CFT description of three-dimensional Hawking-Page phase transition

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Abstract

After a brief review of three-dimensional gravity with $\Lambda < 0$ and of the Hawking-Page phase transition, we consider the three-dimensional transition in terms of the boundary conformal field theory(CFT). We find that the free fermions CFT on the boundary torus has a good description for it. Around the critical temperature where semi-classical approximation does not work, the free fermions model predicts that the transition will smoothly occur through the conical space and a small black hole phase. In classical limit, the most dominant contribution there comes only from the NS-NS sector in free fermions CFT on the boundary torus. We also find that the properties of the semi-classical result can be obtained by neglecting the NS-NS sector and considering only two NS-R and R-NS sectors.

Contents

1 Introduction

Sometimes, the black hole thermodynamics plays one of the most important role of a touchstone to the quantum theory of gravity. Especially, three-dimensional case is important for this purpose because there is no local degrees of freedom and it may be easy to consider a quantum theory[1][2] [3][4][5]. There is a three dimensional black hole solution which is called the BTZ black hole[6][7]. It is an asymptotically AdS_3 black hole and has similar properties of the black hole thermodynamics to higher dimensional asymptotic anti-de Sitter black holes. This fact implies the local degrees of freedom may not be essential for the black hole thermodynamics. Therefore, it is meaningful to consider three-dimensional gravity for obtaining more profound understanding of the black hole thermodynamics.

Almost twenty years ago, Hawking and Page[8] investigated the thermodynamics of asymptotically anti-de Sitter space. They predicted that there is a phase transition between thermal anti-de Sitter space and an anti-de Sitter black hole with thermal radiation. In three-dimensional case, it is the transition between thermal three-dimensional anti-de Sitter space(AdS_3) and a BTZ black hole. This transition includes some topological change which can not occur in classical gravity, and thus it needs quantum physics of gravity.

Asymptotically AdS_3 space has the special property that the asymptotic symmetry of this space is the Virasoro symmetry with the central charge $c = 3l/2G$ [9], where l is the AdS_3 radius and G is the gravitational constant. Asymptotic symmetry includes quantum fluctuations of metric fields, which means the symmetry of the quantum theory of asymptotically AdS_3 gravity is the Virasoro symmetry that is two-dimensional conformal symmetry. The asymptotic topology of AdS_3 is a torus (of course, it becomes a cylinder at zero temperature) and thus, the quantum theory of asymptotically AdS_3 gravity is the conformal field theory on the boundary torus with the central charge $c = 3l/2G$. In this context, Coussert and Henneaux[10] showed that the mass gap between AdS_3 vacuum and a massless BTZ space can be regarded as the casimir energy in the boundary CFT with supersymmetry. Furthermore, Strominger[11] related the entropy of a large BTZ black hole to the number of the microscopic states in conformal field theory using the Cardy's formula[12]. The strominger's argument does not need supersymmetry, which is consistent with the fact that asymptotically AdS_3 does not have global supersymmetry at finite temperature in terms of three-dimensional AdS supergravity.

These studies have deep relations with the recent progress of string theory. It revealed some aspects of microscopic understanding of black holes physics in terms of D-branes. Some D-brane systems become black holes solutions in the supergravity limit and its stringy description is given by gauge theory or conformal field theory. The entropy of a D1/D5 black hole can be given by counting the number of strings attached to D-branes[13], and the Hawking radiation can be understood as the closed string emission from D-branes[14]. These investigation of D-brane black holes is an important precursor to the AdS/CFT correspondence in string theory [15][16][17] which suggests that string theory on asymptotically AdS corresponds to conformal field theory on its boundary. Birmingham et al.[18] showed the quasi-normal frequencies of BTZ black hole can be obtained from the pole of retarded correlators in the boundary CFT. It is also the case for D3-brane black holes[19]. These investigations imply that the thermal radiation on the asymptotically AdS background corresponds to some operator in the boundary CFT, as it is suggested by AdS/CFT correspondence.

In terms of AdS/CFT, Witten suggested that the Hawking-Page phase transition corresponds to the color confinement in dual boundary gauge theory[20]. In [21], the Hawking-Page transition of a BTZ black hole are related to the behavior of correlator on the boundary CFT using AdS/CFT correspondence. And it was also discussed that the three-dimensional Hawking-Page transition can be represented as the behavior of the partition function of the boundary supersymmetric CFT in [22][23], that is, its partition function becomes to gravitational instantons of thermal AdS_3 and a BTZ black hole in low and high temperature limits respectively.

There is one more connection between the three-dimensional asymptotically anti-de Sitter gravity and the boundary conformal field theory, through the Chern-Simons expression of three-dimensional gravity[4][24][25]. Asymptotic AdS_3 gravity can also be expressed in the Chern-Simons gauge theory with proper boundary conditions[26]. Also in this formalism, one can easily see there is only pure gauge in the bulk and no physical degrees of freedom there. On the other hand, the boundary degrees of freedom may not be gauge invariant so that it will be physical. The boundary theory in this case is equivalent to SL(2,R) WZW model[27][28] which is related to the Liouville theory[29][30]. There are many investigation about the relation between the BTZ black hole the WZW CFT or Liouville theory. In particular, Brotz et.al[31] discussed about the Hawking-Page transition in term of mudular invariant structure of the boundary conformal field theory.

The original investigation by Hawking and Page is based on the semi-classical approximation of Euclidean path integral. However such evaluation will break down around the critical temperature, because it needs quantum gravitational effects in order to change spacetime topology. In this thesis, we construct the phenomenological boundary CFT model which describes the three-dimensional Hawking-Page phase transition. It can be regarded as the quantum theory of asymptotically AdS_3 gravity so that it can predict the thermodynamical behavior even for temperature around the critical one. In section 2, we review the classical solutions of asymptotically AdS_3 space and show the boundary topology of these space is a torus T^2. In section 3, we review the Hawking-Page phase transition in the original manner, i.e. in the Euclidean semi-classical formulation for black holes thermodynamics. In section 4, some CFT models are considered. After the brief review of the explanation of the mass gap at the zero temperature and of the Strominger's entropy counting, we introduce CFT on the torus in subsection 4.1 and discuss about the free SCFT model in subsection 4.2. However we show the SCFT model is not so good at the Hawking-Page phase transition because of the bosonic contribution. Thus we consider a free fermions model in subsection 4.3. This model gives us a good description of the transition for temperature $T \gg T_c$ and $T \ll T_c$ in comparison with the result of section 3. This model predict that the phase transition will occur smoothly through a conical space and a small mass black hole configuration around the critical temperature. We also consider the classical limit in section 5. We show around the critical temperature the dominant contribution comes only from the NS-NS sector. The other only two NS-R and R-NS sectors, neglecting the NS-NS sector, give a similar contribution to the Euclidean semi-classical result given in section 3. We give summary and discuss about some unanswered questions in section 6.

2 Classical Solutions of Einstein Gravity

In this section, we briefly review the classical solutions of three dimensional Einstein gravity with a negative cosmological constant and show that the boundary topology of their Euclidean section is a torus T^2. The action which we consider is

$$I = \frac{1}{2\pi} \int \sqrt{-g}(R + 2l^{-1}) d^2x dt + B', \tag{1}$$

where we set the the Newton constant $G = 1/8$ which has the dimensions of an inverse energy. The cosmological constant is related to the AdS radius by $\Lambda = -1/l^2$, and B' is the surface term.

2.1 Classical solutions

The Einstein equations of the action (1) are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R + 2l^{-2}) = 0 \tag{2}$$

which, in a three-dimensional spacetime, determine the full Riemann tensor as

$$R_{\mu\nu\lambda\rho} = -l^{-2}(g_{\mu\lambda} g_{\nu\rho} - g_{\nu\lambda} g_{\mu\rho}), \tag{3}$$

describing a symmetric space of constant negative curvature, because the Weyl tensor equals to zero and Riemann tensor can be expressed by Ricci tensors.

$$\begin{aligned} C_{ijkl} &= R_{ijkl} - 2\left(g_{i[k} R_{l]j} + g_{j[l} R_{k]i} - \frac{1}{2} g_{i[k} g_{l]j} R \right) \\ &= 0. \end{aligned} \tag{4}$$

Here, $A_{i[j} B_{k]l} = A_{ij} B_{kl} - A_{ik} B_{jl}$.

Let us consider the Hamiltonian form of (1) in order to solve the Einstein equation.

$$I = \int \left(\pi^{ij} \dot{g}_{ij} - N^{\perp} \mathcal{H}_{\perp} - N^i \mathcal{H}_i \right) d^2x dt + B. \tag{5}$$

This is sometimes called the (2+1) decomposition or the ADM decomposition. The surface term B differs from the B' in the Lagrangian form (1) because the corresponding volume integral differs by a surface term. The surface deformation generators $\mathcal{H}_\perp, \mathcal{H}_i$ are given by

$$\begin{aligned}
\mathcal{H}_\perp &= 2\pi g^{-1/2}\left(\pi^{ij}\pi_{ij} - (\pi^i_i)^2\right) - (2\pi)^{-1}g^{1/2}(R + 2/l^2), &(6)\\
\mathcal{H}_i &= -2\pi^j_{i|j}. &(7)
\end{aligned}$$

Extremizing the Hamiltonian action with respect to the lapse and shift functions $N^\perp, N^i$ yields the constraint equations $\mathcal{H}_\perp = 0$ and $\mathcal{H}_i = 0$, which are the $(\perp,\perp)$ and $(\perp, i)$ components of (2). We restrict the action principle to a class of fields that possess a rotational Killing vector $\partial/\partial\phi$ and a timelike Killing vector $\partial/\partial t$. If the radial coordinate is properly adjusted, the line element may be written as

$$\begin{aligned}
ds^2 = -\left(N^\perp(r)\right)^2 dt^2 + f^{-2}(r)dr^2 + r^2\left(N^\phi(r)dt + d\phi\right)^2,\\
0 \le \phi < 2\pi, \quad t_1 \le t \le t_2. \qquad (8)
\end{aligned}$$

The form of the momenta π^{ij} may be obtained from (8) through their relation

$$\pi^{ij} = -\frac{1}{2}\pi^i_i g^{-1/2}(K^{ij} - Kg^{ij}) \qquad (9)$$

with the extrinsic curvature K_{ij}, which, for a time independent metric, is simply expressed as $2N^\perp K_{ij} = (N_{i|j} + N_{j|i})$. This gives the only component of the momentum

$$\pi^r_\phi = \frac{l}{2\pi}p(r). \qquad (10)$$

If expressions (8), (10) are introduced in the action, one finds

$$I = -(t_2 - t_1)\int dr\left(N(r)\mathcal{H}(r) + N^\phi\tilde{\mathcal{H}}_\phi\right) + B, \qquad (11)$$

with

$$\begin{aligned}
\mathcal{H} &= 2\pi f(r)\mathcal{H}_\perp = 2l^2\frac{p^2}{r^3} + (f^2)' - 2\frac{r}{l^2}, &(12)\\
\tilde{\mathcal{H}}_\phi &= 2\pi\mathcal{H}_\phi = -2lp', &(13)\\
N(r) &= f^{-1}N^\perp, &(14)
\end{aligned}$$

where a prime denotes the derivative with respect to the radial coordinate.

To find solutions under the assumptions of time independence and axial symmetry, one must extremize the reduced action (11). Variation with respect to N and N^ϕ yields that the generators $\mathcal{H}$ and $\mathcal{H}_\phi$ must vanish. These constraint equations are readily solved to give

$$
\begin{aligned}
p &= -\frac{J}{2l} && (15)\\
f^2 &= -M + \left(\frac{r}{l}\right)^2 + \frac{J^2}{4r^2}. && (16)
\end{aligned}
$$

where M and J are two constants of integration, which is identified as the Abott-Deser mass[34] and angular momentum respectively.

Variation of the action with respect to f^2 and p yields the equations

$$
\begin{aligned}
N' &= 0, && (17)\\
(N^\phi)' + \frac{2lp}{r^3}N &= 0, && (18)
\end{aligned}
$$

which determine N and N^ϕ as

$$
\begin{aligned}
N &= N(\infty), && (19)\\
N^\phi &= -\frac{J}{2r^2}N(\infty) + N^\phi(\infty). && (20)
\end{aligned}
$$

The constants of integration $N(\infty)$ and $N^\phi(\infty)$ are part of the specification of the coordinate system, which is not fully fixed by the form of the line element (8). However, at spatial infinity, the displacement of the timelike Killing vector is $N(\infty)\delta t$ and we should set $N(\infty) = 1$. It is natural to choose $N^\phi(\infty) = 0$ by adjusting the origin of the angular coordinate ϕ. Then the metric is

$$
ds^2 = -N(r)^2 dt^2 + N(r)^{-2} dr^2 + r^2 (N^\phi(r) dt + d\phi)^2, \tag{21}
$$

$$
N(r)^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \qquad N^\phi(r) = -\frac{J}{2r^2}. \tag{22}
$$

2.1.1 Anti-de Sitter space

If we set $M = -1$ and $J = 0$, then the metric becomes the covering space of the three-dimensional anti-de Sitter space (AdS_3). This negative mass can be explained by the casimir energy of the ground state in the supersymmetric CFT [10], as we briefly review in section 4. AdS_3 space can be represented as the hyperboloid

$$u^2 - x^2 - y^2 + v^2 = l^2, \tag{23}$$

in the four-dimensional space with metric

$$ds^2 = -du^2 + dx^2 + dy^2 - dv^2. \tag{24}$$

A system of coordinates covering the whole of the manifold may be introduced by setting

$$u = l \cosh\mu \sin\lambda \tag{25}$$

$$v = l \cosh\mu \cos\lambda \tag{26}$$

with $l \sinh\mu = \sqrt{x^2 + y^2}$ and $0 \leq \mu < \infty, 0 \leq \lambda < 2\pi$. Inserting (26) into (24) gives

$$ds^2 = l^2 \left[-\cosh^2\mu d\lambda^2 + \frac{dx^2 + dy^2}{l^2 + x^2 + y^2} \right]. \tag{27}$$

This expression can be further simplified by introducing polar coordinates in the (x, y) plane

$$x = l \sinh\mu \cos\theta, \quad y = l \sinh\mu \sin\theta, \tag{28}$$

which yields

$$ds^2 = l^2(-\cosh^2\mu d\lambda^2 + d\mu^2 + \sinh^2\mu d\theta^2) \tag{29}$$

for the metric of AdS_3. Because λ is angle, there are closed timelike curves in AdS_3. For this reason, one "unwraps"the λ coordinate, that is, one does not identify λ with $\lambda + 2\pi$. The space thus obtained is the universal covering of anti-de Sitter space. It is this space which, by a common abuse of language, will be called anti-de Sitter space. If the unwrapped λ is denoted by t/l and if one sets $r = l \sinh\mu$, then one obtains

$$ds^2 = -(1 + \frac{r^2}{l^2})dt^2 + (1 + \frac{r^2}{l^2})^{-1}dr^2 + r^2 d\theta^2. \tag{30}$$

which is the metric (22) with $M = -1$ and $J = 0$ and ϕ is replaced by θ.

