# Non-Abelian Walls and Their Moduli Space

12/06/04 at Tagen, Nagoya University

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in collaboration with

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O′ = Ohta, S = Sakai, (+ T= Tachikawa)

1. INOS, Phys.Rev.Lett. **93** (2004) 161601 [`hep-th/0404198`].
   Wall Solutions and Their Moduli Spaces
2. INOS, Phys. Rev. **D** (in press) [`hep-th/0405194`].
   Properties of Walls
3. INOS, `hep-th/0405129`
   Walls + Vortices + Monopoles
4. INOS, `hep-th/0409151`: Review (Proceedings)
5. EINOO′S, `hep-th/0412024` (← last week)
   D-Brane Configuration for Walls
6. EINOS, `hep-th/0412048` (← Today's hep-th)
   Instantons + Vortices
7. EINOO′ST, work in progress
   Connection with the Morse Theory

## §1. Introduction

### Motivation

1) BPS Solitons in Gauge-Higgs System

Gauge theories admit topological solitons.

Supersymmetry (SUSY)

⇒ They preserve the 1/2 SUSY ↔ BPS States

(satisfying 1st order BPS Eqs. as well as 2nd order EOMs)

⇒ no force between them

⇒ Moduli Space

| | codim. | $\pi$ | Moduli Space | quotient |
|---|---|---|---|---|
| Instanton | 4 | $\pi_3$ | ADHM ('78) | HK |
| Monopole | 3 | $\pi_2$ | Nahm ('80) | HK |
| Vortex | 2 | $\pi_1$ | Hanany-Tong ('03) | Kähler |
| Wall | 1 | $\pi_0$ | Our work (NOW) | Kähler |

cf. They naturaly have (hyper-)Kähler structures.
↔ unbroken SUSY

## 2) Brane World Scenario.

Realize our world on solitons.

## 3) D-branes as BPS solitons.

BPS Solitons can realized as D-branes.

⇒ ADHM / Nahm conditions can be obtained from the vacuum (D-, F-flatness) conditions.

| | | |
|---|---|---|
| Instanton | Dp-D(p+4) | Douglas/Witten |
| Monopole | D(p+1)-D(p+3) | Green-Gutpele |
| Vortex | NS5-D3-D1 | Hanany-Tong |
| Wall | kinky Dp-D(p+4) | Our work |

(It's only available method to obtain the vortex moduli.)

# Plan of My Talk.

§1. Introduction.

§2. Review: SUSY, Walls, and BPS Walls.

§3. The Model / The BPS equations

§4. Construction of Walls / The Moduli Space

§5. Properties of Walls / The Wall Algebra

§6. D-Brane Construction for Non-Abelian Walls

§7. 1/4 BPS States (W+V+M, I+V)

§8. Conclusion / Discussion

## §2. Review: SUSY, Walls, and BPS Walls.

### Supersymmetry

$\boxed{\text{Boson}}$ $\Leftrightarrow$ $\boxed{\text{Fermion}}$

Spin: 0,1,2 Spin: 1/2, 3/2 (up to 2)

- Supercharges (fermionic generators) are "square root" of the translation $p_\mu$ of the space-time. $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} \sim \sigma^\mu p_\mu$.
- # of supercharges may be one, two ,three, ...., 32

  There exists the minumum # for each dim. $d$.
  (e.g.; $d = 4 \longrightarrow 4$ supercharges)
- Three kinds of Supermultiplets
  - Scalar multipltes $\ni$ Scalar fields $\phi$
    Spin: 0, 1/2
  - Vector multiplets $\ni$ Gauge fields $A_\mu$
    Spin: (0), 1/2, 1
  - Gravity multiplets $\ni$ Gravity $g_{\mu\nu}$
    Spin: (0,1/2), 3/2, 2

# Nonlinear Sigma Models (NL$\sigma$Ms)

$\phi : M^{1,1}(M^{3,1}$ and so on$) \longrightarrow \mathcal{M}$

($\mathcal{M}$: the target manifold)

Bosons $\phi^i(x)$ = coordinates of the target manifolds

Fermions $\psi^i(x)$ = components of tangent vectors

- $\boxed{\boxed{\text{2 Supercharges}}} \quad \Leftrightarrow \quad \boxed{\boxed{\text{Riemann}}}$

  $\phi^i$: real $\in$ Real scalar multiplets

  $d = 2, \mathcal{N} = (1, 1) \leftarrow d = 3, \mathcal{N} = 1.$

- $\boxed{\boxed{\text{4 Supercharges}}} \quad \Leftrightarrow \quad \boxed{\boxed{\text{Kähler}}}$

  $\phi^i$: complex $\in$ Chiral multiplets

  $d = 2, \mathcal{N} = (2, 2) \leftarrow d = 3, \mathcal{N} = 2 \leftarrow$
  $d = 4, \mathcal{N} = 1.$

- $\boxed{\boxed{\text{8 Supercharges}}} \quad \Leftrightarrow \quad \boxed{\boxed{\text{hyper-Kähler}}}$

  $\phi^i$: quartet $\in$ Hypermultiplets

  $d = 2, \mathcal{N} = (4, 4) \leftarrow d = 3, \mathcal{N} = 4, \leftarrow$
  $d = 4, \mathcal{N} = 2 \leftarrow d = 5, \mathcal{N} = 1 \leftarrow d = 6, \mathcal{N} = 1.$

  (cf. Deligne and Freed, hep-th/9901094)

## Kähler NL$\sigma$Ms $d=4, \mathcal{N}=1$; $d=2, \mathcal{N}=(2,2)$ etc

The bosonic Lagrangian is (Zumino, Wess-Bagger's text)

$$\mathcal{L}_{\rm boson} = -g_{ij^*}\partial_\mu \phi^i \partial^\mu \phi^{*j} - g^{ij^*}\frac{\partial W}{\partial \phi^i}\frac{\partial W^*}{\partial \phi^{*j}} \ . \quad (1)$$

$g_{ij^*}(\phi, \phi^*) = \frac{\partial^2 K(\phi,\phi^*)}{\partial\phi^i \partial\phi^{*j}}$: the target Kähler metric.

$K(\phi, \phi^*)$: the Kähler potential

$W(\phi)$: the superpotential

## (Massive) Hyper-Kähler NL$\sigma$Ms

$d=4, \mathcal{N}=2$; $d=2, \mathcal{N}=(4,4)$ etc

(for mass to exist $D \leq 5$)

The bosonic Lagrangian is (Alvarez-Gaume and Freedman)

$$\mathcal{L}_{\rm boson} = -g_{ij^*}\partial_\mu \phi^i \partial^\mu \phi^{*j} - |\mu|^2 g_{ij^*} k^i k^{*j} \ . \quad (2)$$

$g_{ij^*}(\phi, \phi^*)$: the target HK metric (with isometry).

$k^i(\phi, \phi^*)$: a tri-holomorphic Killing vector.

The form of a potential is severely restrected !!

# Domain Walls ($D = 2$ Kinks)

# Remember Yang-Mills Instantons

$$\begin{aligned} S &= \frac{1}{2}\int d^4x \,\mathrm{tr}\, F\wedge *F \\ &= \int d^4x \,\mathrm{tr}\,(F - *F)^2 + \int d^4x \,\mathrm{tr}\, F\wedge F \\ &\geq \int d^4x \,\mathrm{tr}\, F\wedge F \end{aligned} \tag{3}$$

The action is bounded below by the topological charge $Q = \int d^4x \,\mathrm{tr}\, F\wedge F$.

The inequality is saturated
↔ The selfdual equation: $F = *F$.

Embedd this to $d = 4+1$ $\mathcal{N} = 4$ SUSY YM with 16 supercharges.

- The above bound is for energy density.
- The selfdual eq. = The BPS eq.
- Instantons are 1/2 BPS states, preserving 8 supercharges.
- The moduli space is naturally hyper-Kähler.

# BPS Domain Walls in Massive HK NL$\sigma$Ms

Energy density for a wall (perpendicular to the $y$-axis):

(Abraham and Townsend)

$$\begin{aligned} E &= \int dy[\underbrace{g_{ij^*}\partial_y\phi^i\partial_y\phi^{*j}}_{\text{kinetic energy}} + \underbrace{\mu^2 g_{ij^*}k^i k^{*j}}_{\text{potential energy}}] \\ &= \int dy\, g_{ij^*}(\partial_z\phi^i - \mu k^i)(\partial_z\phi^{*j} - \mu k^{*j}) \\ &\quad + \int dy[\underbrace{\mu g_{ij^*}k^i\partial_z\phi^{*j} + \text{conj.}}_{\text{cross terms}}] \\ &\geq \int dy[\mu g_{ij^*}k^i\partial_z\phi^{*j} + \text{conj.}] \\ &= \mu[D]_{y=-\infty}^{y=\infty} \end{aligned} \tag{4}$$

where $k^i = g^{ij^*}\partial_{j^*}D$ ($D$: Killing potential).

The Bogomol'nyi bound: $\boxed{E \geq \mu[D]_{y=-\infty}^{y=\infty}}$

.... The right hand side is "topological".

The BPS equation: $\partial_z\phi^i - \mu k^i = 0$.

.... This is a 1st-order equation.

Solutions preserve 1/2 SUSY → 1/2 BPS

## §3. The Model / The BPS eq.

We consider BPS domain walls in gauge theories.

$U(N_{\rm C})$ Gauge Theory with $N_{\rm F}$ Hypermultiplets
(8 SUSY: $D=5, \mathcal{N}=1$ or $D=4, \mathcal{N}=2, \cdots$)

### (Bosonic) Field contents:

Hypermultiplets $H^{irA}=(H^{1rA}, H^{2rA})$
with masses $m_A \in \mathbf{R}$

$SU(2)_{\rm R}$ indices: $i=1,2$
color indices: $r,s=1,\cdots,N_{\rm C}$
flavor indices: $A,B=1,\cdots,N_{\rm F}$
$\rightarrow N_{\rm C}\times N_{\rm F}$ matrices.

