Doctoral Thesis

M-branes, D-branes and U-duality from BLG Model

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Abstract

M-theory is considered as one of the most hopeful candidate of unified theory for particle physics. On this theory, it is known that its low-energy limit is supergravity in 11-dim spacetime, and that there are two kinds of spatially spreading stable non-perturbative objects, which are called an M2-brane and an M5-brane. However, the behavior of these M2- and M5-branes still remains a mystery. Then this is an important subject for explication of M-theory at present.

Recently, Bagger, Lambert and Gustavsson [1–4] have written down and proposed an action for multiple (coincident) *M2-branes'* system, which is called 'BLG model.' This model is characteristic in that there is a Chern-Simons-like gauge field on M2-branes' worldvolume, and the gauge group for multiple M2-branes is defined using *Lie 3-algebra*, instead of Lie algebra.

Soon after that, many researchers, including us, began studying this peculiar algebra and the structure of BLG model. We made concrete representations of this algebra, adopted them as examples of the gauge symmetry of multiple M2-branes' system, and analyzed BLG model in each case.

As a result, we [5–8] showed that BLG model does not describe only multiple M2-branes' system, but also a single *M5-brane's* and multiple *Dp-branes'* system in superstring theory. Moreover, we also showed that the relation of M2-branes and Dp-branes, which is known as (a part of) *U-duality*, can be properly written in terms of BLG model's language.

In this Ph.D. thesis, we summarize the history of BLG model and our success in the research on this model.

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Introduction and Summary

M-theory is a very Mysterious theory.[1] It is considered as a Mother theory for all the known fundamental theories of physics, that is, standard model, general relativity, supersymmetric Yang-Mills theory, supergravity, superstring theory and so on, can be reproduced from M-theory by somewhat Magical ways.

In fact, when one compactifies this theory on a 7-dim torus T^7 (which doesn't break any supersymmetry), it becomes a 4-dim theory with $\mathcal{N} = 8$ supersymmetry, whose low-energy limit is the maximal supergravity in 4-dim spacetime. This means that M-theory includes all kinds of fields which we have already known without any contradictions. Besides, when one compactifies M-theory on a circle S^1 or a segment $S^1/\mathbf{Z}_2$, it becomes type IIA or Het $E_8 \times E_8$ superstring theory, which are connected with all other types of superstring theory by S-, T-, U-duality.[2] Therefore, M-theory is also the unified theory of all kinds of superstring theories. (See fig. 1.)

It is well known that Membranes (or M2-branes) play important roles in M-theory, and the formulations with a Matrix for describing M2-branes have been proposed and studied eagerly so far. In this Ph.D. thesis, however, we discuss the newly proposed formulation for M2-branes in M-theory, which is called 'BLG model.' This model is very characteristic in that a Chern-Simons gauge field exists on the M2-branes' worldvolume and the gauge symmetry is defined by some peculiar algebra called Lie 3-algebra, instead of ordinary Lie algebra. The contents are as follows.

Chapter 1 We review the brief history of M-theory before BLG model.

First, we discuss why we consider *supersymmetry* and *higher dimensional spacetime*, since it is a part of answer for the question, why we research on *M-theory* (§1.1). Then we review *11-dim supergravity* as low energy limit of M-theory (§1.2), the property of an M2-brane and an M5-brane (a *dual* object for M2-brane) as non-perturbative objects in M-theory (§1.3), and the relation between M-theory and superstring theory, including U-duality (§1.4). Finally, we discuss the placement of contents of this thesis in the related research on M-theory (§1.5).

The contents are partially based on Kaku's textbook [9] and Ohta's textbook [10].

[1] In Introduction, we dare to use the capital letter 'M' in some words, since these words are considered as the etymology of 'M' of M-theory.

[2] U-duality is the minimal group including both S- and T-duality as its subgroups.

Chapter 2 We review the original BLG model.

First, we collect the clues to construction of multiple M2-branes' worldvolume theory, and suggest that the 3-commutator should be a key idea (§2.1). Then we discuss the structure of *Lie 3-algebra* (the algebra with 3-commutators):

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d \,, \quad \langle T^a, T^b \rangle = h^{ab} \,,$$

where $f^{abc}{}_d$ are structure constants and h^{ab} is a metric (or Killing form). In particular, we show that this algebra can define a natural symmetry of a field theory, when it satisfies the severe conditions of *fundamental identity* and *invariant metric* (§2.2). Next, we gauge the symmetry by introducing a Chern-Simons-like gauge field, and construct a gauge-invariant supersymmetric theory with 16 supercharges. Also, we check that the superalgebra closes on equations of motion for the fields on multiple M2-branes (§2.3). Finally, we show that these equations of motion are derived from a supersymmetric action (BLG action) which is consistent with all the continuous symmetries of the M2-branes' system (§2.4).

The contents are mostly based on Bagger and Lambert's paper [2].

Chapter 3 In order to obtain the concrete form of BLG action, we need to adopt one of representations of Lie 3-algebra. So in this chapter, we classify the various representations of Lie 3-algebra, while the analysis of BLG action with each representation is postponed until Chapter 4-7.

First, we show the famous example of $\mathcal{A}_4$ algebra ($f^{abc}{}_d = \epsilon^{abcd}, h^{ab} = \delta^{ab}$) and its direct sum (§3.1). In fact, these are only examples of finite-dimensional representations of Lie 3-algebra with positive-definite metric. Then we show the proof of this fact (§3.2).

Next, we show the Nambu-Poisson bracket as an infinite-dimensional representation of Lie 3-algebra. We also show that finite-dimensional truncations of Nambu-Poisson bracket are also examples of Lie 3-algebra, whose metric isn't positive-definite, because of the above proof (§3.3). We will analyze BLG model with these examples of Lie 3-algebra in Chapter 4 and 5.

Finally, we classify the representations of Lie 3-algebra with negative-norm generators, and show that there are concrete examples of Lie 3-algebra, such as a central extension of Lie algebra, which can be the symmetry algebra of BLG model (§3.4). We will show that BLG models with these examples of Lie 3-algebra are *physical* theories in spite of the existence of negative-norm generators in Chapter 6 and 7.

The contents are partially based on Papadopoulos' paper [11], Ho, Hou and Matsuo's paper [12] and our papers [6, 7].

Chapter 4 We show that we can obtain a *single M5-brane's* worldvolume theory from BLG model, when we adopt *Nambu-Poisson bracket* as an example of Lie 3-algebra.

First, we summarize the property of Nambu-Poisson bracket defined on 3-dim manifold $\mathcal{N}$ as an infinite-dimensional Lie 3-algebra (§4.1). Then we show that one can define the *local* fields on M5-brane's worldvolume $\mathcal{M} \times \mathcal{N}$, using the BLG fields on original M2-branes' worldvolume $\mathcal{M}$ and the basis functions of Nambu-Poisson bracket. In particular, we can derive the self-dual 2-form gauge field on M5-brane from the Chern-Simons gauge field and some of scalar fields in BLG model (§4.2).

Next, we derive the gauge symmetry of M5-brane's theory from that of BLG model. In particular, we identify the gauge transformation as the volume-preserving diffeomorphism of $\mathcal{N}$ (§4.3). The BLG action is then rewritten in terms of these fields and we can derive the proper action and equations of motion for M5-brane's theory (§4.4). Next, we derive the supersymmetry transformation of the fields on an M5-brane, and show that this transformation indicates that the BLG model in this case describes an M5-brane in a large C-field background (§4.5).

Finally, we derive D4-brane's action from this M5-brane's one by the double dimensional reduction. There, the volume-preserving diffeomorphism is replaced by the area-preserving diffeomorphism, or Nambu-Poisson bracket is replaced by Poisson bracket. As a result, we obtain D4-brane's system in a B-field background (§4.6).

The contents are partially based on Ho and Matsuo's paper [13], and mostly on our paper [5]. *My contribution* in the latter paper is especially to show that one can surely obtain M5-brane's worldvolume action, and discuss the background spacetime of an M5-brane by calculating supersymmetry transformation in this action.

Chapter 5 We discuss BLG model in the case where we adopt the finite-dimensional truncation version of Nambu-Poisson bracket as an example of Lie 3-algebra.

First, we show how to construct the finite-dimensional representations of Lie 3-algebra from Nambu-Poisson bracket by somewhat artificial truncation (§5.1). Then we discuss the structure of these kinds of truncated algebra. In particular, we show that many components of metric in this algebra vanish (§5.2). This means that it is hard to have nontrivial discussion on Lagrangian, so we analyze the equations of motion derived from it. Surprisingly enough, we can show that the $N^{\frac{3}{2}}$ law (the relation between the number of M2-branes and that of degrees of freedom) from AdS/CFT correspondence [14] can be accounted for due to simple algebraic reason, by counting the number of moduli and generators of Lie 3-algebra (§5.3).

The contents are based on our paper [6]. *My contribution* is especially to make sure that this example satisfies fundamental identity and invariant metric condition, and discuss on the counting the number of moduli from obtained equations of motion.

Chapter 6 We review how to obtain *multiple D2-branes'* theory from BLG model with *Lorentzian Lie 3-algebra*, which is a central extension of Lie algebra with one negative-norm generator, as an example of Lie 3-algebra.

First, we summarize the property of Lorentzian Lie 3-algebra and rewrite BLG action in terms of components of this algebra. We also discuss the symmetry of this rewritten action (§6.2). In this action, there is a ghost field which comes from the components of the negative-norm generator. Then in order to obtain a physically meaningful action, we must eliminate this ghost field. Interestingly enough, we can achieve it by inserting VEV's without breaking any gauge symmetry nor supersymmetry. As a result, we obtain 3-dim super Yang-Mills theory, which can be regarded as the low-energy limit of multiple D2-branes' theory (§6.3).

On the other hand, we show that we can also obtain D2-branes' system from BLG model via M5- and D4-brane's theory in Chapter 4. There, we regard Poisson bracket in D4-brane's theory as matrix algebra, and derive multiple D2-branes' theory (§6.4).

The contents are mostly based on Ho, Imamura and Matsuo's paper [15].

Chapter 7 We show that one can obtain *multiple Dp-branes'* theory ($p \geq 3$) from BLG model, when one adopt *general Lorentzian Lie 3-algebra*, which is a central extension of Kac-Moody algebra or loop algebra with more than one negative-norm generators.

First, we analyze BLG model with the simplest nontrivial example of this kind of Lie 3-algebra, and obtain *massive* super Yang-Mills theory with interaction (§7.1). In order to make clear a physical interpretation of this mass in the context of M/string theory, we consider Kac-Moody algebra as a particular example. Then we can show that BLG model in this case describes the multiple D3-branes on a circle, and the mass can be regarded as the Kaluza-Klein mass of the D3-brane winding a circle (§7.2). Next, we consider the Lie 3-algebra with multiple loop algebra, as a generalized Kac-Moody algebra. In this case, BLG model describes the multiple Dp-brane on a $(p-2)$-dim torus ($p \geq 3$) (§7.3). The relation between Dp-brane and M2-brane is well known as U-duality, so finally we discuss that (a part of) U-duality is realized in BLG model. In particular, for D3-brane's case, we show that the full of U-duality (Montonen-Olive duality) is realized (§7.4).

The contents are mostly based on our paper [7, 8]. *My contribution* is especially to classify the examples of Lie 3-algebra with a general number of negative-norm generators, by discussing the conditions of fundamental identity and so on. I also calculate and show that we can obtain Dp-branes' action with $F \wedge F$ term, when we use a central extension of multiple loop algebra as an example of Lie 3-algebra. This enable to discuss how U-duality is realized in BLG model, so I argue this point including what is shortage for discussing whole of U-duality.

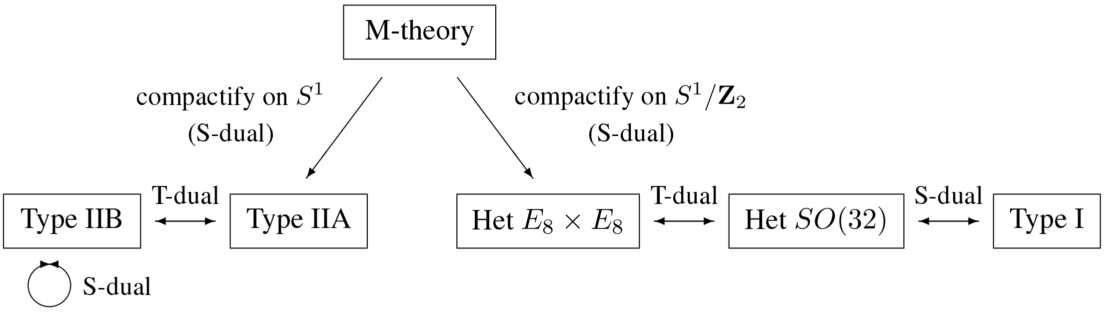

Figure 1: Relation among M-theory and superstring theories

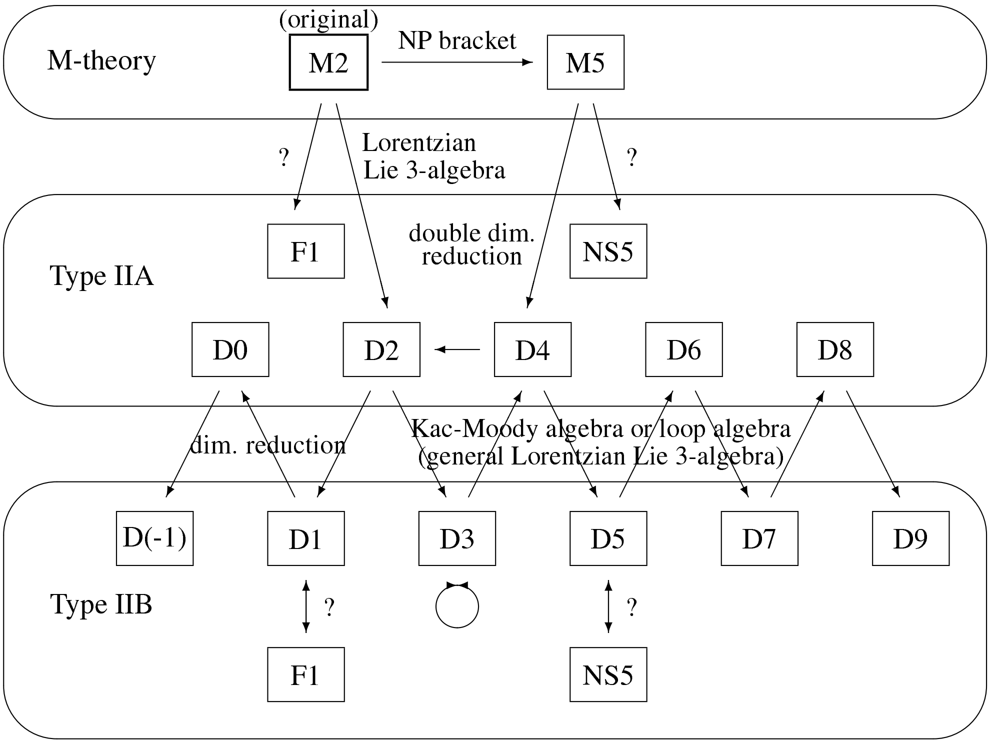

Figure 2: Relation among M-branes and D-branes in BLG model

Part I : Basics and multiple M2-branes

Chapter 1

What is M-theory?

How is the Nature constructed? — This is certainly the most fundamental and commonest question for all human beings. One way, and probably the most profitable way of tackling this question, is the research on *particle physics*. In the particle physics, in the author's understanding, all discussion begin with the following important assumption: all existence in the Nature can be divided into the only one or few kinds of fundamental elements, and if we can fully understand the behavior of each kind of elements, we can also comprehend that of all existence in the Nature. Such way of thinking is called *Reductionism*. So we can say that the ultimate goal for particle physics is the clear understanding of what is the most elementary existence, how it behaves or interacts with each other, and how it creates all existence in our universe. From the view of Reductionism, these pieces of information are *all* of what we need to understand the whole Nature.

Standard model in particle physics — Reductionism is the winner!?

At the moment, the author think, one cannot judge whether or not the Reductionism is valid for tackling our ultimate question. However, at the same time, it must be definitely impossible to deny the brilliant progress of particle physics in the 20th century. The 'standard model' for particle physics is *it*. In this theory, all fundamental elements of the Nature are classified into gauge fields, matter fields and Higgs fields. (Regrettably, the gravity field cannot be included.)

- matter fields (spin 1/2)

quark			lepton		
u_r, u_g, u_b	c_r, c_g, c_b	t_r, t_g, t_b	ν_e	ν_μ	ν_τ
d_r, d_g, d_b	s_r, s_g, s_b	b_r, b_g, b_b	e	μ	τ

The matter fields are classified into the quark and lepton fields. Both fields are composed of three *generations* (which is indicated as three columns). Each generations of fields consist of two types of fields (which is indicated as two rows). Moreover, the quark fields in a single box (which are called *flavor*) are composed of three *colors*.

- gauge fields (spin 1)
  - gluons $(G_1, \cdots, G_8)$
    They mediate the *strong* interaction, that is, the interaction among the quark fields with colors $(*_r, *_g, *_b)$. This interaction is universal for all flavors.
  - weak bosons (W^+, W^-, Z^0)
    They mediate the *weak* interaction, that is, the interaction between matter fields in a same column $(u, d), (c, s), (t, b), (\nu_e, e), (\nu_\mu, \mu), (\nu_\tau, \tau)$.

    On the interaction among different generations, we use Cabbibo-Kobayashi-Maskawa (CKM) matrix which phenomenologically describes the mixing of (mass eigenstates of) quarks of different generations.
  - photon (A)
    It mediates the *electro-magnetic* interaction, that is, the interaction among the matter fields with non-zero electric charge (*i.e.* all the matter fields except neutrinos ν_e, ν_μ, ν_τ.)
- Higgs fields Φ (spin 0)

  These fields interact with the matter fields. This interaction causes the mass of matter fields.
- Gravity field (spin 2)

  It mediates the *gravity* interaction, that is, the interaction among the matter fields with non-zero mass (or energy-momentum). Now we believe that all interactions in the Nature can be classified into these four kinds of interaction above.

  However, as we mentioned, this gravity interaction cannot be dealt in the standard model. It is separately described by general relativity. This is surely an unsatisfactory point, but this causes *no* problem if we study the phenomena in not too high energy scale where the gravity interaction is weaker enough than other interactions.

Concept of 'fields' — *Naive* reductionism is actually invalid.

Let us note here that we use the term 'fields' instead of 'particles.' This is related to the abstruse philosophy based on *quantum theory*. In *quantum mechanics*, the particle-wave duality is the principle. Moreover, in *quantum field theory*, the particles are considered as rippling waves (or 'elemental excitations') of a 'field' which has degrees of freedom in each point of spacetime, and so is defined as an infinitely multi-body system.

This means that we cannot fully understand the Nature when we only research on the 'particles,' and that we must research on the fundamental law for the 'fields' which exist *behind* the particles.

However, the direct observation of 'fields' are very difficult, except the gravity and electro-magnetic fields. Then we still actually research, contradictorily at a glance, on the various interactions among many 'particles,' but we are always conscious of the existence of 'fields' behind these particles. In fact, the standard model is described in terms of quantum field theory.

In the naive reductionism, we assume the existence of only one or a few kinds of fundamental 'particles.' In this meaning, the particle physics now is not in the *naive* reductionism. However, as we will see, we still assume some existence of fundamental 'field,' so the author believe that the particle physics is even now in a kind of reductionism.

Actions of standard model

In quantum field theory, the most important quantity is the action (or Lagrangian). We usually decide the form of action by carefully considering the symmetry which the theory must have.

The standard model is described in terms of quantum field theory with Lorentz symmetry $SO(1,3)$ and gauge (internal) symmetry $SU(3)$ or $SU(2)\times U(1)$. So, in the Lagrangian, all fields must be described by the representations of Lorentz and gauge symmetry group. For Lorentz symmetry, the matter fields are represented by *spinors*, the gauge fields by *vectors* and the Higgs fields by *scalars*. For gauge symmetry, all these fields are described as the representations of the $SU(3) \times SU(2) \times U(1)$ gauge symmetry group, *e.g.* the gauge fields are *adjoint* representations and the matters field are *fundamental* representations.

Now we show the Lagrangian of standard model, which consists of the following two theories:

GWS (Glashow-Weinberg-Salam) theory

It describes the electro-magnetic and weak interaction. The gauge symmetry is defined by $SU(2) \times U(1)$ algebra. The $SU(2)$ algebra is represented as

$$[t^a, t^b] = i\epsilon^{abc} t^c \,, \quad \langle t^a, t^b \rangle = \frac{1}{2}\delta^{ab} \,, \tag{1.1}$$

where $a, b, c = 1, 2, 3$. The Lagrangian is

$$\begin{aligned}
\mathcal{L}_{GWS} \;=\; & -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 \\
& + i\bar{\Psi}^{(f)}_L \gamma^\mu D_\mu \Psi^{(f)}_L + i\bar{E}^{(f)}_R \gamma^\mu D_\mu E^{(f)}_R + i\bar{N}^{(f)}_R \gamma^\mu D_\mu N^{(f)}_R \\
& + (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda \left(|\Phi|^2 + \frac{\mu^2}{2\lambda} \right)^2 \\
& - f_{E^{(f)}} \left(\bar{E}^{(f)}_R (\Phi^\dagger \Psi^{(f)}_L) + (\bar{\Psi}^{(f)}_L \Phi) E^{(f)}_R \right) - f_{N^{(f)}} \left(\bar{N}^{(f)}_R (\Phi^\dagger_c \Psi^{(f)}_L) + (\bar{\Psi}^{(f)}_L \Phi_c) N^{(f)}_R \right) .
\end{aligned} \tag{1.2}$$

For the gauge fields, the covariant derivative $D_\mu = \partial_\mu + ig_w W_\mu^a t^a + i\frac{g_B}{2} B_\mu Y$ and the field strength $F_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g_w f^{abc} W_\mu^b W_\nu^c$, $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. g_w and g_B are the coupling constants. The weak bosons $W^\pm, Z^0$ and photon A are defined as the linear combinations of W_μ^a and B_μ, such that $W^\pm = W^{1,2}$, $Z_\mu^0 = \frac{1}{\sqrt{g_w^2+g_B^2}}(g_w W_\mu^3 - g_B B_\mu)$, $A_\mu = \frac{1}{\sqrt{g_w^2+g_B^2}}(g_w W_\mu^3 + g_B B_\mu)$.

The matter fields are defined as $\Psi_L^{(f)} = (\nu_{eL}, e_L)$, $(\nu_{\mu L}, \mu_L)$, $(\nu_{\tau L}, \tau_L)$, (u_L, d_L), (c_L, s_L), (t_L, b_L), $E_R^{(f)} = e_R$, μ_R, τ_R, d_R, s_R, b_R, $N_R^{(f)} = \nu_{eR}$, $\nu_{\mu R}$, $\nu_{\tau R}$, u_R, c_R, t_R. The index L / R means 'left- / right-handed,' *e.g.* $e_{L,R} = \frac{1\pm\gamma_5}{2} e$ for the electron field e.

Let us note that the potential term for Higgs field Φ (the 7th term) gives a non-zero VEV (Vacuum Expectation Value) to Higgs field. When one expand Φ around the VEV, the Higgs kinetic term (the 6th term) give masses to the weak boson fields $W^\pm$ and Z^0, and the Higgs interaction term (the last two terms) give masses to the matter fields. The index c there means 'charge-conjugated.'

This is well-known *Higgs mechanism*. In general, the *symmetry breaking* occurs in such kinds of mechanism, that is, only a part of the original symmetry is kept after the mechanism works. In fact, the $SU(2) \times U(1)$ symmetry of GWS theory are broken to $U(1)$ symmetry after Higgs mechanism.

QCD (Quantum Chromo-Dynamics)

It describes the strong interaction. The gauge symmetry is defined by $SU(3)$ algebra

$$[t^a, t^b] = if^{ab}{}_c t^c\,, \quad \langle t^a, t^b\rangle = \frac{1}{2}\delta^{ab}\,, \tag{1.3}$$

where $f^{ab}{}_c$ is the $SU(3)$ structure constant, so $a, b, c = 1, \cdots, 8$. The Lagrangian is

$$\mathcal{L}_{QCD} = -\frac{1}{4g_s^2}(F_{\mu\nu}^a)^2 + \bar{\Psi}_a^{(f)}(i\Gamma^\mu D_\mu - m^{(f)})\Psi_a^{(f)}\,, \tag{1.4}$$

where the covariant derivative $D_\mu \Psi_a = \partial_\mu \Psi_a - if^{bc}{}_a G_b \Psi_c$ and the field strength $F_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + f^{abc} G_\mu^b G_\nu^c$. G_μ^a are gluon fields and $\Psi_a^{(f)}$ is a quark field of a particular flavor. g_s is the coupling constant of strong interaction. $m^{(f)}$ is the mass of quarks, which is determined from Higgs interaction in GWS theory.

1.1 Beyond standard model

Towards the deeper truth — We search more fundamental existence in the Nature!

At this moment, the standard model agrees with all the experimental results with marvelous preciseness.[1] In spite of such great success, however, most of researchers don't regard the standard model as ultimate theory, since there are serious problems:

[1]Only the exception is the neutrino vibration, which requires that the mass of neutrinos ν_e, ν_μ, ν_τ should be non-zero, while they are set to zero in the standard model: $f_{\nu_e}, f_{\nu_\mu}, f_{\nu_\tau} = 0$.

- It has too many arbitrary constants.

  There are g_w, g_B, g_s, λ, μ^2, f_e, f_μ, f_τ, f_u, f_d, f_c, f_s, f_t, f_b and four parameters of CKM matrix. Is there anyone who can believe that God carefully adjusted so many parameters simultaneously at the birth of universe...?

- The unification of gravity is impossible.

  The gravity interaction cannot be included in the standard model. So the unification is never possible.

In order to overcome these difficulties, it is nice and natural to follow our fundamental thought that all the existence in our Nature consists of one or few kinds of fundamental elements. It is because if all phenomena can be explained from one or few kinds of elements, the number of parameters (or arbitrary constants) must also be very small. Therefore, what we must consider here is what are the fundamental elements, in other words, how to *unify* all the fields including not only the fields in the standard model but also gravity field.

Let us now consider the theory of only one or a few kinds of elementary fields which contains the degrees of freedom of *all* the standard model fields and gravity field, and which are described as the representations of the *single* symmetry group. This symmetry group must contain the $SU(3) \times SU(2) \times U(1)$ group of the standard model and also the Poincaré symmetry group of general relativity as its subgroup.

Then we assume that this single symmetry group is broken into these two subgroups after symmetry breaking. After the breaking, elementary fields are classified to many kinds of fields, *i.e.* standard model fields and gravity field, which are described as the representations of the subgroups.

Supersymmetry may be the only way that we can choose.

Now we want to know whether or not such a single symmetry group exists. On this discussion, it is useful to note the Coleman-Mandula's theorem [16] and Haag-Łopuszánski-Sohnius' theorem [17].

Coleman-Mandula's theorem

The symmetry G of the S-matrix in quantum theory is restricted to the direct product of Poincaré group and internal symmetries. In terms of algebra, this says that only the direct sum are allowed:

$$(\text{algebra of } G) = (\text{Poincaré algebra}) \oplus (\text{Lie algebra of internal symmetries})\,. \qquad (1.5)$$

This means that there is no nontrivial *single* symmetry group which we are looking for. However, if we allow a graded algebra which includes anti-commutation relations, *e.g. supersymmetry*, the situation changes desirably.

Haag-Łopuszánski-Sohnius' theorem

Now we set the generators of the symmetry algebra as Poincaré generators P_μ, $M_{\mu\nu}$, internal (gauge) symmetry generators T^l, and supersymmetry generators Q^i_α where $\mu = 0, \cdots, D-1$ (D is a spacetime dimension), $i = 1, \cdots, \mathcal{N}$ (number of supersymmetry). Then the nontrivial algebra can be defined as

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\nu\rho}M^{\mu\sigma}) , \\
\{Q^i, \bar{Q}^j\} &= \delta^{ij}\gamma_\mu P^\mu + \mathbf{1}S^{ij} + i\gamma_5 V^{ij} , \\
[Q^i, T^l] &= (T^l)^{ij} Q^j , \\
[T^l, T^m] &= i f^{lm}{}_n T^n , \\
[Q^i_\alpha, M_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu})_{\alpha\beta} Q^i_\beta ,
\end{aligned} \tag{1.6}$$

and otherwise equal 0. Here, $\sigma_{\mu\nu} := \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, and S^{ij}, V^{ij} are central charges.

Therefore, now we must choose only the one way, *i.e.* we consider the symmetry group including the *supersymmetry*.

Supersymmetry brings 'higher dimensional spacetime' naturally.

For simplicity, we consider here the supersymmetric theory of massless fields. Since the supersymmetry generators change the helicity of fields by $1/2$, various fields with different helicity (or spin) can be contained in the same multiplet in supersymmetric theories. The following table shows the multiplicity of each helicity fields in a single multiplet, when we set the maximum and minimum helicity $\lambda_{\max} = 2$, $\lambda_{\min} = -2$:

$\lambda\backslash\mathcal{N}$	1	2	3	4	5	6	7	8
2	1	1	1	1	1	1	1	1
$3/2$	1	2	3	4	5	6	$7+1$	8
1		1	3	6	10	$15+1$	$21+7$	28
$1/2$			1	4	$10+1$	$20+6$	$35+21$	56
0				$1+1$	$5+5$	$15+15$	$35+35$	70
$-1/2$			1	4	$1+10$	$6+20$	$21+35$	56
-1		1	3	6	10	$1+15$	$7+21$	28
$-3/2$	1	2	3	4	5	6	$1+7$	8
-2	1	1	1	1	1	1	1	1

where $\mathcal{N}$ is the number of supersymmetry and $+$ appears from the requirement of CPT invariance. Here we set the maximum limit of spin as 2, since it is widely believed that the existence of fields whose spin is higher than 2 is forbidden by the equivalence principle in general relativity.

Therefore, we can conclude that the single symmetry group which we are looking for must contain the 4-dim $4 \leq \mathcal{N} \leq 8$ supersymmetry group. In these groups, a single multiplet contains all fields, such as gravity field (spin 2), gauge fields (spin 1), matter fields (spin 1/2) and Higgs fields (spin 0). In the following, we concentrate on the 4-dim maximal supersymmetry group $\mathcal{N} = 8$, since the $\mathcal{N} < 8$ supersymmetry groups are only subgroups of $\mathcal{N} = 8$ group.

The supersymmetry generators are described as the smallest representations of spinors in each spacetime dimension, so we see here the degree of freedom of all kinds of spinors in various dimensional spacetime (*i.e.* spinor representations of $SO(D-1,1)$):

rep.\dim.	2	3	4	5	6	7	8	9	10	11	12
Dirac	4	4	8	8	16	16	32	32	64	64	128
Majorana	2	2	4	—	—	—	16	16	32	32	64
Weyl	2	—	4	—	8	—	16	—	32	—	64
Majorana-Weyl	1	—	—	—	—	—	—	—	16	—	—

So the supersymmetry generators of 4-dim $\mathcal{N} = 8$ *supergravity* (*i.e.* the supersymmetric theory which contains a gravity field) have $4 \times 8 = 32$ degrees of freedom. By the way, one can easily see that those of 11-dim $\mathcal{N} = 1$ supergravity also have the same degrees of freedom. This is an important fact for studying 4-dim $\mathcal{N} = 8$ supergravity, because it is hard to deal with this theory, but its concrete form can be easily obtained by the compactification of 11-dim $\mathcal{N} = 1$ supergravity on the 7-dim torus T^7.

Now we naturally introduce *higher dimensional spacetime* as a convenient tool. The discussion on the real existence of extra dimensions should be left to the future research.

1.2 11-dim supergravity as low energy limit

In the previous section, we see that 4-dim $\mathcal{N} = 8$ supergravity may be a candidate of unified theory beyond the standard model, and that it is useful to consider 11-dim $\mathcal{N} = 1$ supergravity for studying this theory. In this section, we briefly review 11-dim $\mathcal{N} = 1$ supergravity and its problematic points.

Fields and their degrees of freedom

In 11-dim $\mathcal{N} = 1$ supergravity, the following (only) three kinds of fields exist and compose the single multiplet.

<u>Graviton g</u>

It is described by the rank-2 symmetric tensor $g_{\mu\nu}$ where $\mu = 0, \cdots, D-1 (= 10)$. So the number of the degrees of freedom is

$$\frac{1}{2}D(D+1) - D - D = \frac{1}{2}D(D-3) \stackrel{D=11}{=} 44\,, \tag{1.7}$$

where the second term is arisen from the invariance under the general coordinate transformation, and the third term is from the gauge fixing for this invariance.

Gravitino ψ

It is described by the representation $\psi_{\mu\alpha}$ with one vector index μ and one spinor index $\alpha = 1, \cdots, 32$. So the number of the degrees of freedom is

$$\frac{1}{2} \cdot 32 \cdot (11 - 1 - 1 - 1) = 128 \,. \tag{1.8}$$

The factor $\frac{1}{2}$ is arisen from the fact that the degree of freedom of fermions is a half of that of bosons, since Dirac equation is the first order differential equation. 32 is the number of spinor elements, 11 is the number of vector elements, the first -1 is arisen from gauge invariance, the second -1 is from $\partial_\mu \psi^\mu = 0$, and the last -1 is from $\gamma_\mu \psi^\mu = 0$.

3-form field $C_{(3)}$

It is described by the rank-3 antisymmetric tensor. So the degree of freedom is

$$_{D-2}C_3 = \frac{(D-2)!}{3!(D-5)!} \overset{D=11}{=} 84 \,. \tag{1.9}$$

Thus the total degree of freedom of bosonic fields $g, C_{(3)}$ is 128, which is the same as that of fermionic fields ψ. It is the common property for all supersymmetric theories that the bosonic and fermionic fields have the same degrees of freedom.

Let us note the degree of freedom of the fields in 4-dim $\mathcal{N} = 8$ supergravity multiplet, which we mentioned in the previous section. That of bosonic fields is $1+28+70+28+1 = 128$ and that of fermionic fields is $(8 + 56) \times 2 = 128$. This means that the fields of 4-dim $\mathcal{N} = 8$ supergravity and 11-dim $\mathcal{N} = 1$ supergravity have the same degrees of freedom, and that each degree of freedom must be in one-to-one correspondence.

Supergravity action and its problems

The Lagrangian is obtained by Cremmer and Julia [18, 19] as

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2\kappa^2} eR - \frac{1}{2} e\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \left(\frac{\omega + \hat{\omega}}{2}\right) \psi_\rho - \frac{1}{48} e (F_{\mu\nu\rho\sigma})^2 \\ & + \frac{\sqrt{2}\kappa}{384} e (\bar{\psi}_\mu \gamma^{\mu\nu\rho\sigma\lambda\tau} \psi_\tau + 12 \bar{\psi}^\nu \gamma^{\rho\sigma} \psi^\lambda)(F + \hat{F})_{\nu\rho\sigma\lambda} - \frac{\sqrt{2}\kappa}{3456} C_{(3)} \wedge F_{(4)} \wedge F_{(4)}\end{aligned} \tag{1.10}$$

where $F_{(4)}$ is a field strength of $C_{(3)}$, and $\hat{F}_{(4)}$ is a super-covariantized $F_{(4)}$. $2\kappa^2 = (2\pi)^8 l_p^9$ and l_p is 11-dim Planck length. e is a vielbein, ω is a spin connection and $\hat{\omega}_{\mu ab} = \omega_{\mu ab} + \frac{1}{8} \bar{\psi}^\rho \gamma_{\rho\mu ab\sigma} \psi^\sigma$, where $a, b, \cdots$ are indices of a tangent flat spacetime.

The various lower dimensional supergravities, including 4-dim $\mathcal{N} = 8$ supergravity, can be obtained by the dimensional reduction of this action on appropriate manifolds. So it seems a promising theory for the ultimate unified one. However, this 11-dim supergravity has the following serious problems:

- This theory may not be renormalizable, *i.e.* the quantum theory cannot be defined.
- If one compactifies this theory on a manifold, the chiral fermions cannot appear. Therefore, the standard model never be reproduced, where left- and right-handed fermions act differently.
- If one compactifies this theory on 7-dim manifold, one can obtain 4-dim theory with only $SO(8)$ gauge symmetry. Again, the standard model cannot be reproduced, since $SU(3) \times SU(2) \times U(1)$ gauge symmetry never be contained in $SO(8)$ symmetry.

The existence of M-branes as the solution of 11-dim supergravity's problems

In fact, these problems can be solved, if we consider the new theory which contains 11-dim supergravity as its low-energy limit.

Let us note that the field $C_{(3)}$ exists in 11-dim supergravity. Then it is natural to consider that there must be the source of *electric* and *magnetic* charge for this field $C_{(3)}$. In 4-dim Maxwell theory, the electric-magnetic field $A_{(1)}$ exists. It is well known that the electric interaction between the point charge and field is described as

$$\int d^4x\, \mathcal{L} \supset \int d^4x\, j^\mu A_\mu \,, \quad j^\mu(x) = \int d\tau\, \delta^{(4)}(x^\mu - X^\mu(\tau))\, \partial_\tau X^\mu(\tau) \,, \tag{1.11}$$

where $X^\mu(\tau)$ is a position of the point charge in 4-dim spacetime, and τ is coordinates on its worldline. Then the analogy goes straightforwardly. In this present case, the electric interaction between the source and field must be described as

$$\begin{aligned} &\int d^{11}x\, \mathcal{L} \supset \int d^{11}x\, j^{\mu\nu\rho} C_{\mu\nu\rho} \,, \\ &j^{\mu\nu\rho}(x) = \int d^3\sigma\, \delta(x^\mu - X^\mu(\sigma^i))\, \epsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho \,, \end{aligned} \tag{1.12}$$

where $X^\mu(\sigma)$ is a position of the source in 11-dim spacetime, and σ^i ($i = 0, 1, 2$) are coordinates on its worldvolume. This means that the sources of electric charge must be the objects which spread for 3-dim spacetime (*i.e.* 2-dim space).

On the other hand, the magnetic interaction for $C_{(3)}$ can be also described easily. In 4-dim Maxwell theory, the source of magnetic charge is called *monopole*, which couples with the dual field $A'_{(1)}$ satisfying $dA'_{(1)} = *_4 dA_{(1)}$. In this case, since the magnetic dual of $C_{(3)}$ is $C'_{(6)}$ which satisfies $dC'_{(6)} = *_{11} dC_{(3)}$, the magnetic interaction between the source and field must be described as $\mathcal{L} \supset j^{\mu\nu\rho\sigma\lambda\tau} C'_{\mu\nu\rho\sigma\lambda\tau}$. From similar discussion, we can conclude that the sources of magnetic charge must be the objects which spread for 6-dim spacetime (*i.e.* 5-dim space).

Therefore, it is natural to consider that there must be these electric and magnetic sources for $C_{(3)}$ field in the new theory, which are called *M2-branes* and *M5-branes*. When a sufficiently large number of coincident M2- and M5-branes' charges are put in 11-dim spacetime, this system must be the solution of 11-dim supergravity, of course. In fact, it is well known that its near-horizon limit is $AdS_4 \times S^7$ and $AdS_7 \times S^4$ spacetime, respectively.

These M-branes are non-perturbative objects, since their masses are l_p^{-3} and l_p^{-6} which go to infinity if $l_p \to 0$ (in the low-energy limit).[2] Moreover, they are 1/2 BPS states which keep a half of supersymmetry if one puts an M-brane (*i.e.* M2- or M5-brane) in 11-dim spacetime, so these M-branes are stable even in quantum theory because of supersymmetry. Therefore, this quantum theory, which contains 11-dim supergravity as its low-energy limit and M-branes as non-perturbative objects, is called *M-theory*.

Then it is widely believed that M-theory has no problems, since such non-perturbative corrections solve the problems which 11-dim supergravity has. For example,

- Renormalizability : M-theory action must have the higher order terms of curvature tensor, and they can make the theory finite.
- Chirality : One can compactify M-theory on a non-manifold space, *e.g.* a segment, then the chiral fermion appears in this case.
- Gauge symmetry : As we saw in fig. 1, for example, when one compactifies M-theory on a circle or segment, it becomes type IIA or Het $E_8 \times E_8$ superstring theories. There, we can obtain a sufficiently large gauge group, such as $U(N \gg 1)$ group in the former case or $E_8 \times E_8$ group in the latter case.

1.3 M2-brane and M5-brane

From the discussion in the previous section, it must be sure that M-theory can be the most profitable unified theory for particle physics, and that in order to understand M-theory, it is indispensable to research the behavior of M-branes in detail.

According to Dirac's discussion on monopoles, M-brane's charge is quantized. It means that the number of charges, *i.e.* the number of M-branes, can be counted. Then, in this section, we first show the action of a *single* M-brane's system. The discussion on *multiple* M-brane's system is put off until §1.5.

A single M2-brane's action

The action of an M2-brane's system is defined on its 3-dim worldvolume, and is invariant under the general coordinate transformation on the worldvolume. It means that the scalar

[2] As we will see in §1.4, M-theory is also the strong interaction limit of superstring theories. In this view of point, M-theory must have no adjustable parameters. It means that only parameter is 11-dim Planck length, so all the physical quantity are determined by its dimension.

fields X^μ which denote the position of an M2-brane in 11-dim spacetime have 8 physical degrees of freedom.

As we mentioned, M2-brane keeps (a half of) supersymmetry, so its action must be also supersymmetric. This requires that there should be the spinor field Ψ as a superpartner of X^μ in M2-brane's action. So this field Ψ is regarded as the fermionic coordinates. Its degrees of freedom is $\frac{1}{2}\cdot\frac{1}{2}\cdot 32 = 8$, where the first $\frac{1}{2}$ comes from the broken half of supersymmetry (or local κ symmetry) and the last $\frac{1}{2}$ comes from equation of motion. This is the same as that of scalars.

Therefore, the supersymmetric action of a single M2-brane can be written in terms of only scalars X^μ and fermions Ψ. The covariant Lagrangian is obtained by Bergshoeff, Sezgin and Townsend [20, 21] as

$$\begin{aligned}\mathcal{L} &= -\sqrt{-\det \Pi_i \cdot \Pi_j} \\ &\quad -\frac{i}{2}(\epsilon^{ijk}\bar{\Psi}\Gamma_{\mu\nu}\partial_i\Psi)\left[\Pi_j^\mu \Pi_k^\nu + i\Pi_j^\mu\bar{\Psi}\Gamma^\nu\partial_k\Psi - \frac{1}{3}(\bar{\Psi}\Gamma^\mu\partial_j\Psi)(\bar{\Psi}\Gamma^\nu\partial_k\Psi)\right] \quad (1.13)\end{aligned}$$

where $\Pi_i^\mu = \partial_i X^\mu - i\bar{\Psi}\Gamma^\mu\partial_i\Psi$. The indices $\mu,\nu,\cdots = 0,\cdots,10$ denote 11-dim spacetime coordinates, while $i,j,\cdots = 0,1,2$ denote M2-brane's worldvolume coordinates.

A single M5-brane's action

From the similar discussion, in the case of M5-brane's system, the scalar fields X^μ have only 5 degrees of freedom. On the other hand, the spinor field on M5-brane's theory has 8 degrees of freedom, just as in M2-brane's case. It means that we need more bosonic fields with 3 degrees of freedom, in order to construct the single M5-brane's supersymmetric action. It is well known that this bosonic field must be the self-dual 2-form field $A_{(2)}$.

Although it is very hard to write down the covariant action including a self-dual field, the covariant Lagrangian for an M5-brane was obtained by Pasti, Sorokin and Tonin [22]. The bosonic part is

$$\mathcal{L} = \sqrt{-\det(g_{mn} + i\tilde{F}_{mn})} + \frac{1}{4}\frac{\sqrt{-g}}{(\partial_r a)^2}F^{*mnl}F_{nlp}\partial^p a\,, \quad (1.14)$$

where the indices $m,n,\cdots = 0,\cdots,5$ denote M5-brane's worldvolume coordinates.
F_{lmn} is a field strength of self-dual 2-form field $A_{(2)}$, and its dual is defined as $F^{*lmn} = \frac{1}{6\sqrt{-g}}\epsilon^{lmnpqr}F_{pqr}$. a is an auxiliary scalar field which can be eliminated by gauge-fixing, and $\tilde{F}_{mn} = \frac{1}{\sqrt{(\partial_p a)^2}}F^*_{mnl}\partial^l a$.

1.4 Relation to superstring theories

In the previous section, we see M-theory as the refined theory of 11-dim $\mathcal{N}=1$ supergravity, which closely relates to 4-dim $\mathcal{N}=8$ supergravity. In this section, we see M-theory as the string coupling limit of superstring theories.

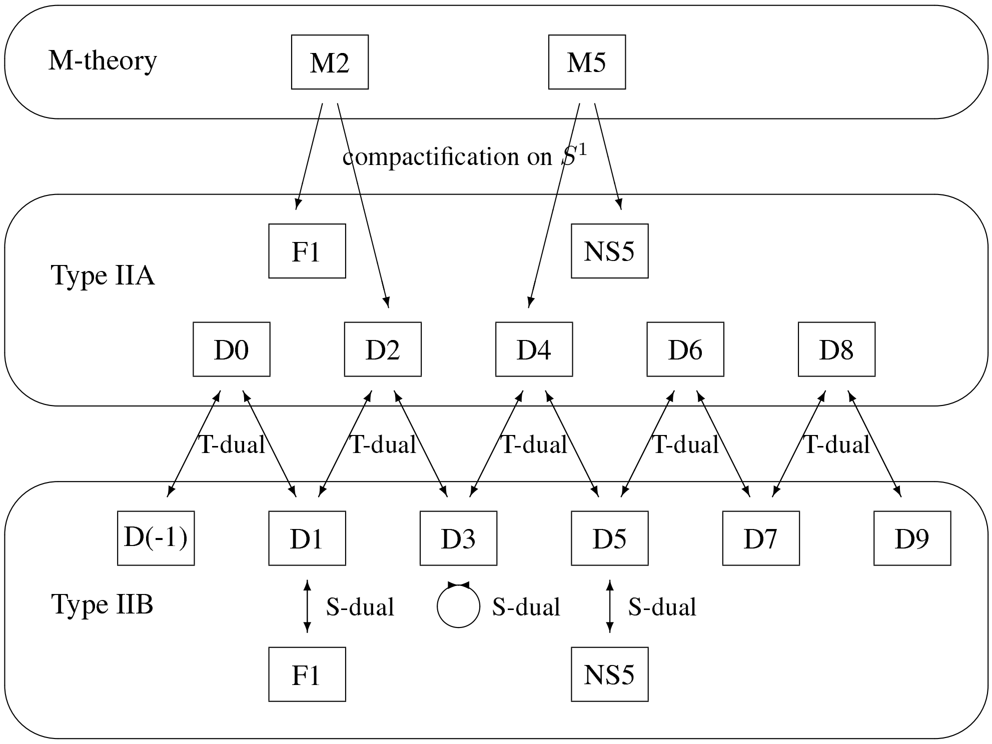

Figure 1.1: Relation among M-branes and D-branes

We already showed the relation among M-theory and superstring theories in fig. 1 in Introduction. It is well known that while the superstring theory is not unique, M-theory can be regarded as the unification of *all* types of superstring theory. In the following, we especially discuss the relation among M-theory and type IIA and IIB superstring theory.

1.4.1 Type IIA superstring theory

Field contents and D-branes

As we saw in fig. 1, type IIA superstring theory is obtained by compactification of M-theory on a circle S^1. In this case, the graviton (or metric) $\hat{g}$ and 3-form field $\hat{C}_{(3)}$ in M-theory are rewritten in terms of the bosonic fields in type IIA superstring theory as

$$\begin{aligned} ds_{11}^2 &= e^{-\frac{2}{3}\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4}{3}\phi}(dy + C_\mu dx^\mu)^2 \,, \\ \hat{C}_{(3)} &= \frac{1}{6} C_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho + \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu \,, \end{aligned} \tag{1.15}$$

where $\mu, \nu, \cdots = 0, \cdots, 9$ and y is a coordinate of the compactified circle. That is, the fields of type IIA superstring theory are

- graviton $G_{\mu\nu} = \sqrt{\hat{g}_{yy}} \left(\hat{g}_{\mu\nu} - \dfrac{\hat{g}_{\mu y}\hat{g}_{\nu y}}{\hat{g}_{yy}} \right)$

- NS-NS B-field $B_{\mu\nu} = \frac{3}{2}\hat{C}_{\mu\nu y}$: F1-string / NS5-brane
- dilaton $\phi = \frac{3}{4}\log \hat{g}_{yy}$
- R-R 3-form field $C_{\mu\nu\rho} = \hat{C}_{\mu\nu\rho}$: D2-brane / D4-brane
- R-R 1-form field $C_\mu = \dfrac{\hat{g}_{\mu y}}{\hat{g}_{yy}}$: D0-brane / D6-brane

Here we also write the objects, called F1-string (or fundamental string), NS5-brane and various dimensional Dp-branes, which couple to each field electrically / magnetically. This can be understood from similar discussion nearby eq. (1.12). The relation to M-branes are showed in fig. 1.1. It is also interesting to compare this figure with fig. 2 in Introduction. The correspondence between them is an important topic in Part III of this thesis.

Compactification radius and string coupling constant

From eq. (1.15), the compactification radius for y-direction is

$$R_{11} = e^{\frac{2}{3}\phi}\sqrt{\alpha'}\,, \tag{1.16}$$

where $\sqrt{\alpha'}$ is the string scale which are usually set to 1. When we rewrite 11-dim supergravity Lagrangian (1.10) as 10-dim type IIA supergravity Lagrangian using the relation (1.15), we can obtain the Kaluza-Klein mass on this circle as

$$\frac{e^{-\frac{1}{3}\phi}}{R_{11}} \sim \frac{1}{e^{\phi}\sqrt{\alpha'}}\,. \tag{1.17}$$

This means that the compactification radius is $\tilde{R}_{11} = e^{\phi}\sqrt{\alpha'}$ when we measure it with 10-dim metric $g_{\mu\nu}$, and that the relation between 11-dim gravity constant κ and 10-dim gravity constant κ_{10} is $\dfrac{2\pi\tilde{R}_{11}}{\kappa^2} = \dfrac{1}{\kappa_{10}^2}$. On the other hand, from the ratio of tension of F1-string and D-brane in string theory, we can obtain κ_{10} as

$$\kappa_{10} = 8\pi^{\frac{7}{2}}\alpha'^2 g\,, \tag{1.18}$$

where g is the string coupling constant, and is determined by the VEV of dilaton field as $g = e^{\langle\phi\rangle}$. Then the 11-dim gravity constant is

$$\kappa_{11}^2 = 2\pi\tilde{R}_{11}\kappa^2 = \frac{1}{2}(2\pi)^8 g^3 \alpha'^{\frac{9}{2}}\,. \tag{1.19}$$

As we saw in eq. (1.10), the relation between κ and 11-dim Planck length l_p is $2\kappa^2 = (2\pi)^8 l_p^9$, so we obtain $l_p = g^{\frac{1}{3}}\sqrt{\alpha'}$.

To summarize above discussion, we find the relations such that

$$g = \left(\frac{R_{11}}{l_p}\right)^3 = \left(\frac{\tilde{R}_{11}}{l_p}\right)^{\frac{3}{2}}, \qquad \alpha' = \frac{l_p^4}{R_{11}^2} = \frac{l_p^3}{\tilde{R}_{11}}\,. \tag{1.20}$$

This means that the strong coupling limit $g \to \infty$ corresponds to $R_{11}, \tilde{R}_{11} \to \infty$, *i.e.* M-theory can be regarded as the strong coupling limit of type IIA string theory.

A single F1-string's action

The bosonic action of an F1-string's system is known as Nambu-Goto action

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma\,(-\det[\partial_i X^\mu \partial_j X^\nu G_{\mu\nu}])^{\frac{1}{2}} =: -\frac{1}{2\pi\alpha'}\int d^2\sigma\,\sqrt{-\det G_{ij}}\,, \quad (1.21)$$

where the scalar fields X^μ denote the position of an F1-string in 10-dim spacetime. The indices $i, j = 0, 1$ denote F1-string's worldvolume coordinates σ^i. We note that the worldvolume $\sqrt{-\det G_{ij}}$ is apparently invariant under the general coordinate transformation.

This action is equivalent to Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-h}\,h^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu}\,, \quad (1.22)$$

where $h = \det h^{ij}$. In fact, this action reduces to Nambu-Goto action, when we use the equation of motion for the auxiliary field h^{ij}, which is a metric on the worldsheet (worldvolume) of an F1-string. One can quantize this action, and all kinds of fields in superstring theory are described as vibrations of an F1-string in the massless spectrum.

The supersymmetric action of an F1-string's system is known as Green-Schwarz action

$$\begin{aligned} S \;=\; & -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\Big[\sqrt{-h}\,h^{ij}\Pi_i\cdot\Pi_j + 2i\epsilon^{ij}\partial_i X^\mu(\bar\Psi^1\Gamma_\mu\partial_j\Psi^1 - \bar\Psi^2\Gamma_\mu\partial_j\Psi^2) \\ & \qquad -2\epsilon^{ij}(\bar\Psi^1\Gamma^\mu\partial_i\Psi^1)(\bar\Psi^2\Gamma_\mu\partial_j\Psi^2)\Big]\,, \end{aligned} \quad (1.23)$$

where $\Pi_i^\mu = \partial_i X^\mu - i\bar\Psi^\alpha\Gamma^\mu\partial_i\Psi^\alpha$ $(\alpha = 1, 2)$.

A single Dp-brane's action

Now let us consider the action of a Dp-brane's system. The discussion goes similarly as M-brane's case in §1.3.

This action is defined on its $(p+1)$-dim worldvolume which are invariant under the general coordinate transformation, so the scalar fields X^μ which denote the position of Dp-brane in 10-dim spacetime have $(9-p)$ physical degrees of freedom. On the other hand, the Dp-brane's system in type IIA and IIB superstring theory have 16 supersymmetries, which means that the superpartner Ψ of the scalar fields must have 8 degrees of freedom, in terms of bosonic degrees of freedom. Therefore, there must be an additional bosonic field with $(p-1)$ degrees of freedom, in order to obtain a supersymmetric theory. It is well known that this field is nothing but the massless vector field A_i on $(p+1)$-dim worldvolume.

The bosonic action of a Dp-brane' system is known as DBI (Dirac-Born-Infeld) action

$$S_p = -T_p\int d^{p+1}\sigma\, e^{-\phi}\left(-\det[G_{ij} + B_{ij} + 2\pi\alpha' F_{ij}]\right)^{\frac{1}{2}}\,, \quad (1.24)$$

where the indices $i, j = 0, \cdots, p$ denote Dp-string's worldvolume coordinates σ^i. G_{ij}, B_{ij} are the pull-back of $G_{\mu\nu}, B_{\mu\nu}$ such that $G_{ij} := \partial_i X^\mu \partial_j X^\nu G_{\mu\nu}$, just as in the F1-string's case. F_{ij} is a field strength of vector field A_i on Dp-brane's worldvolume. We note that this action reduces $U(1)$ Yang-Mills gauge theory in the low-energy limit (where $G_{\mu\nu} = \eta_{\mu\nu}$ and $B_{\mu\nu} = 0$).