By construction, the space has the isometry $SO(2, 2)$, and it is homogeneous and isotropic. The Killing vectors are

$$J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b}, \tag{31}$$

where $x^a = (v, u, x, y)$ or, in detail,

$$\begin{aligned} J_{01} &= v\partial_u - u\partial_v, \quad J_{02} = x\partial_v + v\partial_x, && (32)\\ J_{03} &= y\partial_v + v\partial_y, \quad J_{12} = x\partial_u + u\partial_x, && (33)\\ J_{13} &= y\partial_u + u\partial_y, \quad J_{23} = y\partial_x - x\partial_y. && (34)\\ & && (35) \end{aligned}$$

The vector J_{01} generates "time" displacement ($J_{01} = \partial_\lambda$), whereas J_{23} generates rotations in the (x, y) plane ($J_{23} = \partial_\theta$). The most general Killing vector is given by

$$\frac{1}{2}\omega^{ab} J_{ab}, \quad \omega^{ab} = -\omega^{ba}. \tag{36}$$

The Euclidean metric is obtained by replacing the time t to the Euclidean time $t_E := it$ as

$$ds_E^2 = (1 + \frac{r^2}{l^2})dt_E^2 + (1 + \frac{r^2}{l^2})^{-1} dr^2 + r^2 d\phi^2, \tag{37}$$

which is a solution of the Euclidean action of (1). If the Euclidean time has a periodicity $t_E \sim t_E + \beta$, this Euclidean metric expresses AdS_3 space filled with thermal radiation at the temperature $T = 1/\beta$. This interpretation is justified by the Euclidean path integral formalism for gravitational fields.

2.1.2 Conical space

The next type of classical solutions is a conical space which is a one-parameter family of classical solutions and interpolate between the AdS_3 and massless BTZ spacetime which we will introduce below. The metric is

$$ds^2 = -(\gamma + \frac{r^2}{l^2})dt^2 + (\gamma + \frac{r^2}{l^2})^{-1} dr^2 + r^2 d\phi^2, \tag{38}$$

where γ is the mass of point particle and $0 < \gamma < 1$. At the origin of this coordinates, this spacetime has a conical and naked singularity with defect angle

$$\Delta\phi = 2\pi(1 - \sqrt{\gamma}). \tag{39}$$

The interpretation of this singularity is the point particle sitting at the origin [32] [33].

2.1.3 BTZ black holes

The metric (22) with $M > 0$, $|J| \leq Ml$ means a black hole spacetime, though three dimensional classical gravity has no local degrees of freedom. It is known as the BTZ black hole [6][7][35][36]. The BTZ solution has generally two horizons which is given by the solution of $N(r)^2 = 0$

$$r_\pm^2 = \frac{Ml^2}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}}\right), \tag{40}$$

where r_+ is the event horizon and r_- is the inner horizon. There is one more special value of r

$$r_{erg} = lM^{1/2}, \tag{41}$$

at which g_{00} vanishes. These three special value of r obey the relation

$$r_- \leq r_+ \leq r_{erg}. \tag{42}$$

Just as it happens in 3+1 dimensions for the Kerr metric, r_+ is the black-hole horizon, r_{erg} is the surface of infinite redshift, and the region between r_+ and r_{erg} is the ergosphere.

This spacetime has constant negative curvature, because the Riemann tensor is a constant multiple of an antisymmetrized product of metric tensors, see (3). It is well known that such a spacetime must arise from identifications of points in anti-de Sitter space through a discrete subgroup of its symmetry group SO(2,2). In this case, the discrete subgroup is generated by one element, the exponential of a particular Killing vector

$$\xi = \frac{r_+}{l}J_{12} - \frac{r_-}{l}J_{03} - J_{13} + J_{23}. \tag{43}$$

Throughout anti-de Sitter space this vector can be spacelike, null, or timelike. The whole of the black hole geometry is the region where ξ is spacelike, that is, the regions ξ is null or timelike are cut out. The boundaries of the black hole region are the surfaces $\xi^2 = 0$ which correspond to $r = 0$ in the metric (22). One cannot continue past these boundaries because ξ becomes timelike and the identification would produce closed timelike curves. Of course this black hole space has no closed timelike curves. Therefore, the black hole singularity at $r = 0$ is not curvature singularity, but rather a singularity in the causal structure. This is one of the features for the BTZ black hole.

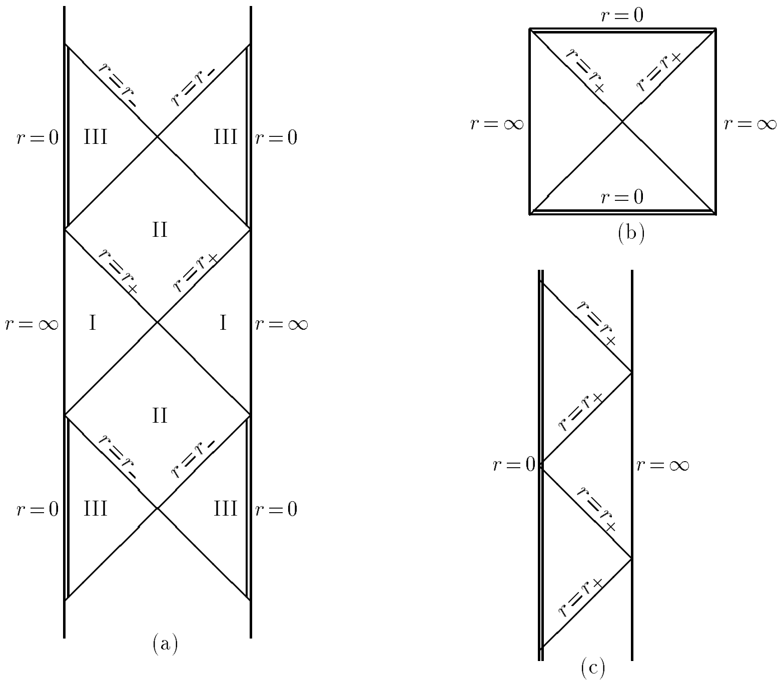

Figure 1

Figure 1: The Penrose diagram for (a) the generic BTZ black hole; (b) the static ($J = 0$) BTZ black hole; and (c) the extreme ($J = \pm Ml$) black hole.

This identification can be represented more explicitly[37][38][39]. We can combine AdS_3 coordinates (x, y, u, v) into a 2×2 matrix,

$$\mathbf{X} = \frac{1}{\ell} \begin{pmatrix} u + x & v + y \\ -v + y & u - x \end{pmatrix}, \qquad \det|\mathbf{X}| = 1, \tag{44}$$

i.e., $\mathbf{X} \in$ SL(2,R). The isometry in AdS_3 may now be represented as elements of the group SL(2,R)$\times$ SL(2,R)$/Z_2 \approx$ SO(2,2): the two copies of SL(2,R) act by left and right multiplication, $\mathbf{X} \rightarrow \rho_{\mathbf{L}} \mathbf{X} \rho_{\mathbf{R}}$, where $\rho_L, \rho_R \in$ SL(2,R), and $\rho_L \mathbf{X} \rho_{\mathbf{R}}$ is also an element of SL(2,R). As $(-\rho_L, -\rho_R)$ act in the same way, $(\rho_L, \rho_R) \sim (-\rho_L, -\rho_R)$, which means the quotient by Z_2. The relevant region of the universal covering space of anti-de Sitter space $\widetilde{AdS_3}$ (see [7]) may be covered by an infinite set of coordinate patches of three types, corresponding to the regions of the Penrose diagram of the figure 1(a):

I. $(r \geq r_+)$

$$x = l\sqrt{\alpha}\sinh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \quad , \quad y = l\sqrt{\alpha - 1}\cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$
$$u = l\sqrt{\alpha}\cosh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \quad , \quad v = l\sqrt{\alpha - 1}\sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$

II. $(r_- \leq r \leq r_+)$

$$x = l\sqrt{\alpha}\sinh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \quad , \quad y = -l\sqrt{1 - \alpha}\sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$
$$u = l\sqrt{\alpha}\cosh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \quad , \quad v = -l\sqrt{1 - \alpha}\cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$

III. $(0 \leq r \leq r_-)$

$$x = l\sqrt{-\alpha}\cosh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \quad , \quad y = -l\sqrt{1 - \alpha}\sinh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right)$$
$$u = l\sqrt{-\alpha}\sinh\left(\frac{r_+}{l}\phi - \frac{r_-}{l^2}t\right) \quad , \quad v = -l\sqrt{1 - \alpha}\cosh\left(\frac{r_+}{l^2}t - \frac{r_-}{l}\phi\right) \tag{45}$$

where

$$\alpha(r) = \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right), \quad -\infty \leq \phi \leq \infty, \ -\infty \leq t \leq \infty. \tag{46}$$

If one put these coordinates into (24), the BTZ metric can be obtained in each patch. However the angle ϕ in (45) has infinite range. To make it into a true angular variable, we must identify ϕ and $\phi + 2\pi$. The displacement $\phi \to \phi + 2\pi$ is a boost in the $x - u$ and the $y - v$ planes and corresponds to an element (ρ_L, ρ_R) of SL(2,R) $\times$ SL(2,R)$/Z_2$ with

$$\rho_L = \begin{pmatrix} e^{\pi(r_+ - r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ - r_-)/\ell} \end{pmatrix}, \qquad \rho_R = \begin{pmatrix} e^{\pi(r_+ + r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ + r_-)/\ell} \end{pmatrix}. \tag{47}$$

The BTZ black hole may thus be viewed as a quotient space $\widetilde{AdS_3}/\langle(\rho_L, \rho_R)\rangle$, where $\langle(\rho_L, \rho_R)\rangle$ denotes the whole transformation generated by (ρ_L, ρ_R).

The vacuum state, namely, what is to be regarded as empty space, is obtained by making the black hole disappear, that is by letting the horizon

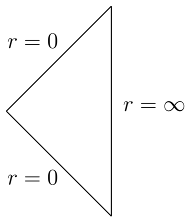

Figure 2: The Penrose diagram for a $M=0, J=0$ BTZ black hole.

size go to zero. This amounts to letting $M \to 0$, which requires $J \to 0$. One thus obtains the line element

$$ds^2 = -(r/l)^2 dt^2 + (r/l)^{-2} dr^2 + r^2 d\phi^2. \tag{48}$$

This space corresponds to the Ramond vacuum state in the boundary supersymmetric CFT, as we show in section4.

In this context, one sees that AdS_3 emerges as a "bound state," separated from the continuous black hole mass spectrum by a mass gap of one unit. This state can not be deformed continuously into the vacuum (48), because the deformation would require going through a sequence of naked singularities which are not included in the classical configuration space. Thus, in order to do it, one has to consider quantum theory of gravity.

The BTZ black hole also has thermodynamical properties just as higher dimensional black holes[6][40][39]. As we show in the next section, the entropy of a BTZ black hole obeys the Bekenstein-Hawking law

$$S = \frac{A}{4G} = 4\pi r_+, \tag{49}$$

where A is the area of the event horizon. The Hawking temperature of this black hole is

$$T_H = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+}. \tag{50}$$

The Euclidean metric of a BTZ black hole is

$$ds_E^2 = N_E(r)^2 dt_E^2 + N_E(r)^{-2} dr^2 + r^2 (N^\phi(r) dt + d\phi)^2, \tag{51}$$

$$N_E(r)^2 = -M + \frac{r^2}{l^2} - \frac{J_E^2}{4r^2}, \qquad N^\phi(r) = -\frac{J_E}{2r^2}, \tag{52}$$

where $t_E = it,\ J_E = -iJ$ is the Euclidean time and the Euclidean angular momentum respectively. This metric is positive definite for the outer region $r > r_+$. For non-extremal case, i.e. $r_+ \neq r_-$, the near horizon topology is $R^2 \times S^1$; the coordinates $r - r_+$ and t_E are the radial and angular coordinates in R^2 respectively, and the event horizon $r = r_+$ is at the origin of it. The apparent singularity at the event horizon is just like the singularity at the origin of polar coordinates and can be removed if the periodicity β of the Euclidean time is the inverse of the Hawking temperature:

$$\beta = \frac{2\pi l^2}{r_+} = \frac{1}{T_H}. \tag{53}$$

Thus, the boundary of the so-called Euclidean section $(r \geq r_+)$ is spatial infinity $r \to \infty$ and does not include the event horizon.

2.2 The boundary topology and coordinates

In the previous subsection, we introduce a general BTZ black hole with angular momentum, but, hereafter, we consider only the spherically symmetric case in order to discuss the Hawking-Page phase transition.