Vector multiplets $(W_M, \Sigma)$

$W_M$: $N_{\rm C}\times N_{\rm C}$ gauge field,
$\Sigma$: $N_{\rm C}\times N_{\rm C}$ real adjoint scalar field

## (Bosonic) Lagrangian

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}\left(F_{MN}F^{MN}\right) + \frac{1}{g^2}\mathrm{Tr}\left(\mathcal{D}_M\Sigma\mathcal{D}^M\Sigma\right)$$
$$+\mathrm{Tr}\left[\mathcal{D}^M H^i(\mathcal{D}_M H^i)^\dagger\right] - V, \qquad (5)$$

$$V = \frac{g^2}{4}\mathrm{Tr}\left[\left(H^1H^{1\dagger} - H^2H^{2\dagger} - c\mathbf{1}_{N_\mathrm{C}}\right)^2\right.$$
$$\left.+4H^2H^{1\dagger}H^1H^{2\dagger}\right]$$
$$+\mathrm{Tr}\left[(\Sigma H^i - H^i M)(\Sigma H^i - H^i M)^\dagger\right], \quad (6)$$

with

$$\mathcal{D}_M\Sigma = \partial_M\Sigma + i[W_M, \Sigma],$$

$$\mathcal{D}_M H^i = (\partial_M + iW_M)H^i,$$

$$F_{MN} = \frac{1}{i}[\mathcal{D}_M, \mathcal{D}_N]$$
$$= \partial_M W_N - \partial_N W_M + i[W_M, W_N].$$

Parameters are

$(M)^A{}_B \equiv m_A\delta^A_B$: diagonal mass matrix,

$g$: gauge coupling,

$c^a = (0, 0, c)$: Fayet-Iliopoulos parameters.

## Vacua in the massless case: $m_A = 0$

In the massless limit $m_A = 0$, the Higgs branch of vacua are obtained as a hyper-Kähler quotient, resulting in the cotangent bundle over the Grassmann manifold

$$\begin{aligned}\mathcal{M}_{\rm vac.}^{M=0} &= T^* G_{N_{\rm F},N_{\rm C}} \\ &= T^*\left[\frac{SU(N_{\rm F})}{SU(N_{\rm F}-N_{\rm C})\times SU(N_{\rm C})\times U(1)}\right], \end{aligned} \quad (7)$$

(cf. No Coulomb branch because of the FI para.)

The moment maps:

$$\mu_{\rm R} = H^1 H^{1\dagger} - H^2 H^{2\dagger} = c\mathbf{1}_{N_{\rm C}},$$

$$\mu_{\rm C} = H^1 H^{2\dagger} = \mathbf{0}_{N_{\rm C}}.$$

(In mathematics, $I = H^1, J = H^{2\dagger}$.)

## Simpler $U(1)$ cases

$N_{\rm C} = 1$: $T^*\mathbf{C}P^{N_{\rm F}-1}$.

$N_{\rm C} = 1$, $N_{\rm F} = 2$: $T^*\mathbf{C}P^1$: the Eguchi-Hanson.

# Vacua in the massive case: $m_A \neq 0$

(Arai-Nitta-Sakai)

Turning on masses $m_A \neq 0$, most points are lifted leaving some discrete points as vacua.
We consider non-degenerate masees with ordering $m_A \geq m_{A+1}$.

$$H^1 \sim \begin{array}{c} \begin{array}{ccccccc} \phantom{0\cdots0} & A_1 & \phantom{0\cdots0} & A_2 & \cdots & A_{N_{\rm C}} & \phantom{0\cdots0} \end{array} \\ \left(\begin{array}{ccccccc} 0\cdots0 & 1 & 0\cdots0 & 0 & 0\cdots0 & 0 & 0\cdots0 \\ 0\cdots0 & 0 & 0\cdots0 & 1 & 0\cdots0 & 0 & 0\cdots0 \\ & & & \vdots & & & \\ 0\cdots0 & 0 & 0\cdots0 & 0 & 0\cdots0 & 1 & 0\cdots0 \end{array}\right) \\ \longrightarrow \text{flavor} \end{array} \downarrow \text{color} \tag{8}$$

$$H^{2rA} = 0,$$

$$\Sigma = \text{diag.}(m_{A_1}, m_{A_2}, \cdots, m_{A_{N_{\rm C}}}). \tag{9}$$

We label this vacuum by

$$\langle A_1 \, A_2 \cdots A_{N_{\rm C}} \rangle. \tag{10}$$

$$\# \text{ of vacua} = {}_{N_{\rm F}}C_{N_{\rm C}} = \frac{N_{\rm F}!}{N_{\rm C}!(N_{\rm F} - N_{\rm C})!} \tag{11}$$

# The 1/2 BPS equations for walls

$\delta_{\rm SUSY}({\rm fermions}) = 0$

$\Downarrow$

$$\begin{aligned} \mathcal{D}_y \Sigma &= \frac{g^2}{2}\left(c\mathbf{1}_{N_{\rm C}} - H^1 H^{1\dagger} + H^2 H^{2\dagger}\right), && (12) \\ 0 &= -g^2 H^2 H^{1\dagger}, && (13) \\ \mathcal{D}_y H^1 &= -\Sigma H^1 + H^1 M, \\ \mathcal{D}_y H^2 &= \Sigma H^2 - H^2 M. && (14) \end{aligned}$$

# The BPS bound

Energy density:

$$\varepsilon \geq \sum |\text{BPS eqs.}|^2 + T_{\text{topological}} \tag{15}$$

The (multi-)BPS wall tension:

$$\begin{aligned} T_w &= \int_{-\infty}^{+\infty} dy \;\; T_{\text{topological}} \\ &= c\,[\mathrm{Tr}\Sigma]_{-\infty}^{+\infty} = c\left(\sum_{k=1}^{N_{\mathrm{C}}} m_{A_k} - \sum_{k=1}^{N_{\mathrm{C}}} m_{B_k}\right). \end{aligned} \tag{16}$$

$y$: the extra dimension perpendicular to walls

$$\begin{aligned} &y = +\infty : \langle A_1\, A_2 \cdots A_{N_{\mathrm{C}}} \rangle \\ &y = -\infty : \langle B_1\, B_2 \cdots B_{N_{\mathrm{C}}} \rangle \\ &(A_r < B_r, \;\; m_{A_r} > m_{B_r}) \end{aligned} \tag{17}$$

# §3. Construction of Walls / The Moduli Space

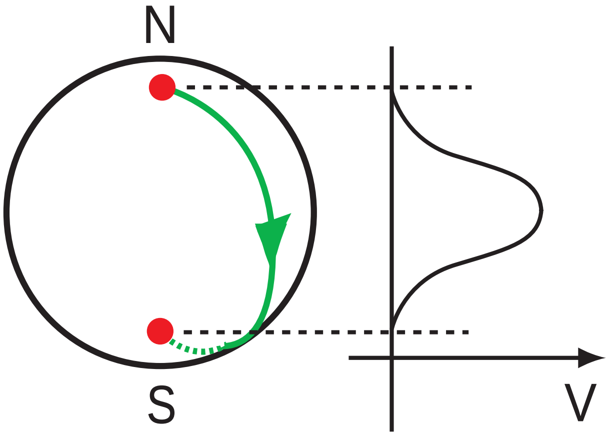
Taking strong gauge coupling limit, $g \to \infty$, the model reduces to the nonlinear $\sigma$ model on the Higgs branch $T^*G_{N_{\rm F},N_{\rm C}}$.

$N_{\rm F} = 2,\ N_{\rm C} = 1$ (the massive Eguchi-Hanson $T^*\mathbf{C}P^1$).

Single wall solution:

$$v = w^* = e^{|\mu|(y-y_0)} e^{i\varphi_0}\ . \tag{18}$$

Two real zero modes $y_0$ (translation) and $\varphi_0$ ($U(1)$ phase). The moduli space is $\mathbf{R} \times S^1$.



## General Solution NEW

$$H^1 = S^{-1} H_0 e^{My}, \quad H^2 = 0 \tag{19}$$

$$S S^\dagger = c^{-1} H_0 e^{2My} {H_0}^\dagger \tag{20}$$

$$S^{-1} \partial_y S \equiv \Sigma + i W_y \tag{21}$$

with $H_0$ an $N_{\rm C} \times N_{\rm F}$ constant complex matrix.

# The Moduli Space

There is an equivalence relation

$$H_0 \simeq V H_0 \,, \qquad V \in GL(N_{\rm C}, \mathbf{C}) \tag{22}$$

This is the Kähler quotient;

Moduli space is the Grassmann manifold.

$$\begin{aligned} H_0 \;&\in\; G_{N_{\rm F},N_{\rm C}} \\ &\simeq \frac{SU(N_{\rm F})}{SU(N_{\rm C}) \times SU(N_{\rm F}-N_{\rm C}) \times U(1)} \,. \end{aligned} \tag{23}$$

This is a special Lagrangian submanifold of massless Higgs branch.

**NOTE**

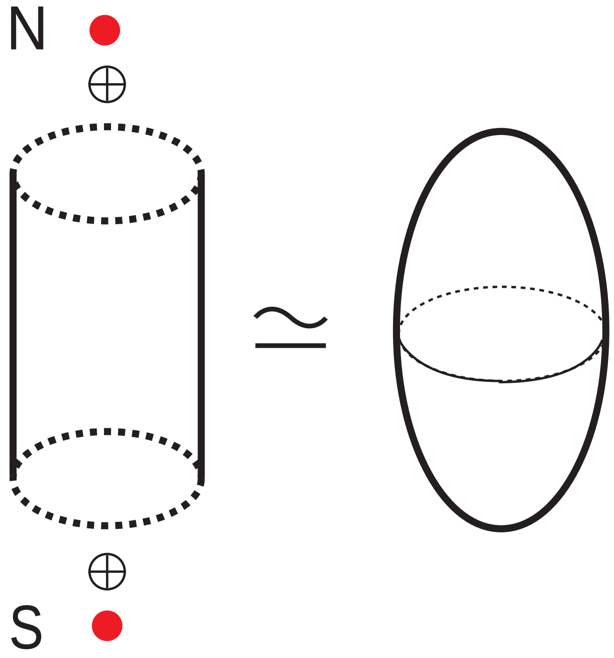
This moduli space is the Total Moduli Space including all topological sectors:

$$\mathcal{M}_{N_{\rm F},N_{\rm C}} = \sum_{\rm BPS} \mathcal{M}_{N_{\rm F},N_{\rm C}}^{\langle A_1 A_2 \cdots A_{N_{\rm C}}\rangle \leftarrow \langle B_1 B_2 \cdots B_{N_{\rm C}}\rangle}, \tag{24}$$

(cf. The total moduli space for instantons / monopoles / vortices is infinite dimensional.)

