through T-duality: $X = A_4, Y = A_5$.

A non-Abelian version of the BI action suffers from the ordering ambiguities of the matrix-valued field components. This ambiguity is fixed in the proposal of Ref. [146] by taking the symmetrized trace operation STr. The action is defined by

$$S = \int d^6x \ \mathrm{STr}\sqrt{-\det(\eta_{ab} + F_{ab})}, \tag{B.0.4}$$

where the 6-dimensional field strengths are given by

$$F_{ab} = \begin{pmatrix} 0 & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & D_0X & D_0Y \\ -\mathcal{E}_1 & 0 & \mathcal{B}_3 & -\mathcal{B}_2 & D_1X & D_1Y \\ -\mathcal{E}_2 & -\mathcal{B}_3 & 0 & \mathcal{B}_1 & D_2X & D_2Y \\ -\mathcal{E}_3 & \mathcal{B}_2 & -\mathcal{B}_1 & 0 & D_3X & D_3Y \\ -D_0X & -D_1X & -D_2X & -D_3X & 0 & -i[X,Y] \\ -D_0Y & -D_1Y & -D_2Y & -D_3Y & i[X,Y] & 0 \end{pmatrix}. \tag{B.0.5}$$

Since the part inside the symmetrized trace in Eq. (B.0.4) can be expanded in a polynomial form in F_{ab}, we can treat this part as if F_{ab} were an Abelian variable before symmetrization. The variation of the action gives

$$\delta S = \frac{1}{2}\mathrm{STr}\left[\left\{\sqrt{-\det(\eta_{cd} + F_{cd})}\ (\eta + F)^{-1}_{ab}\right\}\delta F^{ba}\right]. \tag{B.0.6}$$

In evaluating (B.0.6) for our configurations, we are allowed to substitute Eqs. (3.2.8), (3.2.9), (3.3.1) and (3.3.2) into the curly bracket part $\{\cdots\}$ before the symmetrizing operation. This is because these four equations are linear relations among the components of F_{ab}. Therefore, after a little algebra, the quantity inside the curly bracket of (B.0.6) is simply replaced by $-F_{ab}$. Hence we find a similar situation as in Ref. [64], that is, there is no need to distinguish the symmetrized trace operation from the ordinary one, since we have only two non-commutative elements in the symmetrized trace of (B.0.6). Noticing that $\delta F_{ab} = D_{[a}\delta A_{b]}$ and performing an integration by parts, the equations of motion are reduced to

$$D^a F_{ab} = 0, \tag{B.0.7}$$

which are just the ones from the SYM action. The BPS conditions (3.2.8), (3.2.9), (3.3.1) and (3.3.2) satisfy Eq. (B.0.7). Therefore we have shown that our BPS saturated configuration is also a solution of the equations of motion from the non-Abelian BI action.

Precisely in the same manner, it is possible to show that the solutions satisfying the BPS equations (2.4.27), (2.4.37) and (2.4.38) from the 2-dimensional SYM are also solutions of the 2-dimensional non-Abelian BI equations of motion.

Now we have four scalars, S and X^i Thus the low energy effective action is deduced in the same way by the dimensional reduction from the 6-dimensional BI action with a flat metric, Eq. (B.0.3). We have have appropriately normalized the gauge fields to be consistent with Eq. (2.4.24). Here the fields representing the fluctuations of the D-string are identified with some components of the six-dimensional gauge fields in Eq. (B.0.3) through T-duality as $S = A_2$, $X^i = A_{i+2}$ $(i = 1, 2, 3)$. This identification results in the 6-dimensional field strength as follows:

$$F_{ab} = \begin{pmatrix} 0 & \mathcal{E} & D_0 S & D_0 X^1 & D_0 X^2 & D_0 X^3 \\ -\mathcal{E} & 0 & D_1 S & D_1 X^1 & D_1 X^2 & D_1 X^3 \\ -D_0 S & -D_1 S & 0 & i[S, X^1] & i[S, X^2] & i[S, X^3] \\ -D_0 X^1 & -D_1 X^1 & -i[S, X^1] & 0 & i[X^1, X^2] & i[X^1, X^3] \\ -D_0 X^2 & -D_1 X^2 & -i[S, X^2] & -i[X^1, X^2] & 0 & i[X^2, X^3] \\ -D_0 X^3 & -D_1 X^3 & -i[S, X^3] & -i[X^1, X^3] & -i[X^2, X^3] & 0 \end{pmatrix} . \qquad \text{(B.0.8)}$$

Using the explicit expression (B.0.8) and the BPS equations (2.4.27), (2.4.37) and (2.4.38), it is possible to reduce the equations of motion from the action (B.0.4) to the ordinary Yang-Mills equations of motion (B.0.7). Since the BPS equations (2.4.27), (2.4.37) and (2.4.38) satisfy the Yang-Mills equations of motion (B.0.7), we have shown that our BPS saturated configuration is also a solution of the equations of motion from the non-Abelian BI action.

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