Moreover, when the background R-R fields $C_{(q)}$ exist, we have the additional bosonic terms in the action such that

$$S_{CS} = iT_p \int \exp\left[B_{(2)} + 2\pi\alpha' F_{(2)}\right] \wedge \sum_q C_{(q)} \,, \tag{1.25}$$

where integrated functions must be $(p+1)$-form.

1.4.2 Type IIB superstring theory

T-duality

As we saw in fig. 1, type IIA and IIB superstring theory can be related by *T-duality*. In order to take T-dual for z-direction, we first compactify this direction with the radius R as

$$z \sim z + 2\pi R. \tag{1.26}$$

In this case, the center-of-mass momentum of an F1-string for z-direction is quantized as

$$p_z = \frac{n}{R}\,, \quad n \in \mathbf{Z}. \tag{1.27}$$

in quantum theory, because the translation operator $\exp[2\pi i R p_z]$ should be 1 under the identification (1.26). On the other hand, one of the scalar field X^z which denote the position of a *closed* string in z-direction satisfies

$$X^z(\sigma + 2\pi) = X^z(\sigma) + 2\pi R w\,, \quad w \in \mathbf{Z}. \tag{1.28}$$

where $\sigma \in [0, 2\pi]$ is worldsheet spatial coordinates, and w is the winding number. It only says that a closed string must be *closed* in the compactified space (1.26). Without this compactification, this condition becomes the stronger one such that $X^z(\sigma + 2\pi) = X^z(\sigma)$.

In this case, it can be showed that the mass spectrum of a quantized closed string is invariant under the T-duality transformation

$$R \;\to\; R' = \frac{\alpha'}{R}\,, \quad n \;\leftrightarrow\; w\,, \tag{1.29}$$

which means that the theory is invariant, when one interchanges *long* length and *short* length, and momentum and winding number, simultaneously.

Field contents and D-branes

All the bosonic fields in type IIA superstring theory (in the previous subsection) are various vibration modes of a closed string. These vibration modes, of course, change under the T-duality. Then it is known that if we take T-dual for z-direction, the fields of type IIA superstring theory are transformed to those of type IIB superstring theory:

- graviton $\bar{G}_{\mu\nu}$:

$$\bar{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{z\mu}G_{z\nu} - B_{z\mu}B_{z\nu}}{G_{zz}}, \quad \bar{G}_{z\mu} = \frac{B_{z\mu}}{G_{zz}}, \quad \bar{G}_{zz} = \frac{1}{G_{zz}}. \tag{1.30}$$

- NS-NS B-field $\bar{B}_{\mu\nu}$: F1-string / NS5-brane

$$\bar{B}_{\mu\nu} = B_{\mu\nu} + \frac{2G_{z[\mu}B_{\nu]z}}{G_{zz}}, \quad \bar{B}_{z\mu} = \frac{G_{z\mu}}{G_{zz}}. \tag{1.31}$$

- dilaton $\bar{\phi} = \phi - \frac{1}{2}\log G_{zz}$
- R-R 4-form field $\bar{C}_{\mu\nu\rho\sigma}$: D3-brane

$$\bar{C}_{z\mu\nu\rho} = \frac{3}{8}\left(C_{\mu\nu\rho} - C_{[\mu}B_{\nu\rho]} + \frac{G_{z[\mu}B_{\nu\rho]}C_z}{G_{zz}} - \frac{3}{2}\frac{G_{z[\mu}C_{\nu\rho]z}}{G_{zz}} \right). \tag{1.32}$$

- R-R 2-form field $\bar{C}_{\mu\nu}$: D1-brane / D5-brane

$$\bar{C}_{\mu\nu} = \frac{3}{2}C_{\mu\nu z} - 2C_{[\mu}B_{\nu]z} + \frac{2G_{z[\mu}B_{\nu]z}C_z}{G_{zz}}, \quad \bar{C}_{z\mu} = -C_\mu + \frac{C_z G_{z\mu}}{G_{zz}}. \tag{1.33}$$

- R-R 0-form field $\bar{C} = C_z$: D(-1)-brane / D7-brane

where $\mu, \nu, \cdots = 0, \cdots, 8$. Similarly, we write the objects which couple to each field electrically / magnetically. The relation to D-branes in type IIA superstring are showed in fig. 1.1. It is important that T-duality changes the dimension p of D-brane one by one, *i.e.* if we take T-dual for a transverse direction, the worldvolume of Dp-brane is extended for this direction, and Dp-brane becomes D$(p+1)$-brane, while if we take it for a longitudinal direction, the worldvolume is reduced for this direction, and Dp-brane becomes D$(p-1)$-brane.

$SL(2, \mathbf{Z})$ symmetry and S-duality

It is well known that type IIB superstring theory is invariant under the $SL(2, \mathbf{Z})$ transformation such that

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} \bar{H}_{(3)} \\ \bar{F}_{(3)} \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} \bar{H}_{(3)} \\ \bar{F}_{(3)} \end{pmatrix}, \quad e^{-\frac{\bar{\phi}}{2}}\bar{G}_{\mu\nu},\ \bar{F}_{(5)} : \text{invariant}, \tag{1.34}$$

where $ad - bc \in \mathbf{Z}$, $\tau := \bar{C} + ie^{-\bar{\phi}}$, and

$$\bar{H}_{(3)} = d\bar{B}_{(2)}, \quad \bar{F}_{(3)} = d\bar{C}_{(2)}, \quad \bar{F}_{(5)} = d\bar{C}_{(4)} - \frac{1}{2}\bar{C}_{(2)} \wedge \bar{H}_{(3)} + \frac{1}{2}\bar{B}_{(2)} \wedge \bar{F}_{(3)}. \tag{1.35}$$

While type IIB supergravity is invariant under the $SL(2, \mathbf{R})$ transformation where $ad-bc \in \mathbf{R}$, type IIB string theory is only under $SL(2, \mathbf{Z})$ one, since the all kinds of charge are quantized from the Dirac's monopole discussion which we also mentioned in §1.3.

In particular, in the case of $(a,b,c,d) = (0,-1,1,0)$ and $\bar{C} = 0$, we find

$$\bar{\phi} \to -\bar{\phi}\,, \quad \bar{B}_{(2)} \to \bar{C}_{(2)}\,, \quad \bar{C}_{(2)} \to -\bar{B}_{(2)}\,. \tag{1.36}$$

Since the string coupling constant is $\bar{g} = e^{\bar{\phi}}$, this transformation means the inversion of $\bar{g}$, and it is called *S-duality* as we showed in fig 1.1. In another case of $(a,b,c,d) = (1,n,0,1)$, we find

$$\bar{C}_{(0)} \to \bar{C}_{(0)} + n\,, \quad \bar{C}_{(2)} \to \bar{C}_{(2)} + n\bar{B}_{(2)}\,, \tag{1.37}$$

which means the shift of R-R fields. This is also the symmetry of type IIB superstring theory.

U-duality — the unified group of S- and T-duality

As we mentioned, U-duality is the minimal unified group including S- and T-duality as its subgroups. We note that S- and T-duality do not commute each other, as we can easily see in fig. 1 and 1.1. Therefore, U-duality also includes a generator like the commutator $[S,T]$.

As we saw in the above $SL(2,\mathbf{Z})$ duality, S-duality means the interchange of two compactified directions (y and z in this case), and is always included in $SL(2,\mathbf{Z})$ group. The remaining T- and U-duality group are shown in the following table:

dim.	T-duality group	U-duality group
10 (IIA)	1	1
10 (IIB)	1	$SL(2,\mathbf{Z})$
9	$\mathbf{Z}_2$	$SL(2,\mathbf{Z}) \times \mathbf{Z}_2$
8	$O(2,2;\mathbf{Z})$	$SL(3,\mathbf{Z}) \times SL(2,\mathbf{Z})$
7	$O(3,3;\mathbf{Z})$	$SL(5,\mathbf{Z})$
6	$O(4,4;\mathbf{Z})$	$O(5,5;\mathbf{Z})$
5	$O(5,5;\mathbf{Z})$	$E_{6(6)}(\mathbf{Z})$
4	$O(6,6;\mathbf{Z})$	$E_{7(7)}(\mathbf{Z})$
3	$O(7,7;\mathbf{Z})$	$E_{8(8)}(\mathbf{Z})$
2	$O(8,8;\mathbf{Z})$	$E_{9(9)}(\mathbf{Z})$
1	$O(9,9;\mathbf{Z})$	$E_{10(10)}(\mathbf{Z})$
0	$O(10,10;\mathbf{Z})$?	$E_{11(11)}(\mathbf{Z})$?

It shows that when we compactify and take T-dual for more directions, T- and U-duality group is larger. We will discuss U-duality in detail in the context of BLG model in §7.4.

1.4.3 Multiple D-branes and $U(N)$ symmetry

As we discussed in the previous subsection, there is a $U(1)$ vector field on a single D-brane. It is known that this field is the massless vibration mode of open F1-string which ends on the D-brane and vibrates for its worldvolume direction.

Here we consider the multiple D-branes' system. In this system, there are more than one kinds of the massless vector fields, since the open F1-strings which end on *different* D-branes from each other cause the different kinds of massless vector fields. That is, the vector fields in this system can be labeled as two indices (i, j), where $i, j = 1, \cdots, N$ (N is the number of D-branes).

If the D-branes are separated, only the vector fields on the strings with $i = j$ are massless, and the massive fields on the $i \neq j$ strings are neglected (or integrated out) in low-energy effective theory. On the other hand, if the D-branes are coincident, all the vector fields with (i, j) are massless. As a result, the gauge symmetry is enhanced in this case, and the $U(N)$ Yang-Mills gauge theory are realized on multiple D-branes' worldvolume in low-energy limit, which we mentioned in the end of §1.2.

1.5 M-branes revisited : placement of this research

Towards multiple M-branes' action

Since we know that the $U(N)$ gauge theory is realized on the multiple D-branes' system, it is natural to consider what kinds of theory is realized on the *multiple M-branes'* system. However, this subject has been a very challenging one for a long time.

One difficulty comes from the fact that M-theory is the strong coupling limit of type IIA string theory and hence M2-branes are the strong coupling limit of D2-branes. This implies that the worldvolume theory for N M2-branes is the infra-red fixed point of a maximally supersymmetric 3-dim $U(N)$ super Yang-Mills theory.

However, there is no known Lagrangian description of this system. The only interacting 3-dim Lagrangian with 16 supersymmetries is maximally supersymmetric Yang-Mills, which contains one vector plus seven scalars with an $SO(7)$ symmetry. This is a well-known Lagrangian for multiple D2-branes in low energy. In fact, simple counting suggests that the M2-branes' theory should contain eight scalar fields and an $SO(8)$ symmetry. In the abelian case, corresponding to a single M2-brane, such a theory can be obtained directly from the D2-branes' worldvolume theory by dualizing the vector field into a scalar. In the non-abelian case, however, there is no straightforward way to do this.

Preceding research

Bagger, Lambert [1–3] and Gustavsson [4] have broken through (a part of) this difficulty, by considering Lie 3-algebra as the gauge symmetry of multiple M2-branes' theory. This algebra is defined with a 3-commutator $[*, *, *]$ and trace $\langle *, * \rangle$, such that

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d , \quad \langle T^a, T^b \rangle = h^{ab} , \tag{1.38}$$

where $f^{abc}{}_d$ are structure constants and h^{ab} is metric. In particular, they found that one can write down the consistent action (called *BLG action*) for multiple M2-branes' system, when one adopts an concrete example of Lie 3-algebra which satisfies the conditions of fundamental identity and invariant metric (Chapter 2).

Then, after that, many researchers studied on the structure of Lie 3-algebra, in order to obtain and analyze the concrete form of BLG action.

Analysis on Lie 3-algebra : Ho, Hou and Matsuo [12] studied widely on the examples of Lie 3-algebra, and conjectured that the finite-dimensional representations of Lie 3-algebra with positive-definite metric are only trivial algebra ($f^{abc}{}_d = 0$), $\mathcal{A}_4$ algebra ($f^{abc}{}_d = \epsilon^{abcd}$, $h^{ab} = \delta^{ab}$) and their direct sums. Soon after, this conjecture was proved by Papadopoulos [11], Gauntlett and Gutowski [24] (Chapter 3).

Many researchers were disappointed at this result, since it seems to mean that we can only obtain the concrete forms of BLG action for the infinite number of M2-branes or with non-physical (*e.g.* ghost) fields, except the $\mathcal{A}_4$ case.

M5-brane's system : Ho and Matsuo [13] suggested that when one use the Nambu-Poisson bracket as an infinite-dimensional representation of Lie 3-algebra, BLG action for the infinite number of M2-branes describes a single M5-brane's system. They showed it by deriving the M5-brane's action from BLG action to the quadratic order (Chapter 4).

D2-branes' system : Mukhi and Papageorgakis [25], and Ho, Imamura and Matsuo [15] showed that when one use the central extension of ordinary Lie algebra with one negative-norm generator, BLG action describes the multiple D2-branes' system (Chapter 6).

This is a nice result especially in that the ghost field from the negative-norm generator can be completely removed without breaking any gauge symmetry nor supersymmetry, by a new kind of *Higgs mechanism.*

Towards Dp-branes' system : As a rather mathematical study on Lie 3-algebra, de Medeiros, Figueroa-O'Farrill and Méndez-Escobar [26] classified the Lie 3-algebra with more than one negative-norm generators (Chapter 7).

Placement of our research

Our research have been done based on these preceding researches. We improved them, and make clear the deeper physical meaning of BLG model as multiple M2-branes' system.

M5-brane's system : We improved the discussion of [13], and showed that one can obtain the single M5-brane's action from BLG action in all orders of the fields, when one adopts the Nambu-Poisson bracket as an example of Lie 3-algebra (Chapter 4).

Moreover, we showed that the truncation of Nambu-Poisson bracket can be also an example of Lie 3-algebra which can be used in BLG model. Since this example is finite-dimensional, one can discuss the relation between the number of M2-branes and entropy (degrees of freedom) of the system. Then we showed that we can obtain the consistent result with the celebrated $N^{\frac{3}{2}}$ law from AdS/CFT correspondence (Chapter 5).

Dp-branes' system : We discussed the possibility of obtaining physically meaningful theories from BLG action, based on the discussion of [26] where one considers the Lie 3-algebra with more than one negative-norm generators.

Then we found that when one uses the central extension of Kac-Moody algebra and loop algebra, one can find the physical meaning in the context of M/string theory from BLG model in this case. That is, we can obtain the multiple Dp-branes' system which are compactified on $(p-2)$-dim torus T^{p-2}. As nontrivial checks, we showed that one obtains the proper Kaluza-Klein mass on the torus and reproduce properly the relation of (a part of) U-duality between Dp-branes and M2-branes (Chapter 7).

Related research

ABJM model : Soon after the proposition of BLG model, Aharony, Bergman, Jafferis and Maldacena [27] proposed another $(2+1)$-dim Chern-Simons matter system with $SU(N) \times SU(N)$ gauge symmetry. While it lacks the manifest $\mathcal{N} = 8$ supersymmetry, it has many attractive features such as the brane construction, AdS/CFT correspondence, and relation with the integrable spin chain. In particular, it gives a good description of M2-branes when the coupling constant N/k (k is the level of Chern-Simons term) becomes small.

In $N = 2$ case, ABJM model is equivalent to BLG model with $\mathcal{A}_4$ algebra. However, in other cases, the connection of both models is not still clear. For example, while the U-duality relation in ABJM model is also discussed [28], the correspondence with our discussion in BLG model is unknown.

BFSS/BMN matrix model : As we mentioned in Introduction, BFSS [29] and BMN [30] matrix model are also proposed as the M2-branes' theories. Then it is natural that one wants to discuss the correspondence between these models and BLG model.

These matrix models describe M2-branes in a following state. First, we compactify one of the light-cone directions, and consider the system where all M2-branes have momentum almost only for this compact direction. The momentum for the compact direction is quantized and can be regarded as a number of D0-branes in type IIA superstring theory. When there are more than one D0-branes, the positions of branes become uncertain and seems to spread over 2-dim space. Then these spread D0-branes can be considered as M2-branes.

Since BFSS matrix model describes M2-branes in flat spacetime, it must directly relate to BLG model. However, the analysis of this matrix model is difficult, so it may also be difficult to find the correspondence to BLG model.

On the other hand, BMN matrix model describes M2-branes on 2-dim sphere S^2 in pp-wave background, but the analysis is relatively easy and already studied widely. Fortunately, BLG model on S^2 [31] and in background field [32] have also already studied, the discussion the correspondence with BLG model can be done in the near future.

Chapter 2

BLG model for multiple M2-branes

As we saw in Chapter 1, M-branes are very important but still mysterious objects. While the dynamics of a single M-brane is understood (at least in classical level) as in §1.3, very little is known about that of multiple M-branes. In fact, the construction of theory for multiple M-branes is very challenging for a long time. (See, for example, [33].)

In such circumstances, Bagger, Lambert [1–3] and Gustavsson [4] proposed a model of multiple M2-branes based on Lie 3-algebra which is defined using a totally antisymmetric triple product (or 3-commutator). The examination of the supersymmetry algebra suggested that the theory has a local gauge symmetry that arises from the 3-commutators.

Although their construction with Lie 3-algebra might not be the *only* solution, BLG model is of great value in that it is the *first* (and only, at least at this moment,) example which can describe the multiple M2-branes' system. In this chapter, we review their discussion.

2.1 Clue to construction of theory

2.1.1 Indispensable conditions

The M2-branes' worldvolume theory must have the following continuous symmetries:

1. 16 supersymmetries ($\mathcal{N} = 8$ in 3-dim spacetime)

   M-theory has $\mathcal{N} = 1$ supersymmetry in 11-dim spacetime, *i.e.* 32 supersymmetries. M2-brane is the 1/2 BPS object in M-theory, so must have 16 supersymmetries.

2. $SO(8)$ R-symmetry (which acts on the eight transverse scalars)

   The scalar fields describe the position of M2-branes. If we fix the gauge for world-volume coordinate, the number of transverse scalars X^I is eight (*i.e.* $I = 3, \cdots, 10$). When we consider M2-branes in flat spacetime background, there must be $SO(8)$ rotational symmetry for transverse 8-dim space. So the transverse scalars X^I must have $SO(8)$ symmetry.

3. nontrivial gauge symmetry

   For *multiple* M2-branes' case, there must be gauge (internal) degrees of freedom and the nontrivial symmetry for them. Because of supersymmetry, *i.e.* from the similar discussion in §1.3, the gauge field on M2-branes has no degrees of freedom like Chern-Simons field. (As we will see, this is the case.)

4. conformal symmetry

   M-theory has no adjustable parameters, so no free parameters depending on the energy scale exist, of course. This also suggests that there is no weakly coupled limit that might be described by perturbative quantization of a classical Lagrangian.

Despite the difficulties, it is still interesting to try to construct a classical theory that can capture at least some of the features of multiple M2-branes. In the following, we will show that in fact we can identify the field contents and supersymmetry transformations, and give a geometrical interpretation to the fields.

2.1.2 $N^{\frac{3}{2}}$ law from AdS/CFT correspondence

The algebra of gauge symmetry cannot be a Lie algebra.

There are some peculiar features of the multiple M2-branes' system. Especially, from the discussion of AdS/CFT correspondence, it can be shown that the near horizon limit of N M2-branes is dual to a 3-dim CFT with $N^{\frac{3}{2}}$ degrees of freedom [14]. It is well known that the near horizon limit of the sufficient large number of M2-branes is $AdS_4 \times S^7$, so we can think the AdS/CFT correspondence is satisfied. On the AdS side, we can derive the blackhole entropy made by multiple M2-branes using Bekenstein-Hawking's entropy formula. The relation between this entropy and the number of branes can be regarded as that of the degrees of freedom and the number of branes in CFT side.

When the gauge symmetry is defined by Lie algebra, the degree of freedom is necessarily proportional to N. This is the case for the multiple D-branes, whose gauge symmetry is $U(N)$. This means that the gauge symmetry for multiple M2-branes' system cannot be written by Lie algebra. Actually, there were attempts [34, 35] where the scalar fields were taken to be $U(N)$-valued, transforming under a standard gauge symmetry. The gauge field kinetic term was taken to be of Chern-Simons type, so the vector field did not introduce any propagating degrees of freedom. Under these assumptions, no theory was found with 16 supersymmetries.

2.1.3 Basu-Harvey equation for M2-M5 system

The multiple M2-branes' system is included in well-known Basu-Harvey's M2-M5 system [36]. This system is the BPS state of N coincident M2-branes ending on an M5-brane, such that

	0	1	2	3	4	5	6	$\cdots$	10
M5	–	–	·	–	–	–	–	·	·
M2	–	–	–	·	·	·	·	·	·

where '–' means that the brane expands to this direction, while '·' that it doesn't expand and is pointlike. We note that M2-brane and M5-brane must share one direction (the direction 1 in this case) in order to keep some supersymmetry of M2-M5 system. It is known as the intersection rule.

On the M5-brane's worldvolume, this configuration appears as a self-dual string soliton [37]. Basu and Harvey, however, examined this configuration from the M2-brane point of view. They exploited an analogy with the type IIB string configuration built from N coincident D1-branes ending on a D3-brane. In that case, the end point of the D1-branes appears as a BPS monopole on the D3-brane's worldvolume.

	0	1	2	3	4	5	$\cancel{6}$	$\cdots$	10
D3	–	·	·	–	–	–	$\cancel{-}$	·	·
D1	–	·	–	·	·	·	$\cancel{\cdot}$	·	·

where we compactify the direction 6, and take T-duality for the direction 1, then we obtain the system in type IIB superstring theory. On the D1-branes' worldvolume, the configuration gives rise to a fuzzy funnel soliton [38] which is a fuzzy 2-sphere whose radius grows infinitely as the D3-brane is reached. These two descriptions of the same physical state provide a stringy realization of the Nahm construction [39–41].

$$\frac{\partial X^a}{\partial x^2} + \frac{1}{2}\epsilon^{abc}[X^b, X^c] = 0\,, \tag{2.1}$$

where $a, b, c = 3, 4, 5$. By analogy, we naturally hope that the M2-brane theory might provide a generalized Nahm construction for self-dual string solitons.

The queer commutator appears — the hint for M2-branes' theory?

Then Basu and Harvey proposed that the M2-branes' worldvolume admits a fuzzy funnel solution that satisfies a generalized Nahm equation [36, 42]

$$\frac{dX^a}{d(x^2)} + \frac{K}{4!}\epsilon^{abcd}[G, X^b, X^c, X^d] = 0, \tag{2.2}$$

where $a, b, c = 3, 4, 5, 6$, $K = M^3/8\pi\sqrt{2N}$ is a constant,

$$[A, B, C, D] = ABCD - BACD - ACBD + ACDB + \cdots\,, \tag{2.3}$$

and G is a fixed matrix such that $G^2 = 1$.

Here we use the 4-commutator with one fixed entry, so we can effectively regard it as the 3-commutator. However the 4-commutator is chosen here, because in fact it is known that we cannot *quantize* the n-commutator (n: odd) without losing trace property and by part integration property [43]. The problem of quantization is very important, but also very difficult. We will mention it a little in Chapter 5, but we avoid this problem in this thesis by considering only the *classical* theory, so we use odd commutators without any caution in the following.

The solution of this equation (2.2) describes a fuzzy 3-sphere [44] whose radius grows infinitely as one approaches the M5-brane. Unfortunately, however, it is not known how to derive the Basu-Harvey equation from first principles. In [36] a bosonic theory was constructed, essentially by reversing the Bogomol'nyi procedure of writing the action as a perfect square plus boundary terms. Now what we would like to do is to understand the origin of this theory considering only the geometric and supersymmetric features of M2-branes.

2.1.4 Supersymmetry transformations

The Basu-Harvey equation becomes an important hint for considering the supersymmetry transformations for multiple M2-branes. Now we show this.

For D2-branes' case

We start by considering the supersymmetry transformations of N coincident D2-branes, written so the spacetime symmetries are manifest:

$$\begin{aligned}
\delta X^i &= i\bar{\epsilon}\Gamma^i\Psi \\
\delta A_\mu &= i\bar{\epsilon}\Gamma_\mu\Gamma^{10}\Psi \\
\delta\Psi &= \partial_\mu X^i\Gamma^\mu\Gamma^i\epsilon + \frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\Gamma^{10}\epsilon + \frac{i}{2}[X^i, X^j]\Gamma^{ij}\Gamma^{10}\epsilon.
\end{aligned} \tag{2.4}$$

where $\mu, \nu = 0, 1, 2$ denote the worldvolume coordinates, and $i, j = 3, \cdots, 9$ denote the transverse directions of the D2-branes. There is an $SO(1, 2)$ symmetry of the worldvolume, as well as a manifest $SO(7)$ symmetry of the transverse $\mathbf{R}^7$ that acts on the scalars and on the Γ matrices.

Notice the explicit appearance of Γ^{10}. This matrix ensures that the unbroken supersymmetries satisfy $\Gamma^{012}\epsilon = \epsilon$, while the broken supersymmetries satisfy $\Gamma^{012}\epsilon = -\epsilon$. All the fermions are Goldstinos, and obey the corresponding parity condition $\Gamma^{012}\Psi = -\Psi$.

Toward M2-branes' case

We now attempt to generalize these transformations to the case of multiple M2-branes. The presence of the explicit Γ^{10} forbids a straightforward lift to eleven dimensions. Therefore we simply assume that there is *some* extension of the D2-brane transformations such that, if all the vector fields are set to zero, the D2-brane transformations lift in such a way that the $SO(7)$ symmetry is trivially extended to $SO(8)$. Our transformations capture the fact that the M2-brane theory almost certainly contains eight scalar fields, corresponding to the eight transverse dimensions. Since we have ignored all gauge fields, we cannot expect the corresponding Lagrangian to be invariant under the supersymmetry transformations. Nor can we expect the transformations to close. We will discuss the gauge symmetrization and supersymmetrization later in §2.3.

Thus in what follows we put aside all vector fields and study the scalar-spinor supersymmetry transformations of the multiple M2-branes' theory. As the first step, we propose the lowest order supersymmetry transformations of the following form:

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta\Psi &= \partial_\mu X^I\Gamma^\mu\Gamma^I\epsilon + i\kappa[X^I, X^J, X^K]\Gamma^{IJK}\epsilon, \end{aligned} \tag{2.5}$$

where $I, J, K = 3, 4, 5, \cdots, 10$. In these expressions, κ is a dimensionless constant and $[X^I, X^J, X^K]$ is antisymmetric and linear in each of the fields. These transformations imply that (X^I, Ψ) have dimension $(\frac{1}{2}, 1)$, as required for conformal invariance. We note that there could be other cubic terms that are not totally antisymmetric in I, J, K and that vanish in the D2-brane limit [45], or that correspond to higher-order terms in the Dirac-Born-Infeld effective theory of the D2-branes. Quite a few studies on these possibilities has done, but we do not consider here. Instead, we just stipulate the presence of a Γ^{IJK} term, and we focus on it alone.

Basu-Harvey-like equation appears as BPS equation.

There is another argument for such a Γ^{IJK} term in the supersymmetry transformations. The preserved supersymmetries of N M2-branes in the presence of an M5-brane (*i.e.* Basu-Harvey's M2-M5 system) satisfy $\Gamma^2\epsilon = \Gamma^{3456}\epsilon$, or equivalently

$$\Gamma^{abc}\epsilon = \epsilon^{abcd}\Gamma^2\Gamma^d\epsilon, \tag{2.6}$$

where $a, b, c, d = 3, 4, 5, 6$. From this, one obtains the BPS equation

$$\frac{dX^a}{d(x^2)} = i\kappa\epsilon^{abcd}[X^b, X^c, X^d]. \tag{2.7}$$

The solutions to this equation behave as $X^a \sim 1/\sqrt{x^2}$ as $X^a \to \infty$. Turning this around, we see that $x^2 \sim 1/R^2$ at small R, where $R^2 = (X^3)^2 + (X^4)^2 + (X^5)^2 + (X^6)^2$. This is the correct divergence to reproduce the profile of the self-dual string soliton on the M5-brane [37]. So we can say that the cubic term and the appearance of the Γ^{IJK} are crucial to obtaining a Bogomol'nyi equation (2.2) with the correct features.

2.2 Lie 3-algebra as gauge symmetry

In the proposed supersymmetry transformation (2.5), we use a 3-commutator which is totally antisymmetric and multilinear. In general, for a $\mathcal{D}$-dim linear space $\mathcal{V} = \{\sum_{a=1}^{\mathcal{D}} v_a T^a;$

$v_a \in \mathbf{C}\}$, such a multilinear map $[*, \cdots, *] : \mathcal{V}^{\otimes n} \to \mathcal{V}$ can define the *algebra* which is known as Lie n-algebra or Filippov n-algebra [46]. This can be regarded as a natural generalization of Lie algebra. Here, we usually require that the multilinear map (or n-commutator) should satisfy the following two conditions:

Skew-symmetry

$$[T^{a_{\sigma(1)}}, \cdots, T^{a_{\sigma(n)}}] = (-1)^{|\sigma|}[T^{a_1}, \cdots, T^{a_n}]\,. \tag{2.8}$$

Fundamental identity

$$\begin{aligned}&[T^{a_1}, \cdots, T^{a_{n-1}}, [T^{b_1}, \cdots, T^{b_n}]] \\ &\quad = \sum_{k=1}^{n} [T^{b_1}, \cdots, T^{b_{k-1}}, [T^{a_1}, \cdots, T^{a_{n-1}}, T^{b_k}], T^{b_{k+1}}, \cdots, T^{b_n}]\,,\end{aligned} \tag{2.9}$$

The latter can be regarded as the generalized Jacobi identity. It means that the commutator $[T^{a_1}, \cdots, T^{a_{n-1}}, *]$ acts as a derivative on $\mathcal{V}$, and it will be used to represent a symmetry transformation. In particular, in $n = 3$ case, the *fundamental identity* becomes

$$\begin{aligned}&[T^a, T^b, [T^c, T^d, T^e]] \\ &\quad = [[T^a, T^b, T^c], T^d, T^e] + [T^c, [T^a, T^b, T^d], T^e] + [T^c, T^d, [T^a, T^b, T^e]]\,.\end{aligned} \tag{2.10}$$

Non-associative algebra can make a non-vanishing 3-commutator.

In the following, we concentrate our attention on the $n = 3$ case. Here one may think that T^a can be valued in the Lie algebra $u(N)$, as in the D2-brane theory. If so, the $[T^a, T^b, T^c]$ would be given by a double commutator

$$[T^a, T^b, T^c] = \frac{1}{3!}[[T^a, T^b], T^c] \pm \text{cyclic}\,, \tag{2.11}$$

but it vanishes because of the Jacobi identity. Therefore we must take T^a to be valued in a *non*-associative algebra $\mathcal{A}$, with a product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. We require the algebra to have a 1-dim center generated by I, and define the associator

$$\Big(T^a, T^b, T^c\Big) = (T^a \cdot T^b) \cdot T^c - T^a \cdot (T^b \cdot T^c)\,. \tag{2.12}$$

which vanishes in an associative algebra. We then define

$$[T^a, T^b, T^c] = \frac{1}{2 \cdot 3!}\left(T^{[a}, T^b, T^{c]}\right)\,, \tag{2.13}$$

which is linear and fully antisymmetric, as required. The right hand side is what one finds by expanding out the Jacobi identity $[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b]$. In a non-associative algebra, the antisymmetrized associator leads to a natural 3-commutator structure. With this construction, we have defined the supersymmetry transformations (2.5) for the scalar-spinor sector of the M2-brane theory.

The trace must satisfy the invariant metric condition.

To define an action, we require a trace form on the algebra $\mathcal{A}$. This is a bilinear map $\langle\ ,\ \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ that is symmetric and invariant:

$$\langle T^a, T^b \rangle = \langle T^b, T^a \rangle \,, \qquad \langle T^a \cdot T^b, T^c \rangle = \langle T^a, T^b \cdot T^c \rangle \,. \tag{2.14}$$

We also assume *Hermitian conjugation* $\dagger$ and positivity, which implies $\langle T^{a\dagger}, T^a \rangle \geq 0$ for any $A \in \mathcal{A}$ (with equality if and only if $T^a = 0$). The invariance property implies that

$$\begin{aligned} \langle (T^a, T^b, T^c), T^d \rangle &= \langle (T^a \cdot T^b) \cdot T^c, T^d \rangle - \langle T^a \cdot (T^b \cdot T^c), T^d \rangle \\ &= \langle T^a \cdot T^b, T^c \cdot T^d \rangle - \langle T^a, (T^b \cdot T^c) \cdot T^d \rangle \\ &= -\langle T^a, (T^b, T^c, T^d) \rangle \,. \end{aligned} \tag{2.15}$$

It also follows that

$$\langle [T^a, T^b, T^c], T^d \rangle = -\langle T^a, [T^b, T^c, T^d] \rangle \,, \tag{2.16}$$

which is called the *invariant metric* condition.

The metric and structure constants of Lie 3-algebra

Until now, we consider particularly the case of non-associative algebra. More generally, however, we only require that the algebra admit a totally antisymmetric trilinear product $[*, *, *]$. This means that the antisymmetric product need not arise from a non-associative product on the algebra. We call such an algebra a *Lie 3-algebra*.

To represent this algebra, we introduce the *structure constants*

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d, \tag{2.17}$$

from which is it is clear that $f^{abc}{}_d = f^{[abc]}{}_d$. The trace form (2.14) provides a metric

$$h^{ab} = \langle T^a, T^b \rangle \tag{2.18}$$

that we can use to raise indices: $f^{abcd} = h^{de} f^{abc}{}_e$. On physical grounds, we assume that h^{ab} is positive definite. The condition (2.16) on the trace form implies that

$$f^{abcd} = -f^{dbca} \,, \tag{2.19}$$

and this further implies that $f^{abcd} = f^{[abcd]}$, in analogy with the familiar result in Lie algebras. In terms of the structure constants, the fundamental identity (2.10) becomes

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d \,. \tag{2.20}$$

Hermitian conjugation

On physical grounds, we assume that all the generators are Hermitian, in the sense that $T^{a\dagger} = T^a$. Thus we should make some comments on Hermitian conjugation. A natural definition of the Hermitian conjugate of a 3-commutator is

$$[T^a, T^b, T^c]^\dagger = [T^{c\dagger}, T^{b\dagger}, T^{a\dagger}] . \tag{2.21}$$

This relation determines the reality of structure constants. For the usual Lie algebra, if we choose the generators to be Hermitian, the structure constants $f^{ab}{}_c$ are real numbers, and if the generators are anti-Hermitian, the structure constants are imaginary. This is not the case for 3-commutators. The structure constants are always imaginary when the generators are all Hermitian or all anti-Hermitian.

Fundamental identity is also required by local symmetry.

On physical grounds again, we expand the fields, for example, $X^I = X^I_a T^a$ ($a = 1, \cdots, \mathcal{D}$), where $\mathcal{D}$ is the dimension of $\mathcal{A}$ (and not the number of M2-branes).

When we study on the closure of supersymmetry transformation (2.5), we get the condition that the gauge variation should be [1]

$$\delta X^I \propto i\bar{\epsilon}_2 \Gamma_{JK} \epsilon_1 [X^J, X^K, X^I] , \tag{2.22}$$

which can be viewed as a local version of the global symmetry transformation

$$\delta X = [\alpha, \beta, X] , \tag{2.23}$$

where $\alpha, \beta \in \mathcal{A}$. For (2.23) to be a symmetry, it must act as a derivation

$$\delta([X, Y, Z]) = [\delta X, Y, Z] + [X, \delta Y, Z] + [X, Y, \delta Z] , \tag{2.24}$$

which leads to the fundamental identity (see, for example, [47])

$$[\alpha, \beta, [X, Y, Z]] = [[\alpha, \beta, X], Y, Z] + [X, [\alpha, \beta, Y], Z] + [X, Y, [\alpha, \beta, Z]] . \tag{2.25}$$

In the gauge theory with ordinary Lie algebra, Jacobi identity arises from demanding that the transformation $\delta X = [\alpha, X]$ acts as a derivation. Here the analogous things happen.

In terms of the structure constants, the symmetry transformation (2.23) can be written as

$$\delta X_d = f^{abc}{}_d \alpha_a \beta_b X_c . \tag{2.26}$$

However, the notation allows for the more general transformation

$$\delta X_d = f^{abc}{}_d \Lambda_{ab} X_c , \tag{2.27}$$

which we assume from now on. In particular, the transformation (2.22) corresponds to the choice

$$\Lambda_{ab} \propto i\bar{\epsilon}_1 \Gamma_{JK} \epsilon_2 X^J_a X^K_b . \tag{2.28}$$

Note that the generator Λ_{ab} *cannot* in general be written as $\alpha_{[a}\beta_{b]}$ for a single pair of vectors (α_a, β_b). However, Λ_{ab} can always be written as a sum over $\mathcal{D}$ such pairs.

With the Lie 3-algebra, various fields in BLG model which are symbolically written as $\phi = \sum_a \phi_a T^a$ transform infinitesimally as

$$\delta_\Lambda \phi = \sum_{a,b} \Lambda_{ab} \, [T^a, T^b, \phi] \quad \text{or} \quad \delta_\Lambda \phi_a = \Lambda_{cd} f^{cdb}{}_a \phi_b \tag{2.29}$$

for the gauge parameter Λ_{ab}. The fundamental identity (2.10) implies that this transformation closes in the following sense,

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}]\phi = \delta_{[\Lambda_1,\Lambda_2]}\phi, \quad [\Lambda_1, \Lambda_2]_{ab} := \Lambda_{1de}\Lambda_{2cb} f^{dec}{}_a + \Lambda_{1de}\Lambda_{2ac} f^{dec}{}_b \,. \tag{2.30}$$

As a result of (2.16), the metric (2.18) must also be invariant under the symmetry (2.29),

$$\langle \delta_\Lambda \phi_1, \phi_2 \rangle + \langle \phi_1, \delta_\Lambda \phi_2 \rangle = 0 \,. \tag{2.31}$$

The BLG model, whose action is constructed with the structure constants and the invariant metric, must be a gauge theory associated with this symmetry.

2.3 Gauge and super symmetrization

The minimal Lagrangian

To see that the action is invariant under global symmetries of this form, we observe that for any Y,

$$\begin{aligned} \frac{1}{2}\delta\langle Y, Y\rangle &= \langle \delta Y, Y\rangle = h^{de}\delta Y_d Y_e = h^{de}\Lambda_{ab} f^{abc}{}_d Y_c Y_e = f^{abce}\Lambda_{ab} Y_c Y_e \\ &= 0 \,, \end{aligned} \tag{2.32}$$

by the antisymmetry of f^{abce}. In addition, the fundamental identity ensures that

$$(\delta[X^I, X^J, X^K])_a = f^{cdb}{}_a \Lambda_{cd} [X^I, X^J, X^K]_b \,. \tag{2.33}$$

Thus the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\langle \partial_\mu X^I, \partial^\mu X^I \rangle - 3\kappa^2 \langle [X^I, X^J, X^K], [X^I, X^J, X^K] \rangle \,, \tag{2.34}$$

is invariant under the symmetry $\delta X^I_a = f^{cdb}{}_a \Lambda_{cd} X^I_b$.

To obtain the full Lagrangian with Ψ and A_μ terms, we need to discuss the gauge symmetry and supersymmetry. More concretely, in this section, we show how to gauge the local symmetry and obtain a conformal and gauge-invariant action with all 16 supersymmetries.

2.3.1 Gauge symmetry

We gauge a symmetry that arises from the 3-commutators.

Covariant derivative

Now we wish to promote the global symmetry (2.23) to a local one. To this end, we introduce a covariant derivative $D_\mu X$ such that $\delta(D_\mu X) = D_\mu(\delta X) + (\delta D_\mu)X$. If we let

$$\delta_\Lambda X_a = \Lambda_{cd} f^{cdb}{}_a X_b =: \tilde{\Lambda}^b{}_a X_b \,, \tag{2.35}$$

then the natural choice is to take

$$(D_\mu X)_a = \partial_\mu X_a - \tilde{A}_\mu{}^b{}_a X_b \,, \tag{2.36}$$

where $\tilde{A}_\mu{}^b{}_a \equiv f^{cdb}{}_a A_{\mu cd}$ is a gauge field with two gauge indices. We can therefore think of $\tilde{A}_\mu{}^b{}_a$ as living in the space of linear maps from $\mathcal{A}$ to itself, in analogy with the adjoint representation of a Lie algebra. Then the field X is, in some sense, in the fundamental representation.

Field strength

A little calculation shows that the covariant derivative is obtained by taking

$$\delta_\Lambda \tilde{A}_\mu{}^b{}_a = \partial_\mu \tilde{\Lambda}^b{}_a - \tilde{\Lambda}^b{}_c \tilde{A}_\mu{}^c{}_a + \tilde{A}_\mu{}^b{}_c \tilde{\Lambda}^c{}_a =: D_\mu \tilde{\Lambda}^b{}_a \,. \tag{2.37}$$

Indeed, this is the usual form of a gauge transformation. The field strength is defined as

$$([D_\mu, D_\nu]X)_a = \tilde{F}_{\mu\nu}{}^b{}_a X_b, \tag{2.38}$$

which leads to

$$\tilde{F}_{\mu\nu}{}^b{}_a = \partial_\nu \tilde{A}_\mu{}^b{}_a - \partial_\mu \tilde{A}_\nu{}^b{}_a - \tilde{A}_\mu{}^b{}_c \tilde{A}_\nu{}^c{}_a + \tilde{A}_\nu{}^b{}_c \tilde{A}_\mu{}^c{}_a \,. \tag{2.39}$$

The resulting Bianchi identity is $D_{[\mu} \tilde{F}_{\nu\lambda]}{}^b{}_a = 0$. One also finds that

$$\delta_\Lambda \tilde{F}_{\mu\nu}{}^b{}_a = -\tilde{\Lambda}^b{}_c \tilde{F}_{\mu\nu}{}^c{}_a + \tilde{F}_{\mu\nu}{}^b{}_c \tilde{\Lambda}^c{}_a \,. \tag{2.40}$$

These expressions are identical to what one finds in an ordinary gauge theory based on a Lie algebra, where the gauge field is in the adjoint representation. Here the gauge field takes values in the space of linear maps of $\mathcal{A}$ into itself. The 3-commutator allows one to construct linear maps on $\mathcal{A}$ from two elements of $\mathcal{A}$.

2.3.2 Supersymmetry

Now we show how to supersymmetrize the gauged multiple M2-branes' model. In §2.1.4, we argued that the general form is

$$\begin{aligned} \delta_\epsilon X^I &= i\bar{\epsilon}\Gamma^I \Psi \\ \delta_\epsilon \Psi &= \partial_\mu X^I \Gamma^\mu \Gamma^I \epsilon + \kappa [X^I, X^J, X^K] \Gamma^{IJK} \epsilon, \end{aligned} \tag{2.41}$$

where κ is a constant. However, as we mentioned, this algebra doesn't close, and the closure on the scalars X^I requires the local symmetry $\delta X^I \propto i\bar{\epsilon}_2\Gamma_{JK}\epsilon_1[X^J, X^K, X^I]$.

So let us apply the ideas of the previous subsection to gauge this symmetry. We first introduce the gauge field $\tilde{A}_\mu{}^b{}_a$ with its associated covariant derivative. Then the supersymmetry transformations take the form

$$\begin{aligned}
\delta_\epsilon X^I_a &= i\bar{\epsilon}\Gamma^I\Psi_a \\
\delta_\epsilon \Psi_a &= D_\mu X^I_a \Gamma^\mu\Gamma^I\epsilon + \kappa X^I_b X^J_c X^K_d f^{bcd}{}_a\Gamma^{IJK}\epsilon \\
\delta_\epsilon \tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X^I_c \Psi_d f^{cdb}{}_a.
\end{aligned} \tag{2.42}$$

where the form of $\delta_\epsilon\tilde{A}$ comes from the dimension counting and conformal symmetry (*i.e.* no free dimensionful parameters). In fact, this algebra can be made to close on shell.

Closure of supersymmetry transformations

In order to check the closure of supersymmetry, we use the Fierz identity for arbitrary spinors ϵ_1, ϵ_2 and χ, all of which are real spinors of 11-dim Clifford algebra and have the same chirality with respect to Γ^{012}:

$$\begin{aligned}
&(\bar{\epsilon}_2\chi)\epsilon_1 - (\bar{\epsilon}_1\chi)\epsilon_2 \\
&= -\frac{1}{16}\left[2(\bar{\epsilon}_2\Gamma_\mu\epsilon_1)\Gamma^\mu\chi - (\bar{\epsilon}_2\Gamma_{IJ}\epsilon_1)\Gamma^{IJ}\chi + \frac{1}{4!}(\bar{\epsilon}_2\Gamma_\mu\Gamma_{IJKL}\epsilon_1)\Gamma^\mu\Gamma^{IJKL}\chi\right]
\end{aligned} \tag{2.43}$$

where $m, n, \cdots = 0, \cdots, 10$, $\mu, \nu, \cdots = 0, 1, 2$ and $I, J, \cdots = 3, 4, \cdots, 10$.

Scalar fields

We first consider the scalars. We find that the transformations close into a translation and a gauge transformation

$$[\delta_1, \delta_2]X^I_a = v^\mu D_\mu X^I_a + \tilde{\Lambda}^b{}_a X^I_b\,, \tag{2.44}$$

where $v^\mu = -2i\bar{\epsilon}_2\Gamma^\mu\epsilon_1$ and $\tilde{\Lambda}^b{}_a = 6i\kappa\bar{\epsilon}_2\Gamma_{JK}\epsilon_1 X^J_c X^K_d f^{cdb}{}_a$.

Fermion fields

Next, we consider the fermions. When we evaluate $[\delta_1, \delta_2]\Psi_a$ using eq. (2.43), we find two separate terms involving $\bar{\epsilon}_2\Gamma_\mu\Gamma_{IJKL}\epsilon_1$ that must cancel for closure. This implies

$$\kappa = -1/6, \tag{2.45}$$

so there is no free parameter, as required from the nature of M-theory. Then we find

$$\begin{aligned}
[\delta_1, \delta_2]\Psi_a &= v^\mu D_\mu\Psi_a + \tilde{\Lambda}^b{}_a\Psi_b \\
&\quad + i(\bar{\epsilon}_2\Gamma_\nu\epsilon_1)\Gamma^\nu\left(\Gamma^\mu D_\mu\Psi_a + \frac{1}{2}\Gamma_{IJ}X^I_c X^J_d \Psi_b f^{cdb}{}_a\right) \\
&\quad - \frac{i}{4}(\bar{\epsilon}_2\Gamma_{KL}\epsilon_1)\Gamma^{KL}\left(\Gamma^\mu D_\mu\Psi_a + \frac{1}{2}\Gamma_{IJ}X^I_c X^J_d \Psi_b f^{cdb}{}_a\right).
\end{aligned} \tag{2.46}$$

Closure requires that the second and third lines vanish. This determines the fermionic equation of motion

$$\Gamma^\mu D_\mu \Psi_a + \frac{1}{2}\Gamma_{IJ} X^I_c X^J_d \Psi_b f^{cdb}{}_a = 0\,. \tag{2.47}$$

Thus, as required, we obtain the on-shell equation such that

$$[\delta_1, \delta_2]\Psi_a = v^\mu D_\mu \Psi_a + \tilde{\Lambda}^b{}_a \Psi_b\,. \tag{2.48}$$

Gauge field

We finally consider $[\delta_1, \delta_2]\tilde{A}_\mu{}^b{}_a$. Again we find a term involving $\bar{\epsilon}_2\Gamma_\mu\Gamma_{IJKL}\epsilon_1$, but fortunately, this term vanishes by the fundamental identity. Then we find

$$\begin{aligned}[\delta_1, \delta_2]\tilde{A}_\mu{}^b{}_a &= 2i(\bar{\epsilon}_2\Gamma^\nu\epsilon_1)\epsilon_{\mu\nu\lambda}(X^I_c D^\lambda X^I_d + \frac{i}{2}\bar{\Psi}_c\Gamma^\lambda\Psi_d) f^{cdb}{}_a \\ &\quad - 2i(\epsilon_2\Gamma_{IJ}\epsilon_1) X^I_c D_\mu X^J_d f^{cdb}{}_a.\end{aligned} \tag{2.49}$$

To close the algebra, the equation of motion for $\tilde{A}_\mu{}^b{}_a$ must be

$$\tilde{F}_{\mu\nu}{}^b{}_a + \epsilon_{\mu\nu\lambda}(X^J_c D^\lambda X^J_d + \frac{i}{2}\bar{\Psi}_c\Gamma^\lambda\Psi_d) f^{cdb}{}_a = 0\,, \tag{2.50}$$

so that on shell

$$[\delta_1, \delta_2]\tilde{A}_\mu{}^b{}_a = v^\nu \tilde{F}_{\mu\nu}{}^b{}_a + D_\mu \tilde{\Lambda}^b{}_a. \tag{2.51}$$

is satisfied. Note that $\tilde{A}_\mu{}^c{}_d$ contains no local degrees of freedom, as required.

From the above discussion, we see that the 16 supersymmetries close on shell.

Equations of motion

The equations of motion for Ψ and A_μ are already obtained as eq. (2.47), (2.50). To find the remaining bosonic equation of motion, we take the supervariation of the fermion's one. This gives

$$\begin{aligned} 0 &= \Gamma^I\left(D^2 X^I_a - \frac{i}{2}\bar{\Psi}_c\Gamma^{IJ} X^J_d \Psi_b f^{cdb}{}_a + \frac{1}{2} f^{bcd}{}_a f^{efg}{}_d X^J_b X^K_c X^I_e X^J_f X^K_g\right)\epsilon \\ &\quad + \Gamma^I\Gamma_\lambda X^I_b\left(\frac{1}{2}\varepsilon^{\mu\nu\lambda}\tilde{F}_{\mu\nu}{}^b{}_a - X^J_c D^\lambda X^J_d f^{cdb}{}_a - \frac{i}{2}\bar{\Psi}_c\Gamma^\lambda\Psi_d f^{cdb}{}_a\right)\epsilon\,. \end{aligned} \tag{2.52}$$

The second term vanishes as a consequence of the vector equation of motion (2.50). So the first term determines the scalar equations of motion

$$D^2 X^I_a - \frac{i}{2}\bar{\Psi}_c\Gamma^{IJ} X^J_d \Psi_b f^{cdb}{}_a - \frac{\partial V}{\partial X^{Ia}} = 0\,, \tag{2.53}$$

where $V = \frac{1}{12} f^{abcd} f^{efg}{}_d X^I_a X^J_b X^K_c X^I_e X^J_f X^K_g = \frac{1}{2\cdot 3!}\langle [X^I, X^J, X^K], [X^I, X^J, X^K]\rangle$.

2.4 BLG action and Summary

Let us summarize the results.

Gauge symmetry transformations

$$\delta_\Lambda X^I_a = \tilde{\Lambda}^b{}_a X^I_b\,, \quad \delta_\Lambda \Psi_a = \tilde{\Lambda}^b{}_a \Psi_b\,, \quad \delta_\Lambda \tilde{A}_\mu{}^b{}_a = D_\mu \tilde{\Lambda}^b{}_a\,, \tag{2.54}$$

where $\tilde{A}_\mu{}^b{}_a = A_{\mu cd} f^{cdb}{}_a$, $\tilde{\Lambda}^b{}_a = \Lambda_{cd} f^{cdb}{}_a$, and

$$D_\mu \Lambda_{ab} = \partial_\mu \Lambda_{ab} - f^{cde}{}_a A_{\mu cd} \Lambda_{eb} + f^{cde}{}_b A_{\mu cd} \Lambda_{ea}. \tag{2.55}$$

Supersymmetry transformations

$$\begin{aligned}
\delta X^I_a &= i\bar{\epsilon}\Gamma^I \Psi_a \\
\delta \Psi_a &= D_\mu X^I_a \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK} \epsilon \\
\delta \tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cdb}{}_a\,.
\end{aligned} \tag{2.56}$$

Equations of motion

$$\begin{aligned}
\Gamma^\mu D_\mu \Psi_a + \frac{1}{2}\Gamma_{IJ} X^I_c X^J_d \Psi_b f^{cdb}{}_a &= 0 \\
D^2 X^I_a - \frac{i}{2}\bar{\Psi}_c \Gamma^I{}_J X^J_d \Psi_b f^{cdb}{}_a - \frac{\partial V}{\partial X^{Ia}} &= 0 \\
\tilde{F}_{\mu\nu}{}^b{}_a + \varepsilon_{\mu\nu\lambda}(X^J_c D^\lambda X^J_d + \frac{i}{2}\bar{\Psi}_c \Gamma^\lambda \Psi_d) f^{cdb}{}_a &= 0\,,
\end{aligned} \tag{2.57}$$

where $V = \frac{1}{12}\langle [X^I, X^J, X^K], [X^I, X^J, X^K]\rangle$, and

$$D_\mu X^I_a = \partial_\mu X^I_a - f^{bcd}{}_a A_{\mu bc} X^I_d, \quad D_\mu \Psi_a = \partial_\mu \Psi^I_a - f^{bcd}{}_a A_{\mu bc} \Psi_d\,. \tag{2.58}$$

Closure of supersymmetry

After using the equations of motion, the supersymmetry closes into translations and gauge transformations

$$\begin{aligned}
[\delta_1, \delta_2] X^I_a &= v^\mu \partial_\mu X^I_a + (\tilde{\Lambda}^b{}_a - v^\nu \tilde{A}_\nu{}^b{}_a X^I_b) \\
[\delta_1, \delta_2] \Psi_a &= v^\mu \partial_\mu \Psi_a + (\tilde{\Lambda}^b{}_a - v^\nu \tilde{A}_\nu{}^b{}_a \Psi_b) \\
[\delta_1, \delta_2] \tilde{A}_\mu{}^b{}_a &= v^\nu \partial_\nu \tilde{A}_\mu{}^b{}_a + \tilde{D}_\mu (\tilde{\Lambda}^b{}_a - v^\nu \tilde{A}_\nu{}^b{}_a),
\end{aligned} \tag{2.59}$$

We have explicitly demonstrated that the supersymmetry variation of the fermion equation of motion vanishes, and that the algebra closes on shell. It follows that all the equations of motion are invariant under supersymmetry. Furthermore, one can check using the fundamental identity that the Bianchi identity $\epsilon^{\mu\nu\lambda} D_\mu \tilde{F}_{\nu\lambda}{}^b{}_a = 0$ is satisfied.