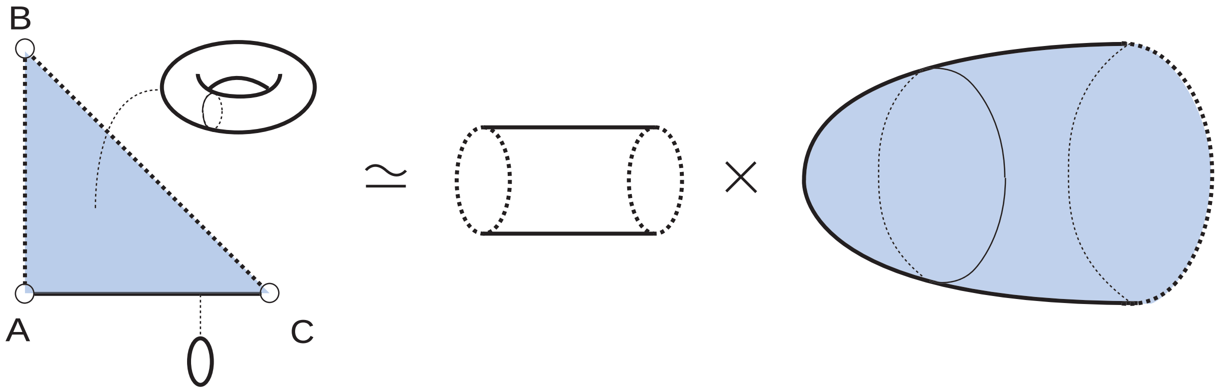
Ex. 1) $N_{\rm F} = 2,\ N_{\rm C} = 1$ ($T^*\mathbf{C}P^1$, EH)

The single-wall moduli + 2 vacua $\simeq S^2 \simeq \mathbf{C}P^1$

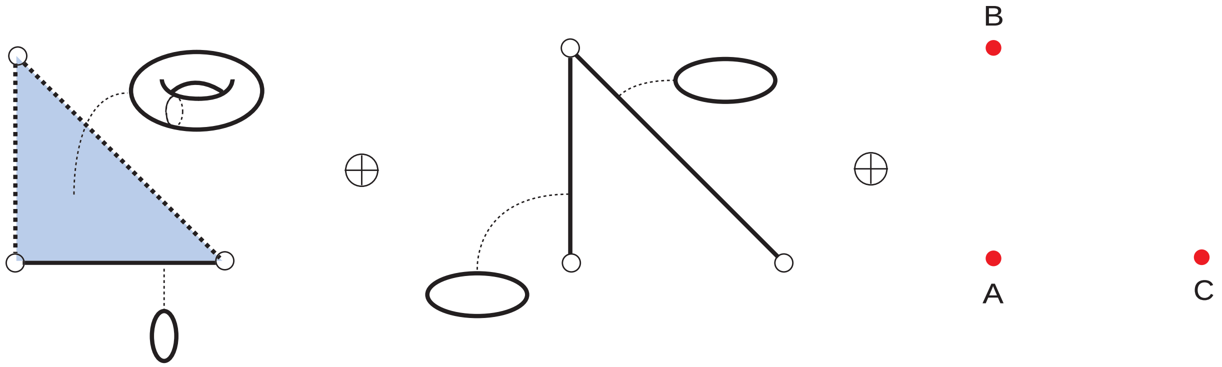


⇒ Natural compactification of the moduli space.

# Ex. 2) $N_{\rm F} = 3,\ N_{\rm C} = 1$ $(T^*\mathbf{C}P^2)$



The double wall moduli + 2 single wall moduli + 3 vacua $\simeq$ $\mathbf{C}P^2$ (toric diagram)



# §4. Properties of Walls / The Wall Algebra

Walls have codimension one.

$$\Downarrow$$

Their positions become important.
There exist two types of property for each pair of walls, impenetrable or penetrable.

(cf. We can smoothly exchange positions of other solitions like instantons.)

## 1. Impenetrable wall pair

1. Positions of two walls do not commute.
2. Taking two walls closer and closer, they constitute a compressed wall in the limit.
3. All wall pairs in Abelian gauge theory are in this type.
4. We can define a level of compressing.

## 2. Penetrable wall pair

1. Positions of two walls do commute.
2. Another vacuum appears between them when we exchange their positions.
3. Characteristic in non-Abelian gauge theory.

# Creation operator of walls

Moduli matrices for vacua are the same with $H^1$:

$$(H_0)_{\rm vac.} = \begin{array}{c} \begin{array}{ccccccc} \phantom{0\cdots0} & A_1 & \phantom{0\cdots0} & A_2 & \cdots & A_{N_{\rm C}} & \phantom{0\cdots0} \end{array} \\ \begin{pmatrix} 0\cdots0 & 1 & 0\cdots0 & 0 & 0\cdots0 & 0 & 0\cdots0 \\ 0\cdots0 & 0 & 0\cdots0 & 1 & 0\cdots0 & 0 & 0\cdots0 \\ & & & \vdots & & & \\ 0\cdots0 & 0 & 0\cdots0 & 0 & 0\cdots0 & 1 & 0\cdots0 \end{pmatrix} \\ \rightarrow \text{flavor} \end{array} \downarrow \text{color} \tag{25}$$

$N_{\rm F} \times N_{\rm F}$ matrices $a, \cdots$ acting on the moduli matrices from the right.

$$\begin{aligned} (H_0)_{\rm vac.} &\rightarrow (H_0)_{\rm vac.} e^a = (H_0)_{1-\rm wall} \\ &\rightarrow (H_0)_{1-\rm wall} e^a = (H_0)_{2-\rm walls} \\ &\rightarrow (H_0)_{2-\rm walls} e^a = (H_0)_{3-\rm walls} \\ &\rightarrow \cdots \end{aligned}$$

Here $a$ is a nilpotent element of $GL(N_{\rm F}, \mathbf{C})$ with the length $N_{\rm F} - 1$, $a^{N_{\rm F}-1} = 0$.

$$a_i = E_{i,i+1} \sim \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

# Wall Algebra

Two walls generated by $a_1$ and $a_2$ are

$$\text{Penetrable} \Leftrightarrow [a_1, a_2] = 0$$

$$\text{Inpenetrable} \Leftrightarrow [a_1, a_2] \neq 0$$

Moreover the compressed wall is generated by $[a_1, a_2]$, a nilpotent operator with the lower length by one !!

$$[a_i, a_j] \sim \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

More compressed wall is generated by $[\cdots, [a_3, [a_1, a_2] \cdots]$.

Level of compressing $\Leftrightarrow$ Length of nilpotent algebra.

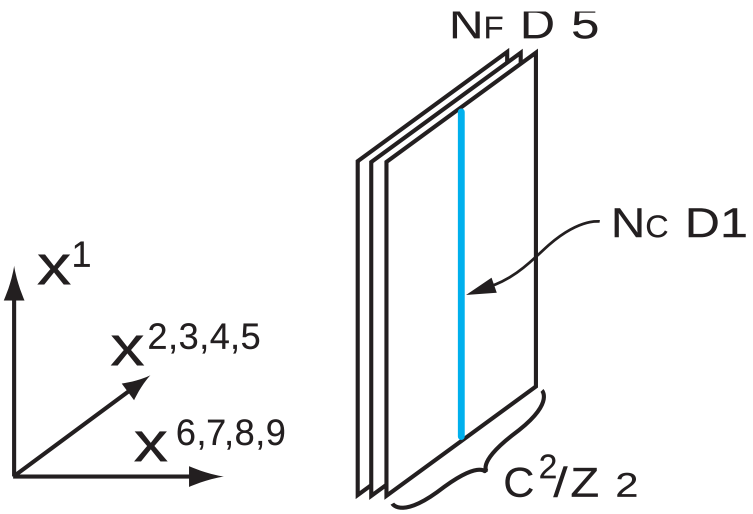
Very interesting and complex phenomena can be understood by the wall algebra.

# §6. D-Brane Construction for Non-Abelian Walls

Consider $d = 2$, $\mathcal{N} = (4,4)$ (8 supercharges) SUSY $U(N_{\rm C})$ gauge theory with $N_{\rm F}$ hypermultiplets (by dimensional reduction from $d = 5$).

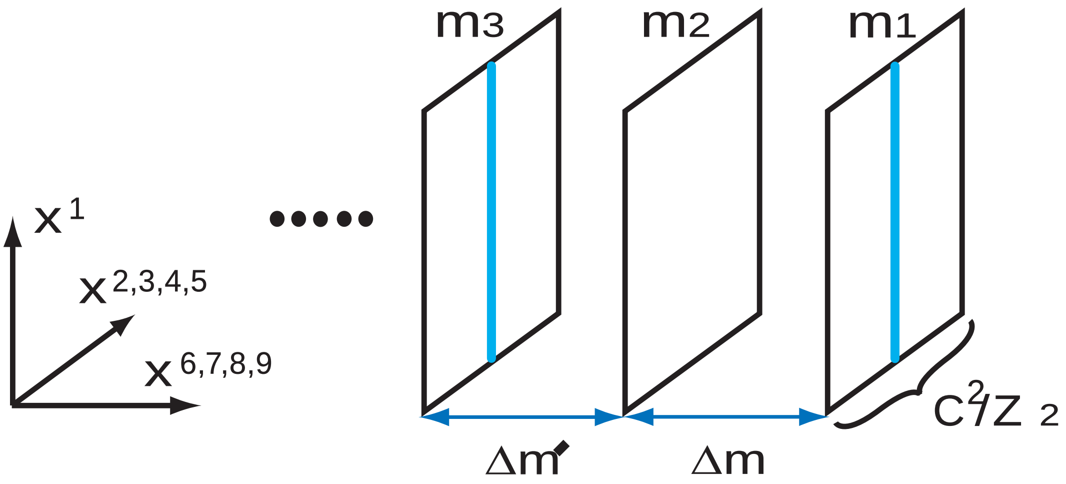
## D-brane Configuration for Massless hypermultiplets

$$\begin{array}{rl} N_{\rm C}\ \mathrm{D1:} & 01 \\ N_{\rm F}\ \mathrm{D5:} & 012345 \\ \mathbf{C}^2/\mathbf{Z}_2\ \mathrm{ALE:} & \quad 2345 \end{array} \tag{28}$$