For the purpose of investigating the asymptotic topology, we introduce following coordinates

$$y := \frac{r_+}{r} \exp\left[\pi l T_H \phi\right] \tag{54}$$

$$z := \left(\frac{r^2 - r_+^2}{r^2}\right) \exp\left[2\pi T_H (l\phi + it_E)\right]. \tag{55}$$

Then the metric of a Euclidean BTZ black hole can be expressed in these coordinates as

$$ds^2 = \frac{l^2}{y^2}\left(dz d\bar{z} + dy^2\right). \tag{56}$$

For spatially asymptotic region $r \to \infty$, these coordinates behave like $y \to 0$ and

$$z \to e^{2\pi T_H (l\phi + it_E)}. \tag{57}$$

In these coordinates, the periodicity of the Euclidean time $t_E \sim t_E + \frac{1}{T_H}$ is satisfied automatically, but the angular periodicity $\phi \sim \phi + 2\pi$ makes z periodic:

$$z \sim z \times e^{4\pi^2 l T_H}. \tag{58}$$

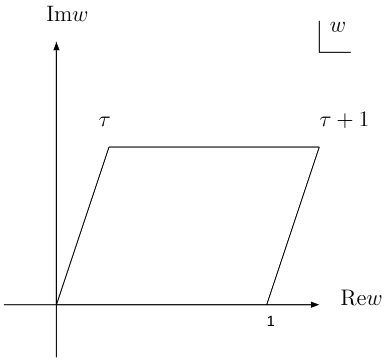

Figure 3: Torus with modular parameter τ.The upper and lower edges are identified and are the right- and left- hand edges.

We introduce the boundary coordinate

$$w := \frac{i}{2\pi} \log z = -T_H t_E + ilT_H \phi, \tag{59}$$

which has two periodicities;

$$w \sim w + 1 \tag{60}$$
$$w \sim w + \tau_{BTZ}, \tag{61}$$

where

$$\tau_{BTZ} := 2\pi l T_H i. \tag{62}$$

This means the topology of asymptotic Euclidean BTZ black hole is a torus T^2 with modular parameter τ_{BTZ}.

On the other hand, the metric of Euclidean AdS_3 with temperature T_H is

$$ds^2 = \left(1 + \frac{r^2}{l^2}\right) dt_E^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2, \tag{63}$$

where the Euclidean time has the periodicity $t_E \sim t_E + \frac{1}{T_H}$. We introduce the asymptotic coordinates of this Euclidean space

$$y' := \left(\frac{l^2}{r^2 + l^2}\right)^{1/2} e^{\frac{t_E}{l}}, \tag{64}$$

$$z' := \left(\frac{r^2}{r^2 + l^2}\right)^{1/2} e^{t_E/l - i\phi}. \tag{65}$$

Then the metric is expressed in these coordinates as

$$ds^2 = \frac{l^2}{y'^2}\left(dz'd\bar{z}' + dy'^2\right) \tag{66}$$

For the asymptotic region $r \to \infty$, these coordinates behave as $y' \to 0$ and

$$z' \to e^{\frac{t_E}{l} - i\phi}. \tag{67}$$

The angular periodicity $\phi \sim \phi + 2\pi$ is trivial, but the periodicity of the Euclidean time makes the coordinate z' periodic in the other direction:

$$z' \sim z' \times e^{1/T_H l}. \tag{68}$$

We introduce the boundary coordinates for the asymptotic AdS_3

$$w' := \frac{i}{2\pi}\log z' = \frac{\phi}{2\pi} + i\frac{t_E}{2\pi l}. \tag{69}$$

This complex coordinate has the periodicity as follows;

$$w' \sim w' + 1 \tag{70}$$

$$w' \sim w' + \tau_{AdS}, \tag{71}$$

where

$$\tau_{AdS} := \frac{i}{2\pi l T_H}. \tag{72}$$

Hence, the boundary topology of thermal AdS_3 is also a torus T^2. The modular parameter of this torus has a relation with that of the boundary BTZ torus:

$$\tau_{AdS} = -\frac{1}{\tau_{BTZ}}. \tag{73}$$

This relation means that these two modular parameter are related by the modular transformation ($\tau \to -\frac{1}{\tau}$) which corresponds to the interchange between two coordinates ϕ and t_E. Therefore the boundary of the Euclidean BTZ black hole and that of thermal AdS_3 with temperature T_H are always the same torus. Thus we can use either modular parameter as that of the boundary torus.

In low temperature limit ($\beta \to 0$), the modular parameter τ_{BTZ} becomes to zero, and the boundary coordinate w is not suitable. On the other hand,

τ_{AdS} goes to infinity. Thus the boundary topology becomes a cylinder. We introduce the coordinates on the cylinder as

$$\sigma_1 = 2\pi \mathrm{Re}\ w', \quad \sigma_2 = 2\pi \mathrm{Im}\ w', \tag{74}$$

where σ_1 is periodic in the spatial direction: $\sigma_1 \sim \sigma_1 + 2\pi$. σ_2 denotes the Euclidean time on the cylinder without periodicity. We call σ_1 and σ_2 the cylinder coordinates. In high temperature limit ($\beta \to 0$), the topology also becomes a cylinder. In this case, the boundary coordinate w can be useful.

3 Hawking-Page phase transition

In this section we briefly review the Hawking-Page phase transition that is a transition between thermal anti-de Sitter space and AdS black holes spacetime with thermal radiation. In subsection 3.1, we consider four-dimensional case following the original paper [8]. Then, our attention will be restricted to three-dimensional case in subsection 3.2.

3.1 Four-dimensional Hawking-Page phase transition

Let us consider the Schwarzschild-anti-de Sitter black hole whose metric is

$$ds^2 \;=\; -V(r)dt^2 + V(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{75}$$

where

$$V(r) \;=\; 1 - \frac{2MG_4}{r} + \frac{r^2}{l^2}, \tag{76}$$

and $l = (-3/\Lambda)^{1/2}$. G_4 is the four-dimensional gravitational constant. This metric has a event horizon at $r = r_+$, where $V(r_+) = 0$. For the Euclidean metric ($t \to t_E = it$), the region $r > r_+$ is positive definite and we call it the Euclidean section. As in the case of a BTZ black hole, the apparent singularity at the event horizon ($r = r_+$) can be removed if the Euclidean time β can be regarded as an angular coordinate with period

$$\beta = \frac{4\pi l^2 r_+}{l^2 + 3r_+^3}. \tag{77}$$

The β is the inverse of the Hawking temperature T_H of this black hole. From this equation (77), we can see that β has a maximum value of $2\pi 3^{-1/2} l$ and therefore T has a minimum value of $T_0 = (2\pi)^{-1} 3^{1/2} l^{-1}$ which is equivalent to $r_+ = r_0 = 3^{-1/2} l$. For $r_+ > r_0$, the temperature T increases with the mass

$$M = \frac{r_+}{2G_4}\left(1 + \frac{r_+^2}{l^2}\right). \tag{78}$$

The argument of Hawking and Page is based on the Euclidean path integral formulation of the black holes thermodynamics[41][42]. This formulation seems not to have strong theoretical foundations because of many difficulties

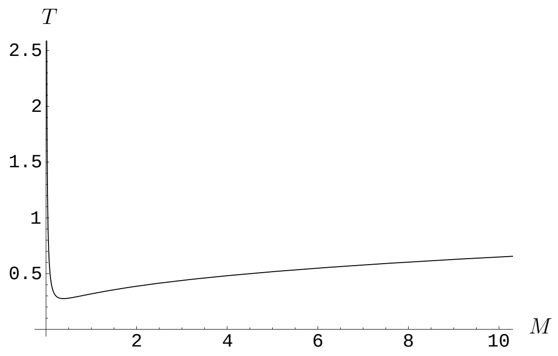

Figure 4: The relation between mass and temperature of Schwarzschild-anti-de Sitter is shown. For sufficiently low temperature, there is no black hole equilibrium, but for high temperature, two black hole equilibrium exists. The lower mass has negative specific heat and the higher mass has positive specific heat. We set $l = G_4 = 1$ in this figure.

to quantizing gravitational fields. However it gives us clear perspective for the black hole thermodynamics, corresponding usual statistical mechanics. One may regard it as one of low energy effective theories which is obtained by integrating out some dynamical degrees of freedom in the quantum theory of gravity such as string theory.

Let us consider the canonical ensemble of black holes, that is, the situation in which a black hole is in an equilibrium with thermal radiation in the bulk at the Hawking temperature. The partition function of this system is defined by a path integral over all matter fields and metric fields which tends asymptotically respectively to zero and to anti-de Sitter space identified periodically in a Euclidean time t_E with period $\beta = 1/T_H$.

$$Z = \int_{t_E \sim t_E + \beta} \mathcal{D}g\mathcal{D}\phi \; e^{-I_E[g] - I_m[\phi]} \tag{79}$$

where I_E, I_m are the Euclidean action of gravitational fields and of all matter fields respectively.

The path integral over the matter fields on the anti-de Sitter background can be regarded as giving the contribution of thermal radiation in anti-de Sitter space to the partition function Z. It will be evaluated as

$$\log Z_{rad} \approx l^3 T^3. \tag{80}$$

where we dropped out the numerical factors. The energy density ρ will be $\rho \approx T^4$ which comes from the Stephan-Boltzmann's law. Then the Schwarzschild radius r_H of the matter in the sphere with the radius l is

$$r_H \sim G_4 T^4 l^3. \tag{81}$$

When this Schwarzschild radius is comparable with the radius of sphere, the gravitational collapse will occur at the temperature

$$T \sim G_4^{-1/4} l^{-1/2}. \tag{82}$$

This is dynamical instability.

For sufficiently lower temperature, the radiation will not have any effect to their background and we can neglect the matter contribution to the path integral if $G_4 \ll l$ and $T \sim 1/l$. In other words, the gravitational contribution to the partition function is dominant there.

We may evaluate the path integral by the semi-classical approximation.

$$Z \approx e^{-I_E}[\hat{g}], \tag{83}$$

where $\hat{g}$ is the metric of classical solutions. However, both the AdS_4 spacetime and the black hole spacetime have infinite volume. One may compute the difference between the Euclidean action of the black hole metric and that of anti-de Sitter space identified with the same physical period in Euclidean time. Let us consider the cutoff R on the radial integrations, and the regularized volume of the AdS_4 is

$$V_1(R) = \int_0^{\beta'} dt_E \int_0^R dr \int_{S^2} d\Omega_2, \tag{84}$$

and the regularized volume of the black hole spacetime is

$$V_2(R) = \int_0^{\beta} dt_E \int_{r_+}^R dr \int_{S^2} d\Omega_2. \tag{85}$$

The black hole spacetime is smooth only if β has the value given in (77), but for the AdS_4, any value of β' is possible. One must adjust β' so that the physical temperature at the hypersurface $r = R$ is the same in the two spacetime. Thus,

$$\beta' \sqrt{1 + r^2/l^2} = \beta \sqrt{1 - 2G_4 M/r + r^2/l^2}. \tag{86}$$

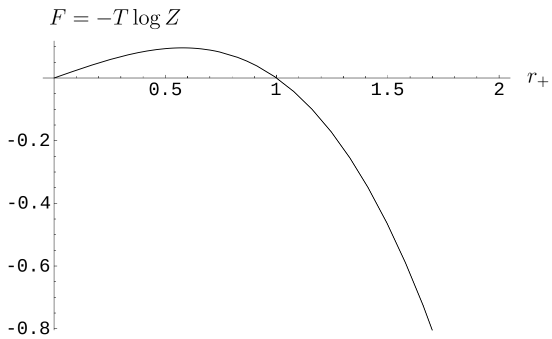

Figure 5: The difference of free energy between Schwarzschild-AdS and thermal AdS_4 are shown. For $r_+ > l \Leftrightarrow T > T_1$, a black hole spacetime are more probable than pure radiation on AdS_4. We set $l = G_4 = 1$ in this figure.

Then the difference of these action is

$$I = \frac{\Lambda}{8\pi G_4} \lim_{R\to\infty} (V_2(R) - V_1(R)) = \frac{\pi r_+^2 (l^2 - r_+^2)}{G_4(l^2 + 3r_+^2)}. \tag{87}$$

Therefore,

$$\log Z = -I = -\frac{\pi r_+^2 (l^2 - r_+^2)}{G_4(l^2 + 3r_+^2)}. \tag{88}$$

The difference of the free energy is

$$F = -T \log Z = -\frac{r_+(l^2 - r_+^2)}{4G_4 l^2}. \tag{89}$$

We show the free energy (89) in Figure 5. The expectation value of energy is

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z = \frac{G_4}{2} r_+ \left(1 + \frac{r_+^2}{l^2}\right) = M. \tag{90}$$

The entropy is

$$S = \beta \langle E \rangle + \log Z = \frac{A}{4G}, \tag{91}$$

where A is the area of the event horizon. Thus the relation between entropy and area is the same as in the case of a asymptotically flat black hole spacetime, that is, the entropy of a black hole equals to one quarter of the area of the event horizon[44][45][46]. This is the well-known Bekenstein-Hawking law.

For temperature $T < T_0$, the only possible equilibrium is thermal radiation without a black hole. If $T > T_0$, there are two possible equilibrium states, that is, two black holes with thermal radiation. The black hole with lower mass has negative specific heat and is therefore unstable to decay either into pure radiation or to the larger mass black hole. The black hole with the higher value of the mass has positive specific heat and therefore at least locally stable. If

$$T_0 < T < T_1 := \frac{1}{\pi l}, \tag{92}$$

the free energy of the pure radiation is less than that of the black hole so that the black hole is less probable than pure thermal radiation , that is, the most dominant contribution to the partition function is the pure thermal radiation state without a black hole. If $T_1 \lesssim T$, the free energy of higher mass black hole will be less than that of pure radiation. The pure radiation will then tend to tunnel to the black hole configuration. This is the Hawking-Page phase transition. The critical temperature in this case is T_1.