BLG action

Finally we obtain the action for this system. The equations of motion (2.57) can be obtained from the Lagrangian

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_X + \mathcal{L}_\Psi + \mathcal{L}_{int} + \mathcal{L}_{pot} + \mathcal{L}_{CS} \\
\mathcal{L}_X &= -\frac{1}{2}\langle D_\mu X^I, D^\mu X^I\rangle \\
\mathcal{L}_\Psi &= \frac{i}{2}\langle \bar{\Psi}, \Gamma^\mu D_\mu \Psi\rangle \\
\mathcal{L}_{int} &= \frac{i}{4}\langle \bar{\Psi}, \Gamma_{IJ}[X^I, X^J, \Psi]\rangle \\
\mathcal{L}_{pot} &= -\frac{1}{12}\langle [X^I, X^J, X^K], [X^I, X^J, X^K]\rangle \\
\mathcal{L}_{CS} &= \frac{1}{2}\epsilon^{\mu\nu\lambda}(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb} A_{\mu ab}A_{\nu cd}A_{\lambda ef})\,.
\end{aligned} \tag{2.60}$$

This theory provides an example of the multiple M2-branes' model. It is invariant under 16 supersymmetries and an $SO(8)$ R-symmetry, and conformal invariant at the classical level. These are all the continuous symmetries that are expected of multiple M2-branes. Moreover, it contains no free parameters, up to a rescaling of the structure constants. This is also appropriate nature for M2-branes' theory.

Notes on Chern-Simons term

It is important to note that the structure constants $f^{abc}{}_d$ enter into the Chern-Simons term in a non-standard way. This 'twisted' Chern-Simons term

$$\Omega = (f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef})\, dx^\mu \wedge dx^\nu \wedge dx^\lambda \tag{2.61}$$

satisfies

$$d\Omega = F_{ab} \wedge \tilde{F}^{ab}, \tag{2.62}$$

where $\tilde{F}_{\mu\nu}{}^b{}_a = F_{\mu\nu cd}f^{cdb}{}_a$. Ω is written in terms of $A_{\mu ab}$ and not the physical field $\tilde{A}_\mu{}^b{}_a = A_{\mu cd}f^{cdb}{}_a$ that appears in the supersymmetry transformations and equations of motion. However, one can check that Ω is invariant under shifts of $A_{\mu ab}$ that leave $\tilde{A}_\mu{}^b{}_a$ invariant. Thus it is locally well defined as a function of $\tilde{A}_\mu{}^b{}_a$.

This Chern-Simons term naively breaks the parity that is expected to be a symmetry of the M2-branes' worldvolume. However, we can make the Lagrangian parity invariant, if we assign an odd parity to f^{abcd}. In particular, if we invert $x^2 \to -x^2$, we must then require that X^I_a and $\tilde{A}_\mu{}^b{}_a$ be parity even for $\mu = 0, 1$, while $\tilde{A}_2{}^b{}_a$ and f^{abcd} be parity odd, and that $\Psi_a \to \Gamma_2\Psi_a$. Note that this assignment implies that $A_{\mu ab}$ is parity odd for $\mu = 0, 1$, while A_{2ab} is parity even.

Finally, one may consider that since the Chern-Simons term exists, it is natural to expect the f^{abcd} to be quantized. However, up to now, the way of quantization is not revealed.

Chapter 3

Examples of Lie 3-algebra and no-go theorem

In the previous chapter, we successfully obtain the action for multiple M2-branes' world-volume theory. This theory is called as BLG model, which is defined with an abstract Lie 3-algebra as gauge symmetry, at this stage. In fact, the concrete examples of BLG model are given by the concrete examples of Lie 3-algebra. So what we have to do next is nothing but the construction and classification of Lie 3-algebra, satisfying all required conditions in §2.2. The trivial solution is to put all structure constants zero ${f^{abc}}_d = 0$. In this chapter, we discuss the various examples except it.

3.1 $\mathcal{A}_4$ algebra and its direct sum

3.1.1 $\mathcal{A}_4$ algebra

The simplest nontrivial solution which satisfy the fundamental identity (2.10) of Lie 3-algebra starts from the number of generators $\mathcal{D} = 4$ such that

$$[T^a, T^b, T^c] = i\epsilon^{abcd}T^d \qquad (a, b, c, d = 1, 2, 3, 4) \tag{3.1}$$

and the metric is fixed by the requirement of invariance (2.16) to be

$$\langle T^a, T^b \rangle = \delta^{ab}. \tag{3.2}$$

Here we normalize the generators by an overall constant factor, and we have a factor of i on the right hand side of (3.1) due to our convention of the 3-commutator's Hermiticity (2.21).

In this case, the space $\mathcal{G}$ generated by all matrices ${\tilde{\Lambda}^c}_d = \Lambda_{ab}{f^{abc}}_d$ is the space of all 4×4 antisymmetric matrices and hence $\mathcal{G} = so(4)$ with the invariant 4-form ϵ^{abcd}. This algebra is invariant under $SO(4)$ and will be denoted as $\mathcal{A}_4$ [48]. The structure constant is given by the totally antisymmetrized epsilon tensor ${f^{abc}}_d = i\epsilon^{abcd}$.

3.1.2 Direct sum of $\mathcal{A}_4$ algebra

From the $\mathcal{A}_4$ algebra, one may obtain higher rank algebras by direct sum as usual. For $n = 3$ case, the algebra $\mathcal{A}_4 \oplus \cdots \oplus \mathcal{A}_4$ (p-times) with $\mathcal{D} = 4p$ is written as,

$$[T^a_{(\alpha)}, T^b_{(\beta)}, T^c_{(\gamma)}] = i\epsilon^{abcd}\delta_{\alpha\beta\gamma\delta}T^d_{(\delta)} \,, \tag{3.3}$$

where $a, b, c, d = 1, \cdots, 4$, $\alpha, \beta, \gamma, \delta = 1, \cdots, p$ and $\delta_{\alpha\beta\gamma\delta} = 1$ (if $\alpha = \beta = \gamma = \delta$) or 0 (otherwise).

A nontrivial question here is whether there exists any Lie 3-algebra which cannot be reduced to the direct sums of the algebra $\mathcal{A}_4$, up to a direct sum with a trivial algebra. For $n = 3$, one may directly solve the fundamental identity by computer for lower dimensions $\mathcal{D}$. Ho, Hou and Matsuo [12] examined the cases $\mathcal{D} = 5, 6, 7, 8$ with the assumption that the metric h^{ab} is invertible and can be set to δ^{ab} after the change of basis. In this case, the structure constants $f^{abc}{}_d$ can be identified with totally anti-symmetric four tensor f^{abcd}.

For $\mathcal{D} = 5, 6$, one can solve directly the fundamental identity algebraically by computer. For $\mathcal{D} = 7, 8$, they assumed the coefficients f^{abcd} are integer and $|f^{abcd}| \leq 3$ and scanned all possible combinations. After all, the solutions can always be reduced to $\mathcal{A}_4$ up to a direct sum with a trivial algebra, or $\mathcal{A}_4 \oplus \mathcal{A}_4$ ($\mathcal{D} = 8$) after a change of basis.

This observation suggests that the Lie n-algebra for $n \geq 3$ is very limited.

3.2 No-go theorem

Unfortunately, the examples of Lie 3-algebra are actually so rare. This means that we can obtain only a few concrete examples of BLG model for multiple M2-branes' system.

On this regrettable fact, Ho, Hou and Matsuo [12] proposed first[1] the conjecture that *all finite-dimensional Lie 3-algebras with positive-definite metrics are direct products of* $\mathcal{A}_4$ *with trivial algebras*, and give a little intuitive proof by noting the resemblance between the fundamental identity and the Plücker relation when a positive-definite metric is assumed.

Soon after, the rigid proof for this conjecture is given by Papadopoulos [11], Gauntlett and Gutowski [24]. In this section, we review the former proof for this *no-go theorem*.

Summary of the proof

In the following, we prove the conjecture that the structure constants f^{ABCD} of a Lie 3-algebra $\mathfrak{a}_{[3]}$ with a positive-definite metric can be written as

$$f_{(4)} := f^{ABCD}\, e_A \wedge e_B \wedge e_C \wedge e_D = \sum_r \mu^r\, d\text{vol}(V_r) \,, \quad V_r \subset \mathfrak{a}_{[3]} \tag{3.4}$$

where the 4-dim planes V_r and $V_{r'}$ are orthogonal for $r \neq r'$ and μ^r are constants. The indices $A, B, C, \cdots = 0, \cdots, \mathcal{D} - 1$, where $\mathcal{D}$ is the dimension of $\mathfrak{a}_{[3]}$. $\{e_0, \cdots, e_{\mathcal{D}-1}\}$

[1]Comment: Gustavsson [49] studied a weaker form of the conjecture.

is the basis of $\mathfrak{a}_{[3]}$. Since we assume the invariant metric condition $f^{ABCD} \overset{!}{=} f^{[ABCD]}$ should be satisfied, we will use the differential form for the structure constants. In addition, of course, we require that the fundamental identity

$$f^{H[ABC}f^{D]EM}{}_H = 0\,. \tag{3.5}$$

To prove this conjecture, we first observe that if a vector $X \in \mathfrak{a}_{[3]}$ is given, one can associate a metric Lie algebra $\mathfrak{a}_{[2]}(X)$ to $\mathfrak{a}_{[3]}$ defined as the orthogonal complement of $X \in \mathfrak{a}_{[3]}$ with structure constants $i_X f_{(4)}$. It is easy to verify that $i_X f_{(4)}$ satisfy the fundamental identity. Then we will demonstrate the following three statements:

- If $\mathfrak{a}_{[3]}$ admits an associated Lie algebra $\mathfrak{a}_{[2]}(X)$ and $\mathfrak{a}_{[2]}(X)$ doesn't have a bi-invariant 4-form,[2] then $f_{(4)}$ is volume form of a 4-dim plane.
- If all the associated metric Lie algebras of $\mathfrak{a}_{[3]}$ are $\mathfrak{a}_{[2]}(X) = \oplus^\ell \mathfrak{u}(1) \oplus \mathfrak{ss}$, for some $\ell \geq 0$, where $\mathfrak{ss}$ is a semi-simple Lie algebra which commutes with $\oplus^\ell \mathfrak{u}(1)$, then $f_{(4)}$ is as in (3.4).
- all metric Lie algebras are isomorphic to $\oplus^\ell \mathfrak{u}(1) \oplus \mathfrak{ss}$.

Step 1 : Lie 3-algebras and invariant 4-forms

Now we prove the first statement. Without loss of generality, we can take the vector field X to be along the 0 direction. Then we split the indices as $A = (0, i)$, $B = (0, j)$ where $i, j, \cdots = 1, \cdots, \mathcal{D} - 1$, and so on.

First, if we set $A = M = 0$ and the rest of the free indices in the range $1, \cdots, \mathcal{D} - 1$ in (3.5), it is easy to see that

$$f^{ijk} := f^{0ijk}\,, \qquad F^{ijkl} := f^{ijkl} \tag{3.6}$$

satisfy the Jacobi identity of ordinary Lie algebras and f^{ijk} are the structure constants of $\mathfrak{a}_{[2]}(X)$. In terms of the differential form, we can rewrite the structure constants (3.6) as

$$f_{(4)} = \frac{1}{3!} f^{ijk}\, e_0 \wedge e_i \wedge e_j \wedge e_k + \frac{1}{4!} F^{ijkl}\, e_i \wedge e_j \wedge e_k \wedge e_l \tag{3.7}$$

where (e_0, e_i) is an orthonormal basis.

[2]Here we see the assumption that $\mathfrak{a}_{[2]}(X)$ doesn't have a bi-invariant 4-form. It is known that bi-invariant forms on the Lie algebra of a group give rise to parallel forms with respect to the Levi-Civita connection on the associated simply connected group manifold. So if $\mathfrak{a}_{[2]}(X)$ admits a bi-invariant 4-form, the associated group manifold G admits a parallel 4-form which is necessarily harmonic. For compact Lie groups, parallel 4-forms represent nontrivial classes in the 4th de Rham cohomology of G. Thus if a compact Lie group admits a parallel 4-form, then $H^4_{dR}(G) \neq 0$. However, for a large class of Lie groups, which includes all semi-simple ones, $H^4_{dR}(G) = 0$.

Therefore, this assumption is always satisfied, as long as an associated Lie algebra $\mathfrak{a}_{[2]}(X)$ to $\mathfrak{a}_{[3]}$ is semi-simple, which is also the assumption in the second statement for $\ell \geq 1$ case.

Next, we set $M = 0$ and the rest of the free indices in the range $1, \cdots, \mathcal{D}-1$ in (3.5). Using the skew-symmetry of $f_{(4)}$, one finds that

$$F^{h[ijk} f^{d]e}{}_h = 0 \, . \tag{3.8}$$

This implies that the 4-form $F_{(4)}$ is bi-invariant with respect to $\mathfrak{a}_{[2]}(X)$. Since such a form cannot exist by assumption, we conclude that $F_{(4)} = 0$. Thus, we find that

$$f_{(4)} = \frac{1}{3!} f^{ijk} e_0 \wedge e_i \wedge e_j \wedge e_k \, . \tag{3.9}$$

Finally, if we take all free indices in (3.5) in the range $1, \cdots, \mathcal{D}-1$, we find that

$$f^{[ijk} f^{h]de} = 0 \, . \tag{3.10}$$

This is the classical Plücker relation which implies that f^{ijk} is a *simple* 3-form, *i.e.* the wedge product of three 1-forms. Thus one concludes that the only solution to (3.5) is

$$f_{(4)} = \mu \, e_0 \wedge e_1 \wedge e_2 \wedge e_3 \, , \tag{3.11}$$

for some constant μ, where we have chosen the four 1-forms, without loss of generality, to lie in the first four directions. This proves the first statement.

Step 2 : Lie 3-algebras and Lie algebras

In fact, we have already proved the second statement in the $\ell = 0$ case at the end of Step 1. So next, let us consider the $\ell = 1$ case, where for some X, the associated Lie algebra is $\mathfrak{a}_{[2]}(X) = \mathfrak{u}(1) \oplus \mathfrak{ss}$. In this case, $H^4_{dR}(G) \neq 0$ and there are bi-invariant 4-forms on $\mathfrak{a}_{[2]}(X)$. Such bi-invariant 4-forms can be written as

$$F_{(4)} = e_{\mathcal{D}-1} \wedge \varphi_{(3)} = \frac{1}{3!} \varphi_{\alpha\beta\gamma} \, e_{\mathcal{D}-1} \wedge e_\alpha \wedge e_\beta \wedge e_\gamma \, , \tag{3.12}$$

where φ is a bi-invariant form on the semi-simple part $\mathfrak{ss}$ of $\mathfrak{a}_{[2]}(X)$. We choose the $\mathfrak{u}(1)$ direction along $e_{\mathcal{D}-1}$, without loss of generality. So the indices $\alpha, \beta, \gamma = 1, \cdots, \mathcal{D}-2$. Thus we have

$$f_{(4)} = \frac{1}{3!} f_{\alpha\beta\gamma} \, e_0 \wedge e_\alpha \wedge e_\beta \wedge e_\gamma + \frac{1}{3!} \varphi_{\alpha\beta\gamma} \, e_{\mathcal{D}-1} \wedge e_\alpha \wedge e_\beta \wedge e_\gamma \, . \tag{3.13}$$

Since φ is a bi-invariant 3-form on a semi-simple Lie algebra, it is a linear combination of the structure constants f_r of the simple Lie algebras in $\mathfrak{ss} = \oplus^r \mathfrak{s}_r$. Thus $f_{(4)}$ can be rewritten as

$$f_{(4)} = \sum_r (\mu^r e_0 + \nu^r e_{\mathcal{D}-1}) \wedge f_r \, , \tag{3.14}$$

where $\mu^r \neq 0$ and ν^r are some constants.

Next, since f_r and $f_{r'}$ ($r \neq r'$) are orthogonal, the fundamental identity (3.5) with $A, B, C \in \mathfrak{s}_r$ and $D, E, M \in \mathfrak{s}_{r'}$ implies that $\mu^r e_0 + \nu^r e_{\mathcal{D}-1}$ and $\mu^{r'} e_0 + \nu^{r'} e_{\mathcal{D}-1}$ are also orthogonal. Moreover, the fundamental identity (3.5) with $A, B, C, D, E, M \in \mathfrak{s}_r$ implies that f_r satisfies the Plücker relation, as in (3.10), and so f_r must be a simple 3-form.

In conclusion, $f_{(4)}$ is the sum of volume forms of at most two orthogonal 4-dim planes, *i.e.* without loss of generality, it can be written as

$$f_{(4)} = \mu\, e_0 \wedge e_1 \wedge e_2 \wedge e_3 + \nu\, e_4 \wedge e_5 \wedge e_6 \wedge e_7\,, \tag{3.15}$$

where μ and ν are constants.

This discussion can be extended to the $\ell > 1$ case, where *all* associated Lie algebras with Lie 3-algebras are $\mathfrak{a}_{[2]}(X) = \oplus^\ell \mathfrak{u}(1) \oplus \mathfrak{ss}$ for $\ell > 1$.

First, we write $f_{(4)} = e_0 \wedge f_{(3)} + F_{(4)}$ as in (3.7), where $f_{(3)}$ are the structure constants of $\mathfrak{ss}$ and $F_{(4)}$ satisfies (3.8). Since semi-simple Lie algebras do not admit bi-invariant 1-, 2- and 4-forms, and $\oplus^\ell \mathfrak{u}(1)$ commutes with $\mathfrak{ss}$, the most general invariant 4-form $F_{(4)}$ is

$$F_{(4)} = \sum_I \rho^I \wedge \varphi_I + \xi_{(4)}\,, \tag{3.16}$$

where ρ^I are the 1-forms along the $\oplus^\ell \mathfrak{u}(1)$ directions, φ_I are bi-invariant 3-forms of $\mathfrak{ss}$ and $\xi_{(4)}$ is a 4-form along the $\oplus^\ell \mathfrak{u}(1)$ directions.

Since the bi-invariant 3-forms of semi-simple Lie algebras are linear combinations of those associated with the structure constants of the simple components $\mathfrak{s}_r$, we have $\varphi_I = {\nu_I}^r f_r$. Then $f_{(4)}$ can be rewritten as

$$f_{(4)} = \sum_r \sigma^r \wedge f_r + \xi_{(4)}\,, \tag{3.17}$$

for some constants $\mu^r \neq 0$ and ${\nu_I}^r$, where $\sigma^r = \mu^r e_0 + \sum_I {\nu_I}^r \rho^I$. Since f_r and $f_{r'}$ ($r \neq r'$) are orthogonal, the fundamental identity (3.5) implies that σ^r and $\sigma^{r'}$ are also orthogonal. Thus there is an orthogonal transformation in $\oplus^\ell \mathfrak{u}(1)$ such that $f_{(4)}$ can be written, without loss of generality, as

$$f_{(4)} = \sum_r \lambda_r\, e_r \wedge f_r + \xi_{(4)}\,, \tag{3.18}$$

for some constants λ_r, where e_r belong to an orthonormal basis in the $\oplus^\ell \mathfrak{u}(1)$ directions, *i.e.* in particular $e_r \perp e_{r'}$ for $r \neq r'$ and $i_{e_r} f_s = 0$ for all r and s. As in the previous cases, the fundamental identity (3.5) with $A, B, C, D, E, M \in \mathfrak{s}_r$ implies that f_r satisfies the Plücker relation (3.10), and so f_r must be a simple 3-form. Thus the component of $f_{(4)}$ *orthogonal* to $\xi_{(4)}$ is as in (3.4).

Furthermore, from the orthogonality of f_r and $\xi_{(4)}$ and the fundamental identity (3.5) with $A, B, C \in \mathfrak{s}_r$ and $D, E, M \in \oplus^\ell \mathfrak{u}(1)$, one finds that

$$i_{e_r} \xi_{(4)} = 0\,. \tag{3.19}$$

This means that the 4-form $\xi_{(4)}$ on $\oplus^\ell \mathfrak{u}(1)$ satisfies (3.5), *i.e.* $\oplus^\ell \mathfrak{u}(1)$ is also a metric Lie 3-algebra $\mathfrak{b}_{[3]}$ with structure constants $\xi_{(4)}$. Since the dimension of $\mathfrak{b}_{[3]} \subset \mathfrak{a}_{[3]}$ is strictly less than that of the original metric Lie 3-algebra $\mathfrak{a}_{[3]}$, the analysis can be repeated and it will terminate after a finite number of steps.

To summarize, we have demonstrated (3.4) under the assumption that all the associated Lie algebras $\mathfrak{a}_{[2]}(X)$ of $\mathfrak{a}_{[3]}$ are isomorphic to $\oplus^\ell \mathfrak{u}(1) \oplus \mathfrak{ss}$.

Step 3 : Metric Lie algebras

Finally, we show that all metric Lie algebras are isomorphic to $\oplus^\ell \mathfrak{u}(1) \oplus \mathfrak{ss}$.

It is known that any Lie algebra $\mathfrak{g}$, which is not semi-simple, contains the *radical* $\mathfrak{r}$, such that $\mathfrak{g}/\mathfrak{r}$ is semi-simple. This follows from the definition of semi-simple Lie algebras.

Now we assume that $\mathfrak{g}$ admits a Euclidean invariant metric $\langle *, * \rangle$. In such a case, one can define the semi-simple algebra $\mathfrak{ss}$ as the *orthogonal complement* of $\mathfrak{r}$ in $\mathfrak{g}$. From this and the invariant metric condition $\langle [\mathfrak{g}, \mathfrak{g}], \mathfrak{g} \rangle + \langle \mathfrak{g}, [\mathfrak{g}, \mathfrak{g}] \rangle = 0$, one finds that

$$[\mathfrak{ss}, \mathfrak{ss}] \subseteq \mathfrak{ss} \ , \quad [\mathfrak{ss}, \mathfrak{r}] \subseteq \mathfrak{r} \ , \quad [\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r} \ . \tag{3.20}$$

So in order to prove the statement, we have first to show that for metric Lie algebras $\mathfrak{g}$, $\mathfrak{r}$ is abelian. From the definition of radical, there is a k such that $\mathfrak{r}^k$ is an abelian Lie algebra, where $\mathfrak{r}^{i+1} = [\mathfrak{r}^i, \mathfrak{r}^i]$ and $\mathfrak{r}^0 = \mathfrak{r}$. The metric restricted on $\mathfrak{r}$ and $\mathfrak{r}^k$ is non-degenerate. Since $\mathfrak{r}^k$ is abelian, it is easy to see that

$$\langle [\mathfrak{r}^k, \mathfrak{r}^k], \mathfrak{r}^{k-1} \rangle + \langle \mathfrak{r}^k, [\mathfrak{r}^k, \mathfrak{r}^{k-1}] \rangle = \langle \mathfrak{r}^k, [\mathfrak{r}^k, \mathfrak{r}^{k-1}] \rangle = 0 \ . \tag{3.21}$$

So if $[\mathfrak{r}^{k-1}, \mathfrak{r}^k] \neq \{0\}$, it is orthogonal to the whole of $\mathfrak{r}^k$. However, since $[\mathfrak{r}^{k-1}, \mathfrak{r}^k] \subseteq \mathfrak{r}^k$ and the metric is non-degenerate, one concludes that $[\mathfrak{r}^{k-1}, \mathfrak{r}^k] = 0$. Similarly, we find

$$\langle [\mathfrak{r}^{k-1}, \mathfrak{r}^k], \mathfrak{r}^{k-1} \rangle + \langle \mathfrak{r}^k, [\mathfrak{r}^{k-1}, \mathfrak{r}^{k-1}] \rangle = \langle \mathfrak{r}^k, [\mathfrak{r}^{k-1}, \mathfrak{r}^{k-1}] \rangle = 0 \ . \tag{3.22}$$

Since $[\mathfrak{r}^{k-1}, \mathfrak{r}^{k-1}] = \mathfrak{r}^k \neq \{0\}$ and the metric is non-degenerate, one concludes that $[\mathfrak{r}^{k-1}, \mathfrak{r}^{k-1}] = 0$. Therefore, $\mathfrak{r}^{k-1}$ is also abelian and $\mathfrak{r}^{k-1} = \mathfrak{r}^k$. Continuing in this way, one can show that $\mathfrak{r} = \mathfrak{r}^k$ is abelian. As a result, without loss of generality, we can write $\mathfrak{r} = \oplus^\ell \mathfrak{u}(1)$.

It remains to show that $[\mathfrak{ss}, \mathfrak{r}] = 0$. This follows again from the invariant metric condition. Indeed,

$$\langle [\mathfrak{r}, \mathfrak{ss}], \mathfrak{r} \rangle + \langle \mathfrak{ss}, [\mathfrak{r}, \mathfrak{r}] \rangle = \langle [\mathfrak{r}, \mathfrak{ss}], \mathfrak{r} \rangle = 0 \ , \tag{3.23}$$

using that $\mathfrak{r}$ is abelian. Thus if $[\mathfrak{r}, \mathfrak{ss}] \neq \{0\}$, the subspace $[\mathfrak{r}, \mathfrak{ss}]$ of $\mathfrak{r}$ is orthogonal to the whole $\mathfrak{r}$. Since the metric is non-degenerate, one again concludes that $[\mathfrak{r}, \mathfrak{ss}] = 0$. Thus all metric Lie algebras can be written as $\oplus^\ell \mathfrak{u}(1) \oplus \mathfrak{ss}$ with $\mathfrak{ss}$ to commute with $\oplus^\ell \mathfrak{u}(1)$.

Then we finally finish the proof.

3.3 Nambu-Poisson bracket and its truncation

As we saw in the previous section, when both of the following conditions are satisfied,

1. the number of generators $\mathcal{D}$ is finite

2. the metric $h^{ab} = \langle T^a, T^b \rangle$ $(a = 1, \cdots, \mathcal{D})$ is positive-definite

it is shown that *all* the examples of Lie 3-algebra are only (1) trivial algebra $f^{abc}{}_d = 0$, (2) the so-called $\mathcal{A}_4$ with $\mathcal{D} = 4$ and the structure constants $f^{abc}{}_d = i\epsilon^{abcd}$, and (3) their direct sums. Therefore, it means conversely that when the constraints (1) $\mathcal{D}$ is finite and / or (2) h^{ab} is positive-definite are replaced by milder ones, there may be many varieties of Lie 3-algebras which satisfy the fundamental identity (see, for example, [12, 50]).

In particular, in this section, we discuss the $\mathcal{D} = \infty$ positive-definite Lie 3-algebra which can be defined on any manifolds with Nambu-Poisson structure [51–57]. In Chapter 4, we will show that one can obtain an M5-brane's system from BLG model, when one adopt this example of Lie 3-algebra as the gauge symmetry algebra.

3.3.1 Nambu-Poisson Brackets

Let $\mathcal{M}_d$ be a d-dim manifold, and $C(\mathcal{M}_d)$ an algebra of functions on $\mathcal{M}_d$. A Nambu-Poisson bracket is a multilinear map from $C(\mathcal{M}_d)^{\otimes 3}$ to $C(\mathcal{M}_d)$ that satisfies the following conditions [58]:

1. Skew-symmetry

$$\{f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}\} = (-1)^{|\sigma|}\{f_1, f_2, f_3\}. \tag{3.24}$$

2. Leibniz rule

$$\{f_1, f_2, gh\} = \{f_1, f_2, g\}h + g\{f_1, f_2, h\}. \tag{3.25}$$

3. Fundamental identity

$$\begin{aligned}\{g, h, \{f_1, f_2, f_3\}\} \\ = \{\{g, h, f_1\}, f_2, f_3\} + \{f_1, \{g, h, f_2\}, f_3\} + \{f_1, f_2, \{g, h, f_3\}\}.\end{aligned} \tag{3.26}$$

The prototype of a Nambu-Poisson bracket is the Jacobian determinant for three variables x^i $(i = 1, 2, 3)$

$$\{f_1, f_2, f_3\} = \epsilon^{ijk}\partial_i f_1\, \partial_j f_2\, \partial_k f_3. \tag{3.27}$$

where $i, j, k = 1, 2, 3$. This is the classical Nambu bracket. More general Nambu-Poisson bracket can be written in terms of the local coordinates as

$$\{f_1, f_2, f_3\} = \sum_{i_1<i_2<i_3} \sum_{\sigma\in S_3} (-1)^{\sigma} P^{i_1 i_2 i_3}(x)\, \partial_{i_{\sigma(1)}} f_1\, \partial_{i_{\sigma(2)}} f_2\, \partial_{i_{\sigma(3)}} f_3. \tag{3.28}$$

It is proved that one can always choose coordinates such that any Nambu-Poisson bracket is locally just a Jacobian determinant [52–56]. Locally we can choose coordinates such that

$$\{f, g, h\} = \epsilon^{ijk}\partial_i f\, \partial_j g\, \partial_k h, \tag{3.29}$$

where $i, j, k = 1, 2, 3$, and $dx^1 dx^2 dx^3$ defines a local expression of the volume form. As a result, it is straightforward to check that the Nambu-Poisson bracket can be used to generate volume-preserving diffeomorphisms on a function f

$$\delta f = \{g_1, g_2, f\} \tag{3.30}$$

specified by two functions g_1 and g_2.

Nambu-Poisson algebra can also be regarded as an infinite-dimensional Lie 3-algebra. For a 3-manifold on which the Nambu-Poisson bracket is everywhere non-vanishing, it is natural to use the volume form picked by the bracket to define an integral $\int_{\mathcal{M}}$, and then the metric can be defined by

$$\langle f, g\rangle = \int_{\mathcal{M}} f\, g. \tag{3.31}$$

Symmetries of the algebra are then automatically preserved by the metric.

Nambu-Poisson bracket with general order

The notion of Nambu-Poisson brackets can be naturally generalized to brackets of order n, as a map from $C(\mathcal{M}_d)^{\otimes n}$ to $C(\mathcal{M}_d)$. The fundamental identity for Nambu-Poisson brackets of order n is

$$\begin{aligned}&\{f_1, \cdots, f_{n-1}, \{g_1, \cdots, g_n\}\} \\ &\qquad = \sum_{k=1}^{n} \{g_1, \cdots, g_{k-1}, \{f_1, \cdots, f_{n-1}, g_k\}, g_{k+1}, \cdots, g_n\}.\end{aligned} \tag{3.32}$$

Both the Leibniz rule and the fundamental identity indicate that it is natural to think of

$$\{f_1, \cdots, f_{n-1}, *\} \;:\; C(\mathcal{M}_d) \to C(\mathcal{M}_d) \tag{3.33}$$

as a derivative on functions.

Each Nambu-Poisson bracket of order n corresponds to a Nambu-Poisson tensor field P through the relation

$$\{f_1, \cdots, f_n\} = P(df_1, \cdots, df_n), \tag{3.34}$$

where

$$P = \sum_{i_1 < \cdots < i_n} P^{i_1 \cdots i_n}(x)\, \partial_{i_1} \wedge \cdots \wedge \partial_{i_n}\,. \tag{3.35}$$

The theorem mentioned above can also be generalized to brackets of order n, which means that any Nambu-Poisson tensor field P is decomposable, *i.e.* one can express P as

$$P = V_1 \wedge \cdots \wedge V_n \tag{3.36}$$

for n-vector fields V_i. For a review of Nambu-Poisson brackets see, e.g. [57].

Let us now focus on the case $n = 3$. When all the coefficients of the Nambu-Poisson tensor field are linear in x, that is, $P^{i_1 i_2 i_3}(x) = \sum_j f^{i_1 i_2 i_3}{}_j x^j$ for constant $f^{i_1 i_2 i_3}{}_j$, we call the bracket a linear Nambu-Poisson bracket, and it takes the form of a Lie 3-algebra on the coordinates

$$\{x^i, x^j, x^k\} = \sum_l f^{ijk}{}_l x^l. \tag{3.37}$$

Apparently, a linear Nambu-Poisson bracket is also a Lie 3-algebra when we restrict ourselves to linear functions of the coordinates x^i. We have to be careful, however, in that the reverse is not true, as they also have some differences. For the Nambu-Poisson bracket, one may change the coordinates by a general coordinate transformation. On the other hand, for Lie 3-algebra, we only allow linear transformations of the basis. Since the requirement of Leibniz rule for the Nambu-Poisson bracket is not imposed on a Lie 3-algebra, we expect that only a small fraction of Lie 3-algebras are also linear Nambu-Poisson algebras. In particular, we don't expect that the Nambu bracket of a generic Lie 3-algebra be decomposable.

It has been shown that any linear Nambu-Poisson tensor of order n on a linear space V_d can be put in one of the following forms by choosing a suitable basis of V_d [59].

Here the choice of coordinates is made such that the Nambu-Poisson tensor field is linear, instead of trying to make its decomposability manifest. When we interpret these brackets as Nambu brackets on the linear space generated by $\{x^i\}$, we are no longer allowed to make general coordinate transformations on the generators x^i, and the decomposability of the Nambu-Poisson tensor field is no longer relevant.

Linear Nambu-Poisson bracket: type I

The Nambu-Poisson tensor field (3.35) for this type of bracket is labeled by a pair of integers (r, s) such that

$$\begin{aligned} P_{(r,s)} &= \sum_{j=1}^{r+1} \pm x^j \partial_1 \wedge \cdots \wedge \partial_{j-1} \wedge \partial_{j+1} \wedge \cdots \wedge \partial_{n+1} \\ &\quad + \sum_{j=1}^{s} \pm x^{n+j+1} \partial_1 \wedge \cdots \wedge \partial_{r+j} \wedge \partial_{r+j+2} \wedge \cdots \wedge \partial_{n+1}, \end{aligned} \tag{3.38}$$

where $-1 \le r \le n$, $0 \le s \le \min(d - n - 1, n - r)$. Explicitly, we have

$$\{x^1, \cdots, x^{j-1}, x^j, \cdots, x^{n+1}\} = \begin{cases} \pm x^j, & 1 \le j \le r+1, \\ \pm x^{j-r+3}, & r+2 \le j \le r+s+1, \\ 0, & r+s+2 \le j \le d. \end{cases} \tag{3.39}$$

$P_{(3,0)}$ in (3.38) with plus signs for $n = 3$ gives $\mathcal{A}_4$ algebra. For other values of (r, s), $P_{(r,s)}$ gives a new algebra.

Linear Nambu-Poisson Bracket: Type II

$$P = \partial_1 \wedge \cdots \wedge \partial_{n-1} \wedge \left(\sum_{i,j=n}^{d} a_{ij} x^i \partial_j \right). \tag{3.40}$$

In other words,

$$\{x^1, \cdots, x^{n-1}, x^j\} = \sum_{i=n}^{d} a_{ij} x^i, \qquad j = n, \cdots, d. \tag{3.41}$$

The linear Nambu-Poisson algebra of type II (3.40), (3.41) for arbitrary constant matrix a_{ij} has the 3-commutator

$$[T^1, T^2, T^j] = \sum_{i=3}^{d} a_{ij} T^i \quad (j = 3, \cdots, d). \tag{3.42}$$

The invariance of the metric implies that

$$h_{i1} = h_{i2} = \sum_{i=3}^{d} h_{ji} a_{ik} = 0 \tag{3.43}$$

for $i, j, k = 3, \cdots, d$. Thus $a = 0$ if h is invertible. Conversely, if a is invertible then $h_{ij} = 0$ for $i, j = 3, \cdots, d$. As T^1 and T^2 don't appear on the right hand side of the 3-commutator, there is no constraint on h_{11}, h_{12} or h_{22}.

As Nambu-Poisson brackets, we can extend the Lie 3-algebra on the space of linear functions $\mathcal{V} = \{\sum_{i=1}^{d} a_i T^i\}$ to all polynomials of T^i's. The product of T^i's defines a commutative algebra.

3.3.2 Truncation of Nambu-Poisson bracket

In this subsection, we make examples of $\mathcal{D} < \infty$ Lie 3-algebra by truncating Nambu-Poisson bracket. From the no-go theorem in §3.2, these examples must not have positive-definite metrics, but have zero-norm generators.

We start from a Nambu-Poisson bracket on d-dim manifold, defined by local coordinates x^μ ($\mu = 1, \cdots, d$) by

$$\begin{aligned} \{f_1, f_2, f_3\} &:= P(f_1, f_2, f_3) \\ &:= \sum_{\mu_1, \mu_2, \mu_3 = 1}^{d} P^{\mu_1 \mu_2 \mu_3}(x) \partial_{\mu_1} f_1 \partial_{\mu_2} f_2 \partial_{\mu_3} f_3 \end{aligned} \tag{3.44}$$

where $P^{\mu_1\mu_2\mu_3}(x)$ is an antisymmetric tensor. In order to apply to the BLG model, it is essential to assume here that the Nambu-Poisson bracket satisfies fundamental identity

$$\begin{aligned}&\{f_1, f_2, \{f_3, f_4, f_5\}\}\\&\qquad = \{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} + \{f_3, f_4, \{f_1, f_2, f_5\}\}\,. \end{aligned} \tag{3.45}$$

The Leibniz rule

$$\{f_0 f_1, f_2, f_3\} = f_0\{f_1, f_2, f_3\} + \{f_0, f_2, f_3\}f_1 \tag{3.46}$$

is usually required in the literature. In the context of BLG model, however, the role of this condition is not very clear at this moment. The fundamental identity imposes a severe constraint on $P^{\mu_1\mu_2\mu_3}(x)$. As we mentioned in the previous subsection, it is known that the fundamental identity implies the decomposability of P, in mathematical literature [57]. Namely it should be rewritten as

$$\begin{aligned}&P = P^{\mu_1\mu_2\mu_3}(x)\partial_{\mu_1} \wedge \partial_{\mu_2} \wedge \partial_{\mu_3} = V_1 \wedge V_2 \wedge V_3\,,\\&V_i(x) = V_i^{\mu}(x)\partial_\mu\,.\end{aligned} \tag{3.47}$$

It implies that the Nambu-Poisson bracket is essentially defined on 3-dim subspace $\mathcal{N}$ specified by the tangent vectors V_i ($i = 1, 2, 3$). As we will discuss in Chapter 4, this can be used to obtain the M5-brane's system from BLG model whose worldvolume is the product $\mathcal{M} \times \mathcal{N}$ ($\mathcal{M}$ is the original M2-branes' worldvolume).

In the following, since we need to restrict $P^{\mu_1\mu_2\mu_3}(x)$ to be *polynomials of a fixed degree* for the consistency of the cut-off, we will not use this decomposability. When $P^{\mu_1\mu_2\mu_3}(x)$ is a homogeneous polynomial of degree p, we call the 3-commutator as the *homogeneous* Nambu-Poisson bracket. Ho, Hou and Matsuo [12] proposed a truncation of the Nambu-Poisson bracket (3.44) which satisfies the fundamental identity. The idea was to truncate the Hilbert space $C(X)$ (functions on X) to polynomials of x^μ of degree $\leq N$. We will write this truncated Hilbert space as $C(X)_N$. For such truncation to work properly, we need to restrict the antisymmetric tensor $P^{\mu_1\mu_2\mu_3}(x)$ to be a homogeneous polynomial of degree $p > 0$.

On $C(X)_N$, we redefine the Nambu-Poisson bracket to project out all the monomials of order $> N$. We denote such projector as π_N which acts on the polynomials of x^μ as

$$\begin{aligned}&\pi_N\left(\sum_{n_1,\cdots,n_d=0}^{\infty} c(n_1, \cdots, n_d)(x^1)^{n_1}\cdots(x^d)^{n_d}\right)\\&\qquad\qquad = \sum_{n_1,\cdots,n_d=0}^{|\vec{n}|\leq N} c(n_1, \cdots, n_d)(x^1)^{n_1}\cdots(x^d)^{n_d}\,,\end{aligned} \tag{3.48}$$

where $|\vec{n}| := \sum_{i=1}^{d} n_i$. The Nambu-Poisson bracket on the truncated Hilbert space $C(X)_N$ is then defined as

$$\{f_1, f_2, f_3\}_N := \pi_N\left(P(f_1, f_2, f_3)\right)\,. \tag{3.49}$$

It satisfies the fundamental identity

$$\begin{aligned}&\{f_1, f_2, \{f_3, f_4, f_5\}_N\}_N \\ &\quad = \{\{f_1, f_2, f_3\}_N, f_4, f_5\}_N + \{f_3, \{f_1, f_2, f_4\}_N, f_5\}_N + \{f_3, f_4, \{f_1, f_2, f_5\}_N\}_N\,,\end{aligned} \tag{3.50}$$

because of the following reason. For simplicity, we assume f_i to be a monomial of degree p_i. Since (3.50) is satisfied trivially if f_i =const, one may assume $p_i > 0$. The fundamental identity becomes nontrivial if the outer bracket is non-vanishing, namely,

$$p_1 + p_2 + p_3 + p_4 + p_5 - 6 + 2p \le N\,. \tag{3.51}$$

The fundamental identity is broken if the inner bracket vanishes due to the projection, but this doesn't happen. For example, for the left hand side of (3.50), the above inequality together with $p_i \ge 1$ implies

$$p_3 + p_4 + p_5 \le N + 6 - 2p - p_1 - p_2 \le N + 4 - 2p \le N + 3 - p\,. \tag{3.52}$$

In the last inequality, we used $p \ge 1$. Therefore whenever the outer bracket doesn't vanish, the value for the outer bracket is identical with the original bracket. So the fundamental identity on the truncated Hilbert space comes from the fundamental identity on the original space.

$C(X)_N$ is generated by finite number of monomials, $(x^1)^{n_1}\cdots(x^d)^{n_d} := T(\vec{n}) = T(n_1, \cdots, n_d)$ where $n_i \ge 0$ and $|\vec{n}| \le N$. Therefore, the truncated Nambu-Poisson bracket defines a Lie 3-algebra such that

$$\{T(\vec{n}_1), T(\vec{n}_2), T(\vec{n}_3)\}_N = \sum_{\vec{n}_4} f^{\vec{n}_1\vec{n}_2\vec{n}_3}{}_{\vec{n}_4} T(\vec{n}_4)\,, \tag{3.53}$$

which satisfies the fundamental identity

$$\begin{aligned}&f^{\vec{n}_3\vec{n}_4\vec{n}_5}{}_{\vec{n}_6} f^{\vec{n}_1\vec{n}_2\vec{n}_6}{}_{\vec{n}_7} \\ &\quad = f^{\vec{n}_1\vec{n}_2\vec{n}_3}{}_{\vec{n}_6} f^{\vec{n}_6\vec{n}_4\vec{n}_5}{}_{\vec{n}_7} + f^{\vec{n}_1\vec{n}_2\vec{n}_4}{}_{\vec{n}_6} f^{\vec{n}_3\vec{n}_6\vec{n}_5}{}_{\vec{n}_7} + f^{\vec{n}_1\vec{n}_2\vec{n}_5}{}_{\vec{n}_6} f^{\vec{n}_3\vec{n}_4\vec{n}_6}{}_{\vec{n}_7}\,.\end{aligned} \tag{3.54}$$

We remark that the geometrical meaning of the algebra becomes clear when one takes the large N limit where the algebra of polynomials can be completed in different ways and this corresponds to different topological spaces.

We note that because of the constraint $p \ge 1$, we cannot define the truncated Lie 3-algebra from the Jacobian

$$P = \partial_1 \wedge \partial_2 \wedge \partial_3\,. \tag{3.55}$$

As for the Leibniz rule (3.46), we have to be careful how to define the product of functions in the truncated Hilbert space. We define

$$f \bullet_N g = \pi_N(f\, g)\,, \tag{3.56}$$

which gives a commutative and associative product on the truncated space. We replace the Leibniz rule by using this product rule

$$\{f_0 \bullet_N f_1, f_2, f_3\}_N = f_0 \bullet_N \{f_1, f_2, f_3\}_N + \{f_0, f_2, f_3\}_N \bullet_N f_1 \,. \tag{3.57}$$

We show that this condition is also satisfied for $p \geq 1$.

Let us assume that f_i are monomials of x with degree $p_i \geq 1$, since the Leibniz rule is trivially satisfied when $p_0 = 0$ or $p_1 = 0$. The condition that the left hand side of (3.57) is non-vanishing is

$$p_0 + p_1 \leq N \,, \quad p_0 + p_1 + p_2 + p_3 + p - 3 \leq N \,. \tag{3.58}$$

Since the second condition gives a stronger condition than the first for $p \geq 1$, we take the second condition. The first term on the right hand side is non-vanishing if

$$p_1 + p_2 + p_3 + p - 3 \leq N \,, \quad p_0 + p_1 + p_2 + p_3 + p - 3 \leq N \,. \tag{3.59}$$

Again the second condition gives a stronger constraint. The second term on the left hand side is non-vanishing with the same condition. To summarize, the conditions for the both sides of equation are the same. So the truncation is compatible with the Leibniz rule (3.57) for $p \geq 1$.

From the discussion above, we show that the truncated Nambu-Poisson bracket $\{*, *, *\}_N$ with the truncated product $\bullet_N$ can make consistent examples of Lie 3-algebra. We will show the concrete examples *i.e.* the truncation version of eq. (3.38) and (3.40) in §5.1.

3.4 Lorentzian Lie 3-algebra

In the previous sections, we avoided considering the examples of Lie 3-algebra which contain negative-norm generators. Now let us consider these cases, which provide other concrete examples of BLG model.

3.4.1 Central extension of a Lie algebra

We first consider the one-generator extension.

For a given Lie algebra $\mathcal{G}$ with generators T^i and structure constants $f^{ij}{}_k$:

$$[T^i, T^j] = f^{ij}{}_k T^k, \tag{3.60}$$

we can introduce a new element u and define a Lie 3-algebra as

$$\begin{aligned} [u, T^i, T^j] &= f^{ij}{}_k T^k, \\ [T^i, T^j, T^k] &= 0\,, \end{aligned} \tag{3.61}$$

for $i, j, k = 1, \cdots, \dim \mathcal{G}$. For a simple Lie algebra $\mathcal{G}$, the invariant metric condition (2.16) demands that

$$\langle [u, T^i, T^j], T^k \rangle + \langle T^j, [u, T^i, T^k] \rangle = 0 \quad \Rightarrow \quad f^{ij}{}_l h^{lk} + f^{ik}{}_l h^{lj} = 0 \,. \tag{3.62}$$

This suggests that h^{ij} should be proportional to the Killing form of $\mathcal{G}$. However, the invariant metric conditions also include

$$\begin{aligned} \langle [T^i, T^j, T^k], u \rangle + \langle T^k, [T^i, T^j, u] \rangle = 0 \;\; &\Rightarrow \;\; f^{ij}{}_l h^{lk} = 0, \\ \langle [T^i, T^j, u], u \rangle + \langle u, [T^i, T^j, u] \rangle = 0 \;\; &\Rightarrow \;\; h^{ku} = 0. \end{aligned} \tag{3.63}$$

Therefore, we cannot use the Killing form of the Lie algebra $\mathcal{G}$ as h^{ij}, but instead the metric should be taken as

$$h^{ij} = \langle T^i, T^j \rangle = 0 \,, \qquad h^{ui} = \langle u, T^i \rangle = 0 \,, \qquad h^{uu} = \langle u, u \rangle = K \,, \tag{3.64}$$

where K is an arbitrary constant. This is a new example, but too simple to use as the gauge symmetry algebra in BLG model.

Lorentzian Lie 3-algebra as a central extension of Lie algebra

Then we consider a more nontrivial example [15, 60, 61] by adding extra generator v such that the 3-commutator is defined as

$$\begin{aligned} [v, T^A, T^B] &= 0 \,, \\ [u, T^i, T^j] &= i f^{ij}{}_k T^k \,, \\ [T^i, T^j, T^k] &= -i f^{ijk} v \,, \end{aligned} \tag{3.65}$$

where $i, j, k = 1, \cdots, \dim \mathcal{G}$, $T^A = \{T^i, u, v\}$, and $f^{ijk} := f^{ij}{}_l h^{lk}$ is totally antisymmetrized. One can check that all these 3-commutators satisfy the fundamental identity

$$\begin{aligned} &[T^a, T^b, [T^c, T^d, T^e]] \\ &\quad = [[T^a, T^b, T^c], T^d, T^e] + [T^c, [T^a, T^b, T^d], T^e] + [T^c, T^d, [T^a, T^b, T^e]]. \end{aligned} \tag{3.66}$$

for all a, b, c, d, e. The requirement of invariance of the metric

$$\langle [T^a, T^b, T^c], T^d \rangle + \langle [T^c, [T^a, T^b, T^d] \rangle = 0 \tag{3.67}$$

implies that the metric has to be defined as

$$\begin{aligned} \langle v, v \rangle = 0, \qquad &\langle v, u \rangle = 1, \qquad \langle v, T^i \rangle = 0, \\ \langle u, u \rangle = K, \qquad &\langle u, T^i \rangle = 0, \qquad \langle T^i, T^j \rangle = h^{ij}, \end{aligned} \tag{3.68}$$

where K is an arbitrary constant. In this case, as compared with the previous case, h^{ij} can be non-zero, so we can obtain some nontrivial form of BLG model by using this algebra. This discussion will be done in Chapter 6.

Note that there is an algebra homomorphism

$$u \to u + \alpha v, \tag{3.69}$$

that preserves the Lie 3-algebra, but changes the metric by a shift of K:

$$K = \langle u, u \rangle \to K - 2\alpha\,. \tag{3.70}$$

Thus one can always choose u such that $K = 0$.

3.4.2 Lie 3-algebra with two or more negative-norm generators

In this subsection, we consider some generalizations of the Lorentzian 3-algebra in the previous subsection by adding pairs of generators with Lorentzian metric. The positive-norm generators are denoted as e^i ($i = 1, \cdots, N$), and the Lorentzian pairs which cause the negative-norm generators as u_a, v_a ($a = 1, \cdots, M$). If we set $M = 1$, it is reduced to the previous case (3.68).

We assume that the invariant metric for them is given by the following simple form

$$\langle e^i, e^j \rangle = \delta^{ij}\,, \quad \langle u_a, v_b \rangle = \delta_{ab}\,. \tag{3.71}$$

In terms of the four-tensor defined by

$$f^{ABCD} := f^{ABC}{}_E h^{ED}\,, \tag{3.72}$$

the invariance of the metric and the skew-symmetry of the structure constants implies that the condition that this 4-tensor is antisymmetric with respect to all indices.

We also assume that the generators v_a are in the center of the Lie 3-algebra. This condition is necessary to obtain the physically meaningful Lagrangian from BLG model (*i.e.* a kind of *Higgs mechanism* to get rid of the ghost fields arising from negative-norm generators works well). We will discuss it in detail in Chapter 6. In terms of the 4-tensor this condition is written as

$$f^{v_a BCD} = 0 \tag{3.73}$$

for arbitrary B, C, D. Therefore, the index in the 4-tensor is limited to e^i and u_a. For the simplicity of the notation, we write i for e^i and a for u_a for indices of the 4-tensor, for example $f^{ijab} := f^{e^i e^j u_a u_b}$ and so on.

Fundamental identities

In the following, we introduce some notation for the 4-tensor

$$f^{ijkl} = F^{ijkl},\ f^{aijk} = f_a^{ijk},\ f^{abij} = J_{ab}^{ij},\ f^{abci} = K_{abc}^i,\ f^{abcd} = L_{abcd}\,. \tag{3.74}$$

Now we rewrite the fundamental identity in terms of this notation (3.74):

$$F^{ijkn}F^{nlmp} + F^{ijln}F^{knmp} + F^{ijmn}F^{klnp} - F^{klmn}F^{ijnp} = 0, \tag{3.75}$$
$$F^{ijkn}f_a^{nlm} + F^{ijln}f_a^{knm} + F^{ijmn}f_a^{kln} - F^{klmn}f_a^{ijn} = 0, \tag{3.76}$$
$$f_a^{ijn}F^{nklm} + f_a^{ikn}F^{jnlm} + f_a^{iln}F^{jknm} - f_a^{inm}F^{jkln} = 0, \tag{3.77}$$
$$(f_a^{ijn}f_b^{nkl} + f_a^{ikn}f_b^{jnl} + f_a^{iln}f_b^{jkn}) + F^{jkln}J_{ab}^{in} = 0, \tag{3.78}$$
$$J_{ab}^{im}F^{mjkl} + J_{ab}^{jm}F^{imkl} + J_{ab}^{km}F^{ijml} + J_{ab}^{lm}F^{ijkm} = 0, \tag{3.79}$$
$$(J_{ab}^{im}f_c^{mjk} + J_{ab}^{jm}f_c^{imk} + J_{ab}^{km}f_c^{ijm}) - F^{ijkm}K_{abc}^{m} = 0, \tag{3.80}$$
$$F^{ijkn}J_{ab}^{nl} - F^{ijln}J_{ab}^{nk} - f_a^{ijn}f_b^{nkl} + f_b^{ijn}f_a^{nkl} = 0, \tag{3.81}$$
$$(J_{ab}^{im}f_c^{mjk} - J_{ac}^{im}f_b^{mjk}) + (f_a^{ijm}J_{bc}^{mk} - f_a^{ikm}J_{bc}^{mj}) = 0, \tag{3.82}$$
$$-K_{abc}^{l}f_d^{lij} + K_{abd}^{l}f_c^{lij} + J_{ab}^{il}J_{cd}^{lj} - J_{cd}^{il}J_{ab}^{lj} = 0, \tag{3.83}$$
$$(f_a^{ikm}J_{bc}^{mi} + f_b^{jkm}J_{ca}^{mi} + f_c^{jkm}J_{ab}^{mi}) + K_{abc}^{m}F^{jkim} = 0, \tag{3.84}$$
$$(J_{ab}^{jl}J_{cd}^{li} + J_{ad}^{jl}J_{bc}^{li} - J_{ac}^{jl}J_{bd}^{li}) - f_c^{jil}K_{abd}^{l} = 0, \tag{3.85}$$
$$-J_{ab}^{ki}K_{cde}^{k} - J_{be}^{ki}K_{acd}^{k} + J_{ae}^{ki}K_{bcd}^{k} + J_{cd}^{ki}K_{abe}^{k} = 0, \tag{3.86}$$
$$f_a^{ijl}K_{bcd}^{l} - f_b^{ijl}K_{acd}^{l} + f_c^{ijl}K_{abd}^{l} - f_d^{ijl}K_{abc}^{l} = 0, \tag{3.87}$$
$$K_{abc}^{i}K_{def}^{i} - K_{ade}^{i}K_{bcf}^{i} + K_{acf}^{i}K_{bde}^{i} - K_{abf}^{i}K_{cde}^{i} = 0. \tag{3.88}$$

There are a few comments which can be made without detailed analysis:

Case of lower M

For lower M (*i.e.* smaller number of Lorentzian pairs (u_a, v_a)), some components of the structure constants (3.74) vanish identically due to the antisymmetry of indices. For example, for $M = 1$, we need to put $J_{ab}^{ij} = K_{abc}^{i} = L_{abcd} = 0$. For $M = 2$, one may put J_{ab}^{ij} nonvanishing but we have to keep $K_{abc}^{i} = L_{abcd} = 0$ and so on.