$\boxed{\text{Vacua}}$ = instantons ($\# = N_{\rm C}$) on $U(N_{\rm F})$ gauge theory on the $A_1$ ALE (Eguchi-Hanson) space.
The ADHM moduli (Hyper-Kähler quotient)
$= T^*G_{N_{\rm F},N_{\rm C}}$. (Kronheimer-Nakajima)

# Massive hypermultiplets

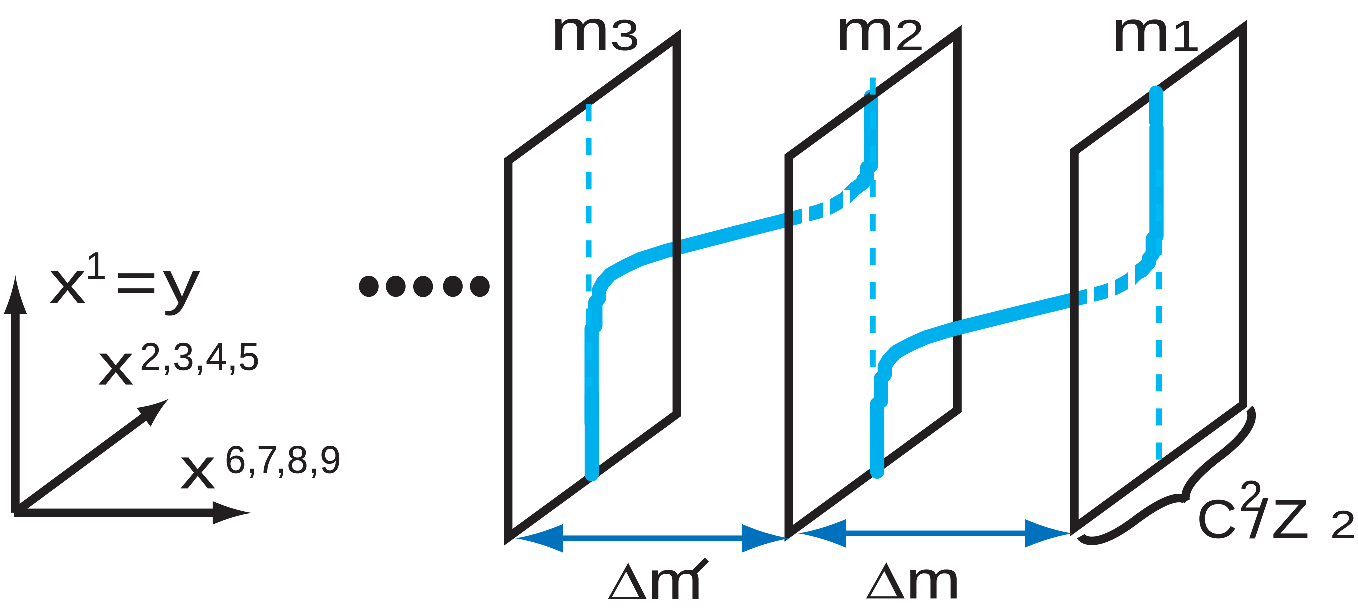


The (T-dualized) S-rule:
At most one Dp-brane can be absorbed into one D(p+4)-brane.

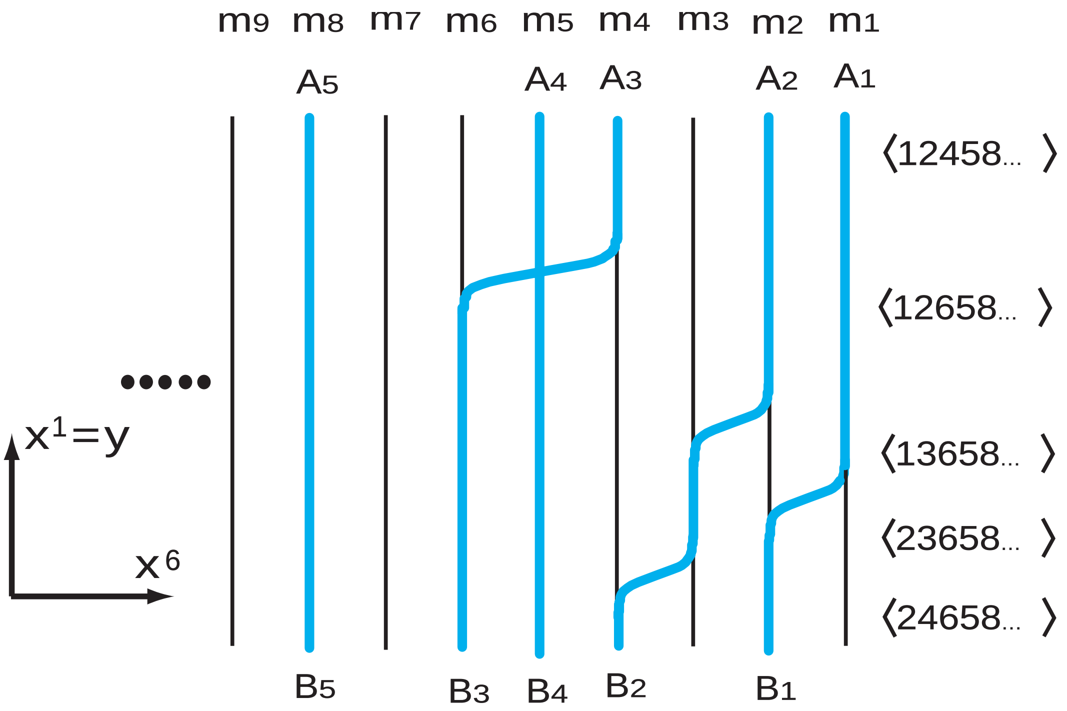
(Hori-Ouguri-Oz, Nakatsu-Ohta-Yokono-Yoshida)

$$\# \text{ of vacua} = {}_{N_{\rm F}}C_{N_{\rm C}} = \frac{N_{\rm F}!}{N_{\rm C}!(N_{\rm F}-N_{\rm C})!} \tag{29}$$

# Kinky D-Branes as Non-Abelian Walls



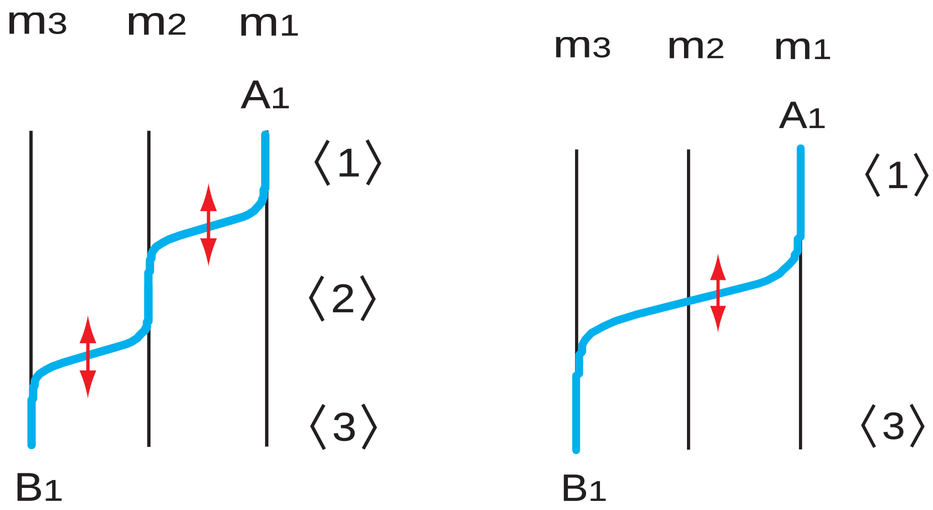
Vacua can be easily read from the D-brane picture.



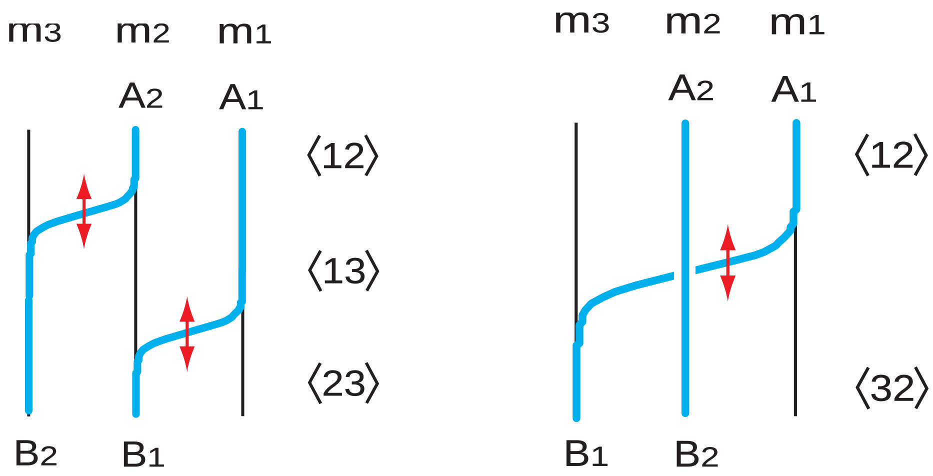
(looking like a braid group?)