3.2 Three-dimensional phase transition

In three-dimensional case, usually the classical actions may be evaluated by choosing the boundary term in Eq.(5) to satisfy $\delta I = 0$ on the boundary as well as on the bulk [6][23][43].

$$\begin{aligned} \delta I_E[\hat{g}_{BTZ}] &= \beta \int_{r_+}^{\infty} \left[N\delta\mathcal{H} + 2\pi N^\phi \delta\tilde{\mathcal{H}}_\phi \right] + \delta B \\ &\quad +(\text{terms vanishing when equations of motion hold}) \qquad (93) \\ &= -\beta\delta M + \beta\delta M(r_+) + \delta B \qquad (94) \\ &= -\beta\delta M + 4\pi\delta r_+ + \delta B \qquad (95) \\ &= -\beta\delta M + \frac{\delta A}{4G} + \delta B, \qquad (96) \end{aligned}$$

where the Euclidean actions are given by $I_E = -iI$. Thus the surface term can be determined as

$$B = \beta M - \frac{A}{4G} \tag{97}$$

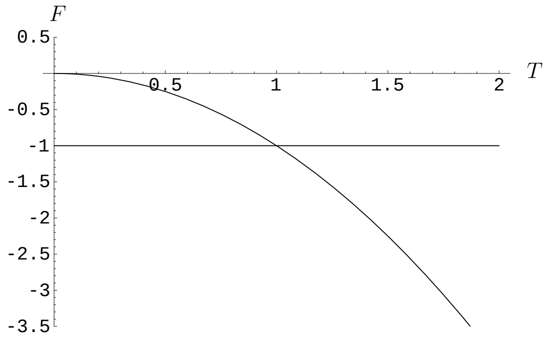

Figure 6: The free energy of a BTZ black hole and of pure radiation in AdS_3 are shown. We set $8G = 1$ and $l = (2\pi)^{-1}$ so that $T_c = 1$.

The action for a BTZ black hole and AdS_3 are:

$$\begin{aligned} I_E[\hat{g}_{BTZ}] &= \beta M - \frac{A}{4G}, &(98)\\ &= \beta M - 4\pi r_+ &(99)\\ I_E[\hat{g}_{AdS_3}] &= -\beta. &(100)\end{aligned}$$

The contributions to the partition function are

$$\begin{aligned} Z_{BTZ}(T) &= \exp(-\beta M + 4\pi r_+) = \exp\left(\frac{(\pi l)^2 T}{2G}\right) &(101)\\ Z_{AdS_3}(T) &= \exp(\beta) = \exp\left(\frac{1}{8GT}\right). &(102)\end{aligned}$$

Sometimes, these contributions are called as gravitational instantons. Then we can obtain the free energy of each thermal spacetime as in the same way with usual statistical mechanics:

$$\begin{aligned} F_{BTZ} &= -\frac{1}{\beta}\log Z_{BTZ} = -(2\pi l)^2 T^2 &(103)\\ F_{AdS_3} &= -1. &(104)\end{aligned}$$

These are shown in the figure 6.

The expectation value of the energy in each space are

$$\langle E\rangle_{BTZ} = -\frac{\partial}{\partial\beta}\log Z = \partial_\beta(-\frac{4\pi^2 l^2}{\beta}) = M = (2\pi l T)^2 \qquad (105)$$

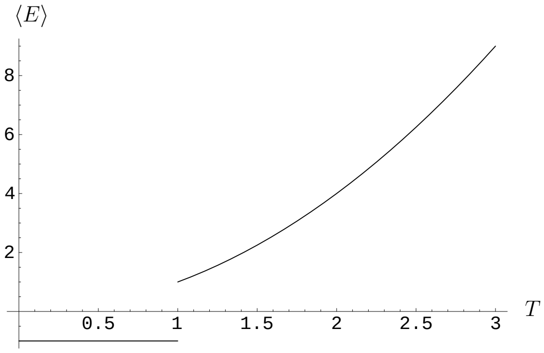

Figure 7: The energy of asymptotically AdS_3 space are shown. The phase transition occur at the critical temperature $T = T_c$. It seems to change discontinuously at the critical temperature, but the semi-classical approximation does not work there. The constants l and G are set to $1/2\pi, 1/8$ respectively so that $T_c = 1$.

$$\langle E\rangle_{AdS_3} = -1. \tag{106}$$

This means the energy of spacetime is given by the Abott-Deser mass in asymptotically anti-de Sitter space. These energy are shown in the figure 7.

The entropy is

$$S_{BTZ} = \beta\langle E\rangle_{BTZ} + \log Z_{BTZ} = 4\pi r_+ = \frac{A}{4G} \tag{107}$$

$$S_{AdS_3} = \beta\langle E\rangle_{AdS_3} + \log Z_{AdS_3} = 0 \tag{108}$$

where A is the area of the event horizon. This is the Bekenstein-Hawking law which is the same as in the four-dimensional case.

For a spherically symmetric BTZ black hole,

$$A = 2\pi r_+ = 2\pi l\sqrt{M}. \tag{109}$$

This means that the density of states $N(M)$ for the black hole grows like $\exp(4\pi l\sqrt{M})$. This is sufficiently slow that the integral defining the partition function

$$Z = \int N(M)e^{-M/T}dM \tag{110}$$

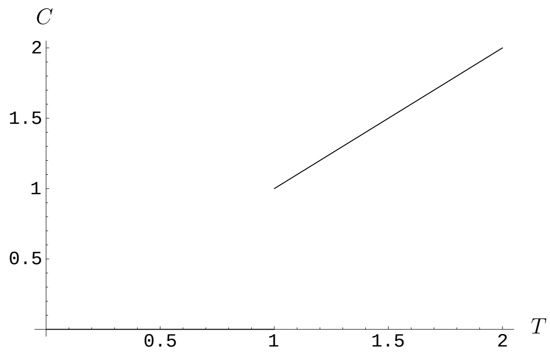

Figure 8: The specific heat of asymptotic AdS_3 are shown. The Euclidean semi-classical approach can not predict at the $T = T_c$. For $T < T_c$, it is zero; for $T > T_c$, it grows linearly. The constants l and G are set to $1/2\pi, 1/8$ respectively.

converges. This means canonical ensemble in asymptotically anti-de Sitter space is well behaved unlike the case of asymptotically flat space in which the canonical ensemble is pathological[42]. It is well-known that a thermal state in flat Minkowski spacetime, no matter how low temperature, is unstable to formation of a black hole[47]. For a Schwarzschild black hole, the Hawking temperature is given by

$$T_S = \frac{1}{8\pi G M_S} \tag{111}$$

where M_S is the mass of the Schwarzschild black hole. Thus the specific heat of this black hole is negative, and a black hole can only be in an unstable equilibrium with radiation at the same temperature in a sufficiently large volume to keep the temperature constant. Any perturbation resulting in an increase of the black hole mass would reduce its temperature below that of its surroundings. Then it cause more absorption than emission so that the black hole continues to grow. This is not the case for a BTZ black hole because of its positive specific heat.

For all temperature, there are two possible equilibrium states which are thermal AdS_3 and a BTZ black hole with thermal radiation. This is one of different points to the four dimensional case in which for sufficiently low temperature the only possible equilibrium is thermal radiation in AdS as we show in the previous subsection. At the temperature $T = T_c := (2\pi l)^{-1}$, the free energy of thermal radiation accords with that of a BTZ black hole.

One can see this from figure 6. For temperatures $T < T_c$, the free energy of thermal radiation is lower than that of a BTZ black hole which means that a BTZ black hole is less possible than pure thermal radiation. On the other hand, if $T > T_c$, the free energy of the BTZ black hole will be less than that of pure radiation. Then pure radiation will tend to tunnel to the BTZ black hole configuration. This is the Hawking-Page phase transition, which means the most dominant classical solution (or state) changes as temperature does.

At the critical temperature $T = T_c$, the energy of spacetime seems to change discontinuously as in the figure 7. However, the evaluation of the partition function in this argument is only depends on the semi-classical approximation which will break near the critical temperature because this phase transition needs a topological change and requires quantum gravitational analysis. Therefore we have to consider alternative approach in order to investigate this phase transition near the critical temperature.

4 CFT description of three dimensional Hawking-Page phase transition

Three dimensional asymptotic anti-de Sitter gravity has a tight relation with conformal field theory. The three dimensional gravity has no local gravitational degrees of freedom, and the boundary theory can determine all the classical gravitational physics. Brown and Henneaux[9] found the asymptotic symmetry of three dimensional anti-de Sitter space is the Virasoro symmetry with central charge $c = 3l/2G$ which is two dimensional conformal symmetry. This fact strongly suggests that the quantum theory of three dimensional anti-de Sitter gravity is the boundary conformal field theory because the asymptotic symmetry also includes off-shell fluctuation of metric fields.

In this context, Coussaert and Henneaux[10] explained the mass gap at zero temperature between AdS_3 vacuum and a massless BTZ in terms of supersymmetric conformal field theory. At zero temperature, the boundary topology of asymptotically AdS_3 is a cylinder. AdS_3 vacuum has supersymmetry and the covariantly constant killing spinors which are antiperiodic under $\phi \to \phi + 2\pi$. This antiperiodicity implies that AdS_3 vacuum corresponds to the Neveu-Schwarz sector in conformal field theory on the boundary cylinder. Let us assume the mass and angular momentum operator of asymptotically AdS_3 are

$$\hat{M} = \frac{1}{l}(L_0 + \bar{L}_0 - \frac{c}{12}), \quad \hat{J} = L_0 - \bar{L}_0, \tag{112}$$

where L_0 and $\bar{L}_0$ are the lowest Virasoro generators. Then the mass of the Neveu-Schwarz ground state is

$$\begin{aligned} \hat{M}|0\rangle_{NS} &= \frac{1}{l}(L_0 + \bar{L}_0 - \frac{c}{12})|0\rangle_{NS} && (113) \\ &= -\frac{c}{12l}|0\rangle_{NS} = -\frac{1}{8G}|0\rangle_{NS}, && (114) \end{aligned}$$

where we use the fact $c = 3l/2G$. On the other hand, massless BTZ space has also supersymmetry and is ground state of AdS supergravity with periodic boundary conditions on the spinor field. Accordingly massless BTZ space corresponds to Ramond sector in CFT on the boundary cylinder. The mass of Ramond ground state is

$$\hat{M}|0\rangle_R = \frac{1}{l}(L_0 + \bar{L}_0 - \frac{c}{12})|0\rangle_R = 0. \tag{115}$$

Thus we can regard the mass gap of asymptotically AdS_3 at zero temperature as the casimir energy in CFT.

Furthermore, Strominger[11] related the entropy of a large BTZ black hole to the number of the microscopic states in conformal field theory through the Cardy's formula, assuming the above relation (112). For the finite temperature system or a massive BTZ black hole, there is no supersymmetry and the Strominger's argument does not use supersymmetry. The Cardy's formula teach us the entropy of Virasoro eigenstates as

$$S = 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{c\bar{\Delta}}{6}}, \qquad (\text{for} \ \ \Delta, \bar{\Delta} \gg c) \tag{116}$$

where $\Delta, \bar{\Delta}$ are the eigenvalue of $L_0, \bar{L}_0$ respectively. We derive this formula in Appendix A. Using (112) and $\Delta, \bar{\Delta} \gg c$, the eigenvalues $\Delta, \bar{\Delta}$ can be expressed as

$$\Delta \simeq \frac{1}{2}(lM + J), \qquad \bar{\Delta} \simeq \frac{1}{2}(lM - J). \tag{117}$$

Thus the entropy of this state is

$$S = \pi\sqrt{\frac{l(lM+J)}{2G}} + \pi\sqrt{\frac{l(lM-J)}{2G}} = \frac{A}{4G}, \tag{118}$$

which is in exact agreement with the Bekenstein-Hawking entropy for the BTZ black hole. This entropy counting can be applied only to a large BTZ black hole: $M \gg 1/8G$.

In this section, we construct CFT models which describe the Hawking-Page phase transition in the statistical meaning. Of course, our interests is restricted to finite temperature systems in which there is no supersymmetry, thus we do not assume a priori supersymmetry. We review the partition function on a torus in subsection 4.1 and consider the free superconformal field theory(SCFT) in subsection 4.2. The free fermions CFT is investigated as a improved model in subsection 4.3.

4.1 The free CFT partition function on a torus

The boundary topology of asymptotically AdS_3 is a torus T^2 as we showed in subsection 2.2. In this subsection we review the partition function of a free CFT on the torus. The details are written in [48]. A torus can be obtained by identifying the two edge of a cylinder: a modular parameter τ denote how to identify them in the cylinder coordinates. The partition function on the torus is given by a trace over all states which are identified at the ends of the cylinder. Thus, the partition function is

$$
\begin{aligned}
Z(\tau) &= \mathrm{Tr}\left[\exp\left(2\pi i\tau_1 P - 2\pi\tau_2 H\right)\right] && (119)\\
&= \mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right). && (120)
\end{aligned}
$$

Here $q := \exp(2\pi i\tau)$, the momentum $P = L_0 - \bar{L}_0$ generates translations of the spatial direction on the cylinder , and the Hamiltonian $H = L_0 + \bar{L}_0 - \frac{c}{12}$ evolves the Euclidean time on the cylinder.

The first example is D free bosons. The central charge of a free boson is 1: it is D for D free bosons. The partition function is

$$
\begin{aligned}
Z_b(\tau) &= \mathrm{Tr}[q^{L_0-\frac{D}{24}}\bar{q}^{\bar{L}_0-\frac{D}{24}}] && (121)\\
&= (q\bar{q})^{-\frac{D}{24}}\mathrm{Tr}\left[q^{\sum_{n=0}^{\infty}\alpha_{-n}\cdot\alpha_n}\bar{q}^{\sum_{n=0}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n}\right] && (122)\\
&= (\tau_2)^{-\frac{D}{2}}(q\bar{q})^{-\frac{D}{24}}\prod_{n=1}^{\infty}\left(1-q^n\right)^{-D}\left(1-\bar{q}^n\right)^{-D} && (123)\\
&= (\tau_2)^{-\frac{D}{2}}|\eta(\tau)|^{-2D}, && (124)
\end{aligned}
$$

where $\eta(\tau)$ is the Dedekind's eta function and $\tau_2 = \mathrm{Im}\tau$. This function is invariant under the modular transformation: $\tau \to \tau + 1, \tau \to -\frac{1}{\tau}$, as one can easily check it by using (231).