On F^{ijkl}

A constraint for F^{ijkl} (3.75) is identical to the fundamental identity of a Lie 3-algebra with the structure constants F^{ijkl}. So if we assume positive definite metric for e^i, it automatically implies that F^{ijkl} is proportional to ϵ^{ijkl} or its direct sums, from the discussion in §3.2.

On L_{abcd}

In the fundamental identity (3.75)–(3.88), there is no constraint on L_{abcd}. It comes from the fact that the contraction with respect to Lorentzian indices automatically vanishes due to the restriction of the structure constants (3.73). So it can take arbitrary value for $M \geq 4$. This term, however, is not physically relevant in BLG model, since they appear only in the interaction terms of the ghost fields which will be erased after Higgs mechanism.

On the change of basis

We note that there is some freedom in the choice of basis when keeping the metric (3.71) and the form of 4-tensor (3.73) invariant:

$$\tilde{e}^i = O^i_j e^j + P^i_a v^a, \quad \tilde{u}^a = Q^a_i e^i + R^a_b u^b + S^a_b v^b, \quad \tilde{v}^a = ((R^t)^{-1})^a_b v^b, \tag{3.89}$$

where

$$O^t O = 1, \quad Q = -RP^t O, \quad R^{-1}S + (R^{-1}S)^t = -P^t P\,. \tag{3.90}$$

The matrices O and R describe the usual rotations of the basis. The matrix P describes the mixing of the Lorentzian generators u_a, v_a with e^i.

By a change of basis (3.89), various components of the structure constants (3.74) mix. For example, if we put $O = R = 1$ for simplicity and keep only the matrix P nontrivial (which implies $S = -\frac{1}{2}P^t P$), the structure constants in terms of the new basis $\{\tilde{e}^i, \tilde{u}^a, \tilde{v}^a\}$ are given as

$$\tilde{F}^{ijkl} = F^{ijkl}, \tag{3.91}$$

$$\tilde{f}^{jkl}_a = f^{jkl}_a + P^i_a F^{ijkl}, \tag{3.92}$$

$$\tilde{J}^{ij}_{ab} = J^{ij}_{ab} + P^k_a f^{ijk}_b - P^k_b f^{ijk}_a + F^{ijkl} P^k_a P^l_b, \tag{3.93}$$

$$\begin{aligned}\tilde{K}^i_{abc} = {} & K^i_{abc} + P^j_a J^{ij}_{bc} - P^j_b J^{ij}_{ac} + P^j_c J^{ij}_{ab}, \\ & + f^{ikl}_c P^k_a P^l_b - f^{ikl}_b P^k_a P^l_c + f^{ikl}_a P^k_b P^l_c + P^j_a P^k_b P^l_c F^{ijkl}\,.\end{aligned} \tag{3.94}$$

We will find that many solutions of the fundamental identities can indeed be identified with well-known Lie 3-algebra after such redefinition of basis. In this sense, the classification of the Lorentzian Lie 3-algebra has a character of cohomology, namely only solutions which can not reduce to known examples after all changes of basis give rise to physically new system.

In the following, we give a somewhat technical analysis of the fundamental identity (3.75)–(3.88). Solutions which we found are summarized in the end of this subsection. We don't claim that our analysis exhausts all the possible solutions. But as we will see in Chapter 7, they play an important physical role in string/M theory compactification.

Case 1 : Lorentzian extension of Nambu-Poisson bracket

Let us examine the case with $F^{ijkl} \neq 0$ first. As we already mentioned, eq. (3.75) implies that $F^{ijkl} \propto \epsilon^{ijkl}$ and its direct sum. So without loss of generality, one may assume $N = 4$ and $F^{ijkl} = \epsilon^{ijkl}$ for the terms which include nontrivial contraction with F^{ijkl}.

Suppose $f^{ijk}_a \neq 0$ for some a. Then by the skew-symmetry of indices they can be written as $f^{ijk}_a = \epsilon^{ijkl} P^a_l$ for some P^a_l. This expression actually solves (3.76) and (3.77). However, this form of f^{ijk}_a is exactly the same as the right hand side of (3.92). It implies that such f^{ijk}_a can be set to zero by a redefinition of basis.

Therefore, at least when Lie 3-algebra is finite-dimensional, it is impossible to construct Lorentzian algebra with nontrivial $F^{ijkl} \neq 0$. The situation is totally different if Lie 3-algebra is infinite-dimensional which we discussed in §3.3.1. The realization of Lie 3-algebra was given as follows. We take $\mathcal{N}$ as a compact 3-dim manifold where Nambu-Poisson bracket [51],

$$\{f_1, f_2, f_3\} = \sum_{a,b,c} \epsilon^{abc}\, \partial_a f_1 \partial_b f_2 \partial_c f_3 \tag{3.95}$$

is well defined. Namely $\mathcal{N}$ is covered by the local coordinate patches where the coordinate transformation between the two patches keeps the 3-commutator (3.95) invariant. If we take $\chi^i(y)$ as the basis of $\mathcal{H}$: the Hilbert space which consists of functions which are globally well-defined on $\mathcal{N}$, and one can choose a basis mutually orthonormal with respect to the inner product

$$\langle \chi^i, \chi^j \rangle := \int_{\mathcal{N}} d^3y\, \chi^i(y)\chi^j(y) = \delta^{ij}\,. \tag{3.96}$$

It is known that the structure constants

$$F^{ijkl} = \langle \left\{ \chi^i, \chi^j, \chi^k \right\}, \chi^l \rangle \tag{3.97}$$

satisfy the fundamental identity (3.75).

We are going to show that it is possible to extend this Lie 3-algebra with the additional generators with the Lorentzian signature. For simplicity, we consider the case $\mathcal{N} = T^3$. The Hilbert space $\mathcal{H}$ is spanned by the periodic functions on T^3. If we write the flat coordinates on T^3 as y^a ($a = 1, 2, 3$), where the periodicity is imposed as $y^a \sim y^a + p^a$, and $p^a \in \mathbf{Z}$. The basis of $\mathcal{H}$ is then given by

$$\chi^{\vec{n}}(y) := e^{2\pi i n_a y^a}\,, \quad \vec{n} \in \mathbf{Z}^3\,, \tag{3.98}$$

with the invariant metric and the structure constants:

$$\langle \chi^{\vec{n}}, \chi^{\vec{m}} \rangle = \delta(\vec{n} + \vec{m})\,, \tag{3.99}$$

$$F^{\vec{n}\vec{m}\vec{l}\vec{p}} = (2\pi i)^3 \epsilon^{abc} n^a m^b l^c \delta(\vec{n} + \vec{m} + \vec{l} + \vec{p})\,. \tag{3.100}$$

The idea to extend the Lie 3-algebra is to introduce the functions which are *not* well-defined on T^3 but the 3-commutators among $\mathcal{H}$ and these generators remains in $\mathcal{H}$. For T^3, such generators are given by the functions $u_a = y^a$. The fundamental identity comes from the definition of derivative and it doesn't matter whether or not the functions in the 3-commutators is well-defined globally. Therefore, even if we include extra generators the analog of fundamental identity holds. More explicitly, we define the extra structure constants as

$$\begin{aligned} f_a^{\vec{n}\vec{m}\vec{l}} &:= \langle \left\{ u^a, \chi^{\vec{n}}, \chi^{\vec{m}} \right\}, \chi^{\vec{l}} \rangle = (2\pi i)^2 \epsilon_{abc} n^b m^c \delta(\vec{n} + \vec{m} + \vec{l}), \\ J_{ab}^{\vec{n}\vec{m}} &:= \langle \left\{ u^a, u^b, \chi^{\vec{n}} \right\}, \chi^{\vec{m}} \rangle = (2\pi i) \epsilon_{abc} n^c \delta(\vec{n} + \vec{m}), \\ K_{abc}^{\vec{n}} &:= \langle \left\{ u^a, u^b, u^c \right\}, \chi^{\vec{n}} \rangle = \epsilon_{abc} \delta(\vec{n}). \end{aligned} \tag{3.101}$$

It is not difficult to demonstrate explicitly that they satisfy all the fundamental identities (3.75)–(3.88).

We have to be careful in the treatment of the new generators. For example, the inner product (3.96) is not well-defined if the function is not globally well-defined on $\mathcal{N}$. The fact that the structure constants (3.100)–(3.101) satisfies the fundamental identities (3.75)–(3.88) implies that we can define the inner product *abstractly* as (3.71). Namely we introduce extra generators v_a ($a = 1, 2, 3$) and define

$$\langle u_a, v_b\rangle = \delta_{ab}, \quad \langle u_a, \chi^{\vec{n}}\rangle = \langle v_a, \chi^{\vec{n}}\rangle = \langle u_a, u_b\rangle = \langle v_a, v_b\rangle = 0 \tag{3.102}$$

while keeping (3.99).

We also need to be careful in the definition of the 3-commutator itself. The naive 3-commutator needs to be modified to make the structure constants F^{ABCD} totally antisymmetric in all four indices. This condition is broken in the original 3-commutator after the introduction of the extra generators u^a. We have to come back to our original definition of Lie 3-algebra where this symmetry is manifest. This implies the following redefinition of Lie 3-algebra:

$$\begin{aligned}
\left[\chi^{\vec{n}}, \chi^{\vec{m}}, \chi^{\vec{l}}\right] &= F^{\vec{n}\vec{m}\vec{l}}{}_{\vec{p}}\chi^{\vec{p}} - f_a^{\vec{n}\vec{m}\vec{l}}v^a,\\
\left[u^a, \chi^{\vec{n}}, \chi^{\vec{m}}\right] &= f_a^{\vec{n}\vec{m}}{}_{\vec{l}}\chi^{\vec{l}} + J_{ab}^{\vec{n}\vec{m}}v^b,\\
\left[u^a, u^b, \chi^{\vec{n}}\right] &= J_{ab\vec{m}}^{\vec{n}}\chi^{\vec{m}} - K_{abc}^{\vec{n}}v^c,\\
\left[u^a, u^b, u^c\right] &= K_{abc\vec{n}}\chi^{\vec{n}}.
\end{aligned} \tag{3.103}$$

This Lie 3-algebra may be regarded as the "central extension" of the Nambu-Poisson bracket. The additional factors which are proportional to v^a on the right hand side is necessary to make the metric invariant. One might worry if the fundamental identity may be violated by the redefinition of the algebra. In this example, fortunately this turns out not to be true. So we have a consistent Lie 3-algebra with Lorentzian signature. It may be useful to repeat our emphasis that, although u^a was originally defined through ill-defined function y^a, we have to neglect this fact to define the metric and Lie 3-algebra.

While the Lie 3-algebra (3.103) is new, the BLG model based on it turns out to be the same as that on the original Nambu-Poisson bracket (*i.e.* the single M5-brane's models) which we will see in Chapter 4, although it was not noticed explicitly [7].

It is straightforward to obtain similar Lorentzian extensions of Nambu-Poisson type Lie 3-algebras defined on different manifolds $\mathcal{N}$ such as S^3 and $S^2 \times S^1$. So far, the only nontrivial Lie 3-algebra with positive-definite metric are $\mathcal{A}_4$ and the Nambu-Poisson type Lie 3-algebras. The examples we consider here would exhaust the Lorentzian extensions which can be obtained from them.

Case 2 : Constraints from the fundamental identities for $F^{ijkl} = 0$

In the following, we restrict ourselves to the case $F^{ijkl} = 0$. The fundamental identities (3.75)–(3.88) are now simplified to be the following:

$$f_a^{ni[j} f_b^{kl]n} = 0\,, \tag{3.104}$$
$$f_{[a}^{ijm} f_{b]}^{mkl} = 0\,, \tag{3.105}$$
$$f_a^{ijm} f_b^{mkl} + f_a^{kim} f_b^{mjl} + f_b^{jkm} f_a^{mil} = 0\,, \tag{3.106}$$
$$J_{ab}^{l[i} f_c^{jk]l} = 0\,, \tag{3.107}$$
$$f_{[a}^{ijk} J_{bc]}^{kl} = 0\,, \tag{3.108}$$
$$J_{a[b}^{il} f_{c]}^{ljk} + J_{bc}^{l[j} f_a^{k]il} = 0\,, \tag{3.109}$$
$$2K_{ab[c}^{k} f_{d]}^{kij} = J_{ab}^{ik} J_{cd}^{kj} - J_{cd}^{ik} J_{ab}^{kj}\,, \tag{3.110}$$
$$f_a^{ijk} K_{bcd}^{k} = 3J_{a[b}^{ik} J_{cd]}^{kj}\,, \tag{3.111}$$
$$3K_{ab[c}^{i} J_{de]}^{ij} = K_{cde}^{i} J_{ab}^{ij}\,, \tag{3.112}$$
$$J_{a[b}^{ij} K_{cde]}^{j} = 0\,, \tag{3.113}$$
$$f_{[a}^{ijk} K_{bcd]}^{k} = 0\,, \tag{3.114}$$
$$3K_{ab[c}^{i} K_{de]f}^{i} = K_{cde}^{i} K_{abf}^{i}\,. \tag{3.115}$$

In the above, we used the notation that all indices in parentheses are fully antisymmetrized. For instance,

$$A_{a[b} B_{cd]e} := \frac{1}{6}\left(A_{ab}B_{cde} + A_{ac}B_{dbe} + A_{ad}B_{bce} - A_{ab}B_{dce} - A_{ac}B_{bde} - A_{ad}B_{cbe}\right). \tag{3.116}$$

The constraints above are not all independent. We can use (3.106) alone to derive (3.104) and (3.105) as follows. Taking (3.106) and replacing the indices as $(ijk) \to (jki)$ and $(ab) \to (ba)$ and subtracting the derived equation from (3.106), we get (3.105). It is also obvious that (3.105) and (3.106) implies (3.104).

Similarly, (3.109) can be easily derived from (3.107) and (3.108).

Detailed analysis

Now we try to solve the fundamental identities displayed above and find a class of solutions.

First, a solution for (3.104) is to use a direct sum of Lie algebras $g = g_1 \oplus \cdots \oplus g_n$. We divide the values of indices into n blocks $I = I_1 \cup \cdots \cup I_n$ and let

$$f_a^{ijk} = \gamma_a^\alpha f_\alpha^{ijk}, \tag{3.117}$$

where f_α^{ijk} is defined by

$$f_\alpha^{ijk} = \begin{cases} f_{g_\alpha}^{ijk} & i,j,k \in I_\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{3.118}$$

Here $f_{g_\alpha}^{ijk}$ are the structure constants for g_α while γ_a^α is a real number.

Note that the number n doesn't have to equal M. It is possible to have some of the sets I_a empty. An example has $g = g_1$ and all $I_{a\neq 1}$ empty. In this case, for $\gamma_a^\alpha = \delta_a^\alpha$, we have $f_1^{ijk} = f_{g_1}^{ijk}$ and $f_a^{ijk} = 0$ for all $a \neq 1$.

If all the other components of Lie 3-algebra structure constants vanish, one obtains from (3.117) a set of solutions to the fundamental identity. The BLG model for this Lie 3-algebra is not new, however. For each range of index, say I_α, we have

$$[e^i, e^j, e^k] = -\sum_a \gamma_a^\alpha v^a\,, \quad [u^a, e^i, e^j] = \sum_k \gamma_a^\alpha f_\alpha^{ijk} e^k\,. \tag{3.119}$$

By a suitable rotation (3.89) with

$$v'^1 = \sum_a \gamma_a^\alpha v^a, \tag{3.120}$$

we always have

$$[e^i, e^j, e^k] = -v'^a\,, \quad [u'^a, e^i, e^j] = \delta_{a1} \sum_k f_\alpha^{ijk} e^k\,. \tag{3.121}$$

Therefore it is reduced to the standard Lorentzian Lie 3-algebra for $M = 1$ after the restriction of indices to I_α.

In order to obtain something new, we have to allow other coefficients to be nonzero.

The simplest class of solutions can be found when $f_a^{ijk} = 0$ for $i, j, k \in I_a$. In this case, for this range I_a, arbitrary antisymmetric matrix J^{ij} $(i, j \in I_a)$ solves the constraints It demonstrates the essential feature that the supersymmetric system acquires mass proportional to eigenvalues of J [7]. However, since we put $f_a^{ijk} = 0$, there is no interaction. In order to have the interacting system, we need nonvanishing f_a^{ijk}.

For simplicity, let us assume that there is a suitable basis of generators such that the solution (3.117) is simplified as

$$f_a^{ijk} = \begin{cases} f_a^{ijk} & i, j, k \in I_a, \\ 0 & \text{otherwise}, \end{cases} \tag{3.122}$$

where the indices are divided into n disjoint sets $I = I_1 \cup \cdots \cup I_n$, and f_a^{ijk} are the structure constants for a Lie algebra g_a.

Starting with (3.122), we can solve all the constraints (3.104)–(3.115) as follows, while (3.122) already solves (3.104)–(3.106).

Eq. (3.108) is trivial if two of the indices a, b, c are identical. Assuming (3.122), eq. (3.108) imposes no constraint on J_{ab}^{ij} if $i \in I_a$ or $i \in I_b$. In general, if $f_c^{ijk} \neq 0$ for $c \neq a$ and $c \neq b$, then $J_{ab}^{ij} = 0$ if $i \in I_c$. Hence we consider the case

$$J_{ab}^{ij} \neq 0 \qquad \text{only if} \qquad i, j \in I_a \quad \text{or} \quad i, j \in I_b. \tag{3.123}$$

Eq. (3.109) is now trivial if all indices a, b, c are different. If two of the indices are the same, it is equivalent to (3.107).

According to (3.107), J_{ab} is a derivation for both Lie algebras g_a and g_b. A derivation $\mathcal{D}$ is a map from g to g such that

$$\mathcal{D}([e^i, e^j]) = [\mathcal{D}(e^i), e^j] + [e^i, \mathcal{D}(e^j)]. \tag{3.124}$$

As a result of (3.107), one can define a derivations $\mathcal{D}_{ab}$ by

$$\mathcal{D}_{ab}(e^i) = J^{ij}_{ab} e^j. \tag{3.125}$$

The simplest case is when J_{ab} corresponds to an inner automorphism, so

$$J^{ij}_{ab} = \Lambda^k_{ab} f_a^{ij}{}_k - \Lambda^k_{ba} f_b^{ij}{}_k, \tag{3.126}$$

where $\Lambda^k_{ab} = 0$ unless $k \in I_a$. (Note that the indices a, b are not summed over in (3.126).) In this case $\mathcal{D}_{ab}(*) = [(\Lambda^k_{ab} - \Lambda^k_{ba})e_k, *]$. It will be more interesting if $\mathcal{D}_{ab}$ instead corresponds to an infinitesimal outer automorphism (an *outer derivation*). [3]

If all indices a, b, c, d are all different, (3.110) is trivial due to (3.123). If $a = d \neq b \neq c$, (3.110) says that the Lie bracket $[J_{ab}, J_{ac}]$ is an inner automorphism. The solution of (3.110) is in general given by

$$K_{abc} := K^i_{abc} e^i = [\mathcal{D}_{ac}, \mathcal{D}_{bc}] + [\mathcal{D}_{ba}, \mathcal{D}_{ca}] + [\mathcal{D}_{cb}, \mathcal{D}_{ab}] + C_{abc}, \tag{3.127}$$

where the antisymmetric tensor $C_{abc} = C^i_{abc}$ is a central element in g. Since all derivations of a Lie algebra is always a Lie algebra, the Lie bracket $[\mathcal{D}_{ab}, \mathcal{D}_{cd}]$ satisfies the Jacobi identity.

For J_{ab} given by an inner automorphism (3.126), K^i_{abc} can be solved from (3.110) to be

$$K^i_{abc} = \Lambda^j_{ab}\Lambda^k_{ac} f_a^{ijk} + \Lambda^j_{bc}\Lambda^k_{ba} f_b^{ijk} + \Lambda^j_{ca}\Lambda^k_{cb} f_c^{ijk} + C^i_{abc}. \tag{3.128}$$

(Indices a, b, c are not summed over in this equation.) The term $\Lambda^j_{ab}\Lambda^k_{ac} f_a^{ijk}$ corresponds to the Lie bracket of the two automorphisms generated by Λ_{ab} and Λ_{ac} on g_a. However, the case of J_{ab} generating an inner automorphism is not interesting because J_{ab} and K^i_{abc} can be both set to zero after a change of basis (3.92) and (3.93),

$$\begin{aligned} e'^i &= e^i - \sum_b \Lambda^i_{ab} v^b \quad \text{for} \quad i \in I_a, && (3.129) \\ u'_a &= u_a - \sum_b \Lambda^i_{ba} e^i. && (3.130) \end{aligned}$$

Therefore, in the following we will focus on the case when J_{ab} is an outer automorphism.

[3] We have to keep in mind that the existence of such automorphisms is quite nontrivial. We will come back to this issue below.

When all indices a, b, c, d, e are different, (3.112) can be easily satisfied if

$$C^i_{abc} = 0 \qquad \text{unless} \qquad i \in I_a \cup I_b \cup I_c. \tag{3.131}$$

Together with (3.123), this implies that K^i_{abc} (3.127) vanishes unless $i \in I_a \cup I_b \cup I_c$.

Due to (3.123) and (3.131), eq. (3.112) is trivial if all indices a, b, c, d, e are different. If $e = a$, it is

$$K^i_{abc}J^{ij}_{ad} + K^i_{acd}J^{ij}_{ab} + K^i_{adb}J^{ij}_{ac} = 0. \tag{3.132}$$

One can then check that this follows from (3.127) and the constraint

$$\mathcal{D}_{ab}(C_{acd}) + \mathcal{D}_{ac}(C_{adb}) + \mathcal{D}_{ad}(C_{abc}) = 0 \tag{3.133}$$

as a result of the Jacobi identity of the Lie bracket of $\mathcal{D}_{ab}$'s. The same discussion applies to (3.111), (3.113), (3.114) and (3.115).

Before closing this analysis, let us comment on infinitesimal outer automorphisms. For finite-dimensional Lie algebra, we have two examples. The first example is when the Lie algebra is abelian, and any nontrivial linear map of the generators is an outer automorphism. The 2nd example is when the Lie algebra is that of matrices composed of upper triangular blocks

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \tag{3.134}$$

where A, B, C are $m\times m$, $m\times n$ and $n\times n$ matrices, respectively. An arbitrary scaling of the off-diagonal block B is an outer automorphism. In both of these examples, the coefficients of e_i in the expansion of X^I or Ψ don't participate in interactions in the BLG model, unless e^i is inert to the outer derivation. Hence the appearance of outer derivation in these cases is irrelevant to physics. A nontrivial example is found when g is an infinite-dimensional Lie algebra. This example will be studied in §7.2.

Summary of solutions

To summarize the result of our construction of a new Lie 3-algebra, the general solution of the fundamental identity for our ansatz

$$\begin{aligned}
[u_a, u_b, u_c] &= K^i_{abc}e_i + L_{abcd}v^d, \\
[u_a, u_b, e^i] &= J^{ij}_{ab}e_j - K^i_{abc}v^c, \\
[u_a, e^i, e^j] &= J^{ij}_{ab}v^b + f^{ijk}_a e_k\,, \\
[e^i, e^j, e^k] &= -f^{ijk}_a v^a,
\end{aligned} \tag{3.135}$$

is given by (3.122), (3.125) and (3.127), which are repeated here for the convenience of the reader,

$$\begin{aligned} f_a^{ijk} &= \begin{cases} f_a^{ijk} & i,j,k \in I_a, \\ 0 & \text{otherwise}, \end{cases} \\ J_{ab}^{ij} e^j &= \mathcal{D}_{ab}(e^i) \qquad \text{for a derivation } \mathcal{D}_{ab}, \\ K_{abc} &:= K_{abc}^i e^i = [\mathcal{D}_{ac}, \mathcal{D}_{bc}] + [\mathcal{D}_{ba}, \mathcal{D}_{ca}] + [\mathcal{D}_{cb}, \mathcal{D}_{ab}] + C_{abc}, \end{aligned} \tag{3.136}$$

where C_{abc} are central elements in g satisfying (3.131) and (3.133)

$$\begin{aligned} & C_{abc}^i = 0 \qquad \text{unless} \qquad i \in I_a \cup I_b \cup I_c, \\ & \mathcal{D}_{ab}(C_{acd}) + \mathcal{D}_{ac}(C_{adb}) + \mathcal{D}_{ad}(C_{abc}) = 0. \end{aligned} \tag{3.137}$$

The nontrivial part of the metric is given by

$$\langle e^i, e^j \rangle = g^{ij}\,, \quad \langle u_a, v^b \rangle = \delta_a^b\,, \tag{3.138}$$

where g^{ij} is the Killing form of the Lie algebra g. Although we have assumed that g^{ij} is positive definite in the derivation above, it is obvious that the Lie 3-algebra can be directly generalized to a generic Killing form which is not necessarily positive-definite.

Compared with the Lie 3-algebra in the previous subsections, the Lie 3-algebra constructed above contains more information. While e^i's are generators of a Lie algebra $g = g_1 \oplus \cdots \oplus g_n$, J_{ab}'s correspond to infinitesimal outer automorphisms (outer derivations), and K_{abc} encodes both the commutation relations among J_{ab}'s and choices of central elements in g.

The concrete form of examples is showed in Chapter 7.

3.5 Summary

In this chapter, we show the various examples of Lie 3-algebra. Then, in the reminder of this thesis (Part II and III), we discuss the BLG model with gauge symmetry of these Lie 3-algebra. While the BLG model is originally proposed as the superconformal $\mathcal{N} = 8$ Chern-Simons theory for multiple M2-branes' system, we will see that this model can also describe a single M5-brane's or multiple Dp-branes' system.

Chapter – $\mathcal{A}_4$ algebra (§3.1.1)

$$[T^a, T^b, T^c] = i\epsilon^{abcd} T^d, \quad \langle T^a, T^b \rangle = \delta^{ab}. \tag{3.139}$$

BLG model with this algebra describes two M2-branes' system.

Chapter 4 Nambu-Poisson bracket (§3.3.1)

$$[f^a, f^b, f^c] = \epsilon^{\mu\nu\rho} \frac{\partial f^a}{\partial y^\mu} \frac{\partial f^b}{\partial y^\nu} \frac{\partial f^c}{\partial y^\rho}, \quad \langle f^a, f^b \rangle = \int_{\mathcal{N}} d^3y\, f^a f^b, \tag{3.140}$$

where $\mathcal{N}$ is a 3-dim manifold where the Nambu-Poisson bracket is equipped. BLG model with this algebra describes a single M5-brane's system.

Chapter 5 Truncation version of Nambu-Poisson bracket (§3.3.2)

$$[f^a, f^b, f^c] = \pi_N \left(P^{\mu\nu\rho}(y) \frac{\partial f^a}{\partial y^\mu} \frac{\partial f^b}{\partial y^\nu} \frac{\partial f^c}{\partial y^\rho} \right), \quad \langle f^a, f^b \rangle = \int_{\mathcal{N}} d^d y \, \pi_N \left(f^a f^b \right), \tag{3.141}$$

where π_N is the projector which neglect the order $> N$ terms. BLG model with this algebra is useful to see the $N^{\frac{3}{2}}$ law.

Chapter 6 Lorentzian Lie 3-algebra (§3.4.1)

$$\begin{aligned} &[u, T^i, T^j] = i f^{ij}{}_k T^k, \quad [T^i, T^j, T^k] = -i f^{ijk} v\,, \quad [v, *, *] = 0\,, \\ &\langle T^i, T^j \rangle = h^{ij}, \quad \langle u, v \rangle = 1\,, \quad \text{otherwise} = 0\,. \end{aligned} \tag{3.142}$$

BLG model with this algebra describes multiple D2-branes' system.

Chapter 7 General Lorentzian Lie 3-algebra (§3.4.2)

$$\begin{aligned} &[u_a, u_b, u_c] = K^i_{abc} e_i + L_{abcd} v^d, \quad [u_a, u_b, e^i] = J^{ij}_{ab} e_j - K^i_{abc} v^c, \\ &[u_a, e^i, e^j] = J^{ij}_{ab} v^b + f^{ijk}_a e_k\,, \quad [e^i, e^j, e^k] = -f^{ijk}_a v^a, \\ &\langle e^i, e^j \rangle = g^{ij}, \quad \langle u_a, v^b \rangle = \delta^b_a, \quad \text{otherwise} = 0\,, \end{aligned} \tag{3.143}$$

where the structure constants f^{ijk}_a, J^{ij}_{ab} and K^i_{abc} are constrained by the conditions (3.136) and (3.137). BLG model with this algebra describes multiple Dp-branes' system on $(p-2)$-dim torus T^{p-2} (for $p \geq 3$).

Part II : M5-brane and applications

Chapter 4

M5-brane as infinite number of M2-branes

As we saw in §3.3.1, Nambu-Poisson bracket is an example of infinite-dimensional positive-definite Lie 3-algebra. Then, in this chapter, we discuss the BLG model with the Lie 3-algebra defined by Nambu-Poisson bracket, and show that it describes a single M5-brane's system whose worldvolume spreads over the original M2-branes' worldvolume $\mathcal{M}$ and a 3-dim manifold $\mathcal{N}$ where Nambu-Poisson bracket is equipped.

It is known that the M5-brane's worldvolume theory contains a self-dual 2-form gauge field, in addition to the scalars corresponding to fluctuations of the M5-brane in the transverse directions, as well as their fermionic superpartners. Especially, we will show that one can nontrivially obtain the self-dual 2-form field in our setup.

4.1 Nambu-Poisson bracket as Lie 3-algebra

Now we examine the BLG model with infinite-dimensional Lie 3-algebras based on 3-dim manifolds $\mathcal{N}$ with Nambu-Poisson structures. We will show that the field theory on the M2-branes' worldvolume $\mathcal{M}$ can be rewritten as field theory on a 6-dim manifold $\mathcal{M} \times \mathcal{N}$ whose bosonic components consist of the self-dual gauge field on $\mathcal{M} \times \mathcal{N}$ and scalar fields which define the embedding. As this is the field content of an M5-brane [22, 62, 63], we interpret it as a model of M5-brane constructed out of infinitely many M2-branes.

We use the simplest example of Nambu-Poisson bracket.

For the construction of M5-brane, we introduce an "internal" 3-dim manifold $\mathcal{N}$ equipped with the Nambu-Poisson brackets. The Nambu-Poisson bracket is a multilinear map from $C(\mathcal{N})^{\otimes 3}$ to $C(\mathcal{N})$ defined as

$$\{f_1, f_2, f_3\} = \sum_{\dot\mu,\dot\nu,\dot\lambda} P_{\dot\mu\dot\nu\dot\lambda}(y)\partial_{\dot\mu} f_1 \partial_{\dot\nu} f_2 \partial_{\dot\lambda} f_3\,, \tag{4.1}$$

where $P_{\dot{\mu}\dot{\nu}\dot{\lambda}}$ is an anti-symmetric tensor. We use the coordinate $y^{\dot{\mu}}$ ($\dot{\mu} = 1, 2, 3$) to parametrize $\mathcal{N}$. As a realization of Lie 3-algebra, the Nambu-Poisson bracket needs to satisfy the fundamental identity

$$\begin{aligned}&\{g, h, \{f_1, f_2, f_3\}\}\\&\qquad = \{\{g, h, f_1\}, f_2, f_3\} + \{f_1, \{g, h, f_2\}, f_3\} + \{f_1, f_2, \{g, h, f_3\}\}\,, \quad (4.2)\end{aligned}$$

which gives severe constraints on $P_{\dot{\mu}\dot{\nu}\dot{\lambda}}(y)$ (see, for example, [57]).

The simplest possible Nambu-Poisson bracket is the Jacobian determinant for 3 variables $y^{\dot{\mu}}$

$$\{f_1, f_2, f_3\} = \sum_{\dot{\mu},\dot{\nu},\dot{\lambda}} \epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\mu}} f_1 \partial_{\dot{\nu}} f_2 \partial_{\dot{\lambda}} f_3. \tag{4.3}$$

This is the classical Nambu bracket. In general, it is known that a consistent Nambu bracket reduces to this Jacobian form locally by the suitable change of local coordinates. This property is referred to as the "decomposability" in the literature [52–56]. So we can use (4.3) in the following for simplicity and also without loss of generality. We also note that the dimension of the internal manifold $\mathcal{N}$ is essentially restricted to 3, because of the decomposability.

Nambu-Poisson bracket may be regarded as the definition of Lie 3-algebra in the infinite dimensional space $C(\mathcal{N})$. We write the basis of functions on $C(\mathcal{N})$ as $\chi^a(y)$ ($a = 1, 2, \cdots, \infty$). So we define the Lie 3-algebra structure constant by Nambu-Poisson bracket as

$$\left\{\chi^a, \chi^b, \chi^c\right\} = \sum_{\dot{\mu},\dot{\nu},\dot{\lambda}} \epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\mu}} \chi^a \partial_{\dot{\nu}} \chi^b \partial_{\dot{\lambda}} \chi^c =: \sum_d f^{abc}{}_d \chi^d(y). \tag{4.4}$$

Then eq. (4.2) implies that the structure constant $f^{abc}{}_d$ here satisfies the fundamental identity.

Inner product can be defined as integration.

As we mentioned in §3.3.1, we write the inner product as integration on the manifold $\mathcal{N}$

$$\langle f_1, f_2 \rangle = \int_{\mathcal{N}} d^p y\, \mu(y) f_1(y) f_2(y)\,. \tag{4.5}$$

The measure factor $\mu(y)$ is chosen such that the inner product is invariant under the Nambu-Poisson bracket as (2.16), namely

$$\langle \{*, *, f_1\}, f_2 \rangle + \langle f_1, \{*, *, f_2\} \rangle = 0\,. \tag{4.6}$$

Here we simply choose $\mu(y)$ as

$$\langle f_1, f_2 \rangle = \frac{1}{g^2} \int_{\mathcal{N}} d^3 y f_1(y) f_2(y)\,, \tag{4.7}$$

where g is a constant. Then we can the invariant metric as

$$h^{ab} = \langle \chi^a, \chi^b \rangle, \quad h_{ab} = (h^{-1})_{ab} \,. \tag{4.8}$$

Here we choose the dual set of basis $\chi_a(Y)$ in $C(\mathcal{N})$ such that $\langle \chi_a, \chi^b \rangle = \delta^a_b$. As we see, the indices of the structure constant can be changed by contraction of the metric. In particular, $\sum_e h^{ae} f^{bcd}{}_e = f^{bcda}$ defines a totally anti-symmetric 4-tensor. In order to have finite metric, we need to restrict $\mathcal{N}$ to a compact manifold. One may, of course, discuss noncompact manifolds by appropriate limits of the compact spaces.

Because we have already fixed the scale of $y^{\dot\mu}$ at (4.3), we cannot in general remove the coefficient g from the metric (4.8). As we will show later, however, if the internal space is $\mathcal{N} = \mathbf{R}^3$, it is possible to set this coupling at an arbitrary value by an appropriate rescaling of variables.

Except for the trivial case ($\mathcal{N} = \mathbf{R}^3$), we have to cover $\mathcal{N}$ by local patches and the coordinates $y^{\dot\mu}$ are the local coordinates on each patch. If we need to go to the different patch where the local coordinates are y', the coordinate transformation between y and y' (say $y'^{\dot\mu} = f^{\dot\mu}(y)$) should keep the Nambu-Poisson bracket (4.3). It implies that

$$\{f^{\dot 1}, f^{\dot 2}, f^{\dot 3}\} = 1 \,. \tag{4.9}$$

Namely $f^{\dot\mu}(y)$ should be the volume-preserving diffeomorphism. As we will see, the gauge symmetry of the BLG model for this choice of Lie 3-algebra is the volume-preserving diffeomorphism of $\mathcal{N}$ which is very natural in this set-up.

We note that we don't need the metric in $y^{\dot\mu}$ space. For the definition of the theory we only need to specify a volume form in $\mathcal{N}$. The gauge symmetry associated with the volume-preserving diffeomorphism is kept not by the metric but the various components of the self-dual 2-form field which comes out from $A_{\mu ab}$ and $X^{\dot\mu}$ (longitudinal components of X) as we will see.

Concrete examples of Nambu-Poisson bracket

It is of some interest to see the algebra itself explicitly, so we present a few examples where explicit computation is possible.

Case of T^3 and $\mathbf{R}^3$

The simplest example of infinite-dimensional Lie 3-algebra is given by T^3 with radius R. The basis of functions are parametrized by $\vec n \in \mathbf{Z}^3$ as (if we take $\mu = (2\pi R)^{-3}$)

$$\chi^{\vec n}(\vec y) = \exp(2\pi i \vec n \cdot \vec y/R) \,, \quad \chi_{\vec n}(\vec y) = \exp(-2\pi i \vec n \cdot \vec y/R) \,. \tag{4.10}$$

The metric and the structure constants are given by

$$\begin{aligned} h^{\vec n_1 \vec n_2} &= \delta(\vec n_1 + \vec n_2), \\ f^{\vec n_1 \vec n_2 \vec n_3}{}_{\vec n_4} &= (2\pi i/R)^3 \vec n_1 \cdot (\vec n_2 \times \vec n_3) \delta(\vec n_1 + \vec n_2 + \vec n_3 - \vec n_4) \,, \\ f^{\vec n_1 \vec n_2 \vec n_3, \vec n_4} &= (2\pi i/R)^3 \vec n_1 \cdot (\vec n_2 \times \vec n_3) \delta(\vec n_1 + \vec n_2 + \vec n_3 + \vec n_4) \,. \end{aligned} \tag{4.11}$$

If we take $R \to \infty$, we obtain the Lie 3-algebra associated with $\mathbf{R}^3$. The label for the basis becomes continuous and the metric becomes the delta function.

Case of S^3

We introduce four variables $y_1, \cdots, y_4$ and the Nambu-Poisson bracket defined by

$$P = -y_1 \partial_2 \wedge \partial_3 \wedge \partial_4 + y_2 \partial_1 \wedge \partial_3 \wedge \partial_4 - y_3 \partial_1 \wedge \partial_2 \wedge \partial_4 + y_4 \partial_1 \wedge \partial_2 \wedge \partial_3 \,. \tag{4.12}$$

If we restrict $C(\mathcal{N})$ to the linear functions of y_i, it agrees with $\mathcal{A}_4$. We impose a constraint $\phi(y) := y_1^2 + y_2^2 + y_3^2 + y_4^2 - 1 = 0$ in $\mathbf{R}^4$ which defines S^3. This restriction is compatible with the Nambu-Poisson bracket in a sense $\{\phi(y) f_1(y), f_2(y), f_3(y)\}\,|_{\phi(y)=0} = 0$ for any $f_i(y)$.

Square integrable functions on S^3 are given by combinations of $y_1^{n_1} y_3^{n_2} y_3^{n_3} y_4^{n_4}$. By the constraint $\phi(y) = 0$, whenever powers of y_4 higher than 2 appears, we can reduce it to zero and one. Therefore the basis of functions are given as

$$T_{\vec{n}} = y_1^{n_1} y_2^{n_2} y_3^{n_3}\,, \quad S_{\vec{n}} = y_1^{n_1} y_2^{n_2} y_3^{n_3} y_4\,, \quad (n_i \geq 0)\,. \tag{4.13}$$

The Lie 3-algebra becomes

$$\begin{aligned}
\{T_{\vec{n}}, T_{\vec{m}}, T_{\vec{\ell}}\} &= \vec{n}\cdot(\vec{m}\times\vec{\ell}) S_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}}\,, \\
\{T_{\vec{n}}, T_{\vec{m}}, S_{\vec{\ell}}\} &= \vec{n}\cdot(\vec{m}\times\vec{\ell}) \left(T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}} - \sum_{i=1}^{3} T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+2\vec{e}_i} \right) \\
&\quad - \sum_{i=1}^{3} \vec{e}_i (\vec{n}\times\vec{m}) T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+2\vec{e}_i}\,, \\
\{T_{\vec{n}}, S_{\vec{m}}, S_{\vec{\ell}}\} &= \vec{n}\cdot(\vec{m}\times\vec{\ell}) \left(S_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}} - \sum_{i=1}^{3} S_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+2\vec{e}_i} \right) \\
&\quad - \sum_{i=1}^{3} \vec{e}_i (\vec{n}\times(\vec{m}-\vec{\ell})) S_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+2\vec{e}_i}\,, \\
\{S_{\vec{n}}, S_{\vec{m}}, S_{\vec{\ell}}\} &= \vec{n}\cdot(\vec{m}\times\vec{\ell}) \left(T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}} - 2\sum_{i=1}^{3} T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+2\vec{e}_i} + \sum_{i,j} T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+2\vec{e}_i+2\vec{e}_j} \right) \\
&\quad - \sum_{i} \vec{e}_i (\vec{n}\times\vec{m} + \vec{m}\times\vec{\ell} + \vec{\ell}\times\vec{n}) (T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+2\vec{e}_i} - T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+4\vec{e}_i}) \\
&\quad + \sum_{i<j} (\vec{e}_i + \vec{e}_j)(\vec{n}\times\vec{m} + \vec{m}\times\vec{\ell} + \vec{\ell}\times\vec{n}) T_{\vec{n}+\vec{m}+\vec{\ell}-\vec{\rho}+2\vec{e}_i+2\vec{e}_j}\,,
\end{aligned} \tag{4.14}$$

where $\vec{\rho} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$.

4.2 Construction of fields on M5-brane

In this section, we redefine the fields of BLG model, in order to obtain an M5-brane's worldvolume theory. Let us first comment on the indices. We consistently use

- $I, J, K, \cdots = 1, \cdots, 8$: label the transverse directions of the M2-branes $\mathcal{M}$.

In the following, we decompose this 8-dim space as a direct product of a 3-manifold $\mathcal{N}$ and remaining 5-dim space. Then we will use

- $\dot{\mu}, \dot{\nu}, \dot{\lambda}, \cdots = \dot{1}, \dot{2}, \dot{3}$: label the directions of $\mathcal{N}$
- $i, j, k, \cdots = 1, \cdots, 5$: label the transverse directions of the M5-brane $\mathcal{M} \times \mathcal{N}$.

Definition of 6-dim fields

By combining the basis of $C(\mathcal{N})$, we can treat $X^I_a(x)$ and $\Psi_a(x)$ as 6-dim local fields

$$X^I(x,y) = \sum_a X^I_a(x)\chi^a(y)\,, \quad \Psi(x,y) = \sum_a \Psi_a(x)\chi^a(y)\,. \tag{4.15}$$

Similarly, the gauge field A^{ab}_λ can be regarded as a bi-local field:

$$A_\lambda(x,y,y') = \sum_{a,b} A^{ab}_\lambda(x)\chi^a(y)\chi^b(y')\,. \tag{4.16}$$

The existence of such a bi-local field doesn't mean the theory is non-local. Let us expand it with respect to $\Delta y^{\dot{\mu}} := y'^{\dot{\mu}} - y^{\dot{\mu}}$ as

$$A_\lambda(x,y,y') = a_\lambda(x,y) + b_{\lambda\dot{\mu}}(x,y)\Delta y^{\dot{\mu}} + \frac{1}{2}c_{\lambda\dot{\mu}\dot{\nu}}(x,y)\Delta y^{\dot{\mu}}\Delta y^{\dot{\nu}} + \cdots\,. \tag{4.17}$$

Because $A_{\lambda ab}$ always appears in the action in the form $f^{bcd}{}_a A_{\lambda bc}$, the field $A_\lambda(y,y')$ is highly redundant, and only the component

$$b_{\lambda\dot{\mu}}(x,y) = \left.\frac{\partial}{\partial y'^{\dot{\mu}}} A_\lambda(x,y,y')\right|_{y'=y} \tag{4.18}$$

contributes to the action. For example, the covariant derivative (2.58) of BLG model is rewritten for our case as,

$$\begin{aligned} D_\lambda X^I(x,y) &:= (\partial_\lambda X^{Ia}(x) - g f^{bcd}{}_a A_{\lambda bc} X^I_d(x))\chi^a(y) \\ &= \partial_\lambda X^I(x,y) - g\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}} \left.\frac{\partial^2 A_\lambda(x,y,y')}{\partial y^{\dot{\mu}}\partial y'^{\dot{\nu}}}\right|_{y=y'} \frac{\partial X^I(x,y)}{\partial y^{\dot{\rho}}} \\ &= \partial_\lambda X^I(x,y) - g\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}(\partial_{\dot{\mu}} b_{\lambda\dot{\nu}}(x,y))(\partial_{\dot{\rho}} X^I(x,y)) \\ &= \partial_\lambda X^I - g\{b_{\lambda\dot{\nu}}, y^{\dot{\nu}}, X^I\}\,. \end{aligned} \tag{4.19}$$

The covariant derivative for the fermion field is similarly,

$$\begin{aligned} D_\lambda \Psi(x,y) &= \partial_\lambda \Psi(x,y) - g\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}(\partial_{\dot{\mu}} b_{\lambda\dot{\nu}}(x,y))(\partial_{\dot{\rho}}\Psi(x,y)) \\ &= \partial_\lambda \Psi - g\{b_{\lambda\dot{\nu}}, y^{\dot{\nu}}, \Psi\}\,. \end{aligned} \tag{4.20}$$

Longitudinal fields

This theory written in terms of fields on 6-dim spacetime is identified with the theory describing a single M5-brane. At this point, only the x^μ part of the metric $g_{\mu\nu} = \eta_{\mu\nu}$ is defined, and we still have $SO(8)$ global symmetry, which is different from the $SO(5)$ symmetry expected in the M5-brane theory.

This is quite similar to the situation in which we consider the D-brane Born-Infeld action. The Born-Infeld Dp-brane action of 10-dim superstring theory possesses $SO(1,9)$ Lorentz symmetry regardless of the worldvolume dimension $p+1$. The rotational symmetry is reduced to $SO(9-p)$ for the transverse directions only after fixing the general coordinate transformation symmetry on the worldvolume with the static gauge condition [1]

$$X^\mu(\sigma) = \sigma^\mu. \tag{4.21}$$

This gauge fixing breaks the global symmetry from $SO(1,9)$ to $SO(9-p)$, and at the same time the worldvolume metric is induced from the target space metric through (4.21).

We can interpret the 6-dim theory we are considering here as a theory obtained from an $SO(1,10)$ symmetric covariant theory by taking a partial static gauge for three among six worldvolume coordinates. As we mentioned above, however, we don't have full diffeomorphism in the $y^{\dot\mu}$ space. The action is invariant only under volume-preserving diffeomorphism. This implies that we cannot completely fix the fields $X^{\dot\mu}$, and there are remaining physical degrees of freedom. For this reason, we should loosen the static gauge condition as

$$X^{\dot\mu}(x,y) = y^{\dot\mu} + b^{\dot\mu}(x,y), \quad b_{\dot\mu\dot\nu} = \epsilon_{\dot\mu\dot\nu\dot\rho} b^{\dot\rho}\,. \tag{4.22}$$

The tensor field $b_{\dot\mu\dot\nu}$ will be identified with a part of the 2-form gauge field on a M5-brane.

Comments on the coupling constant

In the case of ordinary Yang-Mills theories, there are two widely-used conventions for coupling constants and normalization of gauge fields. One way is to normalize a gauge field by the canonical kinetic term $-(1/4)F_{\mu\nu}^2$ and put the coupling constant in the covariant derivative $D = d - igA$. The other choice is to define the covariant derivative $D = d - iA$ without using the coupling constant and instead put $1/g^2$ in front of the kinetic term of the gauge field. Similarly, there are different conventions for coupling constant in the case of the BLG model, too. In the above, we put the coupling constant g in the definition of the metric (4.8). This corresponds to the second convention we mentioned above. We can move the coupling dependence from the overall factor to the interaction terms by rescaling the fields

$$X^I \to cX^I, \quad \Psi \to c\Psi, \quad b_{\mu\dot\mu} \to cb_{\mu\dot\mu}, \tag{4.23}$$

[1] Turning on a background field such as the B-field will of course also break the global symmetry. For the discussion here we are treating the background fields as covariant dynamical fields.

with $c = g$. In general, as ordinary Yang-Mills theories, we cannot remove the coupling constant completely from the action.

If the internal space $\mathcal{N}$ is $\mathbf{R}^3$, however, we have an extra degree of freedom for rescaling, and it is in fact possible to the coupling constant from the action. Let us consider the following rescaling of variables.

$$X^I \to c'^3 X^I, \quad \Psi \to c'^3 \Psi, \quad b_{\mu\dot{\mu}} \to c'^4 b_{\mu\dot{\mu}}, \quad y^{\dot{\mu}} \to c'^2 y^{\dot{\mu}} \,. \tag{4.24}$$

This variable change is associated with an outer automorphism of the algebra, and doesn't change the relative coefficients in the action. The only change in the action is the overall factor. We can thus absorb the coupling constant by (4.24), and this implies that the 6-dim theory doesn't have any coupling constant.

We can adopt an elegant convention in which no coupling constant appears. However, we adopt a different convention below. Because we interpret the 6-dim theory as a theory of an M5-brane, we would like to regard the scalar field X^I as the coordinates of the target space with mass dimension -1. We also give the meaning to the variables $y^{\dot{\mu}}$ as the world-volume coordinates, which also have mass dimension -1. We choose the parametrization in the $y^{\dot{\mu}}$ space so that the linear part of the 6-dim action is invariant under Lorentz transformations in the $(x^\mu, y^{\dot{\mu}})$ space. After fixing the scale of X^I and $y^{\dot{\mu}}$ in this way, we can no longer use the two rescalings (4.23) and (4.24) to change the coupling constant and overall coefficient of the action. These two parameters have physical meaning now.

In the following, in order to express the coupling constant dependence of each term in the action clearly, we separate the coupling constant g from the structure constant. We also introduce an overall coefficient T_6, which is regarded as an effective tension of the M5-brane. This plays an important role in the parameter matching in §4.6, but we will omit this factor in §4.3 – 4.5 for simplicity because it is irrelevant to the analysis in these sections.

Comments on the degrees of freedom

In this section we will show that the BLG model with a Nambu-Poisson structure on a 3-dim manifold contains the low-energy degrees of freedom on an M5-brane. Before going on, let us count the number of degrees of freedom in the bosonic and fermionic sectors in our model.

The fermion Ψ is a Majorana spinor in $(10+1)$-dimensions with a chirality condition, and thus it has 16 real fermionic components, equivalent to 8 bosonic degrees of freedom. For a M5-brane, there are 5 transverse directions corresponding to 5 scalars X^i. For an ordinary 2-form gauge field in 6-dim spacetime, there are 6 propagating modes. But since we don't have the usual kinetic term for A_μ, but rather a Chern-Simons term, there are only 3 propagating modes. The low-energy effective theory of an M5-brane contains the same number of bosonic and fermionic degrees of freedom. But a salient feature of the M5-brane is that the 2-form gauge field is self-dual. Hence our major challenge is to show that the gauge field of the BLG model is equivalent to a self-dual 2-form gauge field.

4.3 Gauge symmetry of M5-brane from Lie 3-algebra

Gauge symmetry transformation

The gauge transformations of the scalar fields X^I and fermion fields Ψ are given by

$$\begin{aligned}\delta_\Lambda X^I(x,y) &= g\Lambda_{ab}(x) f^{abc}{}_d X^I_c(x)\chi^d(y) = g\Lambda_{ab}(x)\{\chi^a,\chi^b,X^I\}\\ &= g(\delta_\Lambda y^{\dot\rho})\partial_{\dot\rho}X^I(x,y),\\ \delta_\Lambda\Psi(x,y) &= g\Lambda_{ab}(x)\{\chi^a,\chi^b,\Psi\} = g(\delta_\Lambda y^{\dot\rho})\partial_{\dot\rho}\Psi(x,y),\end{aligned} \tag{4.25}$$

where we used

$$f^{abc}{}_d = \langle\{\chi^a,\chi^b,\chi^c\},\chi_d\rangle\,,\quad \sum_a \chi^a(y)\chi_a(y') = \delta^{(3)}(y-y')\,, \tag{4.26}$$

and $\delta_\Lambda y^{\dot\mu}$ is defined as

$$\begin{aligned}&\delta_\Lambda y^{\dot\lambda} = \epsilon^{\dot\lambda\dot\mu\dot\nu}\partial_{\dot\mu}\Lambda_{\dot\nu}(x,y)\,,\\ &\Lambda_{\dot\mu}(x,y) = \partial'_{\dot\mu}\tilde\Lambda(x,y,y')|_{y'=y}\,,\quad \tilde\Lambda(x,y,y') := \Lambda_{ab}(x)\chi^a(y)\chi^b(y')\,.\end{aligned} \tag{4.27}$$

We note that although the parameter of a gauge transformation may be expressed as a bi-local function $\tilde\Lambda(x,y,y')$, the gauge transformation induced by it depends only on its component $\Lambda_{\dot\mu}(x,y)$ which is local in $\mathcal{N}$. It comes from the fact that the gauge transformation by Λ_{ab} is always defined through the combination $f^{abc}{}_d\Lambda_{ab}$.

The same argument can be applied to the gauge field $A_\mu(x,y,y')$. As we already mentioned, since it appears only through the combination $A_{\mu ab}f^{abc}{}_d$, the local field $b_{\mu\dot\lambda}(x,y)$ defined as (4.18) shows up in the action.