# Impenetrable Walls (Compressing Walls)

## $N_{\rm F} = 3,\ N_{\rm C} = 1$ (Abelian)



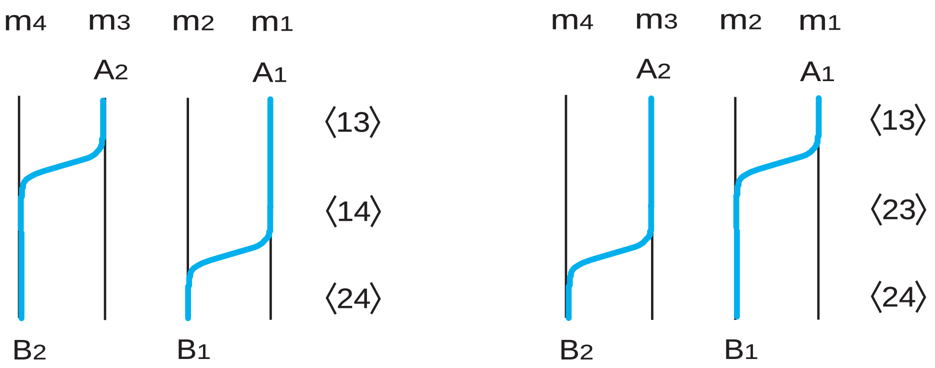
## The dual picture of $N_{\rm F} = 3,\ N_{\rm C} = 2$



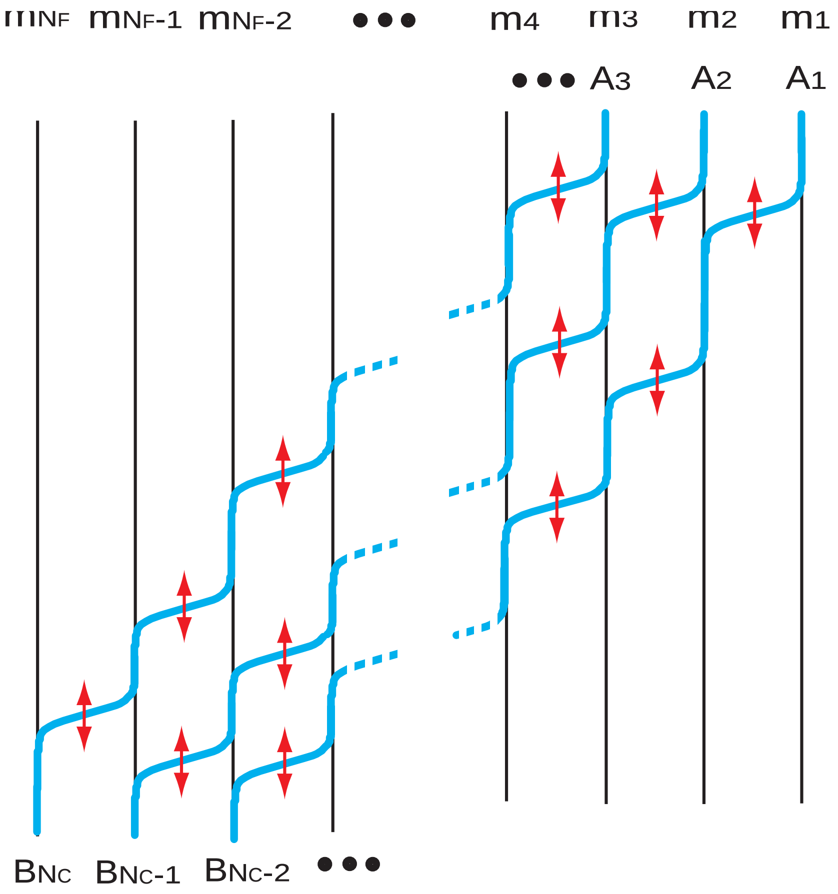
Reconnection (Recombination) occurs in the gauge theory. (K.Hashimoto)

# Penetrable Walls

## $N_{\rm F} = 4,\ N_{\rm C} = 2$



# Wall moduli space

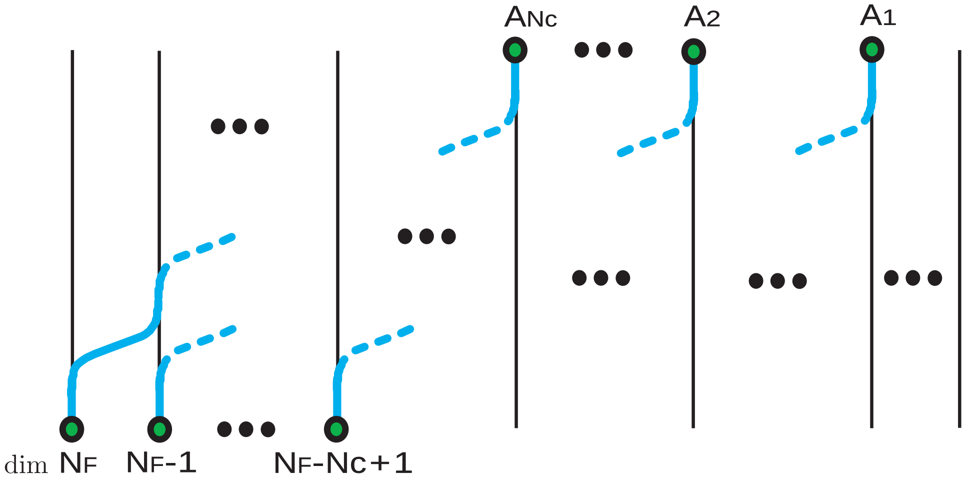


The dimension of wall moduli space is

$$\dim \mathcal{M}_{\rm wall} = 2N_{\rm C}(N_{\rm F} - N_{\rm C}) = \dim G_{N_{\rm F},N_{\rm C}}$$

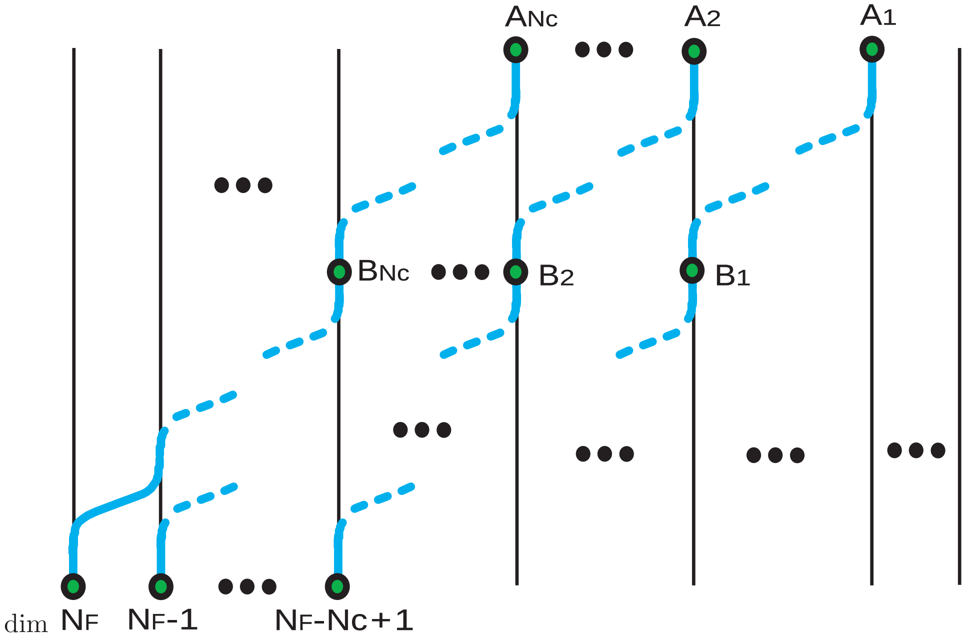
## The Index

$$\nu_{\langle A_1,A_2,\cdots,A_{N_{\rm C}}\rangle} =$$



## The dimension of the topological sector.

$$\dim \mathcal{M}^{\langle A_1,A_2,\cdots,A_{N_{\rm C}}\rangle \leftarrow \langle B_1,B_2,\cdots,B_{N_{\rm C}}\rangle} =$$



$$= \nu_{\langle A_1,A_2,\cdots,A_{N_{\rm C}}\rangle} - \nu_{\langle B_1,B_2,\cdots,B_{N_{\rm C}}\rangle}.$$

## §7. 1/4 BPS states

1/4 BPS equations admitting monopoles, vortices and domain walls.

### The Bogomol'nyi bound

$$\begin{aligned}
\mathcal{E} \geq T_{\text{total}} &= T_{\text{wall}} + T_{\text{vortex}} + T_{\text{monopole}} \\
T_{\text{wall}} &= \text{Tr}[\partial_3(c\Sigma)] \\
T_{\text{vortex}} &= -c\text{Tr}[B_3] \\
T_{\text{monopole}} &= \frac{1}{g^2}\text{Tr}[\partial_a(\Sigma B_a)], \quad B_a \equiv \frac{1}{2}\epsilon_{abc}F_{bc}(W)
\end{aligned}$$