The next example is a free fermion. We shall assume that a free fermion should be periodic or antiperiodic in each two compactified directions. Then, we are restricted to the four boundary conditions on a torus. We associate the names of Ramond (R) to the periodic boundary condition and Neveu-Schwarz (NS) to the antiperiodic one. These conditions are denoted as,

$$
\psi(w+1) = e^{2\pi i\nu}\psi(w), \qquad \psi(w+\tau) = e^{2\pi i\mu}\psi(w) \tag{125}
$$

where $w = \frac{1}{2\pi}(\sigma_1 + i\sigma_2)$ and ν, μ take the value 1 or 1/2. The four boundary conditions are put together as follows:

$$
\begin{aligned}
(\nu,\mu) &= (0,0) \qquad ;(\text{R, R}) && (126)\\
(\nu,\mu) &= (0,\frac{1}{2}) \qquad ;(\text{R, NS}) && (127)
\end{aligned}
$$

$$(\nu,\mu) = (\frac{1}{2},0) \qquad ;(\text{NS, R}) \tag{128}$$

$$(\nu,\mu) = (\frac{1}{2},\frac{1}{2}) \qquad ;(\text{NS, NS}). \tag{129}$$

We shall denote by $Z_{\nu,\mu}$ the partition function associated with the periodicity condition (ν,μ). We suppose the the antiholomorphic component have the same boundary conditions as the holomorphic one. Because of the decoupling between ψ and $\bar{\psi}$, we may consider the partition function obtained by integrating the holomorphic field only, which we call $d_{\nu,\mu}$. It follows that

$$Z_{\nu,\mu} = |d_{\nu,\mu}|^2. \tag{130}$$

The expression for the holomorphic partition functions are $d_{\nu,\mu}$:

$$\begin{aligned}
d_{0,0} &= \frac{1}{\sqrt{2}}\text{Tr}(-1)^F q^{(L_0)_R-\frac{1}{48}} = \frac{1}{\sqrt{2}}\text{Tr}(-1)^F q^{\sum_{k=1}^{\infty} kb_{-k}b_k+\frac{1}{24}} && (131)\\
d_{0,\frac{1}{2}} &= \frac{1}{\sqrt{2}}\text{Tr}\; q^{(L_0)_R-\frac{1}{48}} = \frac{1}{\sqrt{2}}\text{Tr}\; q^{\sum_{k=1}^{\infty} kb_{-k}b_k+\frac{1}{24}} && (132)\\
d_{\frac{1}{2},0} &= \text{Tr}(-1)^F q^{(L_0)_{NS}-\frac{1}{48}} = \text{Tr}(-1)^F q^{\sum_{k=1}^{\infty} kb_{-k}b_k-\frac{1}{48}} && (133)\\
d_{\frac{1}{2},\frac{1}{2}} &= \text{Tr}\; q^{(L_0)_{NS}-\frac{1}{48}} = \text{Tr}\; q^{\sum_{k=1}^{\infty} kb_{-k}b_k-\frac{1}{48}}, && (134)
\end{aligned}$$

where F denotes the fermion number operator. These partition functions may easily be calculated as follows:

$$\begin{aligned}
d_{0,0} &= \frac{1}{\sqrt{2}}q^{1/24}\prod_{n=0}^{\infty}(1-q^n) = 0 && (135)\\
d_{0,\frac{1}{2}} &= \frac{1}{\sqrt{2}}q^{1/24}\prod_{n=0}^{\infty}(1+q^n) = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} && (136)\\
d_{\frac{1}{2},0} &= q^{-1/48}\prod_{r=1/2}^{\infty}(1-q^r) = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}} && (137)\\
d_{\frac{1}{2},\frac{1}{2}} &= q^{-1/48}\prod_{r=1/2}^{\infty}(1+q^r) = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}, && (138)
\end{aligned}$$

where $\theta_i(\tau)$ are the elliptic theta functions, whose definition and properties are put together in the Appendix B. It is now evident that the way to obtain a modular invariant partition function is to include in the theory the three possibilities (NS,R), (R,NS) and (NS,NS), leading to the modular invariant combination

$$\begin{aligned} Z(\tau) &= Z_{0,\frac{1}{2}}(\tau) + Z_{\frac{1}{2},0}(\tau) + Z_{\frac{1}{2},\frac{1}{2}}(\tau) && (139) \\ &= \left|\frac{\theta_2(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|. && (140) \end{aligned}$$

One can easily check the modular invariance by using the equation (229), (230), (231) and (231).

For D free fermions, the modular invariant partition function can be obtained in the same way. The result is

$$Z(\tau) = \left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^D + \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^D + \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^D. \tag{141}$$

For the SCFT including D free bosons and D free fermions, the modular invariant partition function is given by

$$Z_{SCFT}(\tau) = (Z_b(\tau))^D \left[\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^D + \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^D + \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^D\right]. \tag{142}$$

In this subsection, we do not restrict a modular parameter τ up to the present. If we choose it as that of asymptotically AdS_3 (72), the partition function can be expressed as follows:

$$Z(\tau) = \mathrm{Tr}\left[q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right] = \mathrm{Tr}\left[e^{-\beta \hat{M}}\right]. \tag{143}$$

We can interpret this partition function as that of thermal asymptotically AdS_3 in the same sense with the path integral (79). The expression (143) imply the Hamiltonian of this finite temperature system is $\hat{M}$ defined in Eq.(112). If we consider the system with angular momentum as in [23], the modular parameter is

$$\tau = -\frac{\Omega}{2\pi T_H} + \frac{i}{2\pi l T_H}, \tag{144}$$

where Ω is the Euclidean angular velocity. Then the partition function is

$$Z(\tau) = \mathrm{Tr}\left[e^{-\beta\hat{M} + \beta\Omega\hat{J}_E}\right], \tag{145}$$

where $\hat{J}_E = -i\hat{J}$. The operator $\hat{J}$ can be regarded as the charge conjugate to Ω, thus the operator $\hat{J}$ is the angular momentum of the thermal asymptotically AdS_3 system. Therefore it is consistent to set the energy and angular momentum operators as in (112).

4.2 The free SCFT model

As we mentioned before, the mass gap at zero temperature can be understood as the difference of the vacuum energy in the boundary CFT with the superconformal algebra. In this subsection, we consider a free SCFT model. Supersymmetry requires that the number of bosons is the same as that of fermions. The central charge of a free boson and a free fermion is 1 and 1/2 respectively. Thus free SCFT with the central charge c includes $2c/3$ free bosons and $2c/3$ free fermions. The partition function on the torus is

$$\begin{aligned} Z_{SCFT}(\tau) &= (Z_b(\tau))^{\frac{2c}{3}} \left[\sum_{i=2}^{4} \left| \frac{\theta_i(\tau)}{\eta(\tau)} \right|^{\frac{2c}{3}} \right] \\ &= (\tau_2)^{-c/3} |\eta(\tau)|^{-2c} \left[\sum_{i=2}^{4} |\theta_i(\tau)|^{2c/3} \right], \end{aligned} \tag{146}$$

where $\tau = \frac{i}{2\pi l T}$ is the modular parameter of the boundary torus of asymptotically AdS_3. In this case, τ is pure imaginary, then $q = e^{-1/lT}$, $\eta(\tau)$ and $\theta_i(\tau)$ are real.

In low temperature limit $T \to 0$, the Dedekind's eta function and elliptic theta functions are

$$q \to 0, \qquad \eta(\tau) \to q^{1/24}, \tag{147}$$

$$\theta_2(\tau) \to 2q^{1/8}, \qquad \theta_3(\tau),\ \theta_4(\tau) \to 1. \tag{148}$$

Thus, the partition function behaves as

$$Z(T) \to 2(\tau_2)^{-\frac{c}{3}} q^{-\frac{c}{12}} \tag{149}$$

$$= 2(2\pi l T)^{\frac{l}{2G}} e^{\frac{1}{8GT}} = 2(2\pi l T)^{\frac{l}{2G}} Z_{AdS_3}(T). \tag{150}$$

For sufficiently low temperature, the most dominant contribution to the partition function is $Z_{AdS_3}(T)$.

In high temperature limit $T \to \infty$ $(q \to 1)$, it seems difficult to evaluate the Dedekind's Eta function and elliptic theta functions, but we can evaluate it by using formulae (229),(230),(231) and (231), or by using modular invariance of the partition function,

$$Z(\tau) = Z(-1/\tau). \tag{151}$$

If we set $\tilde{q} = e^{-2\pi i/\tau} = e^{-4\pi^2 lT}$, then $\tilde{q} \to 0$ in high temperature limit. In this limit, the Dedekind's eta function and elliptic theta functions are

$$\begin{aligned}
\eta(\tau) &= (-i\tau)^{-\frac{1}{2}}\eta(-1/\tau) \;\to (-i\tau)^{-\frac{1}{2}}(\tilde{q})^{\frac{1}{24}}, & (152)\\
\theta_2(\tau) &= (-i\tau)^{-\frac{1}{2}}\theta_4(-1/\tau) \;\to (-i\tau)^{-\frac{1}{2}}, & (153)\\
\theta_3(\tau) &= (-i\tau)^{-\frac{1}{2}}\theta_3(-1/\tau) \;\to (-i\tau)^{-\frac{1}{2}}, & (154)\\
\theta_4(\tau) &= (-i\tau)^{-\frac{1}{2}}\theta_2(-1/\tau) \;\to 2(-i\tau)^{-\frac{1}{2}}(\tilde{q})^{\frac{1}{8}}. & (155)
\end{aligned}$$

Therefore the partition function behaves as follows;

$$\begin{aligned}
Z_{SCFT}(T) &\to 2(\tau_2)^{c/3}(\tilde{q})^{-\frac{c}{12}} & (156)\\
&= 2(2\pi lT)^{-\frac{l}{2G}}e^{\frac{\pi^2 l^2 T}{2G}} = 2(2\pi lT)^{-\frac{l}{2G}}Z_{BTZ}(T). & (157)
\end{aligned}$$

For sufficiently high temperature, the most dominant contribution comes from $Z_{BTZ}(T)$.

These behaviors of the partition function (146) implies that the Hawking-Page transition will occur in this SCFT model as noted by J.Maldacena, A.Strominger [22] and S.Mano [23]. We can see that the behavior of the partition function changes at the self dual point under the modular transformation $\tau \to -1/\tau$. At this point,

$$\tau = -\frac{1}{\tau} \quad \Leftrightarrow \quad \tau = i \quad \Leftrightarrow \quad T = T_c = \frac{1}{2\pi l}. \tag{158}$$

This temperature is the same as the critical temperature evaluated by the Euclidean semi-classical approach in section 3.

The expectation value of energy in this model is

$$\begin{aligned}
\langle E\rangle_{SCFT} &= -\partial_\beta \log Z_{SCFT} & (159)\\
&= \frac{c}{3}T + \frac{4c}{3}\frac{\partial_\beta \eta}{\eta} - \frac{2c/3}{\sum_i |\theta_i|^{2c/3}}\left[\sum_{i=2}^{4} |\theta_i|^{2c/3}\left(\frac{\partial_\beta \theta_i}{\theta_i} - \frac{\partial_\beta \eta}{\eta}\right)\right] & (160)
\end{aligned}$$

In low temperature limit $T \to 0$, the β derivative of the eta and theta functions are

$$\begin{aligned}
\partial_\beta \eta(\tau) &\to -\frac{1}{24l}q^{\frac{1}{24}}, \quad \partial_\beta \theta_2(\tau) \to -\frac{1}{4l}q^{\frac{1}{8}}, & (161)\\
\partial_\beta \theta_3(\tau) &\to -\frac{1}{l}q^{\frac{1}{2}}, \qquad \partial_\beta \theta_4(\tau) \to \frac{1}{l}q^{\frac{1}{2}}. & (162)
\end{aligned}$$

Thus the expectation value of energy behaves as

$$\begin{aligned}\langle E\rangle_{SCFT} &= -\frac{1}{8G}+\frac{c}{3}T+\mathcal{O}(e^{-\frac{1}{2lT}}) \\ &= M_{AdS_3}+\frac{c}{3}T+\mathcal{O}(e^{-\frac{1}{2lT}}) \qquad (T\to 0).\end{aligned} \tag{163}$$

The leading contribution $-1/8G$ is the energy of AdS_3 and the second linear term means that there is difference to M_{AdS_3} even for $T \ll T_c$. This difference also exists in the classical limit $G \to 0$ or $c \to \infty$. It comes only from the bosonic part of the partition function. In high temperature limit, we can evaluate the behavior as in the case of the partition function. In this limit, the derivative of the eta and elliptic theta functions are

$$\partial_\beta\eta(-1/\tau) \;\to\; \frac{1}{24}(4\pi^2 lT^2)\tilde{q}^{\frac{1}{24}}, \quad \partial_\beta\theta_2(-1/\tau)\to\frac{1}{4}(4\pi^2 lT^2)\tilde{q}^{\frac{1}{8}}, \tag{164}$$

$$\partial_\beta\theta_3(-1/\tau) \;\to\; (4\pi^2 lT^2)\tilde{q}^{\frac{1}{2}}, \qquad \partial_\beta\theta_4(-1/\tau)\to-(4\pi^2 lT^2)\tilde{q}^{\frac{1}{2}}. \tag{165}$$

Therefore,

$$\langle E\rangle_{SCFT} = 4\pi^2 l^2 T^2-\frac{c}{3}T+\mathcal{O}(T^2 e^{-2\pi^2 lT}) \tag{166}$$

$$= M_{BTZ}(T)-\frac{c}{3}T+\mathcal{O}(T^2 e^{-2\pi^2 lT}) \qquad (T\to\infty). \tag{167}$$

there is also a linear term about T even in the classical limit $G \to 0 (c\to\infty)$. We can see these behavior of the energy (160) in the Figure 9.