The transformation (4.25) may be regarded as the infinitesimal reparametrization

$$y'^{\dot\lambda} = y^{\dot\lambda} - g\delta y^{\dot\lambda}\,. \tag{4.28}$$

Since $\partial_{\dot\mu}\delta y^{\dot\mu} = 0$, it represents the volume-preserving diffeomorphism. Since the symmetry is local on $\mathcal{M}$, the gauge parameter is an arbitrary function of x. So what we have obtained is a gauge theory on $\mathcal{M}$ whose gauge group is the volume-preserving diffeomorphism of $\mathcal{N}$. In this sense, the worldvolume of M5-brane may be regarded as the vector bundle $\mathcal{N}\to\mathcal{M}$ but the gauge transformation on each fiber is not merely the linear transformation but the diffeomorphism on the fiber which preserves the volume form

$$\omega = dy^{\dot 1}\wedge dy^{\dot 2}\wedge dy^{\dot 3}\,. \tag{4.29}$$

As we mentioned in the previous section, among 8 scalar fields X^I, the last 5 components X^i are treated as scalar fields representing the transverse fluctuations of the M5-brane. The other 3 $X^{\dot\mu}$ (longitudinal field) are rewritten as as

$$X^{\dot\mu}(y) = \frac{y^{\dot\mu}}{g} + \frac{1}{2}\epsilon^{\dot\mu\dot\kappa\dot\lambda}b_{\dot\kappa\dot\lambda}(y)\,. \tag{4.30}$$

We chose the coefficients so that we obtain Lorentz invariant kinetic terms in the 6-dim action. The gauge transformation of $b_{\dot{\mu}\dot{\nu}}$ can be derived from (4.25) and (4.30) as

$$\delta_\Lambda b_{\dot{\kappa}\dot{\lambda}}(y) = \partial_{\dot{\kappa}}\Lambda_{\dot{\lambda}} - \partial_{\dot{\lambda}}\Lambda_{\dot{\kappa}} + g(\delta_\Lambda y^{\dot{\rho}})\partial_{\dot{\rho}} b_{\dot{\kappa}\dot{\lambda}}(y)\,. \tag{4.31}$$

The gauge transformation of the gauge field $A_\lambda(x,y,y')$ is given by $\delta_\Lambda A_\lambda(x,y,y') = D_\lambda\tilde{\Lambda}(x,y,y')$. The covariant derivative of a bi-local field is defined by tensoring the covariant derivative (4.19) for a local field, and we obtain

$$D_\lambda\Lambda(y,y') = \partial_\lambda\Lambda(y,y') - g\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}[\partial_{\dot{\mu}} b_{\lambda\dot{\nu}}(y)\partial_{\dot{\rho}}\Lambda(y,y') + \partial'_{\dot{\mu}} b_{\lambda\dot{\nu}}(y')\partial'_{\dot{\rho}}\Lambda(y,y')]\,. \tag{4.32}$$

From this we can extract the transformation law of the component field $b_{\lambda\dot{\sigma}}$

$$\delta_\Lambda b_{\lambda\dot{\sigma}} = \partial'_{\dot{\mu}}\delta_\Lambda A_\lambda(y,y')|_{y'=y} = \partial_\lambda\Lambda_{\dot{\sigma}} - g\partial_{\dot{\sigma}}\xi_\Lambda - g\delta_{\rm gc} b_{\lambda\dot{\sigma}}\,, \tag{4.33}$$

where $\delta_{\rm gc} b_{\lambda\dot{\sigma}}$ is the coordinate transformation in y-space

$$\delta_{\rm gc} b_{\lambda\dot{\sigma}} = -\delta_\Lambda y^{\dot{\tau}}\partial_{\dot{\tau}} b_{\lambda\dot{\sigma}} - (\partial_{\dot{\sigma}}\delta_\Lambda y^{\dot{\tau}}) b_{\lambda\dot{\tau}}\,, \tag{4.34}$$

and ξ_Λ is defined by

$$\xi_\Lambda = \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}(\partial_{\dot{\mu}} b_{\lambda\dot{\nu}}\Lambda_{\dot{\rho}} + b_{\lambda\dot{\mu}}\partial_{\dot{\nu}}\Lambda_{\dot{\rho}})\,. \tag{4.35}$$

In addition to these gauge transformations derived from (2.58) and (2.55), there is an additional gauge transformation which acts only on the field $b_{\lambda\dot{\mu}}$. As we can see in (4.19), $b_{\lambda\dot{\mu}}$ appears in the covariant derivative in the form of the rotation in the $y^{\dot{\mu}}$ space. This means that $D_\mu\Phi$ is invariant under

$$\delta b_{\lambda\dot{\mu}} = -\partial_{\dot{\mu}}\Lambda_\lambda. \tag{4.36}$$

We can easily check that the Chern-Simons term is also invariant under this transformation, and thus (4.36) is also a gauge symmetry of the theory.

Now we summarize the gauge transformation of the 6-dim theory.

$$\begin{aligned}
\delta_\Lambda X^i &= g(\delta_\Lambda y^{\dot{\rho}})\partial_{\dot{\rho}} X^i, \\
\delta_\Lambda \Psi &= g(\delta_\Lambda y^{\dot{\rho}})\partial_{\dot{\rho}}\Psi, \\
\delta_\Lambda b_{\dot{\kappa}\dot{\lambda}} &= \partial_{\dot{\kappa}}\Lambda_{\dot{\lambda}} - \partial_{\dot{\lambda}}\Lambda_{\dot{\kappa}} + g(\delta_\Lambda y^{\dot{\rho}})\partial_{\dot{\rho}} b_{\dot{\kappa}\dot{\lambda}}\,, \\
\delta_\Lambda b_{\lambda\dot{\sigma}} &= \partial_\lambda\Lambda_{\dot{\sigma}} - \partial_{\dot{\sigma}}\Lambda_\lambda - g\delta_{\rm gc} b_{\lambda\dot{\sigma}}\,.
\end{aligned} \tag{4.37}$$

We absorbed ξ_Λ in (4.33) into the definition of the parameter Λ_λ. In the weak coupling limit $g \to 0$, we obtain the standard gauge transformation on an M5-brane.

Covariant derivatives in 6-dim theory

An intriguing feature of our 6-dim model is that one may define the covariant derivative in the *fiber* direction.

By using the fundamental identity, it is easy to show that if Φ_1, Φ_2, and Φ_3 are covariant fields (such as X^I or Ψ), not only $D_\mu \Phi_1$ but $\{\Phi_1, \Phi_2, \Phi_3\}$ are also covariant because of the fundamental identity,

$$\delta_\Lambda\{\Phi_1, \Phi_2, \Phi_3\} = \{\delta_\Lambda\Phi_1, \Phi_2, \Phi_3\} + \{\Phi_1, \delta_\Lambda\Phi_2, \Phi_3\} + \{\Phi_1, \Phi_2, \delta_\Lambda\Phi_3\}\,. \tag{4.38}$$

It implies that the following combination defines the "covariant" derivative along the fiber direction,

$$\begin{aligned} \mathcal{D}_{\dot\mu}\Phi &:= \frac{g^2}{2}\epsilon_{\dot\mu\dot\nu\dot\rho}\{X^{\dot\nu}, X^{\dot\rho}, \Phi\} \\ &= \partial_{\dot\mu}\Phi + g(\partial_{\dot\lambda}b^{\dot\lambda}\partial_{\dot\mu}\Phi - \partial_{\dot\mu}b^{\dot\lambda}\partial_{\dot\lambda}\Phi) + \frac{g^2}{2}\epsilon_{\dot\mu\dot\nu\dot\rho}\{b^{\dot\nu}, b^{\dot\rho}, \Phi\}\,. \end{aligned} \tag{4.39}$$

Together with (4.19), which we repeat here again,

$$\mathcal{D}_\mu\Phi \;:=\; D_\mu\Phi = \partial_\mu\Phi - g\{b_{\mu\dot\nu}, y^{\dot\nu}, \Phi\}\,, \tag{4.40}$$

we have a set of covariant derivatives on M5-brane's worldvolume.

These covariant derivatives possess the following important properties.

- Leibniz rule:

$$\mathcal{D}_{\underline{\mu}}\{\Phi_1, \Phi_2, \Phi_3\} = \{\mathcal{D}_{\underline{\mu}}\Phi_1, \Phi_2, \Phi_3\} + \{\Phi_1, \mathcal{D}_{\underline{\mu}}\Phi_2, \Phi_3\} + \{\Phi_1, \Phi_2, \mathcal{D}_{\underline{\mu}}\Phi_3\}\,. \tag{4.41}$$

- Integration by parts:

$$\int d^3x d^3y\, \Phi_1 \mathcal{D}_{\underline{\mu}}\Phi_2 = -\int d^3x d^3y\, (\mathcal{D}_{\underline{\mu}}\Phi_1)\Phi_2\,. \tag{4.42}$$

Here $\mathcal{D}_{\underline{\mu}}$ ($\underline{\mu} = 0, 1, \cdots, 5$) represents both $\mathcal{D}_\mu$ and $\mathcal{D}_{\dot\mu}$.

Field strength

As special cases of these covariant derivatives, we define the following field strengths of the tensor field:

$$\begin{aligned} \mathcal{H}_{\lambda\dot\mu\dot\nu} &= \epsilon_{\dot\mu\dot\nu\dot\lambda}\mathcal{D}_\lambda X^{\dot\lambda} \\ &= H_{\lambda\dot\mu\dot\nu} - g\epsilon^{\dot\sigma\dot\tau\dot\rho}(\partial_{\dot\sigma}b_{\lambda\dot\tau})\partial_{\dot\rho}b_{\dot\mu\dot\nu}, \end{aligned} \tag{4.43}$$

$$\begin{aligned} \mathcal{H}_{\dot1\dot2\dot3} &= g^2\{X^{\dot1}, X^{\dot2}, X^{\dot3}\} - \frac{1}{g} = \frac{1}{g}(V-1) \\ &= H_{\dot1\dot2\dot3} + \frac{g}{2}(\partial_{\dot\mu}b^{\dot\mu}\partial_{\dot\nu}b^{\dot\nu} - \partial_{\dot\mu}b^{\dot\nu}\partial_{\dot\nu}b^{\dot\mu}) + g^2\{b^{\dot1}, b^{\dot2}, b^{\dot3}\}\,, \end{aligned} \tag{4.44}$$

where V is the "induced volume"

$$V = g^3\{X^{\dot{1}}, X^{\dot{2}}, X^{\dot{3}}\}\,, \tag{4.45}$$

and H is the linear part of the field strength

$$\begin{aligned} H_{\lambda\dot{\mu}\dot{\nu}} &= \partial_\lambda b_{\dot{\mu}\dot{\nu}} - \partial_{\dot{\mu}} b_{\lambda\dot{\nu}} + \partial_{\dot{\nu}} b_{\lambda\dot{\mu}}\,, \\ H_{\dot{\lambda}\dot{\mu}\dot{\nu}} &= \partial_{\dot{\lambda}} b_{\dot{\mu}\dot{\nu}} + \partial_{\dot{\mu}} b_{\dot{\nu}\dot{\lambda}} + \partial_{\dot{\nu}} b_{\dot{\lambda}\dot{\mu}}\,. \end{aligned} \tag{4.46}$$

$\mathcal{H}$ are covariantly transformed under the gauge transformation.

Just like the case of ordinary gauge theories, the field strength $\mathcal{H}$ arises in the commutator of the covariant derivatives defined above:

$$[\mathcal{D}_{\dot{\mu}}, \mathcal{D}_{\dot{\nu}}]\Phi = g^2 \epsilon_{\dot{\nu}\dot{\mu}\dot{\sigma}}\{\mathcal{H}_{\dot{1}\dot{2}\dot{3}}, X^{\dot{\sigma}}, \Phi\}\,, \tag{4.47}$$

$$[\mathcal{D}_{\lambda}, \mathcal{D}_{\dot{\lambda}}]\Phi = g^2 \{\mathcal{H}_{\lambda\dot{\nu}\dot{\lambda}}, X^{\dot{\nu}}, \Phi\}\,, \tag{4.48}$$

$$[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]\Phi = -\frac{g}{V}\epsilon_{\mu\nu\lambda}\mathcal{D}_\rho \widetilde{\mathcal{H}}^{\rho\lambda\dot{\kappa}}\mathcal{D}_{\dot{\kappa}}\Phi\,, \tag{4.49}$$

where the dual field strength $\widetilde{\mathcal{H}}$ is defined by

$$\widetilde{\mathcal{H}}^{\lambda\rho\dot{\kappa}} = \frac{1}{2}\epsilon^{\lambda\rho\dot{\kappa}\sigma\dot{\mu}\dot{\nu}}\mathcal{H}_{\sigma\dot{\mu}\dot{\nu}}\,, \quad \widetilde{\mathcal{H}}^{\mu\nu\rho} = \frac{1}{6}\epsilon^{\mu\nu\rho\dot{\mu}\dot{\nu}\dot{\rho}}\mathcal{H}_{\dot{\mu}\dot{\nu}\dot{\rho}}\,. \tag{4.50}$$

By using the explicit form of the covariant derivative in (4.40), we obtain

$$[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]\Phi = -gF^{\dot{\kappa}}_{\mu\nu}\partial_{\dot{\kappa}}\Phi\,, \tag{4.51}$$

where the explicit form of $F^{\dot{\kappa}}_{\mu\nu}$ in terms of the potential is

$$F^{\dot{\kappa}}_{\mu\nu} = \epsilon^{\dot{\kappa}\dot{\mu}\dot{\nu}}\partial_\mu\partial_{\dot{\mu}} b_{\nu\dot{\nu}} - g\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}\partial_{\dot{\mu}} b_{\mu\dot{\nu}}\epsilon^{\dot{\kappa}\dot{\lambda}\dot{\tau}}\partial_{\dot{\rho}}\partial_{\dot{\lambda}} b_{\nu\dot{\tau}} - (\mu \leftrightarrow \nu)\,. \tag{4.52}$$

Because the (non-covariant) derivative appears on the right hand side, $F^{\dot{\kappa}}_{\mu\nu}$ defined by (4.51) is not covariant. We can define the covariantized F by

$$[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]\Phi = -g\mathcal{F}^{\dot{\kappa}}_{\mu\nu}\mathcal{D}_{\dot{\kappa}}\Phi. \tag{4.53}$$

These two fields are related by $F^{\dot{\kappa}}_{\mu\nu}\partial_{\dot{\kappa}}\Phi = \mathcal{F}^{\dot{\kappa}}_{\mu\nu}\mathcal{D}_{\dot{\kappa}}\Phi$, and by substituting $\Phi = X^{\dot{\mu}}$ into this relation and using

$$g\mathcal{D}_{\dot{\mu}}X^{\dot{\sigma}} = V\delta^{\dot{\sigma}}_{\dot{\mu}}\,, \tag{4.54}$$

we obtain

$$V\mathcal{F}^{\dot{\kappa}}_{\mu\nu} = gF^{\dot{\lambda}}_{\mu\nu}\partial_{\dot{\lambda}}X^{\dot{\kappa}}\,. \tag{4.55}$$

$\mathcal{F}$ can be expressed as the covariant derivative of the field strength $\mathcal{H}$.

$$V\mathcal{F}^{\dot{\kappa}}_{\mu\nu} = g\mathcal{F}^{\dot{\lambda}}_{\mu\nu}\mathcal{D}_{\dot{\lambda}}X^{\dot{\kappa}} = \epsilon_{\mu\nu\lambda}\epsilon^{\lambda\rho\sigma}\mathcal{D}_\rho\mathcal{D}_\sigma X^{\dot{\kappa}} = \epsilon_{\mu\nu\lambda}\mathcal{D}_\rho\widetilde{\mathcal{H}}^{\rho\lambda\dot{\kappa}}\,. \tag{4.56}$$

In the first step, we used the relation (4.54). Substituting this into (4.53), we obtain (4.49).

4.4 M5-brane's action and equation of motion

M5-brane's worldvolume action as a final result

We rewrite the various parts of the BLG action in terms of the 6-dim fields and their covariant derivatives, then obtain

$$\begin{aligned} S_X + S_{\text{pot}} &= \int d^3x \Big\langle -\frac{1}{2}(\mathcal{D}_\mu X^i)^2 - \frac{1}{2}(\mathcal{D}_{\dot\lambda} X^i)^2 - \frac{1}{4}\mathcal{H}^2_{\lambda\dot\mu\dot\nu} - \frac{1}{12}\mathcal{H}^2_{\dot\mu\dot\nu\dot\rho} \\ &\quad -\frac{1}{2g^2} - \frac{g^4}{4}\{X^{\dot\mu}, X^i, X^j\}^2 - \frac{g^4}{12}\{X^i, X^j, X^k\}^2 \Big\rangle, \end{aligned} \tag{4.57}$$

$$\begin{aligned} S_\Psi + S_{\text{int}} &= \int d^3x \Big\langle \frac{i}{2}\bar\Psi\Gamma^\mu \mathcal{D}_\mu \Psi + \frac{i}{2}\bar\Psi\Gamma^{\dot\rho}\Gamma_{\dot1\dot2\dot3}\mathcal{D}_{\dot\rho}\Psi \\ &\quad + \frac{ig^2}{2}\bar\Psi\Gamma_{\dot\mu i}\{X^{\dot\mu}, X^i, \Psi\} + \frac{ig^2}{4}\bar\Psi\Gamma_{ij}\{X^i, X^j, \Psi\} \Big\rangle. \end{aligned} \tag{4.58}$$

The scalar kinetic term is manifestly Lorentz symmetric up to the different structure inside the covariant derivatives $\mathcal{D}_\mu$ and $\mathcal{D}_{\dot\mu}$. The Chern-Simons term cannot be rewritten in manifestly gauge-covariant form.

$$\begin{aligned} S_{\text{CS}} &= \int d^3x \epsilon^{\mu\nu\lambda} \Big\langle -\frac{1}{2}\epsilon^{\dot\mu\dot\nu\dot\lambda}\partial_{\dot\mu} b_{\mu\dot\nu}\partial_\nu b_{\lambda\dot\lambda} \\ &\qquad + \frac{g}{6}\epsilon^{\dot\mu\dot\nu\dot\lambda}\partial_{\dot\mu} b_{\nu\dot\nu}\epsilon^{\dot\rho\dot\sigma\dot\tau}\partial_{\dot\sigma} b_{\lambda\dot\rho}(\partial_{\dot\lambda} b_{\mu\dot\tau} - \partial_{\dot\tau} b_{\mu\dot\lambda}) \Big\rangle \\ &= \int d^3x \int_y \epsilon^{\mu\nu\lambda} \left(-\frac{1}{2} db_\mu \wedge \partial_\nu b_\lambda - \frac{g}{6}(*db_\mu) \wedge (*db_\nu) \wedge (*db_\lambda) \right). \end{aligned} \tag{4.59}$$

In the second expression we treat $b_{\mu\dot\mu}$ as a one-form field $b_\mu = b_{\mu\dot\mu} dy^{\dot\mu}$ in the y-space. However, the equation of motion which is derived from these actions turns out to be manifestly gauge-covariant.

Comments on fermion action

In the fermion kinetic terms in (4.58), only the $SO(1,2) \times SO(3)$ subgroup of the Lorentz symmetry is manifest due to the existence of $\Gamma_{\dot1\dot2\dot3}$ in one of two terms. We can remove this unwanted factor from the kinetic term by the unitary transformation

$$\bar\Psi = \bar\Psi' U, \quad \Psi = U\Psi', \tag{4.60}$$

where U is the matrix

$$U = \exp\left(-\frac{\pi}{4}\Gamma_{\dot1\dot2\dot3}\right) = \frac{1}{\sqrt{2}}(1 - \Gamma_{\dot1\dot2\dot3}). \tag{4.61}$$

The supersymmetry parameter ϵ is also transformed in the same way. Note that both Ψ and $\bar\Psi$ are transformed by U. This is consistent with the Dirac conjugation. As the result of the unitary transformation, the fermion terms in the action become

$$
\begin{aligned}
S_\Psi + S_{\rm int} &= \int d^3x \left\langle \frac{i}{2}\bar{\Psi}'\Gamma^\mu \mathcal{D}_\mu \Psi' + \frac{i}{2}\bar{\Psi}'\Gamma^{\dot\rho}\mathcal{D}_{\dot\rho}\Psi' \right. \\
&\left. + \frac{ig^2}{2}\bar{\Psi}'\Gamma_{\dot\mu i}\{X^{\dot\mu}, X^i, \Psi'\} - \frac{ig^2}{4}\bar{\Psi}'\Gamma_{ij}\Gamma_{\dot1\dot2\dot3}\{X^i, X^j, \Psi'\} \right\rangle . \quad (4.62)
\end{aligned}
$$

After the unitary transformation, the condition $\Gamma^{012}\epsilon = \epsilon^{012}\epsilon$, $\Gamma^{012}\Psi = -\epsilon^{012}\Psi$ becomes the chirality condition in 6-dim spacetime,

$$
\Gamma^7\epsilon' = \epsilon', \quad \Gamma^7\Psi' = -\Psi', \quad (4.63)
$$

where the chirality matrix Γ^7 is defined by

$$
\Gamma^{\mu\nu\rho}\Gamma^{\dot1\dot2\dot3} = \epsilon^{\mu\nu\rho}\Gamma^7. \quad (4.64)
$$

This means that the supersymmetry realized in this theory is the chiral $\mathcal{N} = (2,0)$ supersymmetry, which is the same as the supersymmetry on an M5-brane.

Equations of motion

It is easy to obtain the equations of motion for the scalar fields and fermion fields. We also derive the gauge-covariant equations of motion for the gauge fields, as we mentioned.

Scalar fields

$$
\begin{aligned}
0 &= \mathcal{D}_\mu^2 X^i + \mathcal{D}_{\dot\mu}^2 X^i \\
&+ g^4\{X^{\dot\mu}, X^j, \{X^{\dot\mu}, X^j, X^i\}\} + \frac{g^4}{2}\{X^j, X^k, \{X^j, X^k, X^i\}\} \\
&+ \frac{ig^2}{2}\{\bar{\Psi}'\Gamma_{\dot\mu i}, X^{\dot\mu}, \Psi'\} + \frac{ig^2}{2}\{\bar{\Psi}'\Gamma_{ij}\Gamma_{\dot1\dot2\dot3}, X^j, \Psi'\} . \quad (4.65)
\end{aligned}
$$

Fermion fields

$$
0 = \Gamma^\mu\mathcal{D}_\mu\Psi' + \Gamma^{\dot\rho}\mathcal{D}_{\dot\rho}\Psi' + g^2\Gamma_{\dot\mu i}\{X^{\dot\mu}, X^i, \Psi'\} - \frac{g^2}{2}\Gamma_{ij}\Gamma_{\dot1\dot2\dot3}\{X^i, X^j, \Psi'\} . \quad (4.66)
$$

Gauge field $b_{\dot\mu\dot\nu}$

For a variation of $b_{\dot\mu\dot\nu}$, we have the following variations of the action.

$$
\begin{aligned}
\delta S_X &= \int d^3x \langle \delta b^{\dot\mu}\mathcal{D}_\mu\mathcal{D}_\mu X^{\dot\mu}\rangle = \frac{1}{2}\int d^3x\langle \delta b^{\dot\mu}\epsilon^{\dot\mu\dot\rho\dot\sigma}\mathcal{D}_\mu\mathcal{H}_{\mu\dot\rho\dot\sigma}\rangle, \\
\delta S_{\rm pot} &= \frac{g^4}{2}\int d^3x\langle \delta b^{\dot\mu}\{X^I, X^J, \{X^I, X^J, X^{\dot\mu}\}\}\rangle, \\
\delta S_{\rm int} &= \frac{ig^2}{2}\int d^3x\langle \bar{\Psi}, \Gamma_{\dot\mu J}\{\delta b^{\dot\mu}, X^J, \Psi\}\rangle = -\frac{ig^2}{2}\int d^3x\langle \delta b^{\dot\mu}\{\bar{\Psi}\Gamma_{\dot\mu J}, X^J, \Psi\}\rangle, (4.67)
\end{aligned}
$$

and the equation of motion is

$$\begin{aligned}
0 &= \frac{1}{2}\epsilon^{\dot{\mu}\dot{\rho}\dot{\sigma}}\mathcal{D}_\mu\mathcal{H}_{\mu\dot{\rho}\dot{\sigma}} + \frac{g^4}{2}\{X^I, X^J, \{X^I, X^J, X^{\dot{\mu}}\}\} - \frac{ig^2}{2}\langle\delta b^{\dot{\mu}}\{\bar{\Psi}\Gamma_{\dot{\mu}J}, X^J, \Psi\}\rangle \\
&= \frac{1}{2}\epsilon^{\dot{\mu}\dot{\rho}\dot{\sigma}}\mathcal{D}_\mu\mathcal{H}_{\mu\dot{\rho}\dot{\sigma}} + \mathcal{D}_{\dot{\mu}}\mathcal{H}_{\dot{1}\dot{2}\dot{3}} + g^2\epsilon^{\dot{\rho}\dot{\mu}\dot{\tau}}\{X^i, X^{\dot{\rho}}, \mathcal{D}_{\dot{\tau}}X^i\} \\
&\quad + \frac{g^4}{2}\{X^i, X^j, \{X^i, X^j, X^{\dot{\mu}}\}\} - \frac{ig^2}{2}\{\bar{\Psi}\Gamma_{\dot{\mu}J}, X^J, \Psi\}\,,
\end{aligned} \tag{4.68}$$

or, equivalently,

$$\mathcal{D}_\mu\mathcal{H}^{\mu\dot{\rho}\dot{\sigma}} + \mathcal{D}_{\dot{\mu}}\mathcal{H}^{\dot{\mu}\dot{\rho}\dot{\sigma}} = gJ^{\dot{\rho}\dot{\sigma}}, \tag{4.69}$$

where the current is given by

$$\begin{aligned}
J^{\dot{\rho}\dot{\sigma}} &= g(\{X^i, \mathcal{D}_{\dot{\sigma}}X^i, X^{\dot{\rho}}\} - (\dot{\rho}\leftrightarrow\dot{\sigma})) - \frac{g^3}{2}\epsilon^{\dot{\rho}\dot{\sigma}\dot{\mu}}\{X^i, X^j, \{X^i, X^j, X^{\dot{\mu}}\}\} \\
&\quad + \frac{ig}{2}(\{\bar{\Psi}'\Gamma^{\dot{\sigma}}, X^{\dot{\rho}}, \Psi'\} - (\dot{\rho}\leftrightarrow\dot{\sigma})) + \frac{ig}{2}\epsilon^{\dot{\rho}\dot{\sigma}\dot{\mu}}\{\bar{\Psi}'\Gamma_{\dot{\mu}i}, X^i, \Psi'\}\,.
\end{aligned} \tag{4.70}$$

Gauge field $b_{\lambda\dot{\mu}}$

For the variation of the gauge field $b_{\lambda\dot{\mu}}$, we obtain

$$\begin{aligned}
\delta S_X &= -g\int d^3x\langle\delta b_{\lambda\dot{\mu}}\{X^I, \mathcal{D}_\lambda X^I, y^{\dot{\mu}}\}\rangle, \\
\delta S_\Psi &= -\frac{ig}{2}\int d^3x\langle\bar{\Psi}\Gamma^\lambda\{\delta b_{\lambda\dot{\mu}}, y^{\dot{\mu}}, \Psi\}\rangle = -\frac{ig}{2}\int d^3x\langle\delta b_{\lambda\dot{\mu}}\{\bar{\Psi}\Gamma^\lambda, \Psi, y^{\dot{\mu}}\}\rangle, \\
\delta S_{\mathrm{CS}} &= -\frac{1}{2}\int d^3x\langle\epsilon^{\lambda\mu\nu}\delta b_{\lambda\dot{\mu}}F^{\dot{\mu}}_{\mu\nu}\rangle.
\end{aligned} \tag{4.71}$$

The equation of motion for $b_{\lambda\dot{\mu}}$ is

$$\frac{1}{2}\epsilon^{\lambda\mu\nu}F^{\dot{\mu}}_{\mu\nu} + g\{X^I, \mathcal{D}_\lambda X^I, y^{\dot{\mu}}\} + \frac{ig}{2}\{\bar{\Psi}\Gamma^\lambda, \Psi, y^{\dot{\mu}}\} = 0\,. \tag{4.72}$$

This is not covariant, but we can covariantize this by multiplying $g\partial_{\dot{\mu}}X^{\dot{\nu}}$.

$$\frac{V}{2}\epsilon^{\lambda\mu\nu}\mathcal{F}^{\dot{\mu}}_{\mu\nu} + g^2\{X^I, \mathcal{D}_\lambda X^I, X^{\dot{\mu}}\} + \frac{ig^2}{2}\{\bar{\Psi}\Gamma^\lambda, \Psi, X^{\dot{\mu}}\} = 0\,. \tag{4.73}$$

By using (4.43) and (4.56), we can rewrite this equation of motion as follows:

$$\widetilde{\mathcal{D}}_\rho\mathcal{H}^{\rho\lambda\dot{\mu}} + \mathcal{D}_{\dot{\kappa}}\mathcal{H}^{\dot{\kappa}\lambda\dot{\mu}} = gJ^{\lambda\dot{\mu}}, \tag{4.74}$$

where the current is given by

$$J^{\mu\dot{\nu}} = g\{X^i, \mathcal{D}_\mu X^i, X^{\dot{\nu}}\} + \frac{ig}{2}\{\bar{\Psi}'\Gamma^\mu, \Psi', X^{\dot{\nu}}\}\,. \tag{4.75}$$

Bianchi identity and self-duality of $\mathcal{H}$

The Bianchi identity (4.81) is obtained by substituting $\Phi = X^{\dot{\mu}}$ to the commutation relation (4.48). By using the definition of the field strength $\mathcal{H}$, we can rewrite the left hand side as

$$[\mathcal{D}_\lambda, \mathcal{D}_{\dot{\lambda}}]X^{\dot{\mu}} = \delta^{\dot{\mu}}_{\dot{\lambda}} \mathcal{D}_\lambda \mathcal{H}_{\dot{1}\dot{2}\dot{3}} - \frac{1}{2}\epsilon^{\dot{\mu}\dot{\rho}\dot{\sigma}} \mathcal{D}_{\dot{\lambda}} \mathcal{H}_{\lambda\dot{\rho}\dot{\sigma}} \,, \tag{4.76}$$

and the right hand side becomes

$$g^2 \{\mathcal{H}_{\lambda\dot{\nu}\dot{\lambda}}, X^{\dot{\nu}}, X^{\dot{\mu}}\} = \epsilon^{\dot{\nu}\dot{\mu}\dot{\kappa}} \mathcal{D}_{\dot{\kappa}} \mathcal{H}_{\lambda\dot{\nu}\dot{\lambda}} \,. \tag{4.77}$$

Combining these, we obtain the Bianchi identity

$$\mathcal{D}_\lambda \mathcal{H}_{\dot{\lambda}\dot{\rho}\dot{\sigma}} - \mathcal{D}_{\dot{\lambda}} \mathcal{H}_{\lambda\dot{\rho}\dot{\sigma}} - \mathcal{D}_{\dot{\rho}} \mathcal{H}_{\lambda\dot{\sigma}\dot{\lambda}} - \mathcal{D}_{\dot{\sigma}} \mathcal{H}_{\lambda\dot{\lambda}\dot{\rho}} = 0 \,. \tag{4.78}$$

This is equivalent to (4.81).

The equations of motion of gauge fields $b_{\mu\dot{\mu}}$ and $b_{\dot{\mu}\dot{\mu}}$, and the Bianchi identity are combined into the self-dual form:

$$\mathcal{D}_\lambda \mathcal{H}^{\lambda\dot{\mu}\dot{\nu}} + \mathcal{D}_{\dot{\lambda}} \mathcal{H}^{\dot{\lambda}\dot{\mu}\dot{\nu}} = g J^{\dot{\mu}\dot{\nu}}, \tag{4.79}$$

$$\mathcal{D}_\lambda \widetilde{\mathcal{H}}^{\lambda\mu\dot{\nu}} + \mathcal{D}_{\dot{\lambda}} \mathcal{H}^{\dot{\lambda}\mu\dot{\nu}} = g J^{\mu\dot{\nu}}, \tag{4.80}$$

$$\mathcal{D}_\lambda \widetilde{\mathcal{H}}^{\lambda\mu\nu} + \mathcal{D}_{\dot{\lambda}} \widetilde{\mathcal{H}}^{\dot{\lambda}\mu\nu} = 0 \,. \tag{4.81}$$

The first two are equations of motion obtained from the action, while the last one is a Bianchi identity derived from the commutation relation (4.49).

The self-dual tensor field $\mathcal{H}$, chiral fermion field Ψ', and the five scalar fields X^i form a tensor multiplet of $\mathcal{N} = (2, 0)$ supersymmetry [64], which is the same as the field contents on an M5-brane.

4.5 Supersymmetry of M5-brane

Supersymmetry transformation

In this section, we rewrite the supersymmetry transformations (2.56) in terms of the 6-dim covariant derivatives and field strength. The transformation law (2.56) of the gauge field $A_{\mu ab}$ with coupling constant inserted is

$$\widetilde{A}_\mu{}^b{}_a = ig\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cdb}{}_a. \tag{4.82}$$

We cannot determine uniquely the transformation law of the component field $b_{\mu\dot{\nu}}$ from this equation because of the existence of the gauge transformation (4.36), which acts only on $b_{\mu\dot{\mu}}$. In fact, the transformation (4.82) only gives

$$\delta\left(\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}} \partial_{\dot{\mu}} b_{\lambda\dot{\nu}} \partial_{\dot{\rho}} f(y)\right) = ig\bar{\epsilon}\Gamma_\lambda \Gamma_I \{X^I, \Psi, f(y)\}, \tag{4.83}$$

where $f(y)$ is an arbitrary function of $y^{\dot{\mu}}$. One possible choice for $\delta b_{\mu\dot{\mu}}$ is

$$\delta b_{\mu\dot{\nu}} = ig(\bar{\epsilon}\Gamma_I \Gamma_\mu \Psi)\partial_{\dot{\nu}} X^I. \tag{4.84}$$

We can easily check that this transformation law reproduces (4.83).

In some situations an explicit appearance of $b_{\mu\dot{\mu}}$ is not necessary, but all we need is $B_\mu{}^{\dot{\mu}} := \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}\partial_{\dot{\nu}} b_{\mu\dot{\rho}}$, which satisfies the constraint $\partial_{\dot{\mu}} B_\mu{}^{\dot{\mu}} = 0$. The supersymmetry transformation for $B_\mu{}^{\dot{\mu}}$ is uniquely determined from (4.83) as

$$\delta B_\mu{}^{\dot{\mu}} = ig\bar{\epsilon}\Gamma_\mu \Gamma_I \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}}\partial_{\dot{\nu}} X^I \partial_{\dot{\lambda}} \Psi, \tag{4.85}$$

and it is obvious that the constraint is supersymmetry invariant, *i.e.*

$$\delta(\partial_{\dot{\mu}} B_\mu{}^{\dot{\mu}}) = 0. \tag{4.86}$$

The transformation laws rewritten in terms of the 6-dim notation are

$$\begin{aligned}
\delta X^i &= i\bar{\epsilon}'\Gamma^i \Psi', & (4.87)\\
\delta \Psi' &= \mathcal{D}_\mu X^i \Gamma^\mu \Gamma^i \epsilon' + \mathcal{D}_{\dot{\mu}} X^i \Gamma^{\dot{\mu}} \Gamma^i \epsilon' \\
&\quad - \frac{1}{2}\mathcal{H}_{\mu\dot{\nu}\dot{\rho}} \Gamma^\mu \Gamma^{\dot{\nu}\dot{\rho}} \epsilon' - \left(\frac{1}{g} + \mathcal{H}_{\dot{1}\dot{2}\dot{3}}\right)\Gamma_{\dot{1}\dot{2}\dot{3}}\epsilon' \\
&\quad - \frac{g^2}{2}\{X^{\dot{\mu}}, X^i, X^j\}\Gamma^{\dot{\mu}}\Gamma^{ij}\epsilon' + \frac{g^2}{6}\{X^i, X^j, X^k\}\Gamma^{ijk}\Gamma^{\dot{1}\dot{2}\dot{3}}\epsilon', & (4.88)\\
\delta b_{\dot{\mu}\dot{\nu}} &= -i(\bar{\epsilon}'\Gamma_{\dot{\mu}\dot{\nu}}\Psi'), & (4.89)\\
\delta b_{\mu\dot{\nu}} &= -iV(\bar{\epsilon}'\Gamma_\mu \Gamma_{\dot{\nu}} \Psi') + ig(\bar{\epsilon}\Gamma_\mu \Gamma_i \Gamma_{\dot{1}\dot{2}\dot{3}} \Psi')\partial_{\dot{\nu}} X^i. & (4.90)
\end{aligned}$$

The M5-brane in our theory is in a constant C-field background.

A peculiar property of this supersymmetry transformation is that the perturbative vacuum (the configuration with all fields vanishing) is not invariant under this transformation due to the term in $\delta\Psi'$ proportional to $1/g$. We can naturally interpret this term as a contribution of the background C-field. In the M5-brane action coupled to background fields, the self-dual field strength is defined by $H = db + C$ (up to coefficients depending on conventions). The inclusion of C-field in the field strength is required by the invariance of the action under C-field gauge transformations. The shift of the field strength $\mathcal{H}_{\dot{1}\dot{2}\dot{3}}$ by $(1/g)$ in the action as well as in the supersymmetry transformation suggests that the relation $C \propto g^{-1}$ between the Nambu-Poisson structure and the C-field background. This statement of course depends on the normalization of the gauge field C. For more detail about this relation, see §4.6, where we derive the precise form of this relation including the numerical coefficients.

In fact, M5-brane in a constant C-field background is still a 1/2 BPS state. The effect of the C-field is changing which half of 32 supersymmetries remain unbroken. We can find this phenomenon in our 6-dim theory. In addition to 16 supersymmetries we described above, the theory has 16 non-linear fermionic symmetries $\delta^{(\mathrm{nl})}$, which shift the fermion by a constant spinor

$$\delta^{(\mathrm{nl})}\Psi' = \chi, \quad \delta^{(\mathrm{nl})}X^i = \delta^{(\mathrm{nl})}b_{\dot\mu\dot\nu} = \delta^{(\mathrm{nl})}b_{\mu\dot\nu} = 0. \tag{4.91}$$

The action is invariant under this transformation because constant functions in $y^{\dot\mu}$ space are in the center of the Lie 3-algebra. The perturbative vacuum is invariant under the combination of two fermionic symmetries

$$\delta_{\epsilon'} - \frac{1}{g}\delta^{(\mathrm{nl})}_{\epsilon'}\,. \tag{4.92}$$

In the weak coupling limit $g \to 0$, the transformation laws for this combined symmetry agree with those of an $\mathcal{N} = (2, 0)$ tensor multiplet [65].

$$\begin{aligned} \delta X^i &= i\bar\epsilon'\Gamma^i\Psi', & (4.93)\\ \delta\Psi' &= \partial_{\underline{\mu}}X^i\Gamma^{\underline{\mu}}\Gamma^i\epsilon' - \frac{1}{12}H_{\underline{\mu\nu\rho}}\Gamma^{\underline{\mu\nu\rho}}\epsilon', & (4.94)\\ \delta b_{\underline{\mu\nu}} &= -i(\bar\epsilon'\Gamma_{\underline{\mu\nu}}\Psi'). & (4.95) \end{aligned}$$

We obtained the transformation (4.95) only for $\underline{\mu\nu} = \dot\mu\dot\nu$ and $\mu\dot\nu$. To obtain the transformation law of the $b_{\mu\nu}$ components, we first compute the transformation of $\mathcal{H}_{\dot\mu\dot\nu\dot\rho}$ and $\mathcal{H}_{\mu\dot\nu\dot\rho}$ by using the transformation law of $b_{\dot\mu\dot\nu}$ and $b_{\mu\dot\nu}$. Because the field strength is self-dual, it also gives $\delta\widetilde{\mathcal{H}}_{\mu\nu\dot\rho}$ and $\delta\widetilde{\mathcal{H}}_{\mu\nu\rho}$. The equations of motion (4.79) and (4.80) are the Bianchi identities as well for these components of field strength. If we can solve these Bianchi identities on shell and express them by using $b_{\mu\nu}$, we can extract the transformation law of $b_{\mu\nu}$ from $\delta\widetilde{\mathcal{H}}_{\mu\nu\dot\rho}$ and $\delta\widetilde{\mathcal{H}}_{\mu\nu\rho}$. In the free field limit $g = 0$, we can easily carry out this procedure and obtain (4.95) for $b_{\mu\nu}$.

4.6 D4-brane's action from M5-brane

In this section, we demonstrate that the double dimensional reduction of the 6-dim theory correctly reproduces the action of noncommutative $U(1)$ gauge theory, which is realized on a D4-brane in a B-field background.

We here recover the overall factor T_6 in the front of the action. This has mass dimension 6 and can be regarded as the tension of the five-brane, while the coupling constant g is a dimensionless parameter. We should note that this tension T_6 is not necessarily the same as the usual M5-brane tension T_{M5}, because it may be corrected by the background C-field. We will later determine the parameters g and T_6 by comparing the 5-dim action obtained by the double dimensional reduction of the 6-dim theory to the noncommutative $U(1)$ action realized on a D4-brane in a B-field background in type IIA theory. Once we obtain the expression for g and T_6 in terms of type IIA parameters, it will be easy to rewrite them in terms of the M-theory Planck scale and the magnitude of the C-field.

Double dimensional reduction : M5-brane to D4-brane

The double dimensional reduction means that we wrap one leg of the M5-brane on a compactified dimension, so that through Kaluza-Klein reduction we get one fewer dimension for both the target space and the worldvolume. Let us choose the compactified dimension to be $X^{\dot{3}}$. In the double dimensional reduction, we suppress $y^{\dot{3}}$-dependence of all fields except $X^{\dot{3}}$. We have

$$X^{\dot{3}} = \frac{1}{g} y^{\dot{3}}, \qquad b^{\dot{3}} = 0. \tag{4.96}$$

We used a gauge symmetry generated by $\Lambda_{\dot{1}}$ and $\Lambda_{\dot{2}}$ to set $b^{\dot{3}} = 0$. We impose the periodicity condition

$$X^{\dot{3}} \sim X^{\dot{3}} + L_{11} \,. \tag{4.97}$$

The relation (4.96) and (4.97) implies that the compactification period of the coordinate $y^{\dot{3}}$ is gL_{11}, and thus, the overall factor of the 5-dim theory becomes $gL_{11}T_6$.

Let us now first carry out the dimensional reduction for the bosonic terms in the action. Since all the fields except $X^{\dot{3}}$ have no dependence on $y^{\dot{3}}$, we set $\partial_{\dot{3}} = 0$ unless it acts on $X^{\dot{3}}$. We will use the notation that indices $\dot{\alpha}, \dot{\beta}, \cdots$ take values in $\{\dot{1}, \dot{2}\}$, and $a, b, \cdots$ take values in $\{0, 1, 2, \dot{1}, \dot{2}\}$. The antisymmetrized tensor $\epsilon^{\dot{\alpha}\dot{\beta}}$ is defined as $\epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}\dot{3}}$.

Expecting that we will obtain a gauge field theory on a D4-brane, let us define the gauge potentials

$$\hat{a}_\mu = b_{\mu\dot{3}} \qquad \hat{a}_{\dot{\alpha}} = b_{\dot{\alpha}\dot{3}}. \tag{4.98}$$

The covariant derivatives become

$$D_\mu X^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}} \hat{F}_{\mu\dot{\beta}}, \quad D_\mu X^{\dot{3}} = -\tilde{a}_\mu, \quad D_\mu X^i = \hat{D}_\mu X^i, \tag{4.99}$$

where $\hat{F}_{ab}$, $\tilde{a}_\mu$, and $\hat{D}_a$ are defined by

$$\begin{aligned} \hat{F}_{ab} &= \partial_a \hat{a}_b - \partial_b \hat{a}_a + g\{\hat{a}_a, \hat{a}_b\}, & (4.100)\\ \tilde{a}_\mu &= \epsilon^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\alpha}} b_{\mu\dot{\beta}}, & (4.101)\\ \hat{D}_\mu \Phi &= \partial_\mu \Phi + g\{\hat{a}_\mu, \Phi\}. & (4.102) \end{aligned}$$

The Poisson bracket $\{\cdot, \cdot\}$ is defined as the reduction of the Nambu-Poisson bracket

$$\{f, g\} = \{y^{\dot{3}}, f, g\}. \tag{4.103}$$

We note that the components $b_{\mu\dot{\beta}}$ only show up through the form $\tilde{a}_\mu$ in D4-brane's action. Thus we find that, after double dimensional reduction, the scalar kinetic term in the BLG Lagrangian become

$$-\frac{T_6}{2} \int d^3x \langle (D_\mu X^I)^2 \rangle = -\frac{gL_{11}T_6}{2} \int d^3x d^2y \left(\tilde{a}_\mu^2 + \hat{F}_{\mu\dot{\alpha}}^2 + (\hat{D}_\mu X^i)^2 \right). \tag{4.104}$$

The Nambu-Poisson brackets which appear in the potential terms of the BLG action are

$$\begin{aligned}\{X^{\dot{1}}, X^{\dot{2}}, X^{\dot{3}}\} &= \frac{1}{g^2}\hat{F}_{\dot{1}\dot{2}} + \frac{1}{g^3}, \\ \{X^{\dot{3}}, X^{\dot{\alpha}}, X^i\} &= \frac{1}{g^2}\epsilon^{\dot{\alpha}\dot{\beta}}\hat{D}_{\dot{\beta}}X^i, \\ \{X^{\dot{3}}, X^i, X^j\} &= \frac{1}{g}\{X^i, X^j\}. \end{aligned} \tag{4.105}$$

The potential term becomes

$$\begin{aligned}&-\frac{T_6}{12}\int d^3x\langle g^4\{X^I, X^J, X^K\}^2\rangle \\ &= gL_{11}T_6\int d^3xd^2y\left[-\frac{1}{2}\left(\hat{F}_{\dot{1}\dot{2}} + \frac{1}{g}\right)^2 - \frac{1}{2}(D_{\dot{\alpha}}X^i)^2 - \frac{g^2}{4}\{X^i, X^j\}^2\right]. \end{aligned} \tag{4.106}$$

Upon integration over the base space and removing total derivatives, we can replace $(\hat{F}_{\dot{1}\dot{2}} + 1/g)^2$ by $\hat{F}^2_{\dot{1}\dot{2}} + 1/g^2$.

It is also straightforward to show that the Chern-Simons term (4.59) gets simplified considerably as

$$-\frac{gL_{11}T_6}{2}\int d^3xd^2y\,\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}\tilde{a}_\lambda. \tag{4.107}$$

Here again the action depends on $b_{\mu\dot{\beta}}$ only through $\tilde{a}_\mu$. As the action depends on the field $\tilde{a}_\mu$ only algebraically (namely without derivative), we can integrate it out. There are only two terms involving $\tilde{a}_\mu$ and by completing square, we find that the effect of integrating out $\tilde{a}_\mu$ is to replace all terms involving $\tilde{a}_\mu$ by

$$-\frac{gL_{11}T_6}{4}\int d^3xd^2y\,\hat{F}^2_{\mu\nu}. \tag{4.108}$$

The fermion part can be evaluated similarly. The covariant derivatives and bracket are

$$\Gamma^\mu D_\mu\Psi' = \Gamma^\mu(\partial_\mu\Psi' + g\left\{a_\mu, \Psi'\right\}) := \Gamma^\mu\hat{D}_\mu\Psi', \tag{4.109}$$

$$\frac{1}{2}\Gamma_{IJ}\{X^I, X^J, \Psi'\} = \Gamma_{\dot{\alpha}}\Gamma_{\dot{1}\dot{2}\dot{3}}\hat{D}_{\dot{\beta}}\Psi' + \Gamma_{\dot{3}}\Gamma_i\left\{X^i, \Psi'\right\}, \tag{4.110}$$

$$\hat{D}_{\dot{\beta}}\Psi' := \partial_{\dot{\beta}}\Psi' + g\{a_{\dot{\beta}}, \Psi'\}. \tag{4.111}$$

It is quite remarkable that, after collecting all the kinetic, potential and Chern-Simons terms, the $(4+1)$-dim Lorentz invariance is restored (up to the breaking by the noncommutativity). The sum of all these terms is simply

$$\begin{aligned}gL_{11}T_6\int d^3xd^2y\Big[&-\frac{1}{2}(\hat{D}_aX^i)^2 - \frac{1}{4}\hat{F}^2_{ab} - \frac{g^2}{4}\{X^i, X^j\}^2 - \frac{1}{2g^2} \\ &+\frac{i}{2}\left(\bar{\Psi}''\Gamma^a\hat{D}_a\Psi'' + g\bar{\Psi}''\Gamma_i\{X^i, \Psi''\}\right)\Big]. \end{aligned} \tag{4.112}$$

We performed the unitary transformation $\Psi' = (1/\sqrt{2})(\Gamma_{\dot{3}} + \Gamma^7)\Psi''$ to obtain the correct chirality condition $\Gamma_{\dot{3}}\Psi'' = -\Psi''$ for the gaugino on the D4-brane. (Note that $\dot{3}$ is now the "eleventh" direction and $\Gamma_{\dot{3}}$ is the chirality matrix in IIA theory.)

Comparison with the D4-brane's action in the B-field background

Let us compare the action (4.112) with the known result [66,67] for a D4-brane in a B-field background, and match the parameters in this theory and those in type IIA string theory. The noncommutative gauge theory on D4-brane in a B-field background is described with the Moyal product $*$, and the corresponding commutator, the so-called Moyal bracket $[\cdot,\cdot]_{\text{Moyal}}$, defined by

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^j}} f(x+\xi)g(x+\zeta)|_{\xi=\zeta=0}, \tag{4.113}$$

$$[f,g]_{\text{Moyal}} = f*g - g*f = \theta^{ij}\partial_i f \partial_j g + \mathcal{O}(\theta^3). \tag{4.114}$$

The noncommutativity parameter θ^{ij} has the dimension of (length)^2. Because the action (4.112) includes only finite powers of derivatives, it should be compared to the weak coupling limit $\theta \to 0$ of the noncommutative gauge theory. These two match if we truncate the Moyal bracket into the Poisson bracket by

$$[f,g]_{\text{Moyal}} \to \frac{\theta}{T_{\text{str}}}\{f,g\}, \tag{4.115}$$

where we turn on the noncommutativity in the $\dot{1}$-$\dot{2}$ directions by setting

$$\theta^{\dot{1}\dot{2}} = \frac{\theta}{T_{\text{str}}}, \quad \theta^{\mu\dot{\alpha}} = \theta^{\mu\nu} = 0. \tag{4.116}$$

Note that θ is defined as a dimensionless parameter. In the small θ limit, the bosonic part of the action of the noncommutative $U(1)$ gauge theory on a D4-brane is given by [66,67]

$$S = \frac{T_{D4}}{\theta}\int d^3x d^2y \left[-\frac{1}{2}(D_a X^i)^2 - \frac{1}{4T_{\text{str}}}F_{ab}^2 - \frac{\theta^2}{4}\{X^i, X^j\}^2 - \frac{1}{2\theta^2} \right], \tag{4.117}$$

in the open string frame. The worldvolume coordinate $y^{\dot{\alpha}}$ in the open string frame is related to the target space coordinates $X^{\dot{\alpha}}$ by

$$X^{\dot{\alpha}} = \frac{1}{\theta}y^{\dot{\alpha}}. \tag{4.118}$$

The covariant derivative and the field strength are

$$D_a X^i = \partial_\mu X^i + \frac{\theta}{T_{\text{str}}}\{A_a, X^i\}, \quad F_{ab} = \partial_a A_b - \partial_b A_a + \frac{\theta}{T_{\text{str}}}\{A_a, A_b\}. \tag{4.119}$$

We normalize the gauge field A_a so that it couples to the string endpoints by charge 1 through the boundary coupling $S = \int_{\partial \text{F1}} A$ of the fundamental string worldsheet, and this gauge field has mass dimension 1. In the weak coupling limit, the noncommutativity parameter θ is related to the background B-field by

$$B = T_{\text{str}}\theta dX^{\dot{1}} \wedge dX^{\dot{2}} = \frac{T_{\text{str}}}{\theta}dy^{\dot{1}} \wedge dy^{\dot{2}}. \tag{4.120}$$

By comparing two actions (4.112) and (4.117), we obtain the following relations among parameters:

$$T_6 = \frac{T_{M5}}{\theta^2}, \tag{4.121}$$
$$g = \theta. \tag{4.122}$$

To relate quantities in IIA and M-theory, we use the following relations among tensions of M-branes and IIA-branes.

$$T_{D4} = L_{11} T_{M5}, \quad T_{\rm str} = L_{11} T_{D2} = L_{11} T_{M2}. \tag{4.123}$$

The relation $T_{M2}^2 = 2\pi T_{M5}$ is also useful.

In addition to the agreement of the action through the relations (4.121) and (4.122), we can check the consistency in some places.

Firstly, the relation (4.118) between the worldvolume coordinates and the target space coordinates can naturally be lifted to the relation (4.30).

Secondly, the overall factor T_6 agrees with the effective tension of M2-branes induced by the background C-field. The background B-field (4.120) is lifted to the background three-form field

$$C_3 = \theta T_{\rm M2} dX^{\dot 1} \wedge dX^{\dot 2} \wedge dX^{\dot 3} = \frac{T_{\rm M2}}{\theta^2} dy^{\dot 1} \wedge dy^{\dot 2} \wedge dy^{\dot 3}. \tag{4.124}$$

(We use the convention in which the gauge fields B and C couple to the worldvolume of corresponding branes by charge 1 through the couplings $\int_{\rm F1} B$ and $\int_{\rm M2} C$.) Each flux quantum of this background field induces the charge of a single M2-brane on the M5-brane, and effective M2-brane density in the y-space is $\theta^{-2} T_{\rm M2}/(2\pi)$. Thus, if we assume that the tension of M5-brane is dominated by the induced M2-branes, the effective tension becomes $T_{\rm M2} \times \theta^{-2} T_{\rm M2}/(2\pi) = \theta^{-2} T_{\rm M5}$. This agrees with the overall coefficients T_6 given in the relation (4.121).

Finally, the charge of the self-dual strings is consistent with the Dirac's quantization condition. From the comparison of the actions we obtain the relation of gauge fields

$$\hat a_a = \frac{1}{T_{\rm str}} A_a. \tag{4.125}$$

As we mentioned above, the gauge field A couples to string endpoints by charge 1. By the correspondence (4.125) we can determine the strength of the coupling of $\hat a$ and b to boundaries of the corresponding branes. The boundary interactions are given by

$$S = T_{\rm str} \int_{\partial {\rm F1}} \hat a = \frac{T_{\rm M2}}{\theta} \int_{\partial {\rm M2}} b. \tag{4.126}$$

To obtain the second equality in (4.126), we used the fact that a string endpoint is lifted to an M2-brane boundary wrapped on the S^1 along $y^{\dot 3}$ with period $g L_{11}$. The coupling (4.126) shows that the charge of self-dual strings (boundary of M2-branes ending on the M5-brane) is $Q = \theta^{-1} T_{\mathrm{M2}}$. Because the gauge field b is a self-dual field, Q is the electric charge as well as the magnetic charge of a self-dual string, and it must satisfy the Dirac's quantization condition

$$\frac{Q^2}{T_6} = 2\pi. \tag{4.127}$$

We can easily check that this relation certainly holds.

We can now explain the constant shift in the field strength as follows. The M2-brane action includes the following coupling to the bulk 3-form field C and the self-dual 2-form field b:

$$S_{\mathrm{M2}} = \int_{\mathrm{M2}} C_3 + \frac{T_{\mathrm{M2}}}{\theta} \int_{\partial \mathrm{M2}} b. \tag{4.128}$$

The gauge invariance of this action requires that under the gauge transformation $\delta C_3 = d\alpha_2$, the self-dual field on the M5-brane must transform as $\delta b_2 = -\alpha_2/(\theta^{-1} T_{\mathrm{M2}})$. Thus, the gauge invariant field strength H of the tensor field b should be defined by

$$H = db + \frac{\theta}{T_{M2}} C. \tag{4.129}$$

Therefore, the background gauge field (4.124) shifts the field strength as

$$H = db + \frac{1}{\theta} dy^{\dot{1}} \wedge dy^{\dot{2}} \wedge dy^{\dot{3}}. \tag{4.130}$$

This is the same as the constant shift in the definition (4.44) of $\mathcal{H}_{\dot{1}\dot{2}\dot{3}}$.