### A set of 1/4 BPS equations

$$0 = (\mathcal{D}_3 + \Sigma)\, H^1 + H^1 M \tag{30}$$

$$0 = (\mathcal{D}_1 + i\mathcal{D}_2)\, H^1 \tag{31}$$

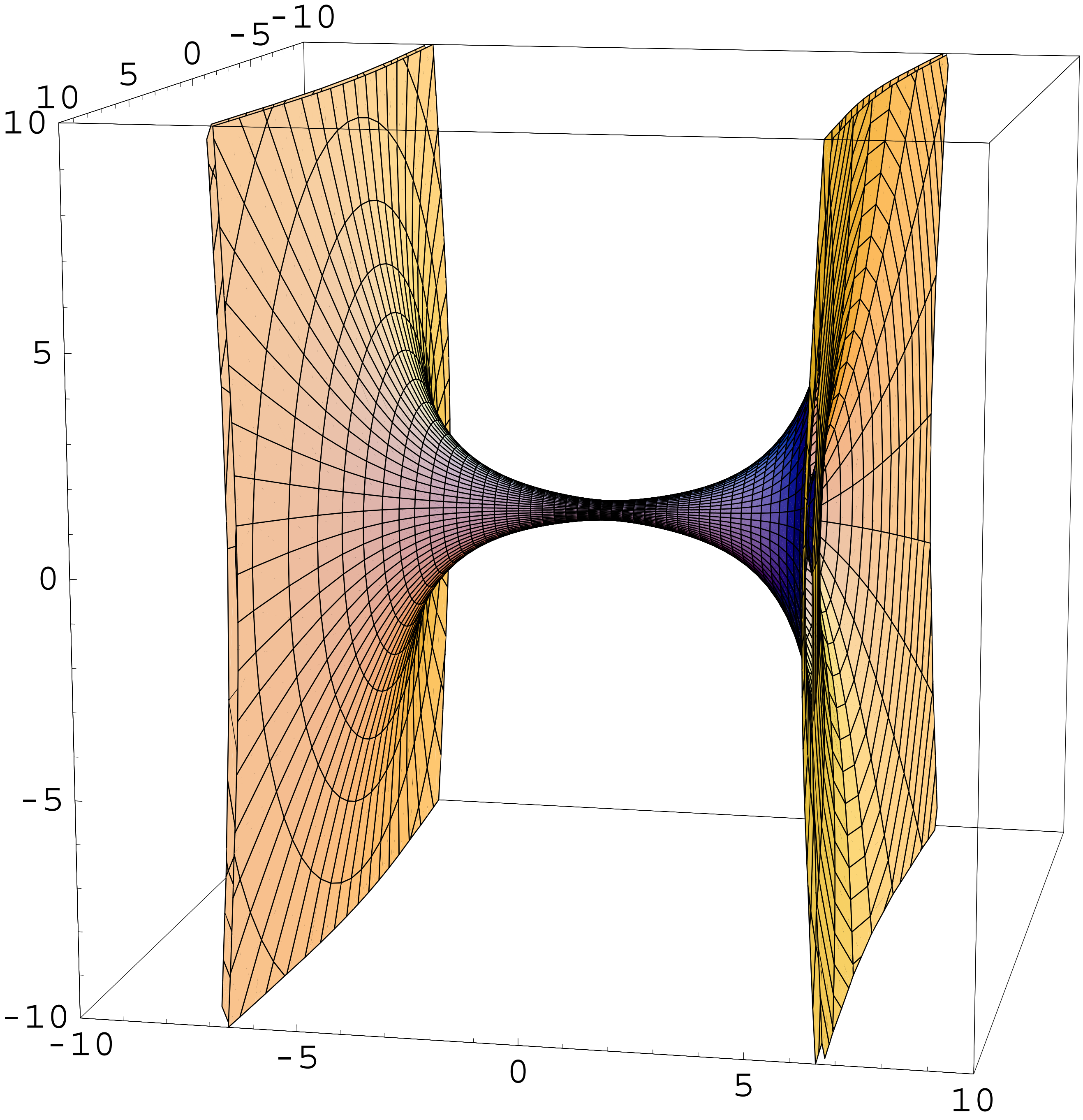
$$0 = F_{12}(W) - \mathcal{D}_3\Sigma + \frac{g^2}{2}(c - H^1 H^{1\dagger}) \tag{32}$$

$$0 = F_{23}(W) - \mathcal{D}_1\Sigma = F_{31}(W) - \mathcal{D}_2\Sigma \tag{33}$$

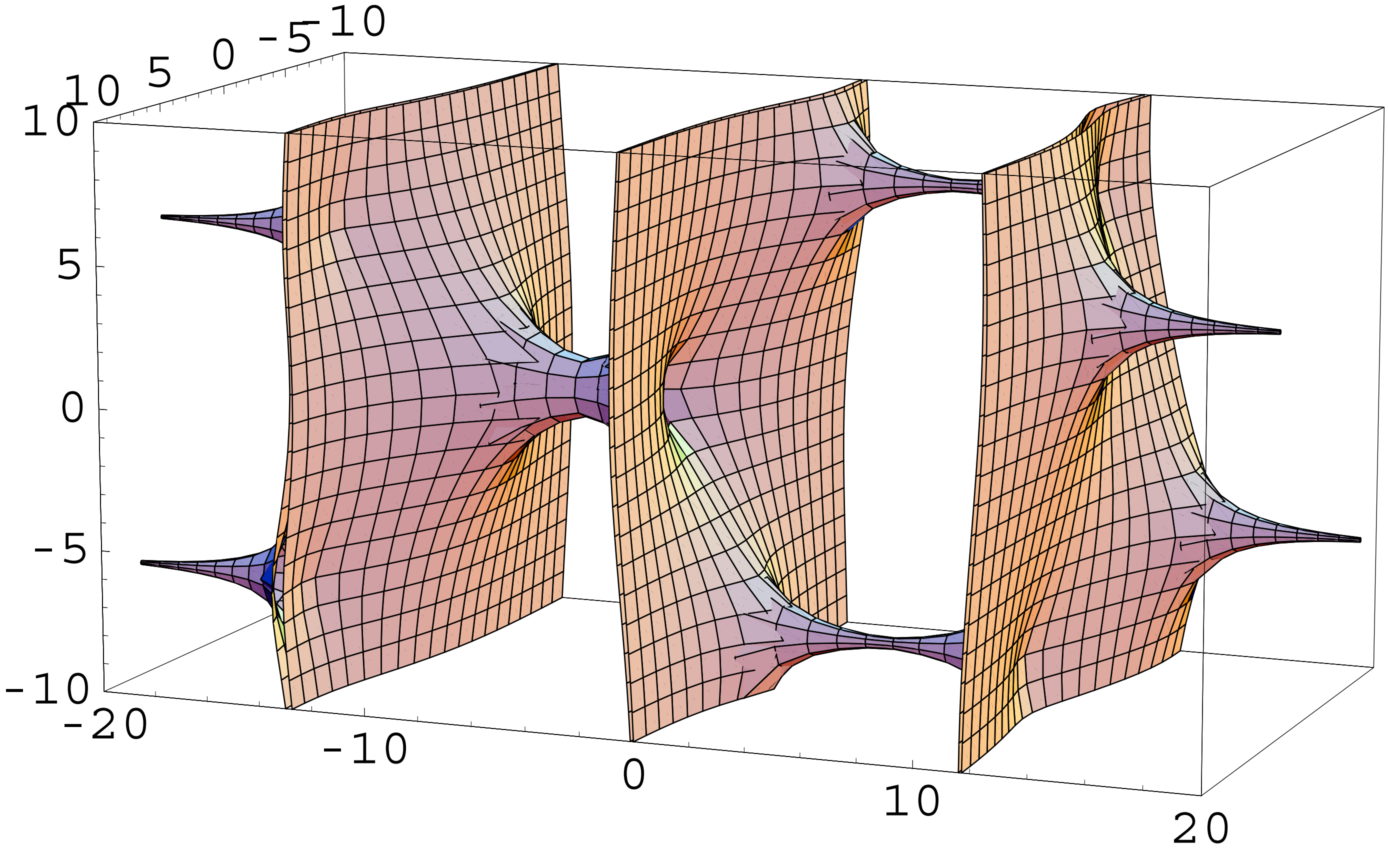
## Exact solutions

can be obtanined by promoting the moduli parameters in $H_0$ for walls to fields depending on the wall world-volume coordinate $z$ holomorphically.