The specific heat of asymptotically AdS_3 is the derivative of the expectation value of energy about temperature,

$$\begin{aligned}C_{SCFT} &:= \frac{d}{dT}\langle E\rangle_{SCFT} \\ &= \frac{c}{3}-\frac{4c}{3T^2}\left(\frac{\partial_\beta^2\eta}{\eta}-(\frac{\partial_\beta\eta}{\eta})^2\right) \\ &\quad +\frac{2c}{3T^2\tilde{Z}}\left[\sum_{i=2}^{4}\theta_i^{2c/3}\left(\frac{\partial_\beta^2\theta_i}{\theta_i}-\frac{\partial_\beta^2\eta}{\eta}-(\frac{\partial_\beta\theta_i}{\theta_i})^2+(\frac{\partial_\beta\eta}{\eta})^2\right)\right] \\ &\quad +\frac{2c}{3T^2\tilde{Z}^2}\left[\tilde{\sigma}_{23}+\tilde{\sigma}_{34}+\tilde{\sigma}_{42}\right],\end{aligned} \tag{168}$$

where we set

$$\tilde{Z} := \theta_2^{2c/3}+\theta_3^{2c/3}+\theta_4^{2c/3}, \tag{169}$$

$$\tilde{\sigma}_{ij} := (\theta_i\theta_j)^{2c/3}\left(\frac{\partial_\beta\theta_i}{\theta_i}-\frac{\partial_\beta\theta_j}{\theta_j}\right). \tag{170}$$

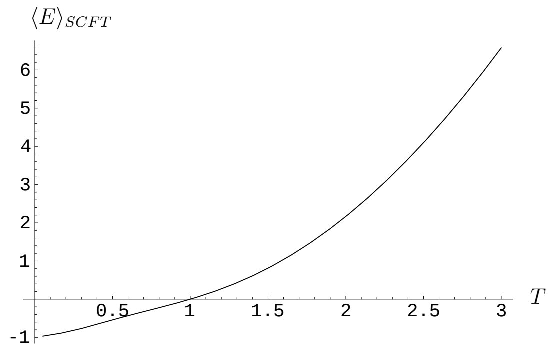

Figure 9: The expectation value of energy (160) in SCFT model. We set $8G = 1$ and $l = (2\pi)^{-1}$ so that $T_c = 1$. For temperature $T < T_c$, $\langle E\rangle$ behaves as (163), and as (167) for $T > T_c$.

It is shown in the Figure 10.

Unlike the case of the Euclidean semi-classical approach, the energy $\langle E\rangle_{SCFT}$ has linear term and the specific heat C_{SCFT} does not equals to zero even for temperature $T \ll T_c$ in classical limit. This implies that the SCFT model is suitable only for the zero temperature and not for the finite temperature system.

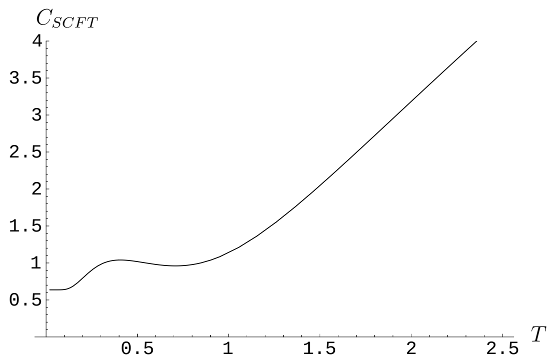

Figure 10: The specific heat in SCFT model. We set $8G = 1$ and $l = (2\pi)^{-1}$ so that $T_c = 1$. This figure shows $C_{SCFT} \neq 0$ even though $T \ll T_c$.

4.3 The free fermion model

The expectation value of energy in SCFT model has a linear term in (163) and in (167) which is different to the Euclidean semi-classical result (105) and (106). This difference comes only from bosonic part of the partition function, thus we consider a pure fermionic model in this subsection.

The central charge of a free fermion is 1/2; for D free fermions it is $D/2$. Thus $2c$ free fermion CFT is also one of candidates for the boundary theory of asymptotic finite temperature AdS_3 system. The partition function of this model on the boundary torus is

$$Z_f(\tau) = \mathrm{Tr}\left[q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right] \tag{171}$$

$$= \left(\frac{\theta_2(\tau)}{\eta(\tau)}\right)^{2c} + \left(\frac{\theta_3(\tau)}{\eta(\tau)}\right)^{2c} + \left(\frac{\theta_4(\tau)}{\eta(\tau)}\right)^{2c}. \tag{172}$$

Of course, this partition function is modular invariant

$$Z_f(\tau) = Z_f(-1/\tau), \tag{173}$$

$$Z_f(\tau + 1) = Z_f(\tau), \tag{174}$$

thus we can evaluate the behavior in low and high temperature limit as in the case of the SCFT model. In these limit, this partition function behaves as follows:

$$Z_f(\tau) \rightarrow 2q^{-\frac{c}{12}} \simeq e^{\frac{1}{8GT}} = Z_{AdS_3}(T), \qquad (T \rightarrow 0) \tag{175}$$

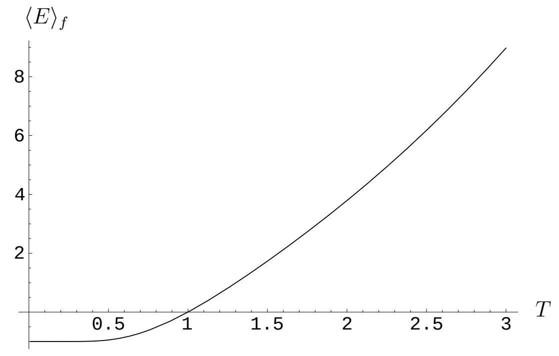

Figure 11: The expectation value of energy (178) in the free fermion model. We set $8G = 1$ and $l = (2\pi)^{-1}$ so that $T_c = 1$. For temperature $T < T_c$, $\langle E\rangle$ behaves as (180), and as (182) for $T > T_c$.

$$Z_f(\tau) \;\;\rightarrow\;\; 2\tilde{q}^{-\frac{c}{12}} \simeq e^{4\pi^2 l^2 T} = Z_{BTZ}(T). \qquad (T \rightarrow \infty) \tag{176}$$

Therefore the Hawking-Page transition will also occur in this model.

The expectation value of energy is

$$\langle E\rangle_f \;=\; -\partial_\beta \log Z_f \tag{177}$$

$$\;=\; -\frac{2c}{\sum_i |\theta_i|^{2c}} \left[\sum_{i=2}^{4} |\theta_i|^{2c} \left(\frac{\partial_\beta \theta_i}{\theta_i} - \frac{\partial_\beta \eta}{\eta} \right) \right]. \tag{178}$$

In low and high temperature limit, this becomes

$$\langle E\rangle_f \;=\; -\frac{1}{8G} + \mathcal{O}(e^{-\frac{1}{2lT}}) \tag{179}$$

$$\;=\; M_{AdS_3}(T) + \mathcal{O}(e^{-\frac{1}{2lT}}), \qquad (T \rightarrow 0) \tag{180}$$

$$\langle E\rangle_f \;=\; 4\pi^2 l^2 T^2 + \mathcal{O}(T^2 e^{-2\pi^2 lT}) \tag{181}$$

$$\;=\; M_{BTZ}(T) + \mathcal{O}(T^2 e^{-2\pi^2 lT}). \qquad (T \rightarrow \infty) \tag{182}$$

There is no linear term in these asymptotic expressions (180) and (182) unlike the case of the SCFT model. For temperature $T \gg T_c$ and $T \ll T_c$, $\langle E\rangle_f$ is approximately the same as that of the Euclidean semi-classical result. The graph of (178) is shown in the Figure 11.

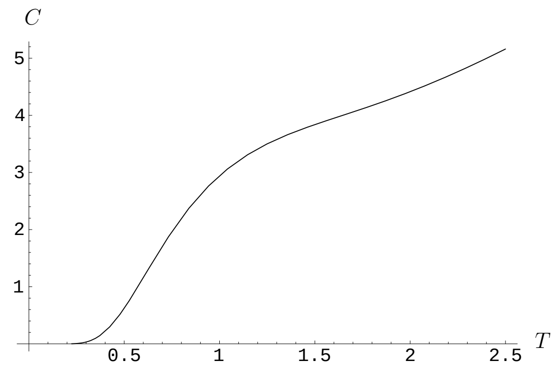

Figure 12: The specific heat in free fermion model. We set $8G = 1$ and $l = (2\pi)^{-1}$ so that $T_c = 1$. For temperature $T \ll T_c$, $C_f = 0$, and this is the same as the case in section3.

The specific heat in this model is

$$\begin{aligned} C_f &:= \frac{d}{dT}\langle E\rangle_f && (183)\\ &= \frac{2c}{T^2\bar{Z}(\tau)}\left[\sum_{i=2}^{4}\theta_i^{2c}\left(\frac{\partial_\beta^2\theta_i}{\theta_i}-\frac{\partial_\beta^2\eta}{\eta}-(\frac{\partial_\beta\theta_i}{\theta_i})^2+(\frac{\partial_\beta\eta}{\eta})^2\right)\right] \\ &\quad +\frac{2c}{T^2\bar{Z}^2(\tau)}\left[\bar{\sigma}_{23}(\tau)+\bar{\sigma}_{34}(\tau)+\bar{\sigma}_{42}(\tau)\right], && (184)\end{aligned}$$

where

$$\begin{aligned} \bar{Z}(\tau) &:= \theta_2^{2c}+\theta_3^{2c}+\theta_4^{2c}, && (185)\\ \bar{\sigma}_{ij}(\tau) &:= (\theta_i\theta_j)^{2c}\left(\frac{\partial_\beta\theta_i}{\theta_i}-\frac{\partial_\beta\theta_j}{\theta_j}.\right). && (186)\end{aligned}$$

The behavior of this specific heat are shown in the Figure 12. It is approximately zero for temperature $T \ll T_c$ and grows as a linear function for $T \gg T_c$ as in the case of the Euclidean semi-classical approach.

Therefore free fermion model is more desirable model to describe the Hawking-Page phase transition than free SCFT model.

5 Classical limit in the free fermion model

For temperature which is far from T_c, the expectation value of energy and specific heat predicted by the free fermion model are almost the same as those of the Euclidean semi-classical result, but not for temperature $T \sim T_c$. One might expect that, in classical limit $G \to 0$, the transition in the free fermion CFT model will become sharp and close to the Euclidean semi-classical result, but this is not the case as we show in this section.

The figure 13 and 14 show the ratio $\theta_2(\tau)/\theta_3(\tau)$ and $\theta_4(\tau)/\theta_3(\tau)$ around the critical temperature T_c. They show $\theta_2/\theta_3 < 1$ and $\theta_4/\theta_3 < 1$ if $T \sim T_c$. Thus, in classical limit $G \to 0$ ($c \to \infty$),

$$\left(\frac{\theta_2}{\theta_3}\right)^{2c} \to 0, \quad \left(\frac{\theta_4}{\theta_3}\right)^{2c} \to 0, \qquad (\text{for } T \sim T_c). \tag{187}$$

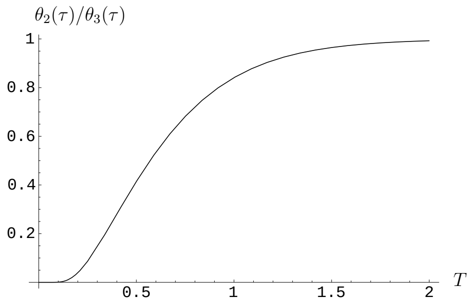

Figure 13: The ratio $\theta_2(\tau)/\theta_3(\tau)$ as a function of temperature T is shown.

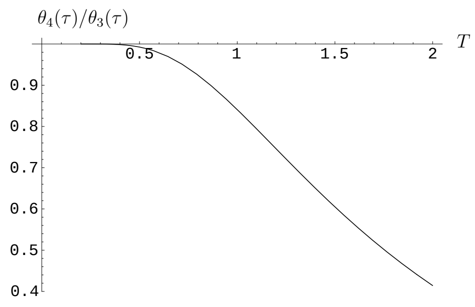

Figure 14: The ratio $\theta_4(\tau)/\theta_3(\tau)$ as a function of temperature T is shown.

The expectation value of energy in the free fermion model(178) can also be rewritten as

$$\begin{aligned} \langle E \rangle_f &= -\frac{2c}{1 + (\frac{\theta_2}{\theta_3})^{2c} + (\frac{\theta_4}{\theta_3})^{2c}} \\ &\quad \times \left[\left(\frac{\theta_3'}{\theta_3} - \frac{\eta'}{\eta} \right) + \left(\frac{\theta_2}{\theta_3} \right)^{2c} \left(\frac{\theta_2'}{\theta_2} - \frac{\eta'}{\eta} \right) + \left(\frac{\theta_4}{\theta_3} \right)^{2c} \left(\frac{\theta_4'}{\theta_4} - \frac{\eta'}{\eta} \right) \right], \end{aligned}$$

where a prime denotes the β derivative. Then, in classical limit, it behaves

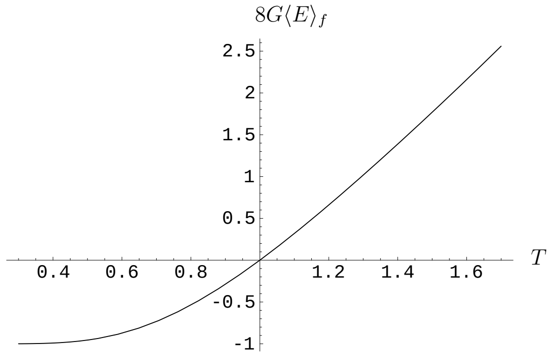

Figure 15: The expectation value of energy (188) around the critical temperature T_c. This is not sharp at the critical temperature unlike the case of the Euclidean semi-classical approach. We set $l := 1/2\pi$, so $T_c = 1$ in this graph.

as

$$8G\langle E\rangle_f \sim -24l\left(\frac{\partial_\beta\theta_3}{\theta_3} - \frac{\partial_\beta\eta}{\eta}\right), \qquad (G \to 0), \tag{188}$$

where we scale $\langle E\rangle_f \to 8G\langle E\rangle_f$ in order to compare it with the non-classical case. This contribution comes only from the NS-NS sector. This means that for temperature $T \sim T_c$ the NS-NS sector becomes dominant relative to the R-NS or NS-R sector in classical limit. The NS-NS sector is invariant by itself under the transformation $\tau \to -1/\tau$. Of course, it is not invariant under the other modular transformation $\tau \to \tau + 1$. The graph of the function (188) is shown in the figure 15. It shows that, even though in the classical limit, the behavior of the energy in the free fermion model does not approach that in the Euclidean semi-classical case around the critical temperature. The Euclidean semi-classical approach includes only the contributions from two classical solutions of the Einstein gravity: AdS_3 and BTZ. This fact imply that the free fermion model includes not only these two classical solutions but also conical space or other non-classical contributions to the partition function which become dominant around the critical temperature. We can see that the transition occur smoothly around the critical temperature through the conical space and small black holes phase. This is the prediction from this model.