Now we have relations between parameters in the BLG model and those in M-theory. The D4-brane action obtained by the double dimensional reduction is the weak coupling ($g = \theta \to 0$) limit of noncommutative $U(1)$ theory because the Moyal bracket is replaced by the Poisson bracket. The coupling constant is determined by the background C-field, and the weak coupling means strong C-field background through the relation (4.124). Our M5-brane theory is expected to apply better to the limit of large C-field background. This is also confirmed in the comparison of the five-brane tension. As we mentioned above, the effective tension T_6 is dominated by the tension of M2-branes induced by the background C-field. This is the case when the background C-field is very large.

For a finite C-field background, we expect that the Nambu-Poisson bracket should be replaced by a quantum Nambu bracket.

4.7 Summary and remarks

In this chapter, we discuss a model of the M5-brane's worldvolume field theory which is constructed as a system of infinitely many M2-branes [5, 13]. In the BLG model, a background configuration of the M2-brane system corresponds to the choice of a Lie 3-algebra [46], and the Lie 3-algebra used for the M5-brane is the Nambu-Poisson algebra [51] on a 3-manifold $\mathcal{N}$ which appears as the internal space from the M2-branes' point of view, but it constitutes the M5-brane's worldvolume together with the M2-branes' worldvolume.

We show that the gauge transformation defined by this Lie 3-algebra can be identified as the diffeomorphism of $\mathcal{N}$ which preserve its volume 3-form. The gauge potential associated with this symmetry can be identified with the self-dual 2-form gauge field $b_{\mu\dot{\nu}}$ (an index μ for the worldvolume and another $\dot{\nu}$ for the internal space $\mathcal{N}$) which is a particular combination of the BLG gauge field $A_{\mu ab}$. We show that only a particular combination of $A_{\mu ab}$ is relevant to define the gauge symmetry, the action and the supersymmetry. We note that the internal space $\mathcal{N}$ may be regarded as the fiber on the 3-dim M2-branes' worldvolume $\mathcal{M}$ in a sense.

Another characteristic feature of the system is that not only the covariant derivative defined by the gauge potential $b_{\mu\dot{\nu}}$ is covariant, the triplet commutator $\{X^{\dot{\mu}}, X^{\dot{\nu}}, \Phi\}$ is also covariant. This follows from the fundamental identity of the Nambu-Poisson structure. From this combination, we obtain the second 2-form field $b_{\dot{\mu}\dot{\nu}}$ by which we can define the covariant derivative in the fiber direction $\mathcal{N}$. By combining two covariant derivatives, one obtains various 6-dim field strengths associated with $b_{\mu\dot{\nu}}, b_{\dot{\mu}\dot{\nu}}$.

The BLG action and the equations of motion are rewritten in terms of these fields. The equations of motion for the tensor field are written in a manifestly gauge-covariant form and combined with the Bianchi identity into a self-dual form.

4.7.1 Seiberg-Witten map

In §4.5, we discuss that the obtained theory describes the M5-brane in large C-field background. Then in our paper [5], we also argue the Seiberg-Witten map [67], which relates the gauge symmetry on a noncommutative space with the gauge symmetry on a classical space:

$$\hat{\delta}_{\hat{\lambda}}\hat{\Phi}(\Phi) = \hat{\Phi}(\Phi + \delta_\lambda\Phi) - \hat{\Phi}(\Phi), \tag{4.131}$$

where $\hat{\Phi}$ and $\hat{\delta}_{\hat{\lambda}}$ are the field and the gauge transformation in the noncommutative gauge theory, while Φ and δ_λ are the correspondence living on the classical space.

Therefore, it is a nontrivial check that we see the Seiberg-Witten map between the gauge theories on spacetimes with and without the Nambu-Poisson structure, corresponding to M5-brane theories in trivial or constant C-field background.

In §4.3, we find the gauge transformations in M5-brane's theory with C-field background as

$$\begin{aligned}
&\delta\hat{X}^i = g\hat{\kappa}^{\dot{\mu}}\partial_{\dot{\mu}}\hat{X}^i, \quad \delta\hat{\Psi} = g\hat{\kappa}^{\dot{\mu}}\partial_{\dot{\mu}}\hat{\Psi}\,, \\
&\hat{\delta}_{\hat{\Lambda}}\hat{b}_{\dot{\mu}\dot{\nu}} = \partial_{\dot{\mu}}\hat{\Lambda}_{\dot{\nu}} - \partial_{\dot{\nu}}\hat{\Lambda}_{\dot{\mu}} + g\hat{\kappa}^{\dot{\lambda}}\partial_{\dot{\lambda}}\hat{b}_{\dot{\mu}\dot{\nu}}\,, \\
&\hat{\delta}_{\hat{\Lambda}}\hat{b}_{\mu\dot{\mu}} = \partial_\mu\hat{\Lambda}_{\dot{\mu}} - \partial_{\dot{\mu}}\hat{\Lambda}_\mu + g\hat{\kappa}^{\dot{\nu}}\partial_{\dot{\nu}}\hat{b}_{\mu\dot{\mu}} + g(\partial_{\dot{\mu}}\hat{\kappa}^{\dot{\nu}})\hat{b}_{\mu\dot{\nu}}\,,
\end{aligned} \tag{4.132}$$

where $\hat{\kappa}^{\dot{\mu}} := \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}}\partial_{\dot{\nu}}\hat{\Lambda}_{\dot{\lambda}}$. On the other hand, in the trivial background, the gauge transformations are

$$\delta X^i = \delta\Psi = 0\,, \quad \delta_\Lambda b_{\dot{\mu}\dot{\nu}} = \partial_{\dot{\mu}}\Lambda_{\dot{\nu}} - \partial_{\dot{\nu}}\Lambda_{\dot{\mu}}\,, \quad \delta_\Lambda b_{\mu\dot{\mu}} = \partial_\mu\Lambda_{\dot{\mu}} - \partial_{\dot{\mu}}\Lambda_\mu\,. \tag{4.133}$$

In this subsection only, we denote all variables in our M5-brane theory by symbols with hats, and those in trivial backgrounds by symbols without hats. As a result, we make sure that the both of gauge transformations are related by the Seiberg-Witten map, up to order $\mathcal{O}(g^1)$. It should be possible to prove it for higher order terms order by order.

4.7.2 Other dimensional Mp-branes from BLG model

One may wonder the possibility of constructing other Mp-branes (which should not exist) in M-theory from multiple M2-branes. However, even if we had considered a higher dimensional manifold $\mathcal{N}$ with Nambu-Poisson structure, due to the decomposability of the Nambu-Poisson bracket, locally one can always choose 3 coordinates $\{y^1, y^2, y^3\}$ in terms of which the bracket is simply

$$\{f, g, h\} = \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\mu}} f\ \partial_{\dot{\nu}} g\ \partial_{\dot{\lambda}} h. \tag{4.134}$$

Hence the rest of the coordinates (y^a for $a > 3$) of $\mathcal{N}$ will not induce derivatives or gauge field components. There can never be more than 3 of the X^I's turning into covariant derivatives. The decomposability of the Nambu-Poisson bracket is thus the mathematical basis of why there are no other Mp-branes with $p \neq 5$.

In order to understand this statement, it may be instructive to consider a straightforward extension

$$P = \partial_1 \wedge \partial_2 \wedge \partial_3 + \partial_4 \wedge \partial_5 \wedge \partial_6 \tag{4.135}$$

which would give us a theory on M8-brane. This doesn't work, however, since this bracket does *not* satisfy the fundamental identity. One may easily confirm this by examining

$$\begin{aligned} &\{y_1 y_4, y_2, \{y_3, y_5, y_6\}\} = 0\,, \qquad \text{but} \\ &\{\{y_1 y_4, y_2, y_3\}, y_5, y_6\} + \{y_3, \{y_1 y_4, y_2, y_5\}, y_6\} + \{y_3, y_5, \{y_1 y_4, y_2, y_6\}\} = 1\,. \end{aligned}$$

The fact that the fundamental identity is so restrictive is helpful here to restrict the branes of M-theory to M2-brane and M5-brane.

4.7.3 Toward multiple M5-branes' theory

In this chapter, we construct a single M5-brane's action from the BLG model. One of the most challenging issue is how to construct the action of *multiple* M5-branes. For that purpose, we need to construct a set of generators $T^A \chi^a(y)$, where T^A $(A = 1, \cdots, d)$ are the generators of an internal algebra and $\chi^a(y)$ is the basis of functions on $\mathcal{N}$. However, as far as we try, it seems difficult to find Lie 3-algebras of this form which satisfies the fundamental identity.

Vortex string and volume-preserving diffeomorphism

As we commented, in our construction of M5-brane's action, we do *not* need the metric on $\mathcal{N}$ but only its volume form, or in other words, the 3-form flux C on it. Our computation further implied that it is natural to assume that there is a very large 3-form flux C on the M5-brane's worldvolume. This set-up reminds us of the open M2-branes in large C-flux. Since we can neglect the Nambu-Goto part (which contains the metric), the action becomes that of the topological M2-branes [68, 69]

$$S \sim \int C_{\mu\nu\rho} dX^\mu \wedge dX^\nu \wedge dX^\rho \,. \tag{4.136}$$

When this M2-branes has the boundary on M5-brane, this topological action gives

$$S \sim \int C_{\mu\nu\rho} X^\mu dX^\nu \wedge dX^\rho \,. \tag{4.137}$$

It gives an action for the string which describes the boundary of the open M2-branes. When the target space has 3 dimensions and $C \sim \epsilon^{\dot\mu\dot\nu\dot\lambda}$, this action is identical to the kinetic term of the vortex string [70], which was found long ago. In the M2-branes' context, it was studied in [47, 71–76]. In particular, it was found that it can be equipped with the Poisson structure with the constraint associated with the diffeomorphism which defines the volume-preserving diffeomorphism naturally [75]

$$\begin{aligned}
&\delta X^{\dot\mu} = \{X^{\dot\mu}, \omega(f,g)\}_D = v^{\dot\mu}(X) + \cdots \,, \\
&v^{\dot\mu} = \epsilon^{\dot\mu\dot\nu\dot\lambda} \partial_{\dot\nu} f \partial_{\dot\lambda} g \,, \quad \partial_{\dot\mu} v^{\dot\mu} = 0 \,, \\
&\omega(f,g) := \int d\sigma\, f(X) dg(X) \,.
\end{aligned} \tag{4.138}$$

Here $\{\,,\,\}_D$ is the Dirac bracket associated with the kinetic term and $\cdots$ in the first line describe the extra variation along the worldsheet which can be absorbed by the reparametrization of the worldsheet. In BLG model, the gauge parameter has an unusual feature that it has two index Λ_{ab}. In this picture, this structure is naturally interpreted as a result of the fact that for the string we can introduce two functions f, g to define the generators on the worldsheet. We hope that this connection with the vortex string would give a new insight into the BLG model.

Chapter 5

Truncation version for finite number of M2-branes

As we saw in §3.3.2, there exists a cut-off version of Nambu-Poisson bracket which defines a finite-dimensional Lie 3-algebra. The algebra still satisfies the fundamental identity and thus produces $\mathcal{N} = 8$ supersymmetric BLG-type equation of motion for multiple M2-branes. Unfortunately, as we will see, this algebra contains a lot of zero-norm generators which make all the terms of BLG Lagrangian vanishing. Thus we can analyze only the equation of motion in BLG model with this truncated algebra.

In this chapter, we discuss the concrete examples of this kind of algebra, and the BLG model with this algebra. As a result, we will derive an entropy formula which scales as $N^{\frac{3}{2}}$ as expected for the multiple (N) M2-branes, by counting the number of the moduli and the degree of freedom.

To see the $N^{\frac{3}{2}}$ law, we need to derive a finite number of M2-brane's system.

As we mentioned in §2.1.2, one of the crucial test of multiple M2-brane theory is whether one can reproduce the celebrated $N^{\frac{3}{2}}$ scaling law for entropy [14] as is predicted by AdS/CFT correspondence. For any theory based on Lie algebra, however, this seems to be difficult. The number of moduli is related to the rank of the Lie algebra and the number of the generators is given by the dimension. It will produce $N^{\frac{3}{2}}$ scaling only if one consider delicately chosen tensor products of Lie groups [77] or so far hidden mechanism changes the degrees of freedom.

In this chapter, we take a different approach to this issue. In the previous chapter, it was proved that BLG model based on infinite-dimensional Lie 3-algebra defined by Nambu-Poisson bracket is equivalent to a single M5-brane's worldvolume theory. What we are going to do is to cut-off this Lie 3-algebra to finite dimensions. It is actually very natural to expect to have $N^{\frac{3}{2}}$ law from the following geometrical reason.

We note that the Nambu-Poisson bracket can be defined on a 3-dim manifold as

$$\{f, g, h\} = \sum_{\mu,\nu,\rho=1}^{3} \epsilon_{\mu\nu\rho}\, \partial_\mu f\, \partial_\nu g\, \partial_\rho h\,, \tag{5.1}$$

where f, g, h are arbitrary functions of three variables x^1, x^2, x^3. Suppose we can truncate this infinite-dimensional Hilbert space into a finite-dimensional one, let us assume that we have N degrees of freedom for each dimension, by truncating all functions to polynomials x^μ of degree $\leq N$. Then the number of independent generators behaves as $\#G \sim N^3$. On the other hand, the number of M2-branes is, roughly speaking, identified with the number of the moduli which are related to mutually commuting degree of freedom. In this case, due to the structure of the Nambu-Poisson bracket, mutually commuting generators may be taken as functions which depend only on two variables, say x^1 and x^2. The number of such generators can be estimated as $\#M \sim N^2$. By combining it, we have the desired scaling $\#G \sim (\#M)^{\frac{3}{2}}$!

In §3.3.2, we ascertained that one can obtain a finite-dimensional Lie 3-algebra from a truncation of the Hilbert space where Nambu-Poisson bracket is defined. The fundamental identity of the Lie 3-algebra is preserved by the cut-off, but, as we will see, it becomes generally difficult to keep a *non-trivial* invariant metric. Therefore, although it is difficult to write BLG action, we can define the $\mathcal{N} = 8$ supersymmetric equation of motion as considered in [78]. The counting of the moduli is given as above and we obtain the $N^{\frac{3}{2}}$ scaling law of entropy rather robustly.

By definition, our truncated algebra becomes the infinite-dimensional Lie 3-algebra from Nambu-Poisson bracket in the large N limit. In this sense, it gives an intermediate geometrical structure between M2-brane and M5-brane. This is somewhat analogous to the fact that D$(p+2)$-brane is obtained by collecting large N limit of Dp-brane. Therefore, it may serve as a candidate of multiple M2-branes, although it requires many improvements to define a realistic theory.[1]

Quantization of Nambu-Poisson bracket is a difficult problem.

Before proceeding to concrete discussion, let us briefly mention the previous studies on the quantum Nambu bracket, since our truncated Nambu-Poisson bracket must be understood as nothing but a candidate for quantum Nambu bracket.

One of the most natural direction is to seek an analog of the Moyal product as a deformation of Poisson bracket. It was studied most extensively by Takhtajan [58] and his collaborators. Despite much efforts, however, the natural analog of the Moyal product has not been

[1] We note that a derivation of $N^{\frac{3}{2}}$ law for M2-branes was considered previously in [42] (see also [33, 79]) in the context of Basu-Harvey equation [36] which describes a "ridge" configuration of M2-M5 system. Their analysis is based on the fuzzy S^3 defined in [80, 81]. Since it appeared before [2], the essential ingredients of the BLG model such as Lie 3-algebra and the fundamental identity were not taken into account.

found so far. At some point, they changed the strategy and found a deformation of Nambu-Poisson bracket which was called "Zariski quantization" [82]. This construction, however, needs to use an analog of the second quantized operators and is infinite-dimensional by its nature.

Another approach is to use a generalization of the matrix commutator (see, for example, [83]). Although it gives rise to a very simple finite-dimensional system, the triple commutator satisfies so called generalized Jacobi identity instead of the fundamental identity. In this sense, it is not obvious how to apply their algebraic structure to the BLG model.

The third approach is to use the cubic matrix (three index object like "A_{ijk}") to represent the 3-algebra (see for example [50, 84]). Although there were some success, for example in the construction of "representations" of $\mathcal{A}_4$ algebra [48], the cubic matrix in general doesn't satisfy the fundamental identity. So it is still mysterious how to apply it to BLG model.

To summarize, although there are some attractive proposals in the quantum Nambu bracket, our simple cut-off procedure of the Nambu-Poisson bracket seems to be the first example which can be readily applicable to BLG model. We do not, of course, mean that other approaches which we mentioned are meaningless in the BLG model. On the contrary, we are trying to find applications of these constructions.

5.1 Homogeneous Nambu-Poisson brackets

In §3.3.2, we saw that one can define a truncated algebra for each N, for any *homogeneous* Nambu-Poisson brackets, *i.e.*

$$\{f, g, h\}_N = \pi_N \left(\sum_{\mu,\nu,\rho=1}^{d} P^{\mu\nu\rho}(x)\, \partial_\mu f\, \partial_\nu g\, \partial_\rho h \right), \tag{5.2}$$

where $P^{\mu\nu\rho}(x)$ is a *homogeneous* polynomial of degree p, and π_N is a projector which acts on the polynomials of x^μ as

$$\pi_N \left(\sum_{n_1,\cdots,n_d=0}^{\infty} c_{n_1,\cdots,n_d} (x^1)^{n_1} \cdots (x^d)^{n_d} \right) = \sum_{n_1,\cdots,n_d=0}^{n_1+\cdots+n_d \le N} c_{n_1,\cdots,n_d} (x^1)^{n_1} \cdots (x^d)^{n_d}\,, \tag{5.3}$$

where $c_{n_1,\cdots,n_d}$ are coefficients. We also define the product $\bullet_N$ of functions in the truncated Hilbert space as

$$f \bullet_N g = \pi_N(f\, g)\,. \tag{5.4}$$

In the following, we give some examples of homogeneous algebra which satisfies the fundamental identity and associate each algebra with a 3-dim manifold. In general, we have descriptions of the homogeneous Nambu-Poisson bracket in terms of d variables, as eq. (5.2). The fact that Nambu-Poisson bracket is defined in 3-dim space can be derived by observing that there are $d-3$ elements $f_a(x)$ which commute with any functions of x, namely,

$$\{f_a, g, h\} = 0 \quad \text{for any } g, h\,. \tag{5.5}$$

So one may use the hyper-surface defined by $f_a(x) = c_a$ ($a = 1, \cdots, d-3$) as the definition of 3-dim submanifold in $\mathbf{R}^d$. If we introduce the cut-off, one may call the corresponding geometry as "fuzzy spaces" by employing the terminology of the noncommutative geometry although our definition of the deformation is very different.

5.1.1 $p = 1$ case

We start from the $p = 1$ case. In this case, we call the bracket as *linear* Nambu-Poisson bracket [59] in the following. We note that the coordinates x^μ define a Lie 3-subalgebra,

$$\{x^{\mu_1}, x^{\mu_2}, x^{\mu_3}\} = \sum_{\mu_4} f^{\mu_1\mu_2\mu_3}{}_{\mu_4} x^{\mu_4}\,, \quad P^{\mu_1\mu_2\mu_3}(x) = \sum_{\mu_4} f^{\mu_1\mu_2\mu_3}{}_{\mu_4} x^{\mu_4}\,. \tag{5.6}$$

As we saw in §3.3.1, the mathematical classification of the linear Nambu-Poisson bracket was already made. It is classified into two groups:

Type I: For each $-1 \le r \le 3$ and $0 \le s \le \min(3-r, d-4)$, one may define the bracket as

$$P^I_{(r,s)} = \sum_{j=1}^{r+1} \pm x^j \partial_1 \wedge \cdots \hat{\partial}_j \cdots \wedge \partial_4 + \sum_{j=1}^{s} \pm x^{n+j+1} \partial_1 \wedge \cdots \hat{\partial}_{r+j+1} \cdots \wedge \partial_4\,, \tag{5.7}$$

where $\hat{\partial}$ means that we delete that element in the wedge product, and we can choose the plus/minus sign freely for each term in the summation.

Type II:

$$P^{II}_a = \partial_1 \wedge \partial_2 \wedge (\sum_{i,j=3}^{d} a_{ij} x^i \partial_j)\,. \tag{5.8}$$

In the following, we pick up interesting examples that come from this classification theorem for each d, the number of coordinates.

$d = 3$ **case**

The only possibility comes from the type II algebra

$$P = \partial_1 \wedge \partial_2 \wedge x^3 \partial_3\,. \tag{5.9}$$

In this case, the x^3 may be taken as a real number or a phase $e^{i\theta_3}$. When x^3 is taken as real, and with an appropriate completion, the truncated algebra can be thought as a deformation of $\mathbf{R}^3$.[2] Due to the extra factor of x^3, the Poisson structure (5.9) breaks $O(3)$ symmetry. In the correspondence with M5-brane's case [5, 13], P represents the 3-form flux on M5-brane's worldvolume. The breakdown of rotational symmetry comes from the fact that the 3-form background doesn't respect the symmetry. When x^3 is a phase, one can think of the truncated algebra as a deformation of $\mathbf{R}^2 \times S^1_+$, where S^1_+ is dual to the algebra of functions with only non-negative Fourier modes. In this case, $P \sim \partial_1 \wedge \partial_2 \wedge \partial_{\theta_3}$ defines a Nambu-Poisson bracket on $\mathbf{R}^2 \times S^1_+$.

$d = 4$ **case**

In this case, a variety of examples comes from type I algebra.

$r = 3$, $s = 0$ case

For this case, a well-known example is

$$P_{(3,0)} = x^1\partial_2 \wedge \partial_3 \wedge \partial_4 - x^2\partial_1 \wedge \partial_3 \wedge \partial_4 + x^3\partial_1 \wedge \partial_2 \wedge \partial_4 - x^4\partial_1 \wedge \partial_2 \wedge \partial_3 \,. \tag{5.10}$$

In this case, the Lie 3-algebra generated by the coordinates is $\mathcal{A}_4$. It defines a Nambu-Poisson bracket on S^3, since $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$ becomes the center of the Lie 3-algebra. Namely,

$$P(r^2 f_1, f_2, f_3) = r^2 P(f_1, f_2, f_3) \,, \tag{5.11}$$

for any f_1, f_2, f_3. So one may put $r^2 = \text{const}$. This means that the truncated algebra defines a fuzzy S^3 in $\mathbf{R}^4$.

From this example, by taking Wick rotation, we obtain other examples. For example, the bracket after $x^4 \to ix^4$

$$P = x^1\partial_2 \wedge \partial_3 \wedge \partial_4 - x^2\partial_1 \wedge \partial_3 \wedge \partial_4 + x^3\partial_1 \wedge \partial_2 \wedge \partial_4 + x^4\partial_1 \wedge \partial_2 \wedge \partial_3 \,, \tag{5.12}$$

defines a bracket on 3-dim de Sitter space dS_3, since $(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2$ becomes the center of the algebra and can be set to a constant.

Similarly after taking the Wick rotation for x^3 and x^4, we obtain

$$P = x^1\partial_2 \wedge \partial_3 \wedge \partial_4 - x^2\partial_1 \wedge \partial_3 \wedge \partial_4 - x^3\partial_1 \wedge \partial_2 \wedge \partial_4 + x^4\partial_1 \wedge \partial_2 \wedge \partial_3 \,. \tag{5.13}$$

In this case, $(x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2$ becomes the center of 3-algebra and can be set to a constant, which defines 3-dim anti-de Sitter space AdS_3.

[2] To avoid possible confusion, we emphasis that this is not the standard R^3 as a Poisson manifold. There the Poisson structure is $SO(3)$ and translationally invariant.

$r = 2$, $s = 0$ case

For this case, we have

$$\begin{aligned} P_{(2,0)} &= x^1\partial_2 \wedge \partial_3 \wedge \partial_4 + x^2\partial_1 \wedge \partial_3 \wedge \partial_4 \pm x^3\partial_1 \wedge \partial_2 \wedge \partial_4 \\ &= (x^1\partial_2 \wedge \partial_3 + x^2\partial_1 \wedge \partial_3 \pm x^3\partial_1 \wedge \partial_2) \wedge \partial_4 \,. \end{aligned} \tag{5.14}$$

The center takes the form $(x^1)^2 + (x^2)^2 \pm (x^3)^2$ and 3-dim manifold associated with it is $S^2 \times \mathbf{R}$ or $(A)dS_2 \times \mathbf{R}$, where $\mathbf{R}$ is described by x^4. For finite N, we have a deformation of these manifolds.

$d > 4$ **case**

From the definition of type I algebra, in order to have $s > 0$, we need to take $d > 4$.

$d = 5$, $r \leq 2$, $s = 1$ case

For example for $s = 1$, we need $d = 5$ and

$$P_{(2,1)} = P_{(2,0)} \pm x^5\partial_1 \wedge \partial_2 \wedge \partial_3 \,. \tag{5.15}$$

In this case, since x^5 doesn't appear in the derivative, it is the center of Lie 3-algebra. Actually, the algebra for the linear functions is identical with the Lorentzian algebra (in §3.4.1) for $g = SU(2)$ or $SL(2)$ where x^4, x^5 play the role of u, v, respectively. In general, the parameter s represents the number of pairs of the Lorentzian generators. For smaller r, we can add more pairs $(3 - r)$ of Lorentzian generators. For $r = 2, s = 1$, the center of the algebra becomes

$$(x^1)^2 + (x^2)^2 \pm (x^3)^2 \pm 2x^4x^5 \quad \text{and} \quad x^5 \,, \tag{5.16}$$

to which we can assign arbitrary value.

For $r = 1$, we obtain $S^1 \times \mathbf{R}^2$ or $\mathbf{R}^3$ and its generalizations with pairs of Lorentzian generators. We note that here we obtain S^1 or $\mathbf{R}^1$ from a constraint $(x^1)^2 \pm (x^2)^2 = \text{const}$.

For $r = 0$, we obtain $\mathbf{R}^3$ with the bracket

$$P = x^1\partial_2 \wedge \partial_3 \wedge \partial_4 \,, \tag{5.17}$$

where x^1 becomes the center of 3-bracket and can be set to a constant.

For $r = -1$, we have only the Lorentzian pairs.

5.1.2 $p > 1$ case

For $p > 1$, we don't have the classification theorem. We have, however, a few interesting examples of Nambu-Poisson bracket where fundamental identity is satisfied.

For $p = 2$, we have, for example,

$$P = \partial_1 \wedge x^2\partial_2 \wedge x^3\partial_3. \tag{5.18}$$

If we take x^2, x^3 real, the we have a deformed $\mathbf{R}^3$ with linear flux introduced in these two directions. By taking x^2 or/and x^3 to be a phase, we can also have deformed $\mathbf{R}^2 \times S^1_+$ or $\mathbf{R} \times T^2_+$ (where T^2_+ represents $S^1_+ \times S^1_+$). Another example is

$$P = (\epsilon_{\mu\nu\lambda} x^\mu \partial_\nu \wedge \partial_\lambda) \wedge x^4\partial_4 \tag{5.19}$$

which can describe the deformation of $S^2 \times \mathbf{R}^1$ or $S^2 \times S^1_+$.

For $p = 3$, we have an example

$$P = x^1\partial_1 \wedge x^2\partial_2 \wedge x^3\partial_3 \tag{5.20}$$

which can describe the deformed $\mathbf{R}^3$, $\mathbf{R}^2 \times S^1_+$, $\mathbf{R} \times T^2_+$ or T^3_+ depending on the interpretation of x^μ.

The last example of deformed T^3_+ will be used in the following, since it has the simplest structure. In particular, the algebra (3.53) takes the following form (after minor change of the normalization factors)

$$\{T(\vec{n}_1), T(\vec{n}_2), T(\vec{n}_3)\} = \vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) T(\vec{n}_1 + \vec{n}_2 + \vec{n}_3)\,. \tag{5.21}$$

Its truncated version becomes

$$\{T(\vec{n}_1), T(\vec{n}_2), T(\vec{n}_3)\}_N = \vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)\, \theta \left(N - |\sum_i \vec{n}_i| \right) T(\vec{n}_1 + \vec{n}_2 + \vec{n}_3)\,, \tag{5.22}$$

where all the elements of $\vec{n}_i \geq 0$ and

$$\theta(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}\,. \tag{5.23}$$

The explicit forms of the algebra for other cases are straightforward to write down. For example, S^3 case (5.10) is given as

$$\{T(\vec{n}_1), T(\vec{n}_2), T(\vec{n}_3)\} = \epsilon_{\mu\nu\lambda\rho} (n_1)_\nu (n_2)_\lambda (n_3)_\rho T(\vec{n}_1 + \vec{n}_2 + \vec{n}_3 - \vec{\sigma} + 2\vec{e}_\mu)\,, \tag{5.24}$$

where $(e^\mu)_\nu = \delta_{\mu\nu}$ and $\vec{\sigma} = \sum_{i=1}^4 \vec{e}_\mu$. The truncated Lie 3-algebra can be obtained by restricting the generators to $|\vec{n}| \leq N$ and including a truncation factor $\theta(N + 2 - \sum_i |\vec{n}_i|)$ on the right hand side.

5.2 Truncated Nambu-Poisson bracket algebra

Before analyzing the BLG model with these truncated Nambu-Poisson algebras, we discuss some details on these kinds of Lie 3-algebra in this section.

5.2.1 Structure of algebra

We note that the truncated 3-algebra, especially the deformed T^3_+ case (5.20), can be decomposed into three subspaces:

$\mathcal{A}_0$: A subspace spanned by generators $T(\vec{k})$ where one or two components of $\vec{k}$ is zero. In the definition of Nambu-Poisson bracket (5.20), we always multiply $x_1x_2x_3$ after taking the derivation. So the generators which belong to $\mathcal{A}_0$ never appear on the right hand side of the 3-commutator. We will denote a generic generator which belongs to $\mathcal{A}_0$ as T_X.

$\mathcal{A}_{-1}$: A subspace spanned by generators $T(\vec{k})$ where $|\vec{k}| = N-1, N$. These generators are the center of the algebra, namely

$$\{T_Y, T(\vec{p}), T(\vec{q})\}_N = 0 \quad \text{for} \, \forall \, \vec{p}, \, \vec{q}. \tag{5.25}$$

where T_Y is a generic generator which belong to $\mathcal{A}_{-1}$. It comes the fact that we need $|\vec{p}|, |\vec{q}| \geq 1$ to have a nonvanishing 3-commutator. These generators can show up on the right hand side of the 3-commutator .

$\hat{\mathcal{A}}$: A subspace spanned by the generators which belong to neither $\mathcal{A}_0$ nor $\mathcal{A}_{-1}$. We will write generic elements of $\hat{\mathcal{A}}$ as T_Z.

We note that there are some elements which belong to $\mathcal{N} = \mathcal{A}_0 \cap \mathcal{A}_{-1}$. Since every element in this subspace has vanishing 3-commutator with anybody else and never appears on the right hand side of the 3-commutator, they decouple from the algebra as $T(\vec{0})$. Therefore, we have to remove them from the algebra. Then we will write

$$\mathcal{A}'_0 = \mathcal{A}_0/\mathcal{N}, \quad \mathcal{A}'_{-1} = \mathcal{A}_{-1}/\mathcal{N}, \tag{5.26}$$

to represent the relevant part of the algebra. The number of generators which belong to each subspace is

$$\#(\hat{\mathcal{A}}) \sim \frac{N^3}{6}, \quad \#(\mathcal{A}_0) \sim \frac{3N^2}{2}, \quad \#(\mathcal{A}_{-1}) \sim N^2, \quad \#(\mathcal{N}) \sim 6N \,. \tag{5.27}$$

In the large N limit, the number of the elements which belong to $\mathcal{A}_0, \mathcal{A}_{-1}$ increase as $O(N^2)$, but it is still much smaller than that of $\hat{\mathcal{A}}$.

5.2.2 Invariant metric

By definition, any element $T^a_Y \in \mathcal{A}'_{-1}$ must appear only on the right hand side of the 3-commutator. It implies

$$\langle T^a_Y, T^b_Y\rangle = \langle [T^P, T^Q, T^R], T^b_Y\rangle = -\langle T^R, [T^P, T^Q, T^b_Y]\rangle = 0$$
$$\langle T^a_Z, T^b_Y\rangle = \langle [T^P, T^Q, T^R], T^b_Y\rangle = -\langle T^R, [T^P, T^Q, T^b_Y]\rangle = 0$$

for some T^P, T^Q, T^R and $T_Z^a \in \hat{\mathcal{A}}$. So all the elements in $\mathcal{A}'_{-1}$ must be orthogonal to any elements in $\mathcal{A}'_{-1}$ and $\hat{\mathcal{A}}$.

Similarly, for two elements $T_X^a, T_X^b \in \mathcal{A}'_0$, since they don't show up in the 3-commutator, there are no constraints for their inner product from the symmetry:

$$\langle T_X^a, T_X^b \rangle = K_{ab} \quad \text{(arbitrary)}\,. \tag{5.28}$$

We can also deduce that any elements in $\mathcal{A}'_{-1}$ and $\hat{\mathcal{A}}$ are orthogonal with the elements of $\mathcal{A}'_0$:

$$\langle T_X^a, T_Y^b \rangle = \langle T_X^a, T_Z^b \rangle = 0. \tag{5.29}$$

A proof is as follows. For the generic elements $T_{k_1k_2k_3} \in \mathcal{A}'_{-1} \cup \hat{\mathcal{A}}$, we have $k_1, k_2, k_3 \neq 0$. So one may write it as a 3-commutator

$$T_{k_1k_2k_3} = \frac{1}{k_1k_2k_3}[T_{k_100}, T_{0k_20}, T_{00k_3}]\,, \tag{5.30}$$

where $T_{k_1k_2k_3} := T(k_1\vec{e}_1 + k_2\vec{e}_2 + k_3\vec{e}_3)$. On the other hand, for any element $T_{p_1p_2p_3} \in \mathcal{A}'_0$, one of p_i must be zero. Let us take it $p_1 = 0$, without loss of generality. Then we have

$$\begin{aligned} \langle T_{0p_2p_3}, T_{k_1k_2k_3} \rangle &\propto \langle T_{0p_2p_3}, [T_{k_100}, T_{0k_20}, T_{00k_3}] \rangle \\ &= -\langle [T_{0p_2p_3}, T_{0k_20}, T_{00k_3}], T_{k_100} \rangle = 0\,. \end{aligned} \tag{5.31}$$

Finally, for any two elements $T_{p_1p_2p_3}, T_{q_1q_2q_3} \in \hat{\mathcal{A}}$, one can derive similarly

$$\begin{aligned} \langle T_{p_1p_2p_3}, T_{q_1q_2q_3} \rangle &\propto \langle T_{p_1p_2p_3}, [T_{q_100}, T_{0q_20}, T_{00q_3}] \rangle \\ &= -\langle [T_{p_1p_2p_3}, T_{0q_20}, T_{00q_3}], T_{q_100} \rangle\,. \end{aligned} \tag{5.32}$$

On the right hand side, $[T_{p_1p_2p_3}, T_{0q_20}, T_{00q_3}]$ is zero or belong to either $\hat{\mathcal{A}}$ or $\mathcal{A}'_{-1}$. Since the inner product between $\hat{\mathcal{A}}$ or $\mathcal{A}'_{-1}$ with any element in $\mathcal{A}'_0$ is already shown to be zero, we arrive at

$$\langle T_Z^a, T_Z^b \rangle = 0 \qquad \text{for } {}^\forall\, T_Z^a, T_Z^b \in \hat{\mathcal{A}}\,. \tag{5.33}$$

As we can easily see, the requirement of invariance of metric

$$\langle T^a, [T^b, T^c, T^d] \rangle + \langle [T^a, T^b, T^c], T^d \rangle = 0 \tag{5.34}$$

imposes very severe constraints on the form of the metric. At the end, the metric has lots of null directions, making it not useful for physical applications. For example, the potential term of the BLG model $\langle [X^I, X^J, X^K], [X^I, X^J, X^K] \rangle$ is identically zero, because non-trivial metric components only exist for elements in $\mathcal{A}'_0$, while elements in $\mathcal{A}'_0$ never appear as the right hand side of a 3-commutator.

5.3 Application to BLG model and entropy counting

As we saw in the previous section, the metric of the truncated Nambu-Poisson bracket has a trivial structure and is useless in the construction of the invariant Lagrangian.[3] Nevertheless, we can write down an $\mathcal{N} = 8$ supersymmetric equation of motion (2.57) in terms of the structure constants of this kind of Lie 3-algebra, which satisfies the fundamental identity [78],

$$\begin{aligned}
&D^2 X^I_a - \frac{i}{2}\bar{\Psi}_c \Gamma^{IJ} X^J_d \Psi_B f^{cdb}{}_a + \frac{1}{2} f^{bcd}{}_a f^{efg}{}_d X^J_b X^K_c X^I_e X^J_f X^K_g = 0\,,\\
&\Gamma^\mu D_\mu \Psi_a + \frac{1}{2}\Gamma_{IJ} X^I_c X^J_d \Psi_b f^{cdb}{}_a = 0\,,\\
&(\tilde{F}_{\mu\nu})^b{}_a + \epsilon_{\mu\nu\lambda}(X^J_c D^\lambda X^J_d + \frac{i}{2}\bar{\Psi}_c \Gamma^\lambda \Psi_d) f^{cdb}{}_a = 0\,.
\end{aligned} \tag{5.35}$$

The supersymmetry transformation (2.42) is

$$\begin{aligned}
&\delta X^I_a = i\bar{\epsilon}\Gamma^I \Psi_a\,,\\
&\delta \Psi_a = D_\mu X^I_a \Gamma^\mu \Gamma_I \epsilon - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma_{IJK}\epsilon\,,\\
&\delta(\tilde{A}_\mu)^b{}_a = i\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cdb}{}_a\,.
\end{aligned} \tag{5.36}$$

The gauge symmetry transformation (2.54) is

$$\delta_\Lambda X^I_a = \tilde{\Lambda}^b{}_a X^I_b\,, \quad \delta_\Lambda \Psi_a = \tilde{\Lambda}^b{}_a \Psi_b\,, \quad \delta_\Lambda \tilde{A}_\mu{}^b{}_a = D_\mu \tilde{\Lambda}^b{}_a\,. \tag{5.37}$$

An essential point here is that the structure constant contracted with metric $f^{abcd} = f^{abc}{}_e h^{ed}$ doesn't appear at all. It enables us to discuss important issues such as the BPS equation or the moduli without knowing the Lagrangian.

The classification of field components : Are they moduli or not?

Let us pick the algebra (5.20) and study the moduli. From the equation of motion, the moduli would be described by solutions of the equation

$$f^{efg}{}_d X^I_e X^J_f X^K_g = 0\,. \tag{5.38}$$

We have to be careful in the structure of the truncated algebra. In the previous section, we show that the algebra (5.20) has a structure which is similar to the Lorentzian algebra. Namely, after removing generators which decouple from the algebra, the set of generators is classified into the following three subsets:

$\mathcal{A}'_0$: The generators which don't appear on the right hand side of 3-commutators, *i.e.* the generator T^d where $f^{abc}{}_d = 0$ for any a, b, c. Such generators have the form $T(\vec{k})$ where one or two components of $\vec{k}$ are zero.

[3] Of course, there may be a chance to add extra generators to obtain a nontrivial and useful metric as in [15, 60, 61].

$\mathcal{A}'_{-1}$: The generators which are in the center of Lie 3-algebra, *i.e.* the generator T^a where ${f^{abc}}_d = 0$ for any b, c, d. Such generators take the form $T(\vec{k})$ where $\sum_i k_i = N-1, N$.

$\hat{\mathcal{A}}$: The generators which don't belong to $\mathcal{A}'_0$ nor $\mathcal{A}'_{-1}$.

Then we study the roles of fields in each subgroup. Let us denote the generic fields which belong to $\mathcal{A}'_0$, $\mathcal{A}'_{-1}$, $\hat{\mathcal{A}}$ as X, Y, Z, respectively. Then the equation of motion is written schematically as

$$\partial^2 X = 0, \quad \partial^2 Y = F_1(X, Z), \quad \partial^2 Z = F_2(X, Z)\,, \tag{5.39}$$

and supersymmetry and gauge symmetry transformations are written similarly as

$$\delta X = 0, \quad \delta Y = G_1(X, Z), \quad \delta Z = G_2(X, Z)\,, \tag{5.40}$$

where $F_{1,2}, G_{1,2}$ represent some nonlinear functions. To find moduli, we can put the left hand side of equation of motion (5.39) to be zero.

First, we note that there is no constraint for Y from (5.38). Besides, the fields Y never appear in the nonlinear terms in the equations of motion. We can take any solutions of Y of their equations of motion, and it will not have any effect on the rest of the fields. In this sense, the fields Y should be viewed as non-physical fields, and we will not treat them as part of the moduli.[4]

Secondly, if we assign VEV to the fields X, the field equation and the symmetry transformations *do* depend on the VEV. On the other hand, the supersymmetry and gauge symmetry transformation (5.40) for the fields X implies that these symmetries are not violated. This behavior is what one expects for a vacuum state. In the Lorentzian BLG model which we will discuss in Chapter 6, however, the VEV for X^I_u was interpreted as the coupling constant of the super Yang-Mills theory on D2-branes and hence is not counted as part of the moduli space. So further analysis is needed to decide whether these fields X are to be counted as part of the moduli space or not. However, we don't care of it here, since, as we will see, whether one includes them or not doesn't affect our entropy counting below.

Finally, the assignment of VEV for Z doesn't seem to have such strange behavior. Therefore, this is the degree of freedom which should be identified with the moduli of M2-brane in ordinary sense.

Counting of moduli and entropy — We can 'derive' the $N^{\frac{3}{2}}$ law!

It turns out that the equation (5.38) can give rise to various solutions. For the Lie 3-algebra (5.20), three polynomials f_1, f_2, f_3 which depend only on two polynomials of x,

[4] On the other hand, if we treat them as part of the moduli, the number of solutions of (5.38) can be of order N^3. We can take 6 of the scalars X^I to be fields Y, and the rest 2 of the X^I's can be arbitrary. For large N, the number of free parameters in these 2 arbitrary fields X^I dominates and it is proportional to N^3.

say $g_1(x), g_2(x)$, in general commute with each other

$$\{f_1(g_1, g_2), f_2(g_1, g_2), f_3(g_1, g_2)\}_N = 0\,. \tag{5.41}$$

Therefore, the moduli space is described by (truncated) polynomials of $g_1(x)$ and $g_2(x)$. Depending on the choice of $g_{1,2}$, we have different type of "Higgs" branches.

If we take both $g_{1,2}$ as function of single variables, say $g_1 = x^1$, $g_2 = (x^2)^m$, all the functions of $g_{1,2}$ belong to the group $\mathcal{A}'_0$. The number of such functions is of the order of N^2. As we explained above, these may or may not be counted as part of the moduli space.

On the other hand, suppose we take $g_{1,2}$ such that their polynomials depend on all the coordinates nontrivially, for example $g_1 = x^1 + x^2$ and $g_2 = (x^3)^2$, the set of polynomials of them contains elements belonging to $\hat{\mathcal{A}}$. In this case, the VEV's are assigned to the fields Z and should be interpreted as the moduli of M2-branes. We can count the number of the M2-branes for given set of $g_{1,2}$. Suppose we choose them such that all the VEV's of fields can be interpreted as the moduli of M2-branes. If the degree of $g_{1,2}$ is $n_{1,2}$ respectively, the number of independent generators are approximately $\frac{N^2}{2n_1 n_2} \sim N^2$, as long as $n_{1,2}$ are much smaller than N. We have the estimate for the number of M2-branes as

$$\#M \sim N^2. \tag{5.42}$$

This permits us to calculate the behavior of the entropy. The number of fields is given as the number of generators ($\#G$). It can be estimated as

$$\#G = \frac{(N+1)(N+2)(N+3)}{6} \sim N^3/6 \sim (\#M)^{\frac{3}{2}}. \tag{5.43}$$

This is nothing but the celebrated $N^{\frac{3}{2}}$ law for M2-branes!

One may do essentially the same counting for other $d = 3$ algebras associated with $\mathbf{R}^3$ (5.9,5.18) which give the same behavior. So one may guess the behavior of $N^{\frac{3}{2}}$ law as a generic feature of the $d = 3$ truncated Nambu-Poisson 3-algebras.

This excellent result can be obtained only in $d = 3$ case.

We note that there are some subtlety if one continues to do the similar analysis for $d > 3$ cases. In these cases, as we have seen, there are $d - 3$ generators $\phi_s(x)$ which satisfy

$$\{\phi_s f_1, f_2, f_3\} = \phi_s\{f_1, f_2, f_3\} \tag{5.44}$$

for any f_1, f_2, f_3. One may set such generators as constant $\phi_s(x) = c_s$ and this constraint gives 3-dim algebra.

For the truncated algebras, since such ϕ_s has nontrivial degree as the polynomial of x. For example, $\phi = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$ which appear for S^3 case has degree two. So the above relation should be modified as

$$\{\phi_s \bullet_N f_1, f_2, f_3\}_N = \phi_s \bullet_N \{f_1, f_2, f_3\}_{N-|\phi_s|} \tag{5.45}$$

where $|\phi_s|$ is the degree of ϕ_s. It implies that we cannot put ϕ_s to a c-number if we want to keep the fundamental identity. If we treat them as the independent generators, we would have different scaling. For example, for any $d = 4$ cases, we have a simple estimate that

$$\#M \sim O(N^3), \quad \#G \sim O(N^4)\,. \tag{5.46}$$

Therefore, we obtain $N^{4/3}$ relation between the number of M2-branes and the number of degrees of freedom. This strange behavior for $d > 3$ signals the breakdown of the truncation process which doesn't properly respect the local factorization of the space into 3-dim and $(d-3)$-dim spaces. Therefore, this anomalous scaling law should be understood as coming from an incorrect regularization of the system.

5.4 Summary

In this chapter, we analyze the BLG model with truncated Nambu-Poisson brackets as examples of Lie 3-algebra. This model has the following two remarkable properties:

- It naturally shows the $N^{\frac{3}{2}}$ scaling of M2-branes with clear geometrical meaning.
- In the large N limit, it can reproduce the M5-brane's theory which we discuss in Chapter 4.

On the other hand, it has obvious shortcomings at this moment, namely we cannot define nontrivial Lagrangian with the current form of the algebra. A hope is that one may cure it by adding some extra generators, just as in [15, 60, 61].

Of course, the truncated algebra which we considered here is rather exotic algebra which was not considered seriously in the literature. For example, it would be much more desirable to do similar truncation by some generalization of the Moyal product or by some generalization of the concept of matrices. We note that, however, our derivation of $N^{\frac{3}{2}}$ law is quite robust and the derivation of the scaling law will be similar even for these cases.

Part III : Multiple Dp-branes and U-duality

Chapter 6

D2-branes from Lorentzian BLG model

As we saw in §3.4.1, we can construct an example of Lie 3-algebra as an extension of an arbitrary Lie algebra by adding zero-norm generators. Then, in this chapter, we analyze the BLG model based on this example of Lie 3-algebra. Since a linear combination of zero-norm generators becomes negative-norm one, the ghost field exists in this case.

Fortunately, however, we show that one can treat the field components corresponding to zero-norm generators as non-dynamical *parameters* without breaking supersymmetry nor gauge symmetry. This interpretation, called a new kind of 'Higgs mechanism,' completely removes the ghost field for our example of Lie 3-algebra.

This is a very unusual procedure for removing the ghost field, so one may doubt if this is a justifiable one from the viewpoint of quantum field theory. The detailed justification must be done in the future research. At this moment, we should regard it as the way to find a special aspect of BLG model for convenience. In fact, this kind of Higgs mechanism also compactifies one transverse spatial dimension for M2-branes, and as a result, we obtain multiple D2-branes' theory from BLG model for multiple M2-branes.

Moreover, we also present another derivation of D2-branes' theory from M2-branes. It is based on the construction of a D4-brane from M2-branes through an M5-brane, which we discussed in Chapter 4. There, we first obtain an M5-brane by using the infinite-dimensional Lie 3-algebra with the Nambu-Poisson bracket on 3-dim manifold. Then we compactify one dimension in the internal 3-dim manifold, and wind one direction of an M5-brane along this direction. In this way, we obtain the noncommutative D4-brane's action where the noncommutativity is infinitesimal.

In this chapter, we show that when the internal 2-dim space of this D4-brane is T^2, by suitably choosing the noncommutativity parameter, one may obtain $U(N)$ symmetry on the D2-branes' worldvolume. As a result, we again obtain multiple D2-branes' theory from BLG model by passing another way.

6.1 Notation

BLG action

In Part III, we write the original BLG action as

$$\begin{aligned}
S &= \int_{\mathcal{M}} d^3x\, L = \int_{\mathcal{M}} d^3x\, (L_X + L_\Psi + L_{int} + L_{pot} + L_{CS}), &(6.1)\\
L_X &= -\frac{1}{2}\langle D_\mu X^I, D^\mu X^I\rangle, &(6.2)\\
L_\Psi &= \frac{i}{2}\langle \bar\Psi, \Gamma^\mu D_\mu \Psi\rangle, &(6.3)\\
L_{int} &= \frac{i}{4}\langle \bar\Psi, \Gamma_{IJ}[X^I, X^J, \Psi]\rangle, &(6.4)\\
L_{pot} &= -\frac{1}{12}\langle [X^I, X^J, X^K], [X^I, X^J, X^K]\rangle, &(6.5)\\
L_{CS} &= \frac{1}{2} f^{ABCD} A_{AB}\wedge dA_{CD} + \frac{i}{3} f^{CDA}{}_G f^{EFGB} A_{AB}\wedge A_{CD}\wedge A_{EF}\,, &(6.6)
\end{aligned}$$

where the indices $\mu = 0, 1, 2$ specify the longitudinal directions of M2-branes, $I, J, K = 3, \cdots, 10$ indicate the transverse directions, and the indices $A, B, C, \cdots$ denote components of Lie 3-algebra generators. $\mathcal{M}$ is the M2-branes' worldvolume.

The covariant derivative is

$$(D_\mu \Phi(x))_A = \partial_\mu \Phi_A + f^{CDB}{}_A A_{\mu CD}(x)\Phi_B \tag{6.7}$$

for $\Phi = X^I, \Psi$. The 3-commutator for the Lie 3-algebra in BLG model

$$[T^A, T^B, T^C] = i f^{ABC}{}_D T^D \tag{6.8}$$

must satisfy the fundamental identity and the invariant metric condition. Note that the notation is slightly different from that in Part I and II, in order to make the field $A_{\mu AB}$ Hermite.

Symmetry transformations

The supersymmetry transformations are

$$\begin{aligned}
\delta_\epsilon X^I_A &= i\bar\epsilon\Gamma^I\Psi_A\,,\\
\delta_\epsilon \Psi_A &= D_\mu X^I_A \Gamma^\mu\Gamma^I\epsilon - \frac{1}{6}X^I_B X^J_C X^K_D f^{BCD}{}_A \Gamma^{IJK}\epsilon\,,\\
\delta_\epsilon \tilde A_\mu{}^B{}_A &= i\bar\epsilon\Gamma_\mu\Gamma_I X^I_C \Psi_D f^{CDB}{}_A\,, \qquad \tilde A_\mu{}^B{}_A := A_{\mu CD} f^{CDB}{}_A\,. &(6.9)
\end{aligned}$$

The gauge symmetry for the bosonic fields are written as

$$\delta_\Lambda X^I_A = \Lambda_{CD} f^{CDB}{}_A X^I_B\,, \quad \delta_\Lambda \tilde A_\mu{}^B{}_A = \partial_\mu \tilde\Lambda^B{}_A - \tilde\Lambda^B{}_C \tilde A_\mu{}^C{}_A + \tilde A_\mu{}^B{}_C \tilde\Lambda^C{}_A\,. \tag{6.10}$$

(The gauge transformation of Ψ is the same as X^I.)

6.2 Lorentzian Lie 3-algebra

As we saw in §3.4.1, for any given Lie algebra $\mathcal{G}$

$$[T^i, T^j] = f^{ij}{}_k T^k \tag{6.11}$$

with structure constants $f^{ij}{}_k$ and Killing form h^{ij}, we can define a corresponding Lie 3-algebra as follows. Let the generators of the Lie 3-algebra be denoted $\{T^A\} = \{T^i, u, v\}$ ($i = 1, \cdots, \dim \mathcal{G}$), where T^i's are one-to-one corresponding to the generators of the Lie algebra $\mathcal{G}$. The 3-commutator is defined by

$$\begin{aligned} [v, T^A, T^B] &= 0\,, \\ [u, T^i, T^j] &= i f^{ij}{}_k T^k, \\ [T^i, T^j, T^k] &= -i f^{ijk} v\,, \end{aligned} \tag{6.12}$$

where $f^{ijk} := f^{ij}{}_l h^{lk}$ is totally antisymmetrized. The invariant metric can be defined as

$$\begin{aligned} \langle v, v\rangle = 0\,, \qquad \langle v, u\rangle = 1\,, \qquad \langle v, T^i\rangle = 0\,, \\ \langle u, u\rangle = 0\,, \qquad \langle u, T^i\rangle = 0\,, \qquad \langle T^i, T^j\rangle = h^{ij}\,. \end{aligned} \tag{6.13}$$

Note that the norm of $u + \alpha v$ is -2α, which is negative for $\alpha > 0$. That is why this algebra is called *Lorentzian* Lie 3-algebra.

Mode expansions of BLG fields

The mode expansions of the fields are

$$\begin{aligned} X^I &:= X^I_A T^A = X^I_u u + X^I_v v + \hat{X}^I, \\ \Psi &:= \Psi_A T^A = \Psi_u u + \Psi_v v + \hat{\Psi}\,, \\ A_\mu &:= A_{\mu AB} T^A \otimes T^B \\ &= v \otimes A_{\mu v} - A_{\mu v} \otimes v + u \otimes \hat{A}_\mu - \hat{A}_\mu \otimes u + A_{\mu ij} T^i \otimes T^j, \end{aligned} \tag{6.14}$$

where

$$\hat{X} := X_i T^i, \quad \hat{\Psi} := \Psi_i T^i, \quad A_{\mu v} := A_{\mu v A} T^A, \quad \hat{A}_\mu := 2A_{\mu u i} T^i. \tag{6.15}$$

We also define

$$A'_\mu := A_{\mu ij} f^{ij}{}_k T^k, \tag{6.16}$$

for the last term of (6.14). We will see below that $A_{\mu v}$ are completely decoupled in the BLG model, and X^I_v and Ψ_v are Lagrange multipliers.