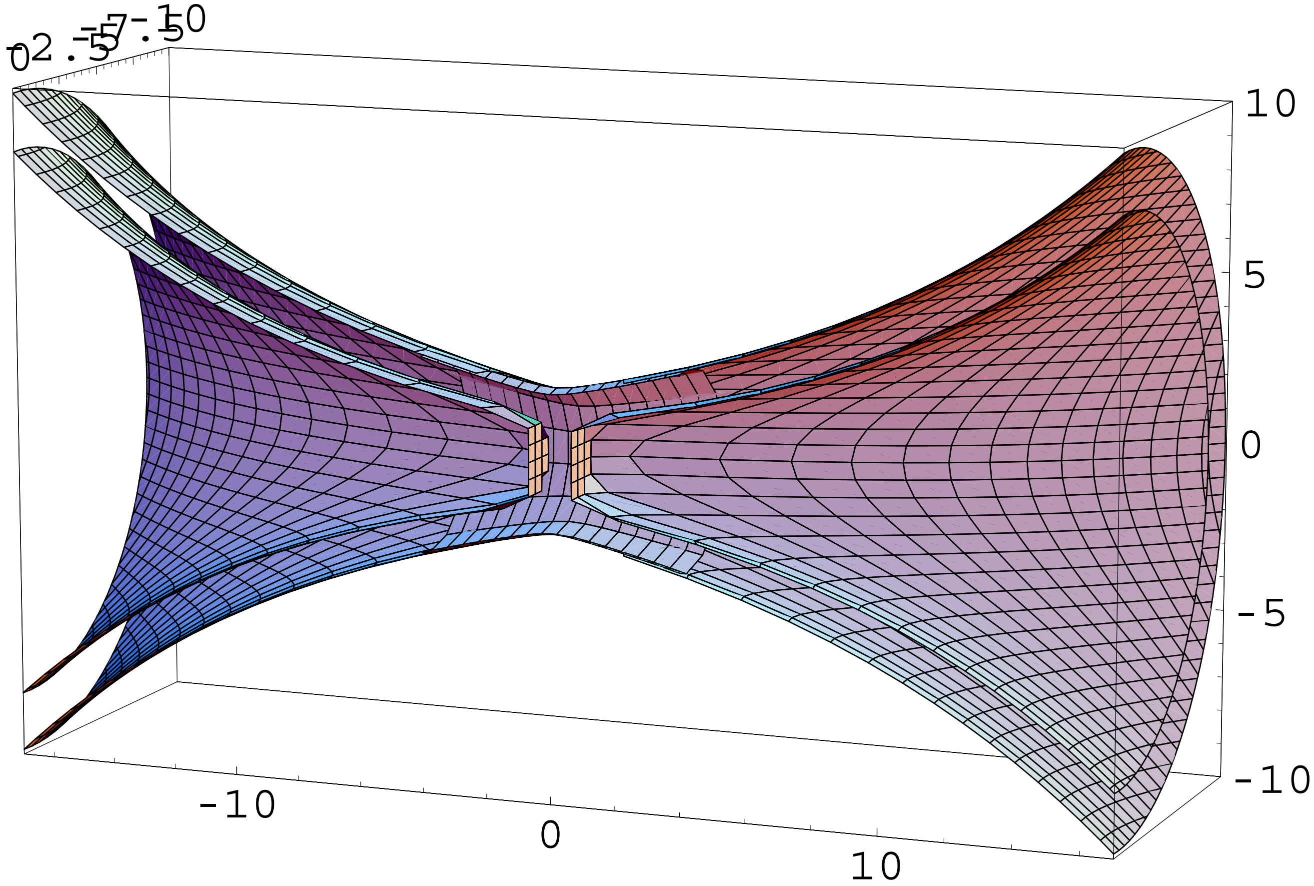
# “BIon” as an exact solution



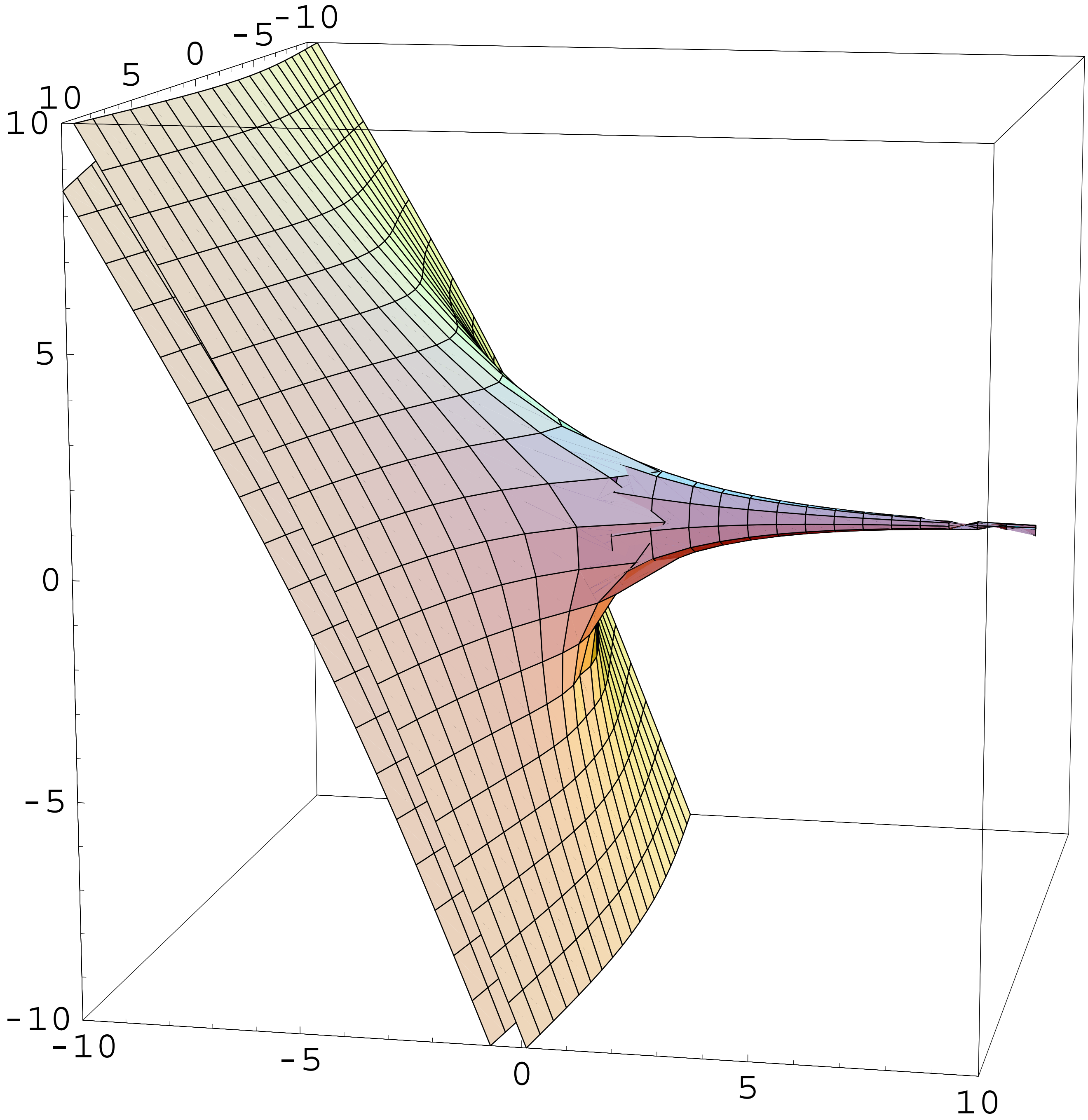
# Amidakuji



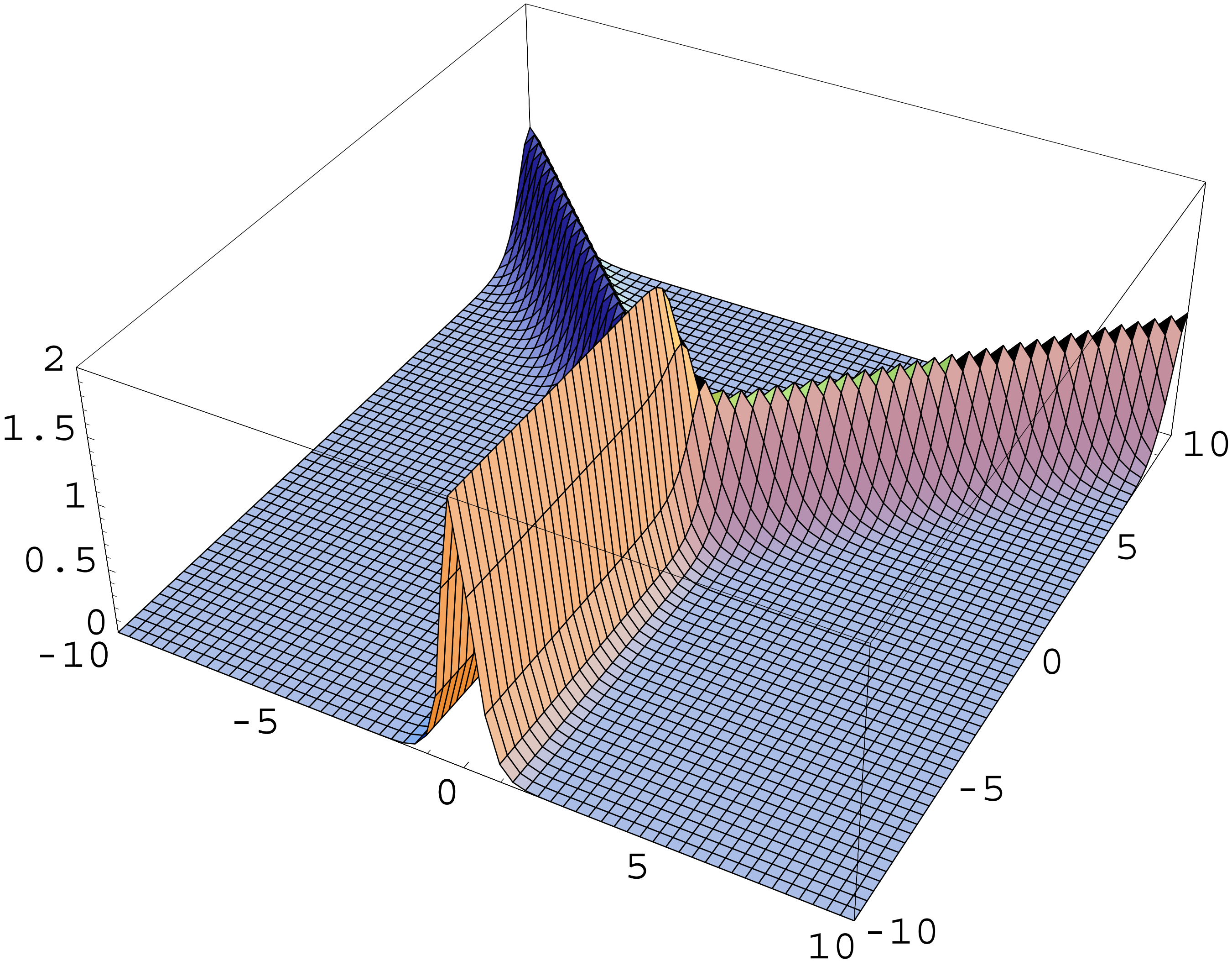
# A monopole in the Higgs phase



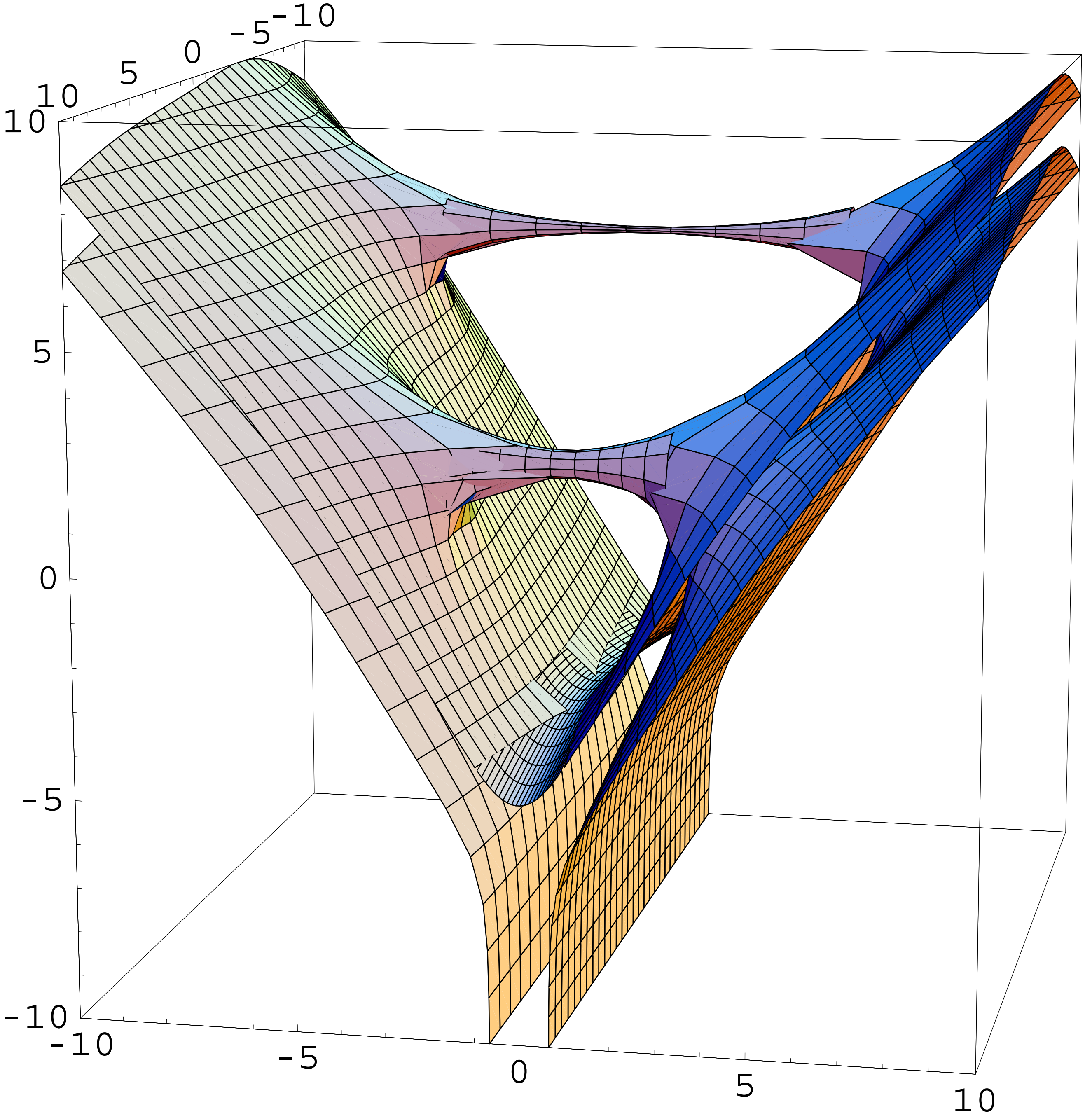
# Wall-vortex junction with constant $B$



# Wall-junction



# A cat's-cradle ( = Ayatori)



Another set of new 1/4 BPS equations admitting vortices and instantons has been discovered.