On the other hands, if we may neglect the NS-NS sectors, the system

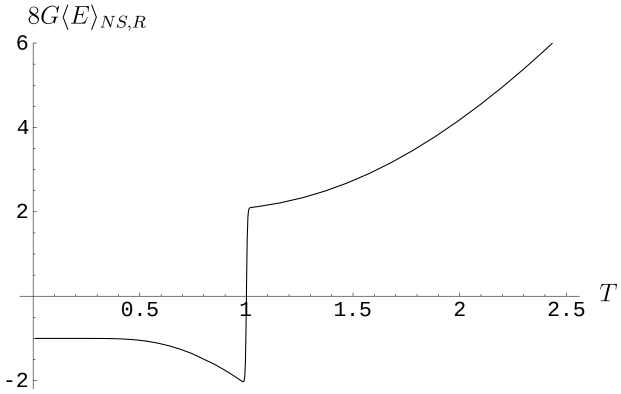

Figure 16: This shows (192). We set $8G = 0.01$ and $l = (2\pi)^{-1}$ so that $T_c = 1$. At the critical temperature T_c, it jumps continuously.

consisting of only two other NS-R and R-NS sectors in the free fermion model behaves like the Euclidean semi-classical result. The partition function of these two sector is also invariant under the transformation $\tau \to -1/\tau$ by interchanging each other,

$$Z_{(\mathrm{R;NS})}(\tau) + Z_{(\mathrm{NS;R})}(\tau) = \left(\frac{\theta_2(\tau)}{\eta(\tau)}\right)^{2c} + \left(\frac{\theta_4(\tau)}{\eta(\tau)}\right)^{2c} \tag{189}$$

$$= \left(\frac{\theta_4(-\frac{1}{\tau})}{\eta(-\frac{1}{\tau})}\right)^{2c} + \left(\frac{\theta_2(-\frac{1}{\tau})}{\eta(-\frac{1}{\tau})}\right)^{2c} \tag{190}$$

$$= Z_{(\mathrm{NS;R})}\left(-\frac{1}{\tau}\right) + Z_{(\mathrm{R;NS})}\left(-\frac{1}{\tau}\right). \tag{191}$$

This partition function, however, is not invariant under $\tau \to \tau + 1$. Thus that of only these two sectors is not modular invariant.

The expectation value of energy and the specific heat in the NS-R and R-NS sector are

$$8G\langle E\rangle_{\mathrm{NS,R}} = -24l\left[\frac{\theta_2^{2c}}{\theta_2^{2c}+\theta_4^{2c}}\left(\frac{\theta_2'}{\theta_2} - \frac{\eta'}{\eta}\right) + \frac{\theta_4^{2c}}{\theta_2^{2c}+\theta_4^{2c}}\left(\frac{\theta_4'}{\theta_4} - \frac{\eta'}{\eta}\right)\right], \tag{192}$$

$$\begin{aligned} 8G\, C_{\mathrm{NS,R}} &= \frac{24l}{T^2}\frac{\theta_2^{2c}}{\theta_2^{2c}+\theta_4^{2c}}\left[\sum_{i=2,4}\left(\frac{\theta_i''}{\theta_i} - \frac{\eta''}{\eta} - (\frac{\theta_i'}{\theta_i})^2 + (\frac{\eta'}{\eta})^2\right)\right] \\ &\quad +24l\frac{2c}{T^2}\frac{(\theta_2\theta_4)^{2c}}{(\theta_2^{2c}+\theta_4^{2c})^2}\left(\frac{\theta_2'}{\theta_2} - \frac{\theta_4'}{\theta_4}\right)^2, \end{aligned} \tag{193}$$

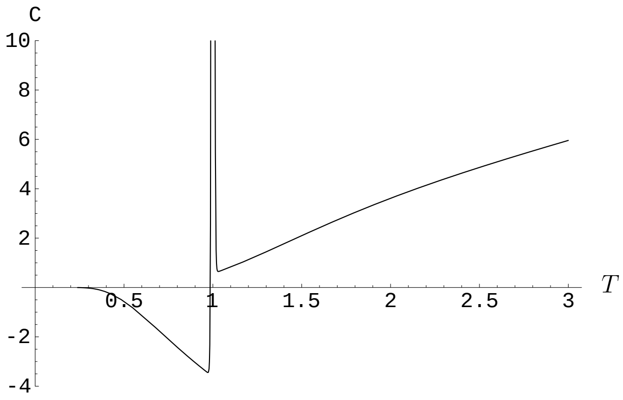

Figure 17: This shows (193). We set $8G = 0.01$ and $l = (2\pi)^{-1}$ so that $T_c = 1$. It diverges at the critical temperature.

which are shown in the figure 16 and 17. In this case, the transition becomes sharp in classical limit. At the critical temperature, the specific heat diverges, but the expectation value of energy change continuously. These behaviors are very similar to that of the Euclidean semi-classical approach. For temperature which is far from T_c, these functions are almost the same as the case of the modular invariant free fermion model.

6 Summary and Discussions

We show that the Hawking-Page phase transition which is originally predicted in the Euclidean semi-classical gravity can also be described in boundary CFT by identifying the boundary CFT partition function with that of Euclidean asymptotic AdS_3 gravity. In the semi-classical gravity, the energy seems to jump from $-1/8G$ to $1/8G$ at the critical temperature discontinuously and the specific heat seems to be divergent. However, one can not discuss thermodynamics near the critical temperature in terms of semi-classical gravity because the transition include some topological change and quantum gravitational physics. In this thesis, we treat the boundary CFT as a effective quantum theory which can predict the thermodynamical behavior even at the critical temperature.

For a CFT model, we consider, at first, CFT with superconformal algebra because it is successful to explain the mass gap between AdS_3 and BTZ space at zero temperature. The partition function of this CFT behaves as that of the gravitational instanton of AdS_3 and a BTZ black hole in the low and high temperature limit respectively. This fact implies the Hawking-Page phase transition will occur in SCFT, and the critical temperature can be determined as the self dual point under the boundary modular transformation $\tau \to -\frac{1}{\tau}$, giving the same temperature as in the Eucledean semi-classical result. However, this model has an unsuitable linear term which does not vanish in classical limit in the expression of energy even for temperature $T \gg T_c$ and $T \ll T_c$. The thermal asymptotically AdS_3 space has no supersymmetry except for the case of extremal BTZ black hole, and we do not have any necessity of assuming a priori supersymmetry for the theory describing finite temperature asymptotic AdS_3 space. And the drawback of the SCFT model comes only from bosonic part, thus we consider free fermions CFT as a improved model. This model has a good description for the Hawking-Page phase transition.

Unlike the transition predicted in the Euclidean semi-classical approach, the fermions model seems to have smooth transition between the AdS_3 phase and the BTZ phase through conical phase and small black holes phase, even in classical limit. This is a new prediction from this fermions model. It implies conical space may play some important role for this transition. Some aspects in this context are discussed in terms of AdS/CFT correspondence in [54].

In classical limit, the dominant contribution to the partition function around the critical temperature comes only from the NS-NS sector in this model. The other R-NS and NS-R sectors are also invariant under the boundary modular transformation $\tau \to -\frac{1}{\tau}$. The contributions from only these two sectors are much like the case of the Euclidean semi-classical approach in classical limit, however only these two sectors can not be modular invariant under the transformation $\tau \to \tau+1$. This fact imply that the main difference between the Euclidean semi-classical approach and the free fermion model comes from the NS-NS sector. It may give us some clue to derive this model from string theory which is regarded as the most hopeful theory including quantum gravity.

There are some unanswered questions.

- The continuum critical Ising model is described by free fermion CFT. Is it possible to understand the Hawking-Page phase transition as Ising model? One might guess a mass generator (112) corresponds to magnetization, and the Hawking-Page transition is the transition between ordering phase and disordering phase.

- Is there some relation between the free fermion model and the WZW or Liouville model which are often considered as the quantum theory of asymptotic AdS_3 space?
  Asymptotic AdS_3 gravity can also be described by the Chern-Simons theory with proper boundary conditions. It is equivalent to the $SL(2,R)$ WZW model, and if one integrate out some dynamical degrees of freedom, the theory will be Liouville theory. Recently, Chen derived the BTZ entropy in the boundary Liouville theory[49]. Thus, we can expect there is some relation our free fermion CFT with these theories.

- Can the Bekenstein-Hawking law be derived from CFT models near the critical temperature T_c ?
  The Strominger's argument does not work for a BTZ black hole with mass $M \sim 1/8G \Leftrightarrow T \sim T_c$.

- What can we say about the relation of free fermion CFT with AdS_3/CFT_2 correspondence in String theory[50][51][52][53]? If anyone construct the complete CFT corresponding to asymptotic AdS_3 string theory, he or she will be able to derive the free fermion model as low energy effective theory.

- How will the phase transition occur?
  The transition includes a topological change, and we do not know how it will occur in terms of the bulk gravity. It will need some study for the connection with Lorentzian bulk gravity.

- What symmetries of CFT are broken in this transition? The free fermion model seems to be helpless to answer this question. It will be needed more fundamental studies to answer this question. The answer will help us to understand the color confinement or other corresponding phenomena in gauge theory through the AdS/CFT in string theory.

- As discussed in [21], the two point function with Lorentzian time has oscillating and exponentially damping behaviors in low and high temperature limit respectively. How can we understand this fact in our context?

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A The Cardy's formula

In this appendix, we derive the Cardy's formula (116) according to [55]. The modular invariant partition function on the torus is

$$Z_0(\tau,\bar{\tau}) = \mathrm{Tr}\left[q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right], \tag{194}$$

which satisfies $Z_0(\tau) = Z_0(-1/\tau)$. We shall denote by $\rho(\Delta,\bar{\Delta})$ the number of all states with the eigenvalue $\Delta,\bar{\Delta}$ of the lowest Virasoro generators $L_0,\bar{L}_0$,

$$\rho(\Delta,\bar{\Delta}) = \frac{1}{(2\pi i)^2}\int \frac{dq}{q^{\Delta+1}}\frac{d\bar{q}}{\bar{q}^{\bar{\Delta}+1}}Z(\tau,\bar{\tau}), \tag{195}$$

where

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\left[q^{L_0}\bar{q}^{\bar{L}_0}\right]. \tag{196}$$

For notational simplicity, I will suppress the $\bar{\tau}$ dependence, and restore it only at the end of the computation. We use the modular invariance of Z_0 to rewrite the contour integral in a form suitable for a saddle-point approximation:

$$Z(\tau) = e^{\frac{2\pi ic}{24}\tau}Z_0(-1/\tau) = e^{\frac{2\pi ic}{24}\tau}e^{\frac{2\pi ic}{24}\frac{1}{\tau}}Z(-1/\tau) \tag{197}$$

and thus

$$\rho(\Delta) = \int d\tau e^{-2\pi i\Delta\tau}e^{\frac{2\pi ic}{24}}e^{\frac{2\pi ic}{24}\frac{1}{\tau}}Z(-1/\tau). \tag{198}$$

Let us assume for a moment that $Z(1/\tau)$ varies slowly near the extremal of the phase. Then

$$\frac{d}{d\tau}\left(-2\pi i\Delta\tau + \frac{2\pi ic}{24}\tau + \frac{2\pi ic}{24}\frac{1}{\tau}\right) \simeq 0. \tag{199}$$

For large Δ, the extremal of the exponent is then

$$\tau \approx i\sqrt{c/24\Delta}. \tag{200}$$

Substituting (200) back into the integral, we obtain

$$\rho(\Delta) \approx \exp\left[2\pi\sqrt{\frac{c\Delta}{6}}\right]Z(i\infty), \tag{201}$$

yielding the Cardy's formula:

$$S = 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{c\bar{\Delta}}{6}} \tag{202}$$

We must now check the saddle-point approximation. From (196),

$$Z(i/\epsilon) = \sum \rho(\Delta) e^{-2\pi\Delta/\epsilon}. \tag{203}$$

If the lowest eigen value of L_0 is $\Delta_0 = 0$, then $Z(i/\epsilon)$ approaches a constant as $\epsilon \to 0$, and the saddle-point approximation is good. But if $\Delta_0 \neq 0$, the factor $Z(-1/\tau)$ in (198) varies like other factor in (198) near the saddle point, and the approximation is not good. This is easily corrected however. Define

$$\tilde{Z}(\tau) = \sum \rho(\Delta) e^{2\pi i(\Delta - \Delta_0)\tau} = e^{-2\pi i \Delta_0 \tau} Z(\tau), \tag{204}$$

which goes to a constant as $\tau \to i\infty$. Then the integral for ρ becomes

$$\rho(\Delta) = \int d\tau e^{-2\pi i \Delta \tau} e^{-2\pi i \Delta_0 \frac{1}{\tau}} e^{\frac{2\pi i c}{24}\tau} e^{\frac{2\pi i c}{24}\frac{1}{\tau}} \tilde{Z}(-1/\tau). \tag{205}$$

For large Δ, this integral can be evaluated in a saddle-point approximation, giving

$$\rho(\Delta) \approx \exp\left[2\pi\sqrt{\frac{(c - 24\Delta_0)\Delta}{6}}\right] \rho(\Delta_0) \tag{206}$$

Thus the corrected Cardy's formula is

$$S = 2\pi\sqrt{\frac{c(n_R - 24\Delta_0)}{6}} + 2\pi\sqrt{\frac{c(n_L - 24\bar{\Delta}_0)}{6}}.(\text{for} \quad n_R + n_L \gg c). \tag{207}$$

For free CFT, the lowest eigenvalue is

$$L_0|0\rangle_{NS} = 0, \tag{208}$$

and the NS ground state is included in the Hilbert space. Therefore the strominger's argument can be applied to free CFT.