Mode expansions of symmetry transformations

The BLG action has $\mathcal{N}=8$ maximal supersymmetry in $d=3$. In terms of the modes, the supersymmetry transformation (6.9) can be written as

$$\begin{aligned}
\delta X^I_u &= i\bar{\epsilon}\Gamma^I\Psi_u\,, \quad \delta X^I_v = i\bar{\epsilon}\Gamma^I\Psi_v\,, \quad \delta\hat{X}^I = i\bar{\epsilon}\Gamma^I\hat{\Psi}\,, \\
\delta\Psi_u &= \partial_\mu X^I_u\Gamma^\mu\Gamma^I\epsilon\,, \\
\delta\Psi_v &= (\partial_\mu X^I_v - \langle A'_\mu \hat{X}^I\rangle)\Gamma^\mu\Gamma^I\epsilon - \frac{1}{3}\langle\hat{X}^I\hat{X}^J\hat{X}^K\rangle\Gamma^{IJK}\epsilon\,, \\
\delta\hat{\Psi} &= \hat{D}_\mu\hat{X}^I\Gamma^\mu\Gamma^I\epsilon - \frac{1}{2}X^I_u[\hat{X}^J,\hat{X}^K]\Gamma^{IJK}\epsilon\,, \\
\delta\hat{A}_\mu &= i\bar{\epsilon}\Gamma_\mu\Gamma_I(X^I_u\hat{\Psi} - \hat{X}^I\Psi_u)\,, \quad \delta A'_\mu = i\bar{\epsilon}\Gamma_\mu\Gamma_I[\hat{X}^I,\hat{\Psi}]\,.
\end{aligned} \tag{6.17}$$

The gauge transformation (6.10) can be written in terms of the modes as

$$\begin{aligned}
&\delta X^I_u = 0\,, \quad \delta X^I_v = \langle\Lambda',\hat{X}^I\rangle\,, \quad \delta\hat{X}^I = [\hat{\Lambda},\hat{X}^I]\,, \\
&\delta\hat{A}_\mu = \partial_\mu\hat{\Lambda} - [\hat{A}_\mu,\hat{\Lambda}]\,, \quad \delta A'_\mu = \partial_\mu\Lambda' - [\hat{A}_\mu,\Lambda'] - [A'_\mu,\hat{\Lambda}]\,,
\end{aligned} \tag{6.18}$$

where

$$\hat{\Lambda} = 2\Lambda_{ui}T^i, \qquad \Lambda' = \Lambda_{ij}f^{ij}{}_kT^k. \tag{6.19}$$

Lagrangian in terms of modes and its symmetry

Plugging the mode expansions (6.14) into the Lagrangian (6.1)–(6.6), we get, up to total derivatives,

$$\begin{aligned}
L &= \Big\langle -\frac{1}{2}(\hat{D}_\mu\hat{X}^I - A'_\mu X^I_u)^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu\hat{D}_\mu\hat{\Psi} + \frac{i}{2}\bar{\Psi}_u\Gamma^\mu A'_\mu\hat{\Psi} \\
&\quad + \frac{i}{2}\bar{\hat{\Psi}}\Gamma_{IJ}X^I_u[\hat{X}^J,\hat{\Psi}] - \frac{i}{2}\bar{\Psi}_u\Gamma_{IJ}\Psi[\hat{X}^I,\hat{X}^J] \\
&\quad + \frac{1}{4}(X^K_u)^2[\hat{X}^I,\hat{X}^J]^2 - \frac{1}{2}(X^I_u[\hat{X}^I,\hat{X}^J])^2 + \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}A'_\lambda\Big\rangle + L_{gh}\,,
\end{aligned} \tag{6.20}$$

$$L_{gh} = -\Big\langle \partial_\mu X^I_u A'_\mu\hat{X}^I + (\partial_\mu X^I_u)(\partial_\mu X^I_v) - \frac{i}{2}\bar{\Psi}_v\Gamma^\mu\partial_\mu\Psi_u\Big\rangle\,, \tag{6.21}$$

where

$$\begin{aligned}
&\hat{D}_\mu X^I := \partial_\mu\hat{X}^I - [\hat{A}_\mu,\hat{X}^I]\,, \qquad \hat{D}_\mu\Psi := \partial_\mu\hat{\Psi} - [\hat{A}_\mu,\hat{\Psi}]\,, \\
&\hat{F}_{\mu\nu} := \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - [\hat{A}_\mu,\hat{A}_\nu]\,.
\end{aligned} \tag{6.22}$$

This Lagrangian is invariant under the *parity transformation*:

$$\begin{aligned}
x^\mu &\to -x^\mu, \qquad \Gamma^\mu \to -\Gamma^\mu, \\
(\hat{X}^I, X^I_u, X^I_v) &\to (\hat{X}^I, -X^I_u, -X^I_v), \\
(\hat{\Psi}, \Psi_u, \Psi_v) &\to (\hat{\Psi}, -\Psi_u, -\Psi_v), \\
(\hat{A}_\mu, A'_\mu) &\to (-\hat{A}_\mu, A'_\mu).
\end{aligned} \tag{6.23}$$

Another symmetry of this model is the *scaling transformation* of the overall coefficient of the Lagrangian. Usually a scaling of the structure constants is equivalent to a scaling of the overall constant factor of the action through a scaling of all fields. This overall factor is then an unfixed coupling, which is undesirable in M-theory. However, the situation is different for Lorentzian Lie 3-algebra. In fact, a scaling of the structure constants

$$f^{ABC}{}_D \;\rightarrow\; g^2 f^{ABC}{}_D \tag{6.24}$$

can be absorbed by the scaling

$$u \rightarrow g^2\, u\,, \quad v \rightarrow g^{-2}\, v\,, \quad T^i \rightarrow T^i, \tag{6.25}$$

which does not change the metric at all. This means that the scaling of the overall coefficient of the Lagrangian is a symmetry. Explicitly, scaling the Lagrangian (6.20) by an overall coefficient $1/g^2$ can be absorbed by the field redefinition

$$\begin{aligned}
(\hat{X}^I, X_u^I, X_v^I) &\rightarrow (g\hat{X}^I,\, g^{-1}X_u^I,\, g^3 X_v^I),\\
(\hat{\Psi}, \Psi_u, \Psi_v) &\rightarrow (g\hat{\Psi},\, g^{-1}\Psi_u,\, g^3\Psi_v),\\
(\hat{A}_\mu, A'_\mu) &\rightarrow (\hat{A}_\mu,\, g^2 A'_\mu).
\end{aligned} \tag{6.26}$$

Hence this Lagrangian has no free parameter at all, which is desirable property for M2-branes' theory.

Ghost fields from $X_{u,v}$ and $\Psi_{u,v}$ can be eliminated by putting a VEV.

Note also that X_v^I and Ψ_v appear only linearly in L_{gh}, and thus they are Lagrange multipliers. Their equations of motion are

$$\partial^2 X_u^I = 0\,, \qquad \Gamma^\mu \partial_\mu \Psi_u = 0\,. \tag{6.27}$$

Hence we treat X_u^I and Ψ_u as classical fields, in the sense that off-shell fluctuations are excluded from the path integral. Actually, we can set

$$X_u^I = \text{const.}, \qquad \Psi_u = 0\,, \tag{6.28}$$

without breaking the supersymmetry (6.17) nor gauge symmetry (6.18).

After we set (6.28), the Lagrangian is given by (6.20) without the last term $L_{\rm gh}$. It is remarkable that the ghost degrees of freedom associated with X_v^I and Ψ_v have totally disappeared for this background. The resulting theory is clearly a well-defined field theory without ghost fields.

The fact that the background (6.28) does not break any symmetry suggests an alternative viewpoint towards the BLG model. That is, we can change the definition of the BLG model by defining X_u^I, Ψ_u as non-dynamical constant parameters fixed by (6.28). The resulting model has as large symmetry as the original definition of the BLG model, but has no ghost fields. In this interpretation, the parameter X_u^I plays the role of coupling constant.

As we pointed out in the beginning of this chapter, this procedure for removing ghost fields is extraordinary. In particular, we must study further whether the process of regarding fields X_u^I, Ψ_u as mere parameters is justified from the viewpoint of quantum field theory.

Comments on why the symmetries are unbroken

Since the assignments (6.28) for a special type of generators preserve all supersymmetry and gauge symmetry, one can take the viewpoint that these variables are non-dynamical *by definition*. We have seen earlier that this interpretation removes the ghost field from the BLG model for Lorentzian Lie 3-algebra.

The origin of the decoupling of the ghost field comes from the specific way that Lorentzian generators appear in the Lie 3-algebra. Namely, the generator v is the center of the Lie 3-algebra and u is not produced in any 3-commutators. This property ensures that the system is invariant under the translation of the scalar fields X_u^I.

The decoupling of the ghost field can be made more rigorous [85, 86] by gauging this global symmetry. Namely, by adding extra gauge fields C_μ and χ through

$$L_{new} = -\bar{\Psi}_u \chi + \partial^\mu X_u^I C_\mu^I \,, \tag{6.29}$$

we have an extra gauge symmetry:

$$\delta X_v^I = \Lambda^I, \qquad \delta C_\mu^I = \partial_\mu \Lambda^I, \qquad \delta \Psi_v = \eta, \qquad \delta \chi = i\Gamma^\mu \partial_\mu \eta \,. \tag{6.30}$$

It enable us to put $X_v^I = \Psi_v = 0$. The equations of motion by variation of C_μ^I and χ give the assignment (6.28) correctly.

6.3 D2-branes' action from M2-branes

In the previous section, we saw that the ghost field in BLG model can be completely removed. Now we derive this *physical* theory, which can be interpreted as D2-branes' theory.

D2-branes' action : M2-branes to D2-branes

Let us now consider the theory defined by Lagrangian (6.20) for the particular background

$$X_u^I = \lambda^I, \qquad \Psi_u = 0 \,, \tag{6.31}$$

where λ is a constant vector. Without loss of generality, for space-like vector λ, we can choose λ to lie on the direction of X^{10}

$$\lambda^I = \lambda \, \delta_{10}^I \,. \tag{6.32}$$

As we mentioned above, fixing the fields X_u^I and Ψ_u by (6.31) removes the ghost term L_{gh} from the Lagrangian. We can now integrate over A' and find

$$L_{\text{eff}} = -\frac{1}{2}(\hat{D}_\mu \hat{X}^I)^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{\lambda^2}{4}[\hat{X}^I, \hat{X}^J]^2 + \frac{i\lambda}{2}\bar{\hat{\Psi}}\Gamma_I[X^I, \hat{\Psi}] - \frac{1}{4\lambda^2}\hat{F}_{\mu\nu}^2, \tag{6.33}$$

where $I, J = 3, \cdots, 9$. This is nothing but supersymmetric Yang-Mills Lagrangian in 3-dim spacetime.

It is very interesting to note that all degrees of freedom in the spatial coordinate X^{10} have totally disappeared from both the kinetic term and the potential term of the action. It is fully decoupled from the Lagrangian for the particular background under consideration. This vanishing X^{10}'s degree of freedom is eaten by the gauge field, which are changed from Chern-Simons to Yang-Mills gauge field.

The VEV λ corresponds to the compactification radius.

Let us now recall that when M-theory is compactified on a circle, it is equivalent to type IIA superstring theory and M2-branes are matched with D2-branes. The background (6.31) considered above is reminiscent of the novel 'Higgs mechanism' in [25]. It was originally proposed to describe the effect of compactification of X^{10}, and later found to correspond to a large k limit of a $\mathbb{Z}_{2k}$ M-fold [87, 88].

The M-theory parameters can be converted to those of type IIA superstring theory via

$$R = g_s l_s \quad \text{and} \quad l_p = g_s l_s^3 \,, \tag{6.34}$$

where R is a radius of the compactified circle, l_p is 11-dim Planck length, g_s is the string coupling and l_s is the string length. It is natural to consider that one set 11-dim Planck length l_p and M2-brane's tension $T_2 = l_p^{-3}$ to 1 in the original BLG Lagrangian. Thus we see that the Lagrangian (6.33) is exactly the same as the low-energy effective action of multiple D2-branes, if λ is given by the radius of the compactified dimension

$$\lambda = R\, l_p^{-3/2}. \tag{6.35}$$

Furthermore, in our setup, the translation symmetry of the center-of-mass coordinates corresponding to the $u(1)$ factor of Lie algebra $\mathcal{G}$ is manifest. This can be regarded as a strong signature of the reduction of M2-branes to D2-branes due to a compactification of the M-theory on S^1.

6.4 D2-branes' action from an M5-brane

In this section, we present a very different derivation of D2-branes from M2-branes. It is based on the derivation of M5-brane from BLG model in Chapter 4.

Step 1 : M2-branes to M5-brane

We consider a 3-dim manifold $\mathcal{N}$ equipped with the Nambu-Poisson structure. By choosing the appropriate local coordinates $y^{\dot{\mu}}$ ($\dot{\mu} = \dot{1}, \dot{2}, \dot{3}$), one may construct an infinite-dimensional Lie 3-algebra from the basis χ^a ($a = 1, 2, 3, \cdots, \infty$) of functions on $\mathcal{N}$ as

$$\left\{\chi^a, \chi^b, \chi^c\right\} = \sum_d f^{abc}{}_d\, \chi^d \,, \quad \{f_1, f_2, f_3\} = \sum_{\dot{\mu},\dot{\nu},\dot{\lambda}} \epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}} \frac{\partial f_1}{\partial y^{\dot{\mu}}} \frac{\partial f_2}{\partial y^{\dot{\nu}}} \frac{\partial f_3}{\partial y^{\dot{\lambda}}} \,. \tag{6.36}$$

From the property of the Nambu-Poisson structure, this Lie 3-algebra satisfies the fundamental identity with positive definite and invariant metric for the generators

$$\langle \chi^a, \chi^b \rangle = \int_{\mathcal{N}} d^3y\, \chi^a(y)\chi^b(y)\,. \tag{6.37}$$

By the summation of these generators with the BLG fields

$$\begin{aligned} X^I(x,y) &= \sum_a X^I_a(x)\chi^a(y)\,, \\ \Psi(x,y) &= \sum_a \Psi_a(x)\chi^a(y)\,, \\ A_\mu(x,y,y') &= \sum_{a,b} A_{\mu ab}(x)\chi^a(y)\chi^b(y'), \end{aligned} \tag{6.38}$$

we obtain the fields on the 6-dim manifold $\mathcal{M}\times\mathcal{N}$ where $\mathcal{M}$ is the worldvolume of the original M2-branes. We note that the gauge field $A_\mu(x,y,y')$ appears to depend on two points on $\mathcal{N}$. However, if we examine the action carefully, one can show that it depends on $A_\mu(x,y,y')$ only through

$$b_{\mu\dot{\nu}}(x,y) = \frac{\partial}{\partial y'^{\dot{\nu}}} A_\mu(x,y,y')\bigg|_{y'=y}\,. \tag{6.39}$$

Therefore, the action can be written in terms of the local fields. It was shown that the BLG Lagrangian, after suitable field redefinitions, describes the field theory on a single M5-brane which properly includes the self-dual 2-form field.

Step 2 : M5-brane to D4-brane

As we discuss in §4.6, in order to obtain a D4-brane from M2-branes, we have to wind $X^{\dot{3}}$ around the compact $y^{\dot{3}}$ direction and impose the constraints that the other fields do not depend on $y^{\dot{3}}$. Other than that, we use the same field configuration:

$$\begin{aligned} &X^{\dot{3}} = y^{\dot{3}}\,, \quad X^{\dot{\alpha}} = y^{\dot{\alpha}} + \epsilon_{\dot{\alpha}\dot{\beta}}a_{\dot{\beta}}(x,y)\,, \\ &a_\mu(x,y) = b_{\mu\dot{3}}(x,y)\,, \quad \tilde{a}_\lambda(x,y) = \epsilon_{\dot{\alpha}\dot{\beta}}\partial_{\dot{\alpha}}b_{\lambda\dot{\beta}}\,, \\ &\partial_{\dot{3}}X^i = \partial_{\dot{3}}\Psi = \partial_{\dot{3}}a_{\dot{\beta}} = \partial_{\dot{3}}a_\mu = \partial_{\dot{3}}\tilde{a}_\lambda = 0\,, \end{aligned} \tag{6.40}$$

where we use the indices $\dot{\alpha}, \dot{\beta}, \cdots$ to denote $\dot{1}, \dot{2}$ such that the worldvolume index of a D4-brane is μ and $\dot{\alpha}$. We use the notation $i = 1, \cdots, 5$ for the transverse directions. Note that $b_{\mu\dot{\nu}}$ appears only through a_μ and $\tilde{a}_\mu$.

Various terms of the D4-brane's action can be computed straightforwardly. After integrating out the auxiliary field $\tilde{a}_\mu$, and neglecting the constant and total derivative terms, we obtain the D4-brane's action as

$$S = 2\pi R \int d^5x \left[-\frac{1}{4}F^2_{\underline{\mu}\underline{\nu}} - \frac{1}{2}(\mathcal{D}_{\underline{\mu}}X^i)^2 + \frac{i}{2}\bar{\Psi}\Gamma^{\underline{\mu}}\mathcal{D}_{\underline{\mu}}\Psi - \frac{1}{4}\{X^i, X^j\}^2 + \frac{i}{2}\bar{\Psi}\Gamma_i\{X^i, \Psi\} \right] \tag{6.41}$$

where $\underline{\mu}, \underline{\nu}, \cdots$ are the integrated indices for μ, ν and $\dot{\alpha}, \dot{\beta}$ running from 0 to 4. R is the radius of the compactified direction. Since the integrand does not depend on $y^{\dot{3}}$, we obtain overall factor of $\int dy^{\dot{3}} = 2\pi R$. The definition of the field strength and the covariant derivatives are

$$\begin{aligned} F_{\underline{\mu}\underline{\nu}} &= \partial_{\underline{\mu}} A_{\underline{\nu}} - \partial_{\underline{\nu}} A_{\underline{\mu}} + \{A_{\underline{\mu}}, A_{\underline{\nu}}\}\,, \\ \mathcal{D}_{\underline{\mu}} X^i &= \partial_{\underline{\mu}} X^i + \{A_{\underline{\mu}}, X^i\}\,, \quad \mathcal{D}_{\underline{\mu}}\Psi = \partial_{\underline{\mu}} X^i + \{A_{\underline{\mu}}, \Psi\}\,. \end{aligned} \tag{6.42}$$

While we expect to have the abelian $U(1)$ gauge field on the D4-brane's worldvolume, we have everywhere the Poisson bracket

$$\{f, g\} = \sum_{\dot{\alpha},\dot{\beta}=\dot{1},\dot{2}} \epsilon_{\dot{\alpha}\dot{\beta}}\, \partial_{\dot{\alpha}} f\, \partial_{\dot{\beta}} g\,. \tag{6.43}$$

We note that $A_{\underline{\mu}} = a_\mu, a_{\dot{\alpha}}$ is not exactly the commutative U(1) gauge field, but it includes noncommutativity in $\underline{\mu} = 3, 4$ directions (originally $\dot{\alpha}$ directions).

We also note that in the computation, there are no ambiguities associated with the inner product. The origin of the noncommutativity is obvious. It comes from the Nambu-Poisson bracket where the space of the function is truncated to

$$\{y^3\} \cup C(\mathcal{N}')\,, \tag{6.44}$$

where we decompose $\mathcal{N}$ into $y^{\dot{3}}$ direction and $\mathcal{N}'$ described by $y^{\dot{1},\dot{2}}$. For $f_i(y^{\dot{1}}, y^{\dot{2}}) \in C(\mathcal{N}')$, the Nambu-Poisson bracket becomes

$$\{y^{\dot{3}}, f_1, f_2\}_{NP} = \{f_1, f_2\}\ , \quad \{f_1, f_2, f_3\}_{NP} = 0\,, \quad \text{otherwise} = 0\,. \tag{6.45}$$

The commutator terms in the Lagrangian come from this algebra. This algebra turns out to be identical to Lie 3-algebra (6.12) if we put λ to zero. The generator that corresponds to u is $y^{\dot{3}}$, which describes the winding of M5-brane's worldvolume around S^1. This discussion is closely related to Lie 3-algebra with the "central extension" of Nambu-Poisson bracket (3.103).

Step 3 : D4-brane to D2-branes

The Poisson bracket $\{f, g\}$ can be obtained from the matrix algebra, when the matrix size N is infinite. By using the standard argument (see, for example, [89–92]), it is easy to claim that the D4-brane's action which we just obtained can be regarded as describing an infinite number of D2-branes.

However, in order to obtain the finite N theory on D2-brane, this is not sufficient. We need to quantize the Nambu bracket. In general, the quantum Nambu bracket is very difficult to define, as we discussed in the beginning of Chapter 5. However, for the truncated Hilbert space (6.44), this is actually possible. We deform the Nambu-Poisson bracket by

$$[f_1, f_2, f_3]_{QN} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} (f_i \star f_j)\, \partial_3 f_k\,, \tag{6.46}$$

where $\star$ is the Moyal product

$$(f \star g)(y^{\dot{1}}, y^{\dot{2}}) = \exp(i\epsilon_{\dot{\alpha}\dot{\beta}}\theta\partial_{y^{\dot{\alpha}}}\partial_{z^{\dot{\beta}}})f(y^{\dot{1}}, y^{\dot{2}})g(z^{\dot{1}}, z^{\dot{2}})\Big|_{z=y} . \tag{6.47}$$

It does not satisfy the fundamental identity, when we consider $C(\mathcal{N})$ as a whole. However, if we restrict the generators to (6.44), we can recover the fundamental identity. If we take $\mathcal{N}'$ as T^2 and quantize θ suitably, the quantum T^2 reduces to the $U(N)$ algebra

$$UV = VU\omega, \quad \omega^N = 1\,, \quad U^N = V^N = 1\,. \tag{6.48}$$

In this case, the quantum Nambu-Poisson bracket reduces to the one-generator extension of $U(N)$ algebra

$$[u, T^i, T^j] = f^{ij}{}_k T^k \quad [T^i, T^j, T^k] = 0\,. \tag{6.49}$$

Thus finally, the multiple D2-branes' action can be obtained by expanding the functions in $y^{\dot{1},\dot{2}}$ directions by U, V and replacing the covariant derivative $\mathcal{D}_{\dot{\alpha}}$ by the commutators

$$\mathcal{D}_{\dot{\alpha}}\Phi \quad \to \quad [X_{\dot{\alpha}}, \Phi] \tag{6.50}$$

for general Φ.

6.5 Summary

In this chapter, we study two approaches to obtain multiple D2-branes' action from the BLG theory. In the first approach, one defines Lie 3-algebra which contains generators of a given Lie algebra. Such an extension inevitably contains a generator with negative norm. We argued that by suitably choosing such extension, one might restrict the field associated with it to constant or zero, while keeping almost all of the symmetry of BLG theory. Such truncation leads to a new kind of 'Higgs mechanism' [25] and generates the standard kinetic term for the Yang-Mills gauge fields on the multiple D2-branes' worldvolume.

In the second derivation of multiple D2-branes, we found that the extra generator has a simple physical origin, *i.e.* the winding of M5-brane around S^1 which defines the reduction from M-theory to type IIA superstring theory.

However, in these approaches, the physical meaning of the extra generators are still not very clear. In particular, our understanding is hindered by the fact that the limit $\lambda \to \infty$ doesn't mean the expansion of 11th direction. We hope that this point will be revealed by future works.

Chapter 7

Dp-branes from General Lorentzian BLG model

In the previous chapter, we discuss the BLG model with Lorentzian Lie 3-algebra, which has a ghost field but it can be completely decoupled by 'Higgs mechanism.' It was realized that the inclusion of the Lorentzian generators is associated with the compactification of a spatial dimension, and this Lorentzian model reproduces the multiple D2-branes' worldvolume theory in type IIA string theory.

In this chapter, we study some generalizations of such Lorentzian Lie 3-algebra for which ghost fields can still be decoupled. We analyze the BLG model with these kinds of Lie 3-algebra, and show that we obtain the multiple Dp-branes' theory on $(p-2)$-dim torus T^{p-2} ($p \geq 3$), when we choose a suitable example of Lie 3-algebra. The general argument about these kinds of algebra has already done in §3.4.2. According to that discussion, we find that there are the following concrete examples.

Here we denote the generators of the general Lorentzian Lie 3-algebra as e^i ($i = 1, \cdots, M$) and u_a, v^a ($a = 1, \cdots, N$). When we require that the algebra should satisfy the fundamental identity and invariant metric condition, the form of 3-commutators becomes

$$\begin{aligned}
[u_a, u_b, u_c] &= K^i_{abc} e_i + L_{abcd} v^d, \\
[u_a, u_b, e^i] &= J^{ij}_{ab} e_j - K^i_{abc} v^c, \\
[u_a, e^i, e^j] &= J^{ij}_{ab} v^b + f^{ijk}_a e_k\,, \\
[e^i, e^j, e^k] &= -f^{ijk}_a v^a,
\end{aligned} \tag{7.1}$$

and, after the suitable change of basis, the nontrivial part of the metric is given by

$$\langle e^i, e^j \rangle = g^{ij}\,, \quad \langle u_a, v^b \rangle = \delta^b_a\,, \tag{7.2}$$

where g^{ij} is the Killing form of the (direct sum of) Lie algebras $g = g_1 \oplus \cdots \oplus g_n$, which is not necessarily positive-definite.

Based on the analysis in §3.4.2, which gives the conditions for the structure constants f_a^{ijk}, J_{ab}^{ij} and K_{abc}^i, we can obtain the following concrete examples of general Lorentzian Lie 3-algebras:

1. $M = 2$, $J_{ab}^{ij} = \epsilon_{ab}J^{ij}$ $(i, j = 1, \cdots, n)$, others $= 0$: This is the simplest finite-dimensional example where some character of the Lorentzian symmetry is displayed. Namely, the BLG model defines the massive $\mathcal{N} = 8$ supersymmetric vector multiplets [7].

2. $M = 2$, $J_{ab}^{ij} = \epsilon_{ab}J^{ij}$, $f_1^{ijk} \neq 0$, others $= 0$ (§7.1) : This is the simplest nontrivial example which contains the interaction. We will present our result by studying the Yang-Mills system where the gauge symmetry is defined by Lorentzian Lie algebra. This is possible since the Lie 3-algebra can be written in the form (7.4). In such case, one can skip the discussion of eliminating one pair of ghost fields. It also illuminate the structure of the Yang-Mills system with Lorentzian Lie algebra.

3. Lie 3-algebra associated with affine Kac-Moody Lie algebra (§7.2) : This is the special case of the previous example, where the Lorentzian Lie algebra is given by the affine Lie algebra. In this case, BLG action describes the multiple D3-branes on a circle. It illuminates how Kaluza-Klein mass (on the circle) is generated by the ghost fields.

4. Lie 3-algebra associated with general loop algebras (§7.3) : This is the generalization of the previous example, and BLG action with this algebra describes the multiple Dp-branes on the $(p-2)$-dim torus with constant B-field flux.

5. Lorentzian Lie 3-algebra with $F^{ijkl} \neq 0$: We have already presented the concrete form of this algebra in eq. (3.103). BLG action in this case describes the single M5-brane's system, just as in Chapter 4 [7].

7.1 Massive super Yang-Mills theory

In this section, we consider the simplest nontrivial (*i.e.* interaction terms exist) example of general Lie 3-algebra, such that

$$f_1^{ijk} \neq 0\,, \quad f_2^{ijk} = 0\,, \quad J^{ij} \neq 0\,. \tag{7.3}$$

In this case, we can rewrite the 3-commutator as

$$\begin{aligned} [u_1, T^A, T^B] &= f^{AB}{}_C T^C\,, \\ [v_1, T^A, T^B] &= 0\,, \\ [T^A, T^B, T^C] &= -h^{CD} f^{AB}{}_D v_1\,, \end{aligned} \tag{7.4}$$

where $A, B, \cdots = \{e^i, u_2, v_2\}$, $f^{ijk} := f_1^{ijk}$ and $f^{u_2 ij} := J^{ij}$. This algebra is similar to that of Lorentzian Lie 3-algebra, that is, a (u_1, v_1)-extension of Lie 3-algebra (6.12). A different point is that this Lie 3-algebra $\{T^A\} = \{e^i, u_2, v_2\}$ has Lorentzian generators, while eq. (6.12) is a standard (positive-definite) Lie algebra.

In the following, we denote generators of this algebra as $\{e^i, u, v\}$, instead of $\{e^i, u_2, v_2\}$. Then the metric (or Killing form) and structure constants are

$$\begin{aligned} &\langle e^i, e^j \rangle = \delta^{ij}, \quad \langle u, v \rangle = 1\,; \\ &f^{ijk}, \quad f^{uij} = J^{ij}, \quad \text{otherwise} = 0\,, \end{aligned} \tag{7.5}$$

where $i = 1, \cdots, N$. The Jacobi identity is written as

$$\begin{aligned} &f^{ijl} f^{lkm} + f^{jkl} f^{lim} + f^{kil} f^{ljm} = 0\,, \\ &f^{ijl} J^{lk} + f^{jkl} J^{li} + f^{kil} J^{lj} = 0\,, \end{aligned} \tag{7.6}$$

which are consistent with the fundamental identity for the Lie 3-algebra $\{T^i, u_{1,2}, v_{1,2}\}$. This is the simplest "Lorentzian extension" of Lie algebra

$$[e^i, e^j] = f^{ij}{}_k e^j + J^{ij} v\,, \quad [u, e^i] = J^{ij} e^j\,. \tag{7.7}$$

This extension is trivial, if J^{ij} is an inner automorphism

$$J^{ij} = f^{ij}{}_k \alpha^k\,, \tag{7.8}$$

for some parameter α^k. One may then redefine the basis

$$e'^i = e^i + \alpha^i v\,, \quad u' = u - \alpha_i e^i\,, \quad v' = v\,, \tag{7.9}$$

such that the algebra becomes the direct sum of the original Lie algebra and Lorentzian pairs:

$$\begin{aligned} &[e'^i, e'^j] = f^{ij}{}_k e'^k\,, \quad \text{other commutators} = 0\,; \\ &\langle e'^i, e'^j \rangle = \delta^{ij}\,, \quad \langle u', v' \rangle = 1\,, \quad \text{other inner products} = 0\,. \end{aligned} \tag{7.10}$$

In the following, we will focus on the nontrivial case where J gives an infinitesimal outer automorphism.

As we discussed in the previous chapter, the BLG model with Lorentzian Lie 3-algebra results in super Yang-Mills theory with Lie algebra. So, let us consider the Yang-Mills theory coupled with scalar fields X^I ($I = 1, \cdots, n$) and spinor fields Ψ based on this extended algebra:

$$\begin{aligned} L \;\; &= \;\; -\frac{1}{2}\langle D_\mu X^I, D^\mu X^I \rangle + \frac{\lambda_1^2}{4}\langle [X^I, X^J], [X^I, X^J] \rangle \\ &\quad\; +\frac{i}{2}\langle \bar{\Psi}, \Gamma^\mu D_\mu \Psi \rangle + \frac{i\lambda_1}{2}\langle \bar{\Psi}, \Gamma_I [X^I, \Psi] \rangle - \frac{1}{4\lambda_1^2}\langle F_{\mu\nu} F^{\mu\nu} \rangle \\ &=: \;\; L_X + L_{pot} + L_\Psi + L_{int} + L_A\,, \end{aligned} \tag{7.11}$$

where X^I takes the adjoint representation

$$\begin{aligned}
X^I &= X^I_i e^i + X^I_u u + X^I_v v\,, \\
(D_\mu X^I)_i &= \partial_\mu X^I_i - f^{jk}{}_i A_{\mu j} X^I_k - J^{ji} C_\mu X^I_j + J^{ji} A_{\mu j} X^I_u \\
&=: (\hat{D}_\mu X^I)_i + J^{ji} A_{\mu j} X^I_u\,, \\
(D_\mu X^I)_u &= \partial_\mu X^I_u\,, \\
(D_\mu X^I)_v &= \partial_\mu X^I_v + J^{ij} A_{\mu i} X^I_j\,, \\
A_{\mu u} &=: C_\mu\,, \quad A_{\mu v} =: B_\mu\,,
\end{aligned} \tag{7.12}$$

and similar expressions for Ψ. The covariant derivative corresponding to the gauge symmetry generated by e^i should thus be defined as

$$\hat{D}_\mu = \partial_\mu - C_\mu \mathcal{D}_u - A_{\mu i} e^i, \tag{7.13}$$

where $\mathcal{D}_u$ is the derivation defined by J:

$$\mathcal{D}_u(e^i) = J^{ij} e^j. \tag{7.14}$$

On the right hand side of (7.13), e^i is used to imply the adjoint action of e^i, namely $e^i(x) = [e^i, x]$. The gauge transformation is written as

$$\begin{aligned}
\delta\Phi_i &= f^{jk}{}_i \epsilon_j \Phi_k + J^{ki} \gamma \Phi_k - J^{ji} \epsilon_j \Phi_u\,, \\
\delta\Phi_u &= 0\,, \\
\delta A_{\mu i} &= \partial_\mu \epsilon_i + f^{jk}{}_i \epsilon_j A_{\mu k} + J^{ki} \gamma A_{\mu k} - J^{ji} \epsilon_j C_\mu \\
&=: (\hat{D}_\mu \epsilon)_j + J^{ji} \gamma A_{\mu j}
\end{aligned} \tag{7.15}$$

for $\Phi = X^I, \Psi$.

Lagrangian : massive super Yang-Mills theory with interaction

The kinetic term for X^I becomes

$$L_X = \frac{1}{2}(\hat{D}_\mu X^I_i + J^{ji} A_{\mu j} X^I_u)^2 + \partial^\mu X^I_u (\partial_\mu X^I_v - J^{ij} A_{\mu i} X^I_j)\,. \tag{7.16}$$

The variation of X^I_v gives $\partial^2 X^I_u = 0$. So we take it as a constant, as in the previous chapter,

$$X^I_u = \lambda_2 \delta_{I1}\,. \tag{7.17}$$

After imposing this VEV, this term becomes

$$L_X = -\frac{1}{2}\sum_{I'=2}^{n} (\hat{D}_\mu X^{I'}_i)^2 - \frac{1}{2\lambda_1^2} F^2_{\mu u}\,, \tag{7.18}$$

where

$$F_{\mu u} := [\hat{D}_\mu, \hat{D}_u]\,, \quad \hat{D}_u := \lambda_1(\lambda_2 \mathcal{D}_u + X^1_i e^i). \tag{7.19}$$

We are thus led to interpret $\mathcal{D}_u$ (or J) as the derivative of a certain noncommutative space in the direction of X_u. The situation here is reminiscent of the result of quotient conditions in the context of Matrix Models in dealing with orbifolds and orientifolds [93].

In analogy, since we have taken the VEV of X_u to be in the direction of X^1, X^1_j plays the role of a gauge potential and J_{ij} that of a covariant derivative on a noncommutative space, and thus $\hat{D}_u$ mimics a covariant derivative. We will see in the next section that for the compactification on a circle, $\hat{D}_u$ is indeed the covariant derivative in the compactified direction.

If we fix the gauge by $X^1_i = 0$, the second term of (7.18) becomes

$$-\frac{\lambda_2^2}{2}(J^2)_{ij}A_{\mu i}A_{\mu j}\,. \tag{7.20}$$

This is the mass term for vector bosons.

The potential term is

$$L_{pot} = \frac{\lambda_1^2}{4}\sum_{I',J'=2}^{n}[X^{I'},X^{J'}]^2 - \frac{1}{2}\sum_{J'=2}^{n}(\hat{D}_u X^{J'})^2. \tag{7.21}$$

If we gauge away X^1_i using the gauge symmetry, the last term above is simply

$$-\frac{\lambda_1^2\lambda_2^2}{2}\sum_{J'=2}^{n}(J^2)_{ij}X_i^{J'}X_j^{J'}, \tag{7.22}$$

which gives the mass term for $X^{J'}$.[1]

The kinetic term for the gauge field becomes

$$-\frac{1}{4\lambda_1^2}\langle F_{\mu\nu},F^{\mu\nu}\rangle = -\frac{1}{4\lambda_1^2}\left\{(F_{\mu\nu i})^2 + F_{\mu\nu u}F^{\mu\nu}{}_v\right\}, \tag{7.23}$$

where

$$\begin{aligned} F_{\mu\nu i} &= \partial_\mu A_{\nu i} - \partial_\nu A_{\mu i} - f^{jk}{}_i A_{\mu j}A_{\nu k} + J^{ij}(C_\mu A_{j\nu} - C_\nu A_{j\mu})\,, \\ F_{\mu\nu u} &= \partial_\mu C_\nu - \partial_\nu C_\mu\,, \\ F_{\mu\nu v} &= \partial_\mu B_\nu - \partial_\nu B_\mu - J^{ij}A_{\mu i}A_{\nu j}\,. \end{aligned} \tag{7.24}$$

Variation of gauge field B_μ gives a free equation of motion for C_μ,

$$\partial^\mu\partial_{[\mu}C_{\nu]} = 0\,. \tag{7.25}$$

If we start from the original BLG action (6.1)–(6.6), we have slightly different Lagrangian

$$L_{A'C} = \epsilon_{\mu\nu\lambda}A'_\mu\partial_\nu C_\lambda, \tag{7.26}$$

[1]If J is an inner automorphism, *i.e.* $J^{ki} = f^{jk}{}_i\mu_j$, one may shift $X^1_j = -\mu_j$ to absorb J in X^1. This is consistent with our comment above that J can be redefined away if it corresponds to an inner automorphism.

where A'_μ is an auxiliary field.

From the viewpoint of the super Yang-Mills theory, although it is not presented from the beginning, one can add this term as a way to gauge the global symmetry of translation of C_μ, analogous to (6.29), where we gauged the translation of X_u and Ψ_u. By variation of A'_μ, C_μ becomes topological and pure gauge. Hence we should set C_ν to be a constant. It can be interpreted as the projection of the "u"-direction on the D-branes' worldvolume, while X_u^I is the projection of the u-direction in the transverse directions.

On the fermionic parts, after setting the VEV to $\Psi_u = 0$ as in the previous chapter, they become

$$L_\Psi = \frac{i}{2}\langle\bar{\Psi}, \Gamma^\mu \hat{D}_\mu \Psi\rangle, \tag{7.27}$$

and

$$L_{int} = \sum_{I'=2}^{n} \frac{i\lambda_1}{2}\langle\bar{\Psi}_i, \Gamma_{I'}[X_j^{I'}, \Psi_k]\rangle + \frac{i}{2}\bar{\Psi}_i\Gamma_1\hat{D}_u\Psi_i. \tag{7.28}$$

In the gauge $X_i^1 = 0$, the second term becomes the mass term for the fermions with their masses given by the matrix $\lambda_1\lambda_2 J$.

To summarize, in the gauge $X^1 = 0$,

$$\begin{aligned}
L &= L_X + L_\Psi + L_{int} + L_A\,, \\
L_X &= \sum_{I',J'=2}^{n} -\frac{1}{2}(\hat{D}_\mu X_i^{I'})^2 + \frac{\lambda_1^2\lambda_2^2}{2}X_i^{I'}(J^2)_{ij}X_j^{I'}\,, \\
L_\Psi &= \sum_{I'=2}^{n} \frac{i}{2}\bar{\Psi}\Gamma^\mu\hat{D}_\mu\Psi - \frac{\lambda_1\lambda_2}{2}\bar{\Psi}_i(i\Gamma_1)J^{ij}\Psi_j\,, \\
L_{int} &= \sum_{I',J'=2}^{n} \frac{\lambda_1^2}{4}[X^{I'}, X^{J'}]^2 + \frac{i\lambda_1}{2}\langle\bar{\Psi}, \Gamma_{I'}[X^{I'}, \Psi]\rangle\,, \\
L_A &= -\frac{1}{4\lambda_1^2}F_{\mu\nu}^2 - \frac{\lambda_2^2}{2}(J^2)_{ij}A'_{\mu i}A'_{\mu j}\,,
\end{aligned} \tag{7.29}$$

which is of the form of a massive super Yang-Mills theory with the mass matrix $\lambda_1\lambda_2 J_{ij}$.

7.2 Dp-branes to D$(p+1)$-branes

In this section, we consider an example of the general theory studied in the previous section. We consider the Kac-Moody algebra as an example of the Lorentzian extension of a Lie algebra, and show that the super Yang-Mills theory with the gauge symmetry generated by the Kac-Moody algebra is equivalent to a super Yang-Mills theory with a finite-dimensional gauge group on a base space of higher dimensions.

We consider Kac-Moody algebra as an example.

Here we consider a concrete example of Lie 3-algebra which is defined as (7.4) where Lie algebra $\mathcal{G}$ itself is a Lorentzian Lie algebra. The simplest example is when $\mathcal{G}$ is the affine Lie algebra $\hat{g}$,

$$\begin{aligned} &[u, T^a_m] = mT^a_m\,, \\ &[T^a_m, T^b_n] = mvg^{ab}\delta_{m+n} + if^{ab}{}_c T^c_{m+n}\,, \\ &[v, u] = [v, T^a_m] = 0\,, \end{aligned} \tag{7.30}$$

where $a, b, c = 1, \cdots, \dim g$, $m, n \in \mathbf{Z}$ and g^{ab} is the Killing form of a compact Lie algebra g. This algebra has an invariant metric

$$\langle T^a_m, T^b_n\rangle = g^{ab}\delta_{m+n}\,, \quad \langle u, v\rangle = 1\,. \tag{7.31}$$

We note that the generator v is the center of Kac-Moody algebra and usually taken as a quantized c-number. Here we identify it as a nontrivial generator. On the other hand, the generator u gives the level (or $-L_0$ in the Virasoro algebra). While T^a_n has a positive-definite metric, the generators u, v have a negative-norm generator.[2]

We follow the method in §7.1, where we use the super Yang-Mills system on D2-branes with gauge symmetry $\hat{g}$ by using the Higgs mechanism for one Lorentzian pair.

In fact, the following analysis can be carried out for any Dp-branes' system and provides a general mechanism of the gauge theory with affine gauge symmetry. What we are going to show is that the Dp-branes' system whose gauge symmetry is $\hat{g}$ can be identified with D$(p+1)$-branes' system with Lie algebra g.

We start here from $(p+1)$-dim super Yang-Mills theory.

If we start from the BLG model directly, we have a different perspective in which we will treat more general argument given in the next section. Then we start from the ordinary super Yang-Mills action

$$\begin{aligned} L \;=\; & -\frac{1}{4\lambda^2}\langle F_{\mu\nu}, F^{\mu\nu}\rangle - \frac{1}{2}\langle D_\mu X^I, D^\mu X^I\rangle + \frac{\lambda^2}{4}\langle [X^I, X^J], [X^I, X^J]\rangle \\ & + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu \Psi + \frac{i\lambda}{2}\bar{\Psi}\Gamma_I [X^I, \Psi]\,, \end{aligned} \tag{7.32}$$

where $X^I(x)$ $(I = 1, \cdots, D)$ are the scalar fields and $\Psi(x)$ is the spinor field. Both are in the adjoint representation of g. The worldvolume index is given as $\mu, \nu = 0, \cdots, p$. The covariant derivative and the field strength are defined (only in this subsection) as

$$D_\mu \Phi := \partial_\mu \Phi - i[A_\mu, \Phi]\,, \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\,, \tag{7.33}$$

for $\Phi = X^I, \Psi$.

[2] We note that a different type of Lie 3-algebra based on Kac-Moody symmetry was obtained in [94].

Mode expansions

We consider the following component expansion

$$\begin{aligned}
A_\mu &= A_{\mu(a,n)}T^a_n + B_\mu v + C_\mu u\,, \\
X^I &= X^I_{(a,n)}T^a_n + X^I_u u + X^I_v v\,, \\
\Psi &= \Psi_{(a,n)}T^a_n + \Psi_u u + \Psi_v v\,.
\end{aligned} \tag{7.34}$$

Various components of the covariant derivative and the field strength are given as

$$\begin{aligned}
(D_\mu X^I)_{(an)} &= \partial_\mu X^I_{an} + f^{bc}{}_a \sum_m A_{\mu(b,m)}X^I_{(c,n-m)} - nC_\mu X^I_{(a,n)} \\
&\quad + inA_{\mu(a,n)}X^I_u \\
&=: (\hat{D}_\mu X^I)_{(a,n)} + inA_{\mu(a,n)}X^I_u\,, \\
(D_\mu X^I)_u &= \partial_\mu X^I_u\,, \\
(D_\mu X^I)_v &= \partial_\mu X^I_v + \sum_m img^{ab}A_{\mu(a,m)}X^I_{(b,-m)}\,,
\end{aligned} \tag{7.35}$$

$$\begin{aligned}
(F_{\mu\nu})_{(a,n)} &= \partial_\mu A_{\nu(a,n)} - \partial_\nu A_{\mu(a,n)} + f^{bc}{}_a \sum_m A_{\mu(b,m)}A_{\nu(c,n-m)}\,, \\
(F_{\mu\nu})_u &= \partial_\mu C_\nu - \partial_\nu C_\mu\,, \\
(F_{\mu\nu})_v &= \partial_\mu B_\nu - \partial_\nu B_\mu + \sum_m img^{ab}A_{\mu(a,m)}A_{\nu(b,-m)}\,,
\end{aligned} \tag{7.36}$$

and similar expressions for $D_\mu\Psi$. From the kinetic part for u, v components, the equations of motion for X_u, Ψ_u and C_μ are free,

$$\partial^\mu\partial_\mu X^I_u = \Gamma^\mu\partial_\mu\Psi_u = \partial^\mu(\partial_\mu C_\nu - \partial_\nu C_\mu) = 0\,. \tag{7.37}$$

We fix their values as

$$X^I_u = \text{const.} =: \lambda'\delta^{ID}, \quad \Psi_u = 0\,, \quad \partial_\mu C_\nu - \partial_\nu C_\mu = 0\,. \tag{7.38}$$

For the first two relations, we need to use the method of 'Higgs mechanism' which we discussed in the previous chapter. We need to introduce the extra gauge symmetry as commented in the paragraph after (7.25) to derive the last one. For general worldvolume dimensions, the additional action is

$$S_{additional} = -\frac{1}{4\lambda^2}D_{\mu\nu}(\partial_\mu C_\nu - \partial_\nu C_\mu)\,, \tag{7.39}$$

where $D_{\mu\nu}$ is a new field. It gives rise to a new gauge symmetry,

$$\delta D_{\mu\nu} = \partial_\mu\Xi_\nu - \partial_\nu\Xi_\mu\,, \quad \delta B_\mu = -\Xi_\mu\,, \tag{7.40}$$

by which we can gauge fix $B_\mu = 0$. The equation of motion by the variation of $D_{\mu\nu}$ gives the flatness condition of C_μ. Since the gauge field C_μ is essentially flat, we can ignore it for simplicity (namely, set $C_\mu = 0$) in the following. After this, the ghost fields $C_\mu, B_\mu, X^I_u, X^I_v, \Psi_u, \Psi_v$ disappear from the action, and the system becomes unitary.

Redefinition of fields

We identify the infinite components of the scalar, spinor and gauge fields as fields in $p+2$ dimensions,

$$\begin{aligned}
&\tilde{X}^I_a(x,y) = \sum_m X^I_{(a,n)}(x)e^{-iny/R}\,, \quad \tilde{\Psi}_a(x,y) = \sum_m \Psi_{(a,n)}(x)e^{-iny/R}\,, \\
&\tilde{A}_{\mu a}(x,y) = \sum_m A_{\mu(a,n)}(x)e^{-iny/R}, \qquad (7.41)
\end{aligned}$$

where an extra coordinate y is introduced to parametrize S^1 with the radius R. We also rename

$$\tilde{X}^D_a(x,y) \to \frac{1}{\lambda}\tilde{A}_{ya}(x,y)\,. \qquad (7.42)$$

We finally obtain $(p+2)$-dim super Yang-Mills theory.

The kinetic term of the scalar field X^I can be rewritten as

$$-\frac{1}{2}\int\frac{dy}{2\pi R}\left[\sum_{I=1}^{D-1}(\partial_\mu\tilde{X}^I_a - f^{bc}{}_a\tilde{A}_{\mu b}\tilde{X}^I_c)^2 + \frac{1}{\lambda^2}\tilde{F}^2_{\mu y a}\right], \qquad (7.43)$$

where

$$\tilde{F}_{\mu y a} := \partial_\mu\tilde{A}_{ya} - \partial_y\tilde{A}_{\mu a} + f^{bc}{}_a\tilde{A}_{\mu b}\tilde{A}_{yc}\,. \qquad (7.44)$$

Here the second term can be produced properly if we identify

$$R = 1/\lambda\lambda'\,. \qquad (7.45)$$

This relation seems strange if we compare with (6.35). It can be fixed by applying the T-duality transformation [95].

The second term of (7.43), when combined with the kinetic term for gauge fields, properly reproduces the kinetic term for $(p+2)$-dim worldvolume. The Kaluza-Klein mass from the compactification radius (7.45) is $n\lambda\lambda'$ which is consistent with the result in the previous section.

Similarly, we can rewrite the commutator term as

$$\begin{aligned}
\frac{\lambda^2}{4}\sum_{I,J=1}^{D}\langle[X^I,X^J],[X^I,X^J]\rangle &= \frac{\lambda^2}{4}\sum_{I,J=1}^{D-1}\int\frac{dy}{2\pi R}\langle[\tilde{X}^I,\tilde{X}^J],[\tilde{X}^I,\tilde{X}^J]\rangle \\
&\quad -\frac{1}{2}\sum_{I=1}^{D-1}\int\frac{dy}{2\pi R}(D_y\tilde{X}^I)^2. \qquad (7.46)
\end{aligned}$$

Here again the second term can be combined with the kinetic term for X^I to give the kinetic energy on $(p+2)$-dim worldvolume.

Finally, we can rewrite the interaction term as

$$\frac{i\lambda}{2}\sum_{I=1}^{D}\bar{\Psi}\Gamma_I[X^I,\Psi]=\frac{i\lambda}{2}\sum_{I=1}^{D-1}\int\frac{dy}{2\pi R}\bar{\tilde{\Psi}}\Gamma_I[\tilde{X}^I,\tilde{\Psi}]+\frac{i}{2}\int\frac{dy}{2\pi R}\bar{\tilde{\Psi}}\Gamma^y D_y\tilde{\Psi}. \tag{7.47}$$

Here, in this time, the second term can be combined with the kinetic term for Ψ.[3]

In the end, the Lagrangian thus obtained is the same as the original Lagrangian (7.32) except that we change the dimension parameter $D\to D-1$ and $p\to p+1$ and the gauge symmetry $\mathcal{G}=\hat{g}\to g$:

$$\begin{aligned}
L &= L_A+L_X+L_\Psi+L_{pot}+L_{int}\,,\\
L_A &= -\frac{1}{4\lambda^2}\int\frac{dy}{2\pi R}(\tilde{F}_{\mu\nu}^2+2\tilde{F}_{\mu y}^2)\,,\\
L_X &= -\frac{1}{2}\int\frac{dy}{2\pi R}\sum_{I=1}^{D-1}\left[(D_\mu\tilde{X}^I)^2+(D_y\tilde{X}^I)^2\right]\,,\\
L_\Psi &= \frac{i}{2}\int\frac{dy}{2\pi R}\bar{\tilde{\Psi}}(\Gamma^\mu D_\mu+\Gamma^y D_y)\tilde{\Psi}\,,\\
L_{pot} &= \frac{\lambda^2}{4}\sum_{I,J=1}^{D-1}\int\frac{dy}{2\pi R}\langle[\tilde{X}^I,\tilde{X}^J],[\tilde{X}^I,\tilde{X}^J]\rangle\,,\\
L_{int} &= \frac{i\lambda}{2}\sum_{I=1}^{D-1}\int\frac{dy}{2\pi R}\bar{\tilde{\Psi}}\Gamma_I[\tilde{X}^I,\tilde{\Psi}]\,.
\end{aligned} \tag{7.48}$$

7.3 Dp-branes' action from M2-branes

In this section, we consider the generalized system from the previous section's one, namely the compactification and T-dualization of D2-branes' worldvolume on torus. However, in this time, we start from the original BLG model for multiple M2-branes corresponding to an example of the Lie 3-algebra. The formulation here will be more general than above as we will turn on noncommutativity and a gauge field background.

7.3.1 Summary of procedure

Lie 3-algebra with multiple loop algebra (Kac-Moody algebra)

We start by defining a Lie algebra g_0 with generators $T^i_{\vec{m}}$, structure constants

$$f^{(i\vec{l})(j\vec{m})(k\vec{n})}=f^{ijk}_{\vec{l}\vec{m}}\delta_{\vec{0}}^{\vec{l}+\vec{m}+\vec{n}}, \tag{7.49}$$

[3] We should notice the definition of Γ_μ and Γ_I here. We see from the kinetic term of Ψ in the Lagrangian (7.32) that Γ_μ satisfies $\{\Gamma_\mu,\Gamma_\nu\}=\text{diag.}(+-\cdots-)$. On the other hand, Γ_I should satisfy $\{\Gamma_I,\Gamma_J\}=\delta_{IJ}$ as usual. So we choose $\Gamma^D=-i\Gamma^y$ and obtain (7.47).

and metric

$$g^{(i\vec{m})(j\vec{n})} = g^{ij}_{\vec{m}}\delta^{\vec{m}+\vec{n}}_{\vec{0}}, \tag{7.50}$$

where $\vec{m}$ is a d-dim vector of integers.

The simplest example of g_0 has

$$T^i_{\vec{m}} = T^i e^{i\vec{m}\cdot\vec{x}}, \tag{7.51}$$

where T^i is the generator for $U(N)$ and $\vec{x}$ is the coordinate on a d-dim torus. More generally, one can consider a twisted bundle on a noncommutative torus T^d_θ. In this case

$$T^i_{\vec{m}} = T^i Z_1^{m_1}\cdots Z_d^{m_d}, \tag{7.52}$$

where T^i denotes a generator of the $U(N)$ gauge group, and Z_i are noncommutative algebraic elements satisfying

$$Z_i Z_j = e^{i\theta'_{ij}} Z_j Z_i. \tag{7.53}$$

The parameter θ' is in general not the same as the noncommutative parameter θ of the noncommutative torus T^d_θ, and it depends on the rank of the gauge group and its twisting. Z_i maps a section of the twisted bundle to another section. For the trivial bundle, $Z_i = e^{ix_i}$ and (7.52) reduces to (7.51). The case of $d = 2$ was studied in [96,97]. It is straightforward to generalize it to arbitrary dimensions.