Massless gauge theories with $M = 0$.

## The Bogomol'nyi bound

$$\mathcal{E} \geq \mathrm{tr}\left[\underbrace{-c(F_{13}+F_{24})}_{\text{vortices}} + \underbrace{\frac{1}{2g^2}F_{mn}\tilde{F}_{mn}}_{\text{instantons}}\right], \quad (34)$$

## A set of 1/4 BPS equations

$$\begin{aligned} &F_{12} = F_{34}, \quad F_{23} = F_{14}, \\ &F_{13} + F_{24} = -\frac{g^2}{2}\left[c\mathbf{1}_{N_\mathrm{C}} - HH^\dagger\right] \\ &\bar{\mathcal{D}}H = 0, \quad \bar{\mathcal{D}}'H = 0. \end{aligned} \quad (35)$$

- Setting $c = 0$, $H = 0$
  $\Rightarrow$ The self-dual eq. for instantons
- Ignoring $x^2, x^4$ dependence and $W_2$ and $W_4$
  $\Rightarrow$ The BPS eq. for vortices in the $(x^1, x^3)$-plane
- Ignoring $x^1, x^3$ dependence and $W_1$ and $W_3$
  $\Rightarrow$ The BPS eq. for vortices in the $(x^2, x^4)$-plane

## §7. Conclusion / Discussion

1. We have given an elegant construction method for exact solutions of BPS non-Abelian domain walls in 8 SUSY gauge theories (in the strong coupling limit $g \to \infty$), which is comparable with the ADHM / Nahm constructions.

2. The moduli space for non-Abelian walls is the (deformed) Grassmann manifold $G_{N_{\rm F},N_{\rm C}}$ which is a special Lagrangian submanifold of massless Higgs branch vacua $T^*G_{N_{\rm F},N_{\rm C}}$.

3. The properties of walls can be understood by a beautiful algebraic structure on the moduli space, composed of creation / anihilation operators of walls.

4. A brane construction is given by the kinky D-branes, explaining properties of walls easily.

5. We have generalized our method to
   1) 1/4 BPS of walls, vortices and monopoles,
   2) 1/4 BPS of vortices and instantons,
   and determined their moduli spaces.

# Discussion

1. Relation with other soliton's moduli space.

| | codim. | $\pi$ | Moduli Space | quotient |
|---|---|---|---|---|
| Instanton | 4 | $\pi_3$ | ADHM | HK |
| Monopole | 3 | $\pi_2$ | Nahm | HK |
| Vortex | 2 | $\pi_1$ | Hanany-Tong | Kähler |
| Wall | 1 | $\pi_0$ | Our work | Kähler |

Vortex moduli $= \frac{1}{2}$ (Instanton moduli)

($\Uparrow$ Hanany-Tong)

Wall moduli $= \frac{1}{2}$ (Monopole moduli) (?)

2. Relation with integrable systems like KDV?.

3. 2nd Quntization of Walls $\Leftarrow$ Wall Algebra

4. Application to the Hori-Vafa's mirror symmetry between the sigma models and the sine-Gordon models (Affine Toda).

5. Other gauge groups / matter contents.
   We conjecture that the moduli of walls is always a special Lagrangian submanifold of massless Higgs branch.

6. There exist two other sets of 1/4 BPS equations.
   1. Vortex$^3$
   2. Wall$^2$-Vortex

7. Classification of BPS equations.

   There exist two kinds of 1/8 BPS equations in five-dimensions, $\boxed{\text{now in progress}}$.
   1. Wall$^3$-Vortex$^3$-Monopole
   2. Vortex$^6$-Instanton

8. Maybe there exist lots of application / extention / variation / ... like other BPS solitons.

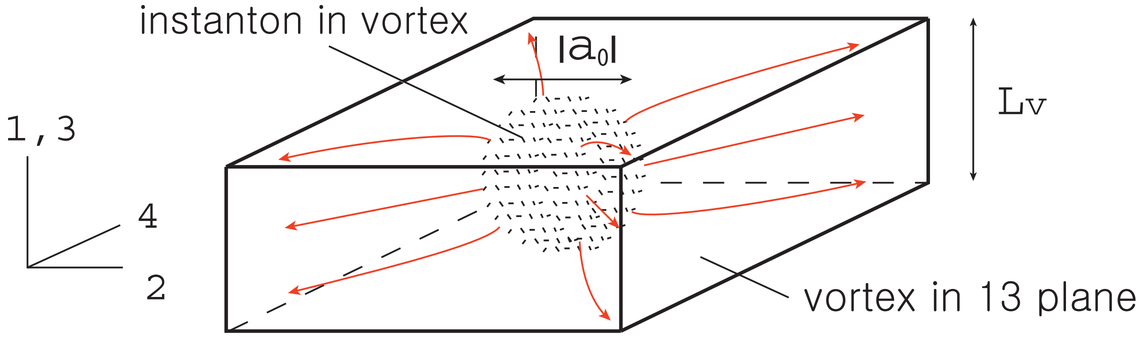
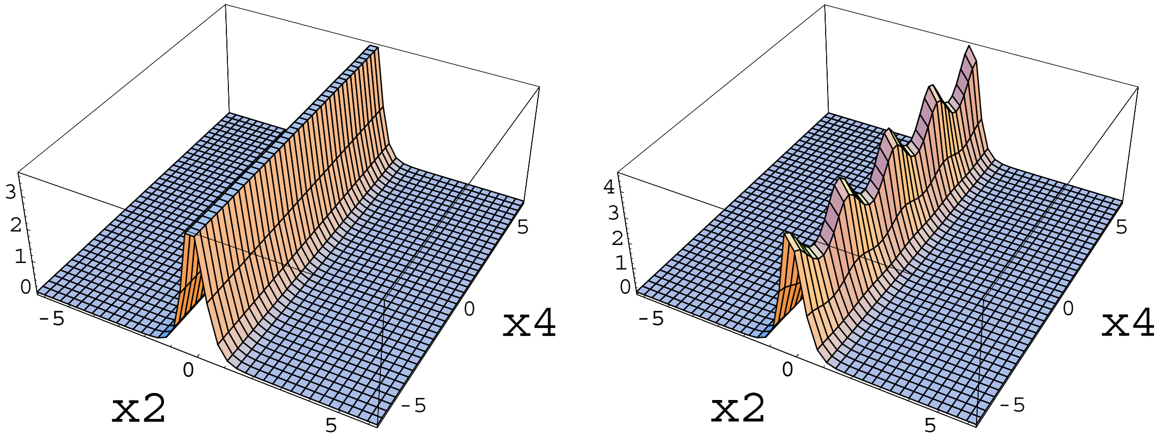
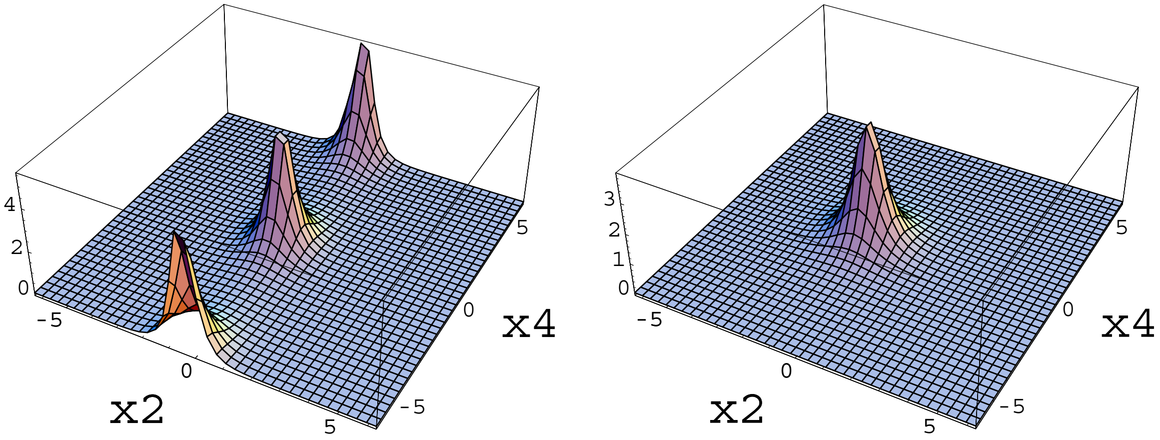


Figure 1: Single instanton in the Higgs phase. The size of the vortex is given by $L_{\rm v} \sim 1/g\sqrt{c}$.



(a) monopole-string
$\mu = 2,\ \theta = 5 \times 10^{-4}$

(b) caloron
$\mu = 2,\ \theta = 0.1$



(c) caloron
$\mu = 1,\ \theta = 1$

(d) instanton
$\mu = 0.2,\ \theta = 5$

Figure 2: Energy density of the calorons in terms of the vortex theory.