B Theta function

Jacobi's theta function are defined as follows:

$$\theta_1(z|\tau) = -i \sum_{r\in\mathbb{Z}+1/2} (-1)^{r-1/2} y^r q^{r^2/2} \tag{209}$$

$$\theta_2(z|\tau) = \sum_{r\in\mathbb{Z}+1/2} y^r q^{r^2/2} \tag{210}$$

$$\theta_3(z|\tau) = \sum_{n\in\mathbb{Z}} y^n q^{n^2/2} \tag{211}$$

$$\theta_4(z|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{n^2/2} \tag{212}$$

where z is a complex variable and τ a complex parameter living in the upper half-plane. We have defined $q = \exp 2\pi i\tau$ and $y = \exp 2\pi i z$.

Jacobi's triple product allows us to rewrite these functions in the form of infinite products:

$$\theta_1(z|\tau) = -iy^{1/2}q^{1/8} \prod_{n=1}^{\infty}(1-q^n) \prod_{n=0}^{\infty}(1-yq^{n+1})(1-y^{-1}q^n) \tag{213}$$

$$\theta_2(z|\tau) = y^{1/2}q^{1/8} \prod_{n=1}^{\infty}(1-q^n) \prod_{n=0}^{\infty}(1+yq^{n+1})(1+y^{-1}q^n) \tag{214}$$

$$\theta_3(z|\tau) = \prod_{n=1}^{\infty}(1-q^n) \prod_{r\in\mathbb{Z}+1/2}^{\infty} (1+yq^r)(1+y^{-1}q^r) \tag{215}$$

$$\theta_4(z|\tau) = \prod_{n=1}^{\infty}(1-q^n) \prod_{r\in\mathbb{Z}+1/2}^{\infty} (1-yq^r)(1-y^{-1}q^r) \tag{216}$$

For instance, the equivalence of the two expressions for θ_1 is obtained by setting $t = yq^{1/2}$ in Jacobi's triple product:

$$\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2}t)(1+q^{n-1/2}/t) = \sum_{n\in\mathbb{Z}} q^{n^2/2}t^n. \tag{217}$$

Jacobi's theta function satisfies the following equation:

$$\frac{\partial^2\theta_i}{\partial v^2} = 4\pi i \frac{\partial \theta_i}{\partial \tau} \qquad (i = 1, 2, 3, 4) \tag{218}$$

We shall mainly use the theta function at $z=0$:

$$\theta_i(\tau) = \theta_i(0|\tau) \tag{219}$$

for $i = 2,3,4$ (one can easily check that $\theta_1(0|\tau) = 0$). their explicit expressions, in terms of sums and products, are

$$\begin{aligned}
\theta_2(\tau) &= \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2/2} = 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 && (220)\\
\theta_3(\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2 && (221)\\
\theta_4(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2. && (222)
\end{aligned}$$

We shall also use the Dedekind's Eta function:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{223}$$

These function satisfies the following equations:

$$\begin{aligned}
\theta_2(\tau+1) &= e^{i\pi/4}\theta(\tau), && (224)\\
\theta_3(\tau+1) &= \theta_4(\tau), && (225)\\
\theta_4(\tau+1) &= \theta_3(\tau), && (226)\\
\eta(\tau+1) &= e^{i\pi/12}\eta(\tau), && (227)
\end{aligned}$$

$$\begin{aligned}
\theta_2(-1/\tau) &= \sqrt{-i\tau}\theta_4(\tau) && (228)\\
\theta_3(-1/\tau) &= \sqrt{-i\tau}\theta_3(\tau) && (229)\\
\theta_4(-1/\tau) &= \sqrt{-i\tau}\theta_2(\tau) && (230)\\
\eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau) && (231)
\end{aligned}$$

These formulas can be proofed by using the Poisson resummation formula.

$$\sum_{n\in\mathbb{Z}}\exp(-\pi a n^2 + bn) = \frac{1}{\sqrt{a}}\sum_{k\in\mathbb{Z}}\exp\left[-\frac{\pi}{a}\left(k+\frac{b}{2\pi i}\right)^2\right]. \tag{232}$$

This formula is easily demonstrated by using the identity

$$\sum_{n \in \mathbb{Z}} \delta(x-n) = \sum_{k \in \mathbb{Z}} e^{2\pi i k x} \tag{233}$$

and by integrating it over $\exp(-\pi a x^2 + bx)$.

References

[1] A.Hosoya and K.Nakao
"(2+1)-Dimensional Quantum Gravity"
Prog.Theor.Phys. 84 (1990)739-748

[2] A.Hosoya and K.Nakao
"(2+1)-dimensional pure gravity for an arbitrary closed initial surface"
Class.Quanntum Grav.7(1990)163-176

[3] V.Moncrief
"Reduction of the Einstein equations in (2+1)-dimensions to a Hamiltonian system over Teichmuller space"
J.Math.Phys.30:2907-2914,1989

[4] E.Witten
"2+1 Dimensional Gravity as an Exactly Soluble System"
Nucl.Phys.B 311(1988)46-78

[5] S.Carlip
"Observable, gauge invariance, and time in (2+1)-dimensional quantum gravity"
Phys.Rev.D 42 (1990)2647

[6] M.Banados, C.Teitelboim and J.Zanelli
"Black Hole in Three-Dimensional Spacetime"
Phys.Rev.Lett. 69(1992)1849-1851

[7] M.Banados, M.Henneaux, C.Teitelboim and J.Zanelli
"Geometry of the (2+1) black hole"
Phys.Rev.D 48(1993)1506-1525

[8] S.W.Hawking and D.N.Page
"Thermodynamics of Black Holes in Anti-de Sitter Space"
Commun.Math.Phys. 87, 577-588 (1983)

[9] J.D.Brown and M.Henneaux
"Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three Dimensional Gravity"
Commun.Math.Phys. 104, 207-226 (1986)

[10] O.Coussaert and M.Henneaux
"Supersymmetry of the (2+1)-Dimensional Black Holes"
Phys.Rev.Lett. 72 (1994)183-186

[11] A.Strominger
"Black Hole Entropy from Near-Horizon Microstates"
JHEP 9802(1998)009

[12] J.L. Cardy
"Operator Content of two-dimensional Conformally Invariant Theory"
Nucl.Phys.B 270(1986)186-204

[13] A.Strominger and C.Vafa
"Microscopic Origin of the Bekenstein-Hawking Entropy"
Phys.Lett.B361(1995)59-65

[14] C.G.Callan and J.Maldacena
"D-brane Approach to Black hole Quantum Mechanics"
Nucl.Phys.B472(1996)591-610

[15] J.Maldacena
"The Large N Limit of Superconformal Field Theories and Supergravity"
Adv.Theor.Math.Phys.2(1998)231-252;
Int.J.Theor.Phys.38(1999)1113-1133

[16] E.Witten
"Andi De Sitter Space And Holography"
Adv.Theor.Math.Phys.2(1998)253-291

[17] S.S.Gubser, I.R.Klebanov and A.M.Polyakov
"Gauge Theory Correlators from Non-Critical String Theory"
Phys.Lett.B428(1998)105-114

[18] D.Birmingham,I.Sachs and S.N.Solodukhin
"Conformal Field Theory Interpretation of Black Hole Quasinormal Modes" Phys.Rev.Lett. 88(2002)151301

[19] Y.Kurita and M.Sakagami
"Quasi-normal modes of D3-brane black holes"
Phys.Rev.D 67(2003)024003

[20] E.Witten
"Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories"
Adv.Theor.Math.Phys.2(1998)505-532

[21] D.Birmingham,I.Sachs and S.N.Solodukhin
"Relaxation in Conformal field Theory, Hawking-Page Transition, and Quasinormal/normal Modes"
Phys.Rev.D67(2003)104026

[22] J.Maldacena and A.Strominger
"AdS_3 black holes and a stringy excluion principle"
JHEP 12(1998)005

[23] S.Mano
"AdS_3 Gravitational Instantons from Conformal Field Theory"
Mod.Phys.Lett.A 14(1999)1961-1981

[24] A.Achucarro and P.K.Townsend
"A Chern-Simons action for three-dimensional anti-de sitter supergravity theories"
Phys.Lett.B180(1986)89

[25] For a review see: S.Carlip
"Quantum Gravity in 2+1 Dimensions"
Cambridge University Press(1998)

[26] M.Banados
"Global charges in Chern-Simons theory and the 2+1 black hole"
Phys.Rev.D 52(1995)5816

[27] G.Moore and N.Seiberg
"Taming the conformal zoo"
Phys.Lett.B220(1989)422

[28] S.Elitzur, G.Moore, A.Schwimmer and N.Seiberg
"Remarks on the canonical quantization of the Chern-Simons-Witten theory"
Nucl.Phys.B326(1989)108-134

[29] K.Behrndt, I.Brunner and I.Gaida
"AdS_3 Gravity and Conformal Field Theories"
Nucl.Phys.B546(1999)65-95

[30] O.Coussaert M.Henneaux and P.van Driel
"The asymptotis dynamics of three-dimensional Einstein gravity with a negative cosmological constant"
Class.Quant.Grav.12(1995)2961-2966

[31] T.Brotz, M.E.Ortiz and A.Ritz
"Modular invariance and 3D gravitational instantons"
Phys.Rev.D 60(1999)064012

[32] S.Deser, R.Jackiw, and G.'t Hooft
"Three-Dimensional Einstein Gravity: Dynamics of Flat Space"
Ann.Phys.(N.Y.) 152 (1984) 220-235

[33] J.D.Brown
Ph.D. thesis, University of Texas at Austin, (1985)
published as a book,
"Lower Dimensional Gravity"
World Scientific, Singapore,1988

[34] L.F.Abbott and S.Deser
"Stability of Gravity with a Cosmological constant"
Nucl.Phys.B195(1982)76-96

[35] S.Carlip
"The (2+1)-dimensional black hole"
Class.Quantum Grac.12(1995)2853-2879

[36] D.Birmingham, I.Sachs and S.Sen
"Exact Result for the BTZ Black Hole"
Int.J.Mod.Phy,D10(2001)833-858

[37] D.Cangemi, M.Leblanc and R.B.Mann
"Gauge formulation of the spinning black hole in (2+1)-dimensional anti-de Sitter space"
Phys.Rev.D 48(1993)3606

[38] A.R.Steif
"Supergravity of three-dimensional black holes"
Phys.Rev.D 53(1996)5521

[39] S.Carlip and C.Teitelboim
"Aspects of black hole quantum mechanics and thermodynamics in 2+1 dimensions"
Phys.Red.D 51(1995)622-631

[40] B.Reznik
"Thermodynamics and evapolation of the (2+1)-dimensional black hole"
Phys.Rev.D51(1995)1728-1732

[41] G.W.Gibbons and S.W.Hawking
"Cosmological event horizons, thermodynamics, and particle creation"
Phys.Rev.D 15(1977)2738

[42] G.W.Gibbons and S.W.Hawking
"Action Integrals and partition functions in quantum gravity"
Phys.Rev.D 15(1977)2752

[43] C.Teitelboim
"Action and entropy of extreme and nonextreme black holes"
Phys.Rev.D 51(1995)4315-4318

[44] J.Bekenstein
"Black Holes and Entropy"
Phys.Rev.D 7(1973)2333

[45] J.Bekenstein
"Generalized second law of thermodynamics in black-hole physics"
Phys.Rev.D 9(1974)3292

[46] S.W.Hawking
"Black holes and thermodynamics"
Phys.Rev.D 13(1976)191

[47] D.J.Gross, M.J.Perry and L.G.Yaffe
"Instability of flat space at finite temperature"
Phys.Rev.D 25(1982)330

[48] See, for example, P.Di Francesco, P.Mathieu and D.Sénéchal
"Conformal Field Theory"
New York, USA: Springer(1997)

[49] Y.Chen
"Quantum Liouville theory and BTZ black hole entropy"
hep-th/0310234

[50] A.Giveon, D.Kutasov and N.Seiberg
"Comments on String Theory on AdS_3"
Adv.Theor.Math.Phys.2(1998)733-780

[51] J.de Boer, H.Ooguri, H.Robins and N.Seiberg
"String Theory On AdS_3"
JHEP 9812(1998)026

[52] D.Kutasov and N.Seiberg
"More Comments On String Theory On AdS_3"
JHEP 9904(1999)008

[53] J.R.David, G.Mandal and S.R.Wadia
"Microscopic Formulation of Black Holes in String Theory"
Phys.Rept.369(2002)549-686

[54] J.R.David, G.Mandal, S.Vaidya and S.Wadia
"Point Mass Geometries, Spectral Flow and $AdS_3 - CFT_2$ correspondence"
Nucl.Phys.B564(2000) 128-141

[55] S.Carlip
"What we don't know about BTZ black hole entropy"
Class.Quantum Grav.15(1998)3609-3625