Since the structure constant (7.49) of g_0 has the property

$$f^{(i\vec{l})(j\vec{m})(k\vec{n})} \propto \delta^{\vec{l}+\vec{m}+\vec{n}}_{\vec{0}}. \tag{7.54}$$

g_0 has derivations

$$J_{0a}^{(i\vec{m})(j\vec{n})} = m_a\delta^{(i\vec{m})(j\vec{n})}. \tag{7.55}$$

Now we consider the Lie 3-algebra with the underlying Lie algebra $g = g_0$ and $I_{a\neq 0}$'s empty. We take $J_{ab} = 0$ if $a, b \neq 0$, and J_{0a} given by (7.55). It follows that the first 3 terms in (3.127) vanish, hence

$$K^{(i\vec{m})}_{abc} = \delta^i_0\delta^{\vec{m}}_{\vec{0}} B_{abc}\,, \tag{7.56}$$

assuming that T^0 is the identity of $U(N)$, so that $T^{(0\vec{0})}$ is the identity of g_0. In the following, we choose

$$\begin{aligned} K^{i\vec{m}}_{0ab} &= \delta^i_0\delta^{\vec{m}}_{\vec{0}} B_{ab}\,, \\ K^{i\vec{m}}_{abc} &= 0, \qquad \text{otherwise.} \end{aligned} \tag{7.57}$$

It will be shown below that the constants B_{ab} corresponds to a nontrivial gauge field background.

To summarize, the Lie 3-algebra is defined by the 3-commutators

$$
\begin{aligned}
[u_0, u_a, u_b] &= B_{ab}T^0_{\vec{0}} + L_{0abc}v^c, \\
[u_0, u_a, T^i_{\vec{m}}] &= m_a T^i_{\vec{m}} - \delta^i_0 \delta^{\vec{0}}_{\vec{m}} B_{ab} v^b, \\
[u_0, T^i_{\vec{m}}, T^j_{\vec{n}}] &= m_a g^{ij}_{\vec{m}} \delta^{\vec{0}}_{\vec{m}+\vec{n}} v^a + f^{ijk}_{\vec{m}\vec{n}} T^k_{\vec{m}+\vec{n}}\,, \\
[T^i_{\vec{l}}, T^j_{\vec{m}}, T^k_{\vec{n}}] &= -f^{ijk}_{\vec{l}\vec{m}} \delta^{\vec{0}}_{\vec{l}+\vec{m}+\vec{n}} v^0,
\end{aligned}
\tag{7.58}
$$

where $a, b, c = 0, 1, 2, \cdots, d$ and $i, j, k = 1, 2, \cdots, N$. This Lie 3-algebra is actually precisely the Lorentzian algebra (7.4) which is constructed from the (multiple) loop algebra defined by

$$
\begin{aligned}
[u_a, u_b] &= B_{ab}T^0_{\vec{0}} + L_{0abc}v^c, \\
[u_a, T^i_{\vec{m}}] &= m_a T^i_{\vec{m}} - K^i_{0ab} v^b, \\
[T^i_{\vec{m}}, T^j_{\vec{n}}] &= m_a g^{ij} \delta^{\vec{0}}_{\vec{m}+\vec{n}} v^a + f^{ij}_{\vec{m}\vec{n}}{}_k T^k_{\vec{m}+\vec{n}}, \\
[v^a, T^i_{\vec{m}}] &= 0\,,
\end{aligned}
\tag{7.59}
$$

where $a, b = 1, \cdots, d$. In the sense that one can construct the Lie 3-algebra (7.58) from a Lie algebra by adjoining two elements (u_0, v^0), this Lie 3-algebra is not a good representative of the new class of Lie 3-algebras. However, it is still a good example because it demonstrates the roles played by the new parameters J_{ab} and K_{abc}, which encode the information about derivatives of the Lie algebra g, which is a subalgebra of the loop algebra (7.59).

We derive D$(d+2)$-branes' theory on d-dim torus from BLG model.

It follows from the discussion in the previous chapter that the BLG model with the Lie 3-algebra (7.58) is exactly equivalent to the super Yang-Mills theory defined with the Lie algebra (7.59). In §7.2, we showed explicitly that for $d = 1$, the resulting super Yang-Mills theory is the low-energy theory for D3-branes. Now we briefly sketch the derivation for generic d to obtain the super Yang-Mills theory for D$(d+2)$-branes. The concrete calculation will be done in the next subsection.

Expanding the fields in the BLG model, we have

$$
\begin{aligned}
X^I &= \sum_{a=0}^{d} X^I_a u_a + \hat{X}^I(Z) + Y^I_a v_a, \\
\Psi &= \sum_{a=0}^{d} \Psi_a u_a + \hat{\Psi}(Z) + \Phi_a v_a, \\
A_\mu &= \frac{1}{2}\sum_{a,b=0}^{d} A_{\mu ab} u_a \wedge u_b + \sum_{a=0}^{d} u_a \wedge \hat{A}_{\mu a}(Z) + \sum_{a=0}^{d} v^a \wedge \hat{A}'_{\mu a}(Z) \\
&\quad + \frac{1}{2}\sum_{a,b=0}^{d} A'_{\mu ab} v^a \wedge v^b + \frac{1}{2}\sum_{ij} A_{\mu(i\vec{m})(j\vec{n})} T^i_{\vec{m}} \wedge T^j_{\vec{n}},
\end{aligned}
\tag{7.60}
$$

where we have used (7.52) and the notation

$$\begin{aligned}
\hat{X}^I(Z) &:= \sum_{\vec{m}} X^I_{(i\vec{m})} T^i Z^{m_1} \cdots Z^{m_d}, \\
\hat{\Psi}(Z) &:= \sum_{\vec{m}} \Psi_{(i\vec{m})} T^i Z^{m_1} \cdots Z^{m_d}, \\
\hat{A}_{\mu a}(Z) &:= \sum_{\vec{m}} A_{\mu a(i\vec{m})} T^i Z^{m_1} \cdots Z^{m_d}, \\
\hat{A}'_{\mu a}(Z) &:= \sum_{\vec{m}} A'_{\mu a(i\vec{m})} T^i Z^{m_1} \cdots Z^{m_d},
\end{aligned} \tag{7.61}$$

and $X_i^I(Z)$, $\Psi_i(Z)$ $\hat{A}_{\mu a}(Z)$ and $\hat{A}'_{\mu a}(Z)$ are sections of a twisted bundle on T^d_θ.

As we have done it many times already, we fix the coefficients of u_a as

$$X^I_a = \text{const.}, \qquad \Psi_a = 0\,, \qquad A_{\mu ab} = 0\,, \qquad \text{where} \quad a, b = 0, 1, \cdots, d, \tag{7.62}$$

and the coefficients of v_a can be ignored. Here $A_{\mu ab}$ is chosen to be zero for simplicity. If $A_{\mu ab}$'s are nonzero, it corresponds to turning on a constant background field strength with nonvanishing components of $F_{\mu I}$.

To proceed, we first define covariant derivatives $\mathcal{D}_a$ on the noncommutative torus, such that

$$[\mathcal{D}_a, Z_1^{m_1} \cdots Z_d^{m_d}] = m_a Z_1^{m_1} \cdots Z_d^{m_d}, \tag{7.63}$$

$$[\mathcal{D}_a, \mathcal{D}_b] = B_{ab}\,, \tag{7.64}$$

where B_{ab} is the constant background field strength that determines the twisting of the bundle on T^d_θ.

The rest of the derivation is essentially the same as §7.2. Finally, after integrating out the field $\tilde{A}$, the BLG Lagrangian turns into that of a super Yang-Mills theory

$$\mathcal{L} = -\frac{1}{4} \sum_{A,B=0}^{9} \langle F_{AB}, F^{AB} \rangle + \frac{i}{2} \langle \bar{\Psi}, \Gamma^A \hat{D}_A \Psi \rangle, \tag{7.65}$$

where

$$F_{\mu\nu} := [\hat{D}_\mu, \hat{D}_\nu]\,, \qquad F_{\mu I} := [\hat{D}_\mu, \hat{D}_I]\,, \qquad F_{IJ} := [\hat{D}_I, \hat{D}_J] + B_{IJ}\,, \tag{7.66}$$

and

$$\hat{D}_\mu := \partial_\mu - \hat{A}_{\mu 0}(Z)\,, \qquad \hat{D}^I := X^I_a \mathcal{D}_a - \hat{X}^I(Z)\,, \qquad C^{IJ} := X^I_a X^J_b B_{ab}\,. \tag{7.67}$$

Roughly speaking, only d of the $\hat{D}^I$'s are covariant derivatives and the rest $7-d$ are scalar fields. To turn on the background field $B_{\mu I}$, we can assign nonzero values to $A_{\mu 0a}$ and $A_{\mu ab}$.

7.3.2 Concrete calculation

In this subsection, we pick up an example of Lie 3-algebras which produces the worldvolume theory of Dp-brane ($p = d + 2$) : [4]

$$
\begin{aligned}
&[u_0, u_a, u_b] = 0\,,\\
&[u_0, u_a, T^i_{\vec m}] = m_a T^i_{\vec m}\,,\\
&[u_0, T^i_{\vec m}, T^i_{\vec n}] = m_a v^a \delta_{\vec m+\vec n}\delta_{ij} + i f^{ij}{}_k T^k_{\vec m+\vec n}\,,\\
&[T^i_{\vec l}, T^j_{\vec m}, T^k_{\vec n}] = -i f^{ijk}\delta_{\vec l+\vec m+\vec n} v^0\,.
\end{aligned}
\tag{7.68}
$$

where $a, b = 1, \cdots, d, \vec l, \vec m, \vec n \in \mathbf{Z}^d$ and f^{ijk} $(i, j, k = 1, \cdots, \dim \mathcal{G})$ is a structure constant of an arbitrary Lie algebra g which satisfies Jacobi identity. The nonvanishing part of the metric is given as

$$
\langle u_A, v_B\rangle = \delta_{AB}, \quad \langle T^i_{\vec m}, T^j_{\vec n}\rangle = \delta_{ij}\delta_{\vec m+\vec n}\,. \tag{7.69}
$$

where $A, B = 0, 1, \cdots, d$.

We note that this Lie 3-algebra can be regarded as original Lorentzian metric Lie 3-algebra (7.142) where Lie algebra is replaced by

$$
\begin{aligned}
&[u_a, u_b] = 0, \quad [u_a, T^i_{\vec m}] = m_a T^i_{\vec m},\\
&[T^i_{\vec m}, T^j_{\vec n}] = m_a v^a \delta_{\vec m+\vec n}\delta_{ij} + i f^{ij}{}_k T^k_{\vec m+\vec n}\,.
\end{aligned}
\tag{7.70}
$$

For $d = 1$, this is the standard Kac-Moody algebra with degree operator u and the central extension v. Therefore, this algebra is its higher loop generalization.

Since this is a generalized case of §7.1, where BLG model reduces to super Yang-Mills theory, one might guess that BLG model based on the Lie 3-algebra (7.68) should be equivalent to super Yang-Mills whose gauge group is the loop algebra (7.70). It turns out that this is not the case. As we explain below, BLG Lagrangian contains extra topological term such as $\theta \int F\tilde F$, which can not be reproduced from super Yang-Mills action.

Component Expansion

For the Lie 3-algebra (7.68), we expand various fields as

$$
\begin{aligned}
X^I &= X^I_{(i\vec m)} T^i_{\vec m} + X^I_A u_A + \underline{X}^I_A v_A\,,\\
\Psi &= \Psi_{(i\vec m)} T^i_{\vec m} + \Psi_A u_A + \underline{\Psi}^I_A v_A\,,\\
A_\mu &= A_{\mu(i\vec m)(j\vec n)} T^i_{\vec m}\wedge T^j_{\vec n} + \frac{1}{2}A_{\mu(i\vec m)} u^0 \wedge T^i_{\vec m} + \frac{1}{2}A_{\mu a(i\vec m)} u^a \wedge T^i_{\vec m}\\
&\quad + \frac{1}{2}A_{\mu a} u_0 \wedge u_a + A_{\mu ab} u_a \wedge u_b + (\text{terms including } v_A)\,.
\end{aligned}
\tag{7.71}
$$

[4] In the previous subsection, more general Lie 3-algebra is considered with the anti-symmetric tensor B_{ab}, *i.e.* $[u_0, u_a, u_b] = B_{ab} T^0_{\vec 0}$ instead of eq. (7.68). This tensor is related with the noncommutativity parameter on Dp-brane. In this subsection, we omit this factor for the simplicity of the argument.

Now we will rewrite the BLG action (6.1) as an action for Dp-branes ($p = d + 2$). More precisely, if we denote the original M2-branes' worldvolume as $\mathcal{M}$, the worldvolume of Dp-brane is given by a flat T^d bundle over $\mathcal{M}$. The index $\vec{m} \in \mathbf{Z}^d$ which appears in some components represents the Kaluza-Klein momentum along the T^d.

In this geometrical setup, each bosonic components plays the following roles:

- $X^I_{(i\vec{m})}$: These are split into three groups. Some are the collective coordinates which describe the embedding into the transverse directions, others are the gauge fields on the worldvolume, and the other is the degree of freedom which can be absorbed when M-direction disappears. The concrete expression is eq. (7.108).
- X^I_A : Higgs fields whose VEV's determine either the moduli of T^d or the compactification radius in M-direction.
- $A_{\mu(i\vec{m})}$: gauge fields along the M2-branes' worldvolume $\mathcal{M}$.
- $A_{\mu a}$: a connection which describes the fiber bundle $T^d \to \mathcal{M}$. The equation of motion implies that it is always flat $\partial_{[\mu}A_{\nu]a} = 0$.

The other bosonic components become Lagrange multiplier or do not show up in the action at all. In the following, we set $A_{\mu a} = A_{\mu ab} = 0$ for simplicity.

Solving the ghost sector

The components of ghost fields $\underline{X}$ and $\underline{\Psi}$ appear in the action only through the following terms:

$$L_{gh} = -(D_\mu X^I)_{u_A}(D_\mu X^I)_{v_A} + \frac{i}{2}\left(\bar{\Psi}_{u_A}\Gamma^\mu D_\mu \Psi_{v_A} + \bar{\Psi}_{v_A}\Gamma^\mu D_\mu \Psi_{u_A}\right) \tag{7.72}$$

where

$$\begin{aligned}
(D_\mu X^I)_{u_A} &= \partial_\mu X^I_A\,, \\
(D_\mu X^I)_{v_0} &= \partial_\mu \underline{X}^I_0 + i m_a (A_{\mu a(i\vec{m})} X_{(i,-\vec{m})} + A_{\mu(i\vec{m})(i,-\vec{m})} X_a) \\
&\quad - f^{ijk} A_{\mu(i\vec{m})(j\vec{m})} X_{(k,-\vec{m}-\vec{n})}\,, \\
(D_\mu X^I)_{v_a} &= \partial_\mu \underline{X}^I_a - i m_a (A_{\mu(i\vec{m})} X_{(i,-\vec{m})} + A_{\mu(i\vec{m})(i,-\vec{m})} X_0)\,,
\end{aligned} \tag{7.73}$$

and similar for Ψ. The variation of $\underline{X}^I_A$ and $\underline{\Psi}_A$ always give the *free* equations of motion for X^I_A and Ψ_A, namely

$$\partial^\mu \partial_\mu X^I_A = 0\,, \quad \Gamma^\mu \partial_\mu \Psi_A = 0\,. \tag{7.74}$$

By introducing extra gauge fields $C^I_{\mu A}$ and χ through [85, 86]

$$L_{new} = C^I_{\mu A}\partial_\mu X^I_A - \chi \bar{\Psi}_A\,, \tag{7.75}$$

one may modify the equations of motion for X_A^I and Ψ_A to

$$\partial_\mu X_A^I = 0\,, \quad \Psi_A = 0\,, \tag{7.76}$$

and absorb the ghosts $\underline{X}_A^I$ and $\underline{\Psi}$ by gauge fixing. This is how the ghost fields can be removed, as we did in many times. Since the equations of motion for X_A^I (7.76) imply that they are constant vectors in $\mathbf{R}^8$, we fix these constants as

$$\vec{X}_A = \vec{\lambda}_A \in \mathbf{R}^{d+1} \subset \mathbf{R}^8\,. \tag{7.77}$$

In Chapter 6, there is only one $\vec{\lambda} = \vec{\lambda}_0$ which specifies the M-direction compactification radius. This time, we have extra VEV's $\vec{\lambda}_a$ which give the moduli of the toroidal compactification T^d.

In the following, we prepare some notations for the later discussion. We write the dual basis to $\vec{\lambda}_A$ as $\vec{\pi}^A$, which satisfy

$$\vec{\lambda}_A \cdot \vec{\pi}^B = \delta_A^B\,. \tag{7.78}$$

We introduce a projector into the subspace of $\mathbf{R}^8$ which is orthogonal to all $\vec{\lambda}_A$ as

$$P^{IJ} = \delta^{IJ} - \sum_A \lambda_A^I \pi^{AJ}\,, \tag{7.79}$$

which satisfies $P^2 = P$. We define 'metric' as

$$G_{AB} = \vec{\lambda}_A \cdot \vec{\lambda}_B\,, \tag{7.80}$$

where λ_A^I play the role of vierbein. Using this metric, $\vec{\pi}^0$ can be written as

$$\vec{\pi}^0 = \frac{1}{G_{00}}\,\vec{\lambda}_0 - \frac{G_{0a}}{G_{00}}\,\vec{\pi}^a\,, \tag{7.81}$$

and from now we use $\{\vec{\lambda}_0, \vec{\pi}^a\}$ as the basis of $\mathbf{R}^{d+1}$ spanned by $\vec{\lambda}_A$. Note that $\vec{\lambda}_0 \perp \vec{\pi}^a$ for all a. Our claim that the $\mathbf{R}^{d+1}$ is compactified on T^{d+1} will be deduced from the Kaluza-Klein mass which is generated by the Higgs mechanism. This will be demonstrated below.

Comments on Higgs potential

Since $\vec{X}_A$ plays the role of Higgs fields, it is natural to wonder if one may introduce a potential for them and fix the value of VEV's. This seems to be physically relevant since they are related to the moduli of torus. One naive guess is to add a potential $-V(\vec{X}_A)$ to the action. Since the supersymmetry and gauge transformations of $\vec{X}_A$ are trivial, this potential breaks neither supersymmetry nor gauge symmetry. However, the kinetic term is given in the mixed form $\partial\vec{X}_A\partial\underline{\vec{X}}_A$, the potential does not fix $\vec{X}_A$ but physically irrelevant $\underline{\vec{X}}_A$.

Derivation of Dp-brane action

We finally rewrite the BLG action (6.1) in terms of Lie 3-algebra components and by putting VEV's to ghost fields X^I_A and Ψ_A.

Kinetic terms for X^I and Ψ

The covariant derivative becomes, after the assignment of VEV's to ghosts,

$$(D_\mu X^I)_{(i\vec{m})} = (\hat{D}_\mu X^I)_{(i\vec{m})} + A'_{\mu(i\vec{m})}\lambda^I_0 - im_a A_{\mu(i\vec{m})}\lambda^I_a \tag{7.82}$$

where

$$\begin{aligned}(\hat{D}_\mu X^I)_{(i\vec{m})} &= \partial_\mu X^I_{(i\vec{m})} + f^{jk}{}_i A_{\mu(k\vec{n})} X^I_{(j,\vec{m}-\vec{n})}\,, &(7.83)\\ A'_{\mu(i\vec{m})} &= -im_a A_{\mu a(i\vec{m})} + f^{jk}{}_i A_{\mu(j,\vec{m}-\vec{n})(k\vec{n})}\,. &(7.84)\end{aligned}$$

We decompose this formula into the components into the orthogonal spaces $\mathbf{R}^{7-d}$ and $\mathbf{R}^{d+1}$ by using the projector P^{IJ} as

$$(D_\mu X^I)_{(i\vec{m})} = P^{IJ}(\hat{D}_\mu X^J)_{(i\vec{m})} + \sum_A \lambda^I_A (F_{\mu A})_{(i\vec{m})} \tag{7.85}$$

where

$$\begin{aligned}(F_\mu{}^0)_{(i\vec{m})} &= \vec{\pi}^0\cdot(\hat{D}_\mu\vec{X})_{(i\vec{m})} + A'_{\mu(i,\vec{m})}\\ &= \frac{1}{G_{00}}\hat{D}_\mu(\vec{\lambda}_0\cdot\vec{X})_{(i\vec{m})} - \frac{G_{0a}}{G_{00}}\hat{D}_\mu(\vec{\pi}^a\cdot\vec{X})_{(i\vec{m})} + A'_{\mu(i\vec{m})}\,, &(7.86)\\ (F_\mu{}^a)_{(i\vec{m})} &= \hat{D}_\mu(\vec{\pi}^a\cdot\vec{X})_{(i\vec{m})} - im_a A_{\mu(i\vec{m})}\,. &(7.87)\end{aligned}$$

We will rewrite $\vec{\pi}^a\cdot\vec{X}$ as A_a below, since they play the role of gauge fields along the fiber T^d as we mentioned. $F_\mu{}^a$ will be regarded as the field strength with one leg in $\mathcal{M}$ and the other in T^d. $F_\mu{}^0$ seems to be the field strength in a similar sense with one leg in M-direction. However, the gauge field $A'_{\mu(i\vec{m})}$ is an auxiliary field as we see below, and after it is integrated out, $F_\mu{}^0$ will completely disappear from the action. In this sense, $F_\mu{}^0$ do not have any geometrical meaning. We suspect, however, that it may give a hint to keep the trace of the compactification of M-theory to type IIA superstring theory.

Finally, using eq. (7.85), the kinetic term for X^I becomes

$$L_X = -\frac{1}{2}\hat{D}_\mu X^I_{(i\vec{m})} P^{IJ}\hat{D}_\mu X^J_{(i,-\vec{m})} - \frac{1}{2}G_{AB}F_{\mu}{}^{A}_{(i\vec{m})}F_{\mu}{}^{B}_{(i,-\vec{m})}\,. \tag{7.88}$$

Similarly, the kinetic term for Ψ becomes

$$L_\Psi = \frac{i}{2}\bar{\Psi}_{(i\vec{m})}\Gamma^\mu\hat{D}_\mu\Psi_{(i,-\vec{m})}\,. \tag{7.89}$$

Chern-Simons term and integration of A'

The Chern-Simons term is written as

$$\begin{aligned} L_{CS} &= \frac{1}{2}\left(A'_{(i\vec{m})} \wedge dA_{(i,-\vec{m})} + A_{(i,-\vec{m})} \wedge dA'_{(i\vec{m})}\right) \\ &\quad -if^{ijk}A'_{(i\vec{m})} \wedge A_{(j\vec{n})} \wedge A_{(k,-\vec{m}-\vec{n})}\,, \end{aligned} \tag{7.90}$$

or, up to the total derivative terms,

$$L_{CS} = \frac{1}{2}A'_{(i\vec{m})} \wedge F_{(i,-\vec{m})} + \text{(total derivative)}\,, \tag{7.91}$$

where

$$F_{\mu\nu(i\vec{m})} = \partial_\mu A_{\nu(i\vec{m})} - \partial_\nu A_{\mu(i\vec{m})} + f^{jk}{}_i A_{\mu(j\vec{n})} A_{\nu(k,\vec{m}-\vec{n})}\,. \tag{7.92}$$

Since the gauge field A' shows up only in L_{CS} and L_X, one may algebraically integrate over it. Variation of A' gives the equation of motion gives

$$\begin{aligned} A'_{\mu(i,\vec{m})} &= -\frac{1}{G_{00}}\hat{D}_\mu(\vec{\lambda}_0 \cdot \vec{X})_{(i\vec{m})} + \frac{G_{0a}}{G_{00}}\hat{D}_\mu A_{a(i\vec{m})} - \frac{G_{0a}}{G_{00}}(F_\mu{}^a)_{(i\vec{m})} \\ &\quad -\frac{1}{2G_{00}}\epsilon_{\mu\nu\lambda}(F_{\nu\lambda})_{(i\vec{m})}\,, \end{aligned} \tag{7.93}$$

where $A_a := \vec{\pi}^a \cdot \vec{X}$. By putting back this value to the original action (7.90),

$$\begin{aligned} L_X + L_{CS} &= -\frac{1}{2}\hat{D}_\mu X^I P^{IJ} \hat{D}_\mu X^J - \frac{1}{4G_{00}}(F_{\nu\lambda})^2 - \frac{1}{2}\tilde{G}_{ab}F_\mu{}^a F_\mu{}^b \\ &\quad -\frac{G_{0a}}{2G_{00}}\epsilon^{\mu\nu\lambda}F_\mu{}^a F_{\nu\lambda} + L_{td}\,, \end{aligned} \tag{7.94}$$

where

$$\tilde{G}_{ab} := G_{ab} - \frac{G_{a0}G_{b0}}{G_{00}}\,, \tag{7.95}$$

$$L_{td} = -\frac{1}{2G_{00}}\epsilon_{\mu\nu\lambda}\partial_\mu\left[\left(-i\hat{D}_\nu(\vec{\lambda}_0 \cdot \vec{X}) + \frac{1}{2}\epsilon_{\nu\rho\sigma}F_{\rho\sigma}\right)A_\lambda\right]\,. \tag{7.96}$$

Here we omit the indices $(i\vec{m})$ for simplicity. Note that the redefinition of the metric $G_{ab} \to \tilde{G}_{ab}$ is very similar to that of T-duality transformation in M-direction. The term L_{td} is total derivative which does not vanish in the limit $G_{0a} \to 0$. Since we know that the total derivative terms do not play any role for the case $G_{0a} = \vec{\lambda}_0 \cdot \vec{\lambda}_a = 0$, we will neglect them in the following. In a sense, this is equivalent to redefine the BLG action,

$$S_{BLG} = \int d^3x\,(L_{BLG} - L_{td})\,, \tag{7.97}$$

where L_{BLG} is the original BLG Lagrangian. On the other hand, while the fourth term in eq. (7.94) is also total derivative, we must *not* neglect it. This is because this term is proportional to G_{0a} and becomes essential to understand the U-duality. For $d = 1$ case, it becomes the θ term of the super Yang-Mills action and it should be involved in the S-duality transformation in the complex coupling constant $\tau = C_0 + ie^{-\phi}$. We note that this is the term which does not show up if we analyze the Yang-Mills system with loop algebra symmetry (7.70).

Kaluza-Klein mass by Higgs mechanism

At this point, it is easy to understand how compactification occurs after the Higgs mechanism. Note that in the definition of ${F_\mu}^a$ (7.87), we have a factor with m_a in front of $A_{\mu(i\vec{m})}$. In the language of D2-branes' worldvolume, it gives rise to the mass proportional to $\mathcal{O}(\vec{m}^2)$ for $A_{\mu(i\vec{m})}$. Also, we will be able to see that similar mass term exists for all fields with index $\vec{m}$. Because of the dependence of mass on $\vec{m}$ with the correct radius dependence, it is natural to regard these terms as the mass terms for the winding modes on the torus T^d.

In order to be more explicit, we will use the T-dual picture [95] in the following. We identify the various fields with index $\vec{m}$ with the higher $(3+d)$-dim fields by the identification

$$\Phi_{\vec{m}}(x) \to \tilde{\Phi}(x,y) := \sum_{\vec{m}} \Phi_{\vec{m}}(x) e^{i\vec{m}\vec{y}} \tag{7.98}$$

where $\vec{y} \in [0, 2\pi]^d$ are coordinates of T^d. ${F_\mu}^a$ can be identified with the field strength by

$$(\tilde{F}^a_\mu)_i = \hat{D}_\mu \tilde{A}^a_i - \frac{\partial}{\partial y_a} \tilde{A}_{\mu i} \tag{7.99}$$

where $\tilde{A}^a_i(x,y) := \vec{\pi}^a \cdot \vec{\tilde{X}}_i(x,y)$. The kinetic terms of gauge fields in eq. (7.94) imply that we have a metric in $\vec{y}$ direction as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{ab} dy^a dy^b \quad \text{where} \quad g_{ab} := G_{00} \tilde{G}_{ab}\,. \tag{7.100}$$

When $\vec{\lambda}_A$ are all orthogonal, one may absorb the metric g_{ab} in the rescaling of y^a as $y'^a = |\vec{\lambda}_0||\vec{\lambda}_a|\, y^a$ or $y'_a = (|\vec{\lambda}_0||\vec{\lambda}_a|)^{-1} y_a$ (where $y_a = g_{ab} y^b$). Since y_a has the radius 1, y'_a has the radius $\frac{1}{|\vec{\lambda}_0||\vec{\lambda}_a|}$. This is consistent with the analysis in §7.2. In this scaling $y_a \to y'_a$, the kinetic terms for gauge fields in eq. (7.94) become

$$-\frac{1}{4G_{00}} \left((F_{\nu\lambda})^2 + 2F^2_{\mu a} \right) , \tag{7.101}$$

which is also consistent with our previous study for $d = 1$.

We note that the use of Kac-Moody algebra as the symmetry of the Kaluza-Klein mode is not new. See, for example, [98–101]. Here the novelty is to use the Higgs mechanism to obtain the Kaluza-Klein mass.

Worldvolume as a flat fiber bundle

So far, since we put $A_{\mu a} = 0$ for the simplicity of the argument, the worldvolume of Dp-brane is the product space $\mathcal{M} \times T^d$. In order to see the geometrical role of $A_{\mu a}$, let us keep it nonvanishing for a moment. The covariant derivative (7.82) get an extra term, $m_a A_{\mu a}(x) X^I_{(i\vec{m})}$, which becomes on $\mathcal{M} \times T^d$,

$$iA_{\mu a}(x)\frac{\partial}{\partial y^a}\tilde{X}^I_i(x,y)\,. \tag{7.102}$$

$A_{\mu a}$ turns out to be the gauge field for the gauge transformation from those of BLG:

$$\delta\tilde{X}^I_i(x,y) = i\gamma_a(x)\frac{\partial}{\partial y^a}\tilde{X}^I_i(x,y)\,. \tag{7.103}$$

The existence of the gauge coupling implies that the worldvolume is not the direct product $\mathcal{M} \times T^d$ but the fiber bundle Y:

$$\begin{tikzcd} T^d \arrow[r] & Y \arrow[d] \\ & M \end{tikzcd}$$

where T^d act as the translation of y^a.

The kinetic term for the connection comes from the Chern-Simons term:

$$L_{fiber} = \epsilon^{\mu\nu\lambda} D_{\mu a}\partial_\nu A_{\lambda a}\,, \quad D_{\mu a} := \sum_{\vec{n}} n_a A_{\mu(i\vec{n})(i,-\vec{n})}\,. \tag{7.104}$$

Since $D_{\mu a}$ does not appear in other place in the action, its variation gives,

$$\partial_{[\mu}A_{\nu]a} = 0\,. \tag{7.105}$$

Therefore Y must be a flat bundle as long as we start from BLG model.

There seem to be various possibilities to relax this constraint to the curved background. One naive guess is to replace L_{fiber} to

$$L'_{fiber} = \epsilon^{\mu\nu\lambda} D_{\mu a}(\partial_\nu A_{\lambda a} - \frac{1}{2}F^{(0)}_{a\nu\lambda})\,, \tag{7.106}$$

for an appropriate classical background $F^{(0)}_{a\nu\lambda}$.

Interaction terms

The compactification picture works as well in the interaction terms. For the fermion interaction term L_{int}, we use

$$[X^{[I}, X^{J]}, \Psi]_{(i,-\vec{m})} = -m_a \lambda_0^{[I}\lambda_a^{J]}\Psi_{(i,-\vec{m})} + if^{jk}{}_i\lambda_0^{[I}X^{J]}_{(j\vec{n})}\Psi_{(k,-\vec{m}-\vec{n})} \tag{7.107}$$

and from eq. (7.81),

$$\begin{aligned} X^I &= P^{IJ}X^J + \lambda^I_A(\vec{\pi}^A \cdot \vec{X}) \\ &= P^{IJ}X^J + \lambda^I_0(\vec{\lambda}_0 \cdot \vec{X}) + \left(-\frac{G_{0a}}{G_{00}}\lambda^I_0 + \lambda^I_a\right) A_a \,. \end{aligned} \tag{7.108}$$

Then L_{int} can be written as

$$\begin{aligned} L_{int} &= \frac{i}{4}\bar{\Psi}_{(i\vec{m})}(\Gamma_{IJ}\lambda^I_0\lambda^J_a)\left(-m_a\Psi_{(i,-\vec{m})} + if^{jk}{}_iA_{a(j\vec{n})}\Psi_{(k,-\vec{m}-\vec{n})}\right) \\ &\quad + \frac{i}{4}\bar{\Psi}_{(i\vec{m})}(\Gamma_{IJ}\lambda^I_0)\left(if^{jk}{}_iP^{JK}X^K_{(j\vec{n})}\Psi_{(k,-\vec{m}-\vec{n})}\right) \\ &= \int \frac{d^dy}{(2\pi)^d}\sqrt{g}\left(\frac{i}{2}\bar{\tilde{\Psi}}\Gamma^a\hat{D}_a\tilde{\Psi} + \frac{i\sqrt{G_{00}}}{2}\bar{\tilde{\Psi}}\Gamma_I[P^{IJ}\tilde{X}^J, \tilde{\Psi}]\right) \,, \end{aligned} \tag{7.109}$$

where $g = \det g_{ab}$, $\hat{D}_a\tilde{\Psi} := \partial_a\tilde{\Psi} - i[\tilde{A}_a, \tilde{\Psi}]$ and

$$\Gamma^a := \frac{i}{2}\Gamma_{IJ}\lambda^I_0\lambda^J_a \,, \quad \Gamma_J := \frac{1}{2\sqrt{G_{00}}}\Gamma_{IJ}\lambda^I_0 \,, \tag{7.110}$$

which satisfy $\{\Gamma^a, \Gamma^b\} = g_{ab}$ and $\{\Gamma^I, \Gamma^J\} = \delta_{IJ}$.

On the other hand, the potential term for the boson L_{pot} is the square of a 3-commutator:

$$[X^I, X^J, X^K]_{(i,\vec{m})} = m_a\lambda^{[I}_0\lambda^J_aX^{K]}_{(i,\vec{m})} + f^{ij}{}_k\lambda^{[I}_0X^J_{(j,\vec{n})}X^{K]}_{(i,\vec{m})}. \tag{7.111}$$

The square of the first term gives

$$\left(m_a\lambda^{[I}_0\lambda^J_aX^{K]}_{(i,\vec{m})}\right)^2 = 6g_{ab}m_am_bX^I_{\vec{m}}P^{IJ}_{\vec{m}}X^J_{-\vec{m}} \,, \tag{7.112}$$

where

$$\begin{aligned} P^{IJ}_{\vec{m}} &:= \delta^{IJ} - \frac{|\vec{\lambda}_0|^2\lambda^I_{\vec{m}}\lambda^J_{\vec{m}} + |\lambda_{\vec{m}}|^2\vec{\lambda}^I_0\vec{\lambda}^J_0 - (\vec{\lambda}_0, \vec{\lambda}_{\vec{m}})(\lambda^I_0\lambda^J_{\vec{m}} + \lambda^J_0\lambda^I_{\vec{m}})}{(|\vec{\lambda}_0|^2|\vec{\lambda}_{\vec{m}}|^2 - (\vec{\lambda}_0, \vec{\lambda}_{\vec{m}})^2)} \,, \\ \vec{\lambda}_{\vec{m}} &:= m_a\vec{\lambda}_a \,, \end{aligned} \tag{7.113}$$

which satisfy

$$P^{IJ}_{\vec{m}}\lambda^J_0 = P^{IJ}_{\vec{m}}\lambda^J_{\vec{m}} = 0 \,, \quad P^2_{\vec{m}} = P_{\vec{m}} \,. \tag{7.114}$$

The mixed term vanishes and does not contribute to the action. The commutator part is

$$(f^{ij}{}_k\lambda^{[I}_0X^J_{(j,\vec{n})}X^{K]}_{(i,\vec{m})})^2 = 3\left(G_{00}\langle[X^J, X^K]^2\rangle - 2\langle[(\vec{\lambda}_0 \cdot \vec{X}), X^I]^2\rangle\right) \tag{7.115}$$

which is identical to the similar term in the previous chapter and it produces the standard commutator terms. Using eq. (7.108), these terms can be summarized as

$$\begin{aligned} L_{pot} &= \int \frac{d^dy}{(2\pi)^d}\sqrt{g}\left(-\frac{1}{2}g_{ab}\hat{D}_a\tilde{X}^IP^{IJ}\hat{D}_b\tilde{X}^J - \frac{1}{4G_{00}}g_{ac}g_{bd}\tilde{F}_{ab}\tilde{F}_{cd}\right. \\ &\qquad \left. + \frac{G_{00}}{4}[P^{IK}\tilde{X}^K, P^{JL}\tilde{X}^L]^2\right) \,, \end{aligned} \tag{7.116}$$

where $\hat{D}_a\tilde{X}^I = \partial_a\tilde{X}^I - i[\tilde{A}_a, \tilde{X}^I]$ and $\tilde{F}_{ab} = \partial_a\tilde{A}_b - \partial_b\tilde{A}_a - i[\tilde{A}_a, \tilde{A}_b]$.

Summary

By collecting all the results in this subsection, the BLG action (6.1) becomes

$$
\begin{aligned}
L &= L_A + L_{FF} + L_X + L_\Psi + L_{pot} + L_{int} + L_{td}\,,\\
L_A &= -\frac{1}{4G_{00}}\int \frac{d^d y}{(2\pi)^d}\sqrt{g}\left(\tilde{F}_{\mu\nu}^2 + 2g_{ab}\tilde{F}_{\mu a}\tilde{F}_{\mu b} + g_{ac}g_{bd}\tilde{F}_{ab}\tilde{F}_{cd}\right)\,,\\
L_{FF} &= -\frac{G_{0a}}{8G_{00}}\int \frac{d^d y}{(2\pi)^d}\sqrt{g}\left(4\epsilon^{\mu\nu\lambda}\tilde{F}_{\mu a}\tilde{F}_{\nu\lambda}\right)\,,\\
L_X &= -\frac{1}{2}\int \frac{d^d y}{(2\pi)^d}\sqrt{g}\left(\hat{D}_\mu\tilde{X}^I P^{IJ}\hat{D}_\mu\tilde{X}^J + g_{ab}\hat{D}_a\tilde{X}^I P^{IJ}\hat{D}_b\tilde{X}^J\right)\,,\\
L_\Psi &= \frac{i}{2}\int \frac{d^d y}{(2\pi)^d}\sqrt{g}\,\bar{\tilde{\Psi}}\left(\Gamma^\mu\hat{D}_\mu + \Gamma^a\hat{D}_a\right)\tilde{\Psi}\,,\\
L_{pot} &= \frac{G_{00}}{4}\int \frac{d^d y}{(2\pi)^d}\sqrt{g}\,[P^{IK}\tilde{X}^K, P^{JL}\tilde{X}^L]^2\,,\\
L_{int} &= \frac{i\sqrt{G_{00}}}{2}\int \frac{d^d y}{(2\pi)^d}\sqrt{g}\,\bar{\tilde{\Psi}}\Gamma_I[P^{IJ}\tilde{X}^J, \tilde{\Psi}]\,.
\end{aligned}
\tag{7.117}
$$

It is easy to see that this is the standard Dp-brane action ($p = d+2$) on $\Sigma \times T^d$ with the metric (7.100). Interpretation and implications of this action is given in the next section.

7.4 U-duality in BLG model

In the previous section, we saw that we obtain the multiple Dp-branes' system on a torus from BLG model, when we consider a particular example of general Lorentzian Lie 3-algebra. On the other hand, the original BLG model describes the multiple M2-branes.

Since the relation between Dp-branes and M2-branes is well known as *U-duality*, which we mentioned in §1.4, we now examine how this U-duality relation [102] is realized in the theory (7.117).

7.4.1 D3-branes case

For $d = 1$, if we write $\vec{\lambda}_0 = \vec{e}_0$, $\vec{\lambda}_1 = \tau_1\vec{e}_0 + \tau_2\vec{e}_1$ (where the basis $\{\vec{e}_0, \vec{e}_1\}$ is orthonormal), the action for the gauge field is given as

$$
\begin{aligned}
L_A + L_{FF} &= -\frac{1}{4G_{00}}\int \frac{dy}{2\pi}\sqrt{g}\,F^2 - \frac{G_{01}}{8G_{00}}\int \frac{dy}{2\pi}\,F\widetilde{F}\\
&= -\frac{1}{8\pi}\int dy\left(\frac{\tau_1}{2}F\widetilde{F} + \tau_2 F^2\right)
\end{aligned}
\tag{7.118}
$$

where now $g = g_{11}$ and

$$
\begin{aligned}
F^2 &= \tilde{F}_{\mu\nu}^2 + 2g_{11}\tilde{F}_{\mu 1}\tilde{F}_{\mu 1}\,,\\
F\widetilde{F} &= (4\sqrt{g_{11}}\,\epsilon^{\mu\nu\lambda})\tilde{F}_{\mu 1}\tilde{F}_{\nu\lambda}\,.
\end{aligned}
\tag{7.119}
$$

This shows that the action (7.117) in this case is the standard D3-brane action with the θ term.

Under the $SL(2,\mathbf{Z})$ transformation

$$\begin{pmatrix} \vec{\lambda}_1 \\ \vec{\lambda}_0 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{\lambda}_1 \\ \vec{\lambda}_0 \end{pmatrix}, \tag{7.120}$$

the moduli parameter τ is transformed as,

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{7.121}$$

For $b = -c = 1$, $a = d = 0$, it gives rise to the standard S-duality transformation $\tau \to -1/\tau$. On the other hand, $a = d = 1$, $b = n$ and $c = 0$ gives the translation $\tau \to \tau + n$.

On S-duality

We do not claim that we have proven S-duality symmetry from our model. At the level of Lie 3-algebra (7.68), there is obvious asymmetry between u_0, v_0 and u_1, v_1. Nevertheless, it is illuminating that the S-duality symmetry can be interpreted in terms of Higgs VEV's $\vec{\lambda}_{0,1}$ in so simple manner.

On T-duality shift

On the other hand, the translation symmetry reduces to the automorphism of the Lie 3-algebra (7.68),

$$\begin{aligned} &u_0 \to u_0 - nu_1\,, \quad u_1 \to u_1\,, \\ &v_0 \to v_0\,, \quad v_1 \to v_1 + nv_0\,. \end{aligned} \tag{7.122}$$

It is easy to see that the transformation changes neither Lie 3-algebra nor their metric. It induces the redefinition the ghost fields as,

$$X^I = X^I_{u_0} u_0 + X^I_{u_1} u_1 + \cdots = X^I_{u_0}(u_0 - nu_1) + (X^I_{u_1} + nX^I_{u_0})u_1 + \cdots\,. \tag{7.123}$$

It implies the transformation $\vec{\lambda}_0 \to \vec{\lambda}_0\,, \vec{\lambda}_1 \to \vec{\lambda}_1 + n\vec{\lambda}_0\,$. Of course, at the classical level, there is no reason that the parameter n must be quantized. It is interesting anyway that part of the duality transformation comes from the automorphism of Lie 3-algebra.

On T-duality

The T-duality symmetry $\mathbf{Z}_2$ of D3-brane comes from the different identification of component fields. Namely, we have constructed 4-dim field $\tilde{X}^I(x,y)$ from the component fields $X^I_{(i\vec{m})}(x)$ by Fourier series (7.98). One may instead interpret $X^I_{(i\vec{m})}(x)$ as the 3-dim field and interpret $\vec{m}$ index as describing open string mode which interpolate mirror images of a point in $T^1 = \mathbf{R}/\mathbf{Z}$. This is the standard T-duality argument [95].

The relation between the coupling constant and the radius in T-duality transformation is given as follows. Let us assume for a moment that $\vec{\lambda}_0 \perp \vec{\lambda}_1$ for simplicity. It is well known [25] that putting a VEV $\vec{X}_{u_0} = \vec{\lambda}_0$ means the compactification of M-direction with the radius

$$R_0 = \lambda_0 \, l_p^{3/2} \,, \tag{7.124}$$

where l_p is 11-dim Planck length. From the symmetry of $X_{u_0} \leftrightarrow X_{u_1}$, putting a VEV $\vec{X}_{u_1} = \vec{\lambda}_1$ must imply the compactification of another direction with the similar radius $\tilde{R}_1 = \lambda_1 \, l_p^{3/2}$ before taking T-duality along $\vec{\lambda}_1$. At this point, we have D2-branes' worldvolume theory with string coupling

$$g_s = g_{YM}^2 \, l_s = \lambda_0^2 \, l_s \,. \tag{7.125}$$

where l_s is the string length, satisfying $l_p^3 = g_s l_s^3$. In the previous section, we obtain D3-branes since we compactify the $\vec{\lambda}_1$ direction with radius $\tilde{R}_1$ and simultaneously take T-duality for the same direction. Thus the D3-branes are compactified on S^1 with the radius

$$R_1 = \frac{l_s^2}{\tilde{R}_1} = \frac{l_s^2}{\lambda_1 \sqrt{\lambda_0^2 l_s^4}} = \frac{1}{\lambda_0 \lambda_1} \,, \tag{7.126}$$

and the string coupling for D3-branes' worldvolume theory is

$$g_s' = g_s \frac{l_s}{\tilde{R}_1} = \frac{\lambda_0}{\lambda_1} \,. \tag{7.127}$$

This result is consistent with our results.

To summarize, the U-duality transformation of D3-brane action is

$$SL(2, \mathbf{Z}) \bowtie \mathbf{Z}_2 \,, \tag{7.128}$$

where the first factor is described by the rotation of Higgs VEV's and the second factor is described by the different representation as the field theory.

7.4.2 U-duality for $d > 1$

We consider M-theory compactified on T^{d+1} (where $d = p - 2$). This theory has U-duality group $E_{d+1}(\mathbf{Z})$ and scalars taking values in E_{d+1}/H_{d+1} where H_{d+1} is the maximal compact subgroup of E_{d+1}. See, for example, [103] for detail. We call the space of these scalars 'parameter space' in the following.

Parameters in U-duality group from L-BLG models

In this subsection, we compare the parameters obtained from L-BLG model with that in the parameter space. We can extract various parameters on Dp-brane from the action obtained in §7.3.2 which are all determined by the Higgs VEV's $\vec{\lambda}_A$. The first one is the Yang-Mills coupling :

$$g_{YM}^2 = \frac{(2\pi)^d G_{00}}{\sqrt{g}}\,, \quad g := \det(g_{ab})\,. \tag{7.129}$$

Secondly, the metric

$$g_{ab} = G_{00}G_{ab} - G_{0a}G_{0b} \tag{7.130}$$

gives the moduli of the torus T^d. Finally, L_{FF} gives a generalization of θ term for $d = 1$ case. Since the θ term may be regarded as the axion coupling, a natural generalization for general d is the R-R field $B_{(d-1)}$, which appears in the Dp-brane Lagrangian of string theory like as $C_{(d-1)} \wedge F \wedge F$. Such term was discussed in the literature, for example, in [103].

In our setup in the previous section, the existence of such coupling $C \wedge F \wedge F$ can be understood as follows. There the compactification of the M-direction was determined by $\vec{\lambda}_0$ and we took T-duality on T^d specified by $\{\vec{\lambda}_a\} = \{\vec{\lambda}_1, \cdots, \vec{\lambda}_d\}$. If $G_{0a} = \vec{\lambda}_0 \cdot \vec{\lambda}_a \neq 0$, we obtain the non-zero $C_{(0)}$ field, after the compactification of M-direction and the T-duality transformation along only y^a. After taking T-duality in the remaining $d-1$ directions on T^d too, we obtain the nonzero $C_{(d-1)}$ field whose nonvanishing component is $C_{1\cdots\hat{a}\cdots d}$, where the index with $\hat{}$ should be erased. This component of R-R field must interact with gauge fields on D-brane as $\epsilon^{\mu\nu\lambda 1\cdots d}C_{1\cdots\hat{a}\cdots d}F_{\mu\nu}F_{\lambda a}$. In our action (7.117), L_{FF} describes this coupling. It determines the components of $C_{(d-1)}$ as

$$C_{\hat{a}} := C_{1\cdots\hat{a}\cdots d} = \frac{G_{0a}}{4(2\pi)^d G_{00}}\frac{\sqrt{g}}{\sqrt{g_{aa}}}\,, \tag{7.131}$$

where no sum is taken on a.

$SL(d+1, \mathbf{Z})$ transformations in U-duality can be reproduced.

The number of parameters thus obtained is $1 + \frac{d(d+1)}{2} + d = \frac{(d+1)(d+2)}{2}$ which coincides with the number of metric $G_{AB} = \vec{\lambda}_A \cdot \vec{\lambda}_B$. As is $d = 1$ case, it is natural to guess the $SL(d+1, \mathbf{Z})$ transformation

$$\vec{\lambda}'_A = \Lambda_{AB}\,\vec{\lambda}_B\,, \quad \Lambda_{AB} \in SL(d+1, \mathbf{Z})\,, \tag{7.132}$$

is related to the first factor of $SL(d+1, \mathbf{Z}) \bowtie O(d, d; \mathbf{Z})$ in U-duality transformation. Now we derive the transformation law of these parameters explicitly.

$SL(d+1,\mathbf{Z})$ is generated by the following two kinds of $(d+1)\times(d+1)$ matrices:

$$S(i,j)\quad:\quad \begin{cases} \Lambda_{AB}=\delta_{AB}\quad (\text{for } A,B\neq i,j)\,, \\ \begin{pmatrix}\Lambda_{ii} & \Lambda_{ij}\\ \Lambda_{ji} & \Lambda_{jj}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \end{cases}$$

$$T(i,j;n)\quad:\quad \begin{cases} \Lambda_{AB}=\delta_{AB}\quad (\text{for } A,B\neq i,j)\,, \\ \begin{pmatrix}\Lambda_{ii} & \Lambda_{ij}\\ \Lambda_{ji} & \Lambda_{jj}\end{pmatrix}=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}. \end{cases}$$

where $i,j=0,1,\cdots,d$ $(i<j)$ and $n\in\mathbf{Z}$. Obviously, $S(i,j)$ is a generalization of S-duality transformation and $T(i,j;n)$ is the generalization of translation generator.

(I) $\Lambda=S(0,i)$ $(i\neq 0)$**:** This transformation interchanges $\vec{\lambda}_0$ and $\vec{\lambda}_i$, *i.e.* M-direction and one of the torus directions. It is a generalization of S-duality transformation for $d=1$ case. G_{0A} and g_{ab} are transformed as

$$\begin{aligned} &G'_{00}=G_{ii}\,,\quad G'_{0i}=-G_{0i}\,,\quad G'_{0a}=G_{ia}\,,\\ &g'_{ii}=g_{ii},\quad g'_{ia}=-(G_{ii}G_{0a}-G_{i0}G_{ia})\,,\quad g'_{ab}=G_{ii}G_{ab}-G_{ia}G_{ib}\,, \end{aligned}\tag{7.133}$$

for $a,b\neq 0,i$. In the simple case of $G_{0a}=G_{0i}=G_{ia}=0$,

$$g_{YM}^2=\sqrt{\frac{G_{00}}{G_{ii}}}\,\frac{(2\pi)^d}{G_{00}^{(d-1)/2}}\,\frac{1}{\sqrt{\hat{G}_i}}\quad\to\quad g'^2_{YM}=\sqrt{\frac{G_{ii}}{G_{00}}}\,\frac{(2\pi)^d}{G_{ii}^{(d-1)/2}}\,\frac{1}{\sqrt{\hat{G}_i}}\,,\tag{7.134}$$

where $\hat{G}_i$ is the minor determinant of G_{ab} excluding the i'th row and column. On the other hand, $C_{\hat{a}}$ remains zero in this simple case.

(II) $\Lambda=T(0,i;n)$ $(i\neq 0)$**:** This transformation shifts the M-direction as $\vec{\lambda}_0\to\vec{\lambda}_0+n\vec{\lambda}_i$ and should be a generalization of T-duality transformation. G_{0A} and g_{ab} are transformed as

$$\begin{aligned} &G'_{00}=G_{00}+2nG_{0i}+n^2G_{ii}\,,\quad G'_{0a}=G_{0a}+nG_{ia}\,,\\ &g'_{ab}=g_{ab}+n(2G_{0i}G_{ab}-G_{0a}G_{ib}-G_{ia}G_{0b})+n^2(G_{ii}G_{ab}-G_{ia}G_{ib}) \end{aligned}\tag{7.135}$$

for $a,b\neq 0$. In the simple case of $G_{0a}=G_{ab}=0$ (where $a\neq b$),

$$g_{YM}^2\;\to\;g'^2_{YM}=\left(1+n^2\frac{G_{ii}}{G_{00}}\right)^{1-\frac{d}{2}}g_{YM}^2\,.\tag{7.136}$$

On the other hand, R-R field $C_{(d-1)}$is shifted as in the D3-branes case,

$$C_{\hat{i}}=0\;\to\;C'_{\hat{i}}=nG_{ii}\cdot\frac{\sqrt{g'}}{4(2\pi)^dG'_{00}\sqrt{g'_{ii}}}\,.\tag{7.137}$$

(III) $\Lambda = S(i,j)$ $(i,j \neq 0)$ **:** This transformation interchanges $\vec{\lambda}_i$ and $\vec{\lambda}_j$ and should make no physical change. In fact,

$$\begin{aligned} &G'_{00} = G_{00}\,, \quad G'_{0i} = G_{0j}\,, \quad G'_{0j} = -G_{0i}\,, \\ &g'_{ii} = g_{jj}\,, \quad g'_{ij} = -g_{ji}\,, \quad g'_{ji} = -g_{ij}\,, \quad g'_{jj} = g_{ii}\,, \end{aligned} \tag{7.138}$$

and other G_{0a} and g_{ab} remain the same. The coupling constant g_{YM}^2 is invariant under this transformation. The components of $C_{(d-1)}$ is shuffled by the interchange of the basis $\{\vec{\lambda}_a\}$, but this doesn't mean any physical changes.

(IV) $\Lambda = T(i,j;n)$ $(i,j \neq 0)$ **:** This transformation shifts the torus direction as $\vec{\lambda}_i \to \vec{\lambda}_i + n\vec{\lambda}_j$. In this case, G_{0A} and g_{ab} are transformed as

$$\begin{aligned} &G'_{00} = G_{00}\,, \quad G'_{0i} = G_{0i} + nG_{0j}\,, \quad G'_{0a} = G_{0a}\,, \\ &g'_{ii} = g_{ii} + 2ng_{ij} + n^2 g_{jj}\,, \quad g'_{ia} = g_{ia} + ng_{ja}\,, \quad g'_{ab} = g_{ab}\,, \end{aligned} \tag{7.139}$$

for $a, b \neq 0, i$. So g_{YM}^2 varies only by the change of $\sqrt{g}$ (or the volume of T^d) caused by the shift of $\vec{\lambda}_i$. The components of $C_{(d-1)}$, just as in the case of $S(i,j)$, is effected by the transformation of the basis $\{\vec{\lambda}_a\}$, but it is not physically meaningful.

The transformation laws are less illuminative compared with $d = 1$ case, since the parameters g_{YM}^2 and $C_{\hat{a}}$ depend on G_{AB} in a complicated way. However, since the number of the parameters is the same, it is straightforward to obtain the inverse relation $G_{AB} = G_{AB}(g_{YM}^2, g_{ab}, C_{\hat{a}})$. This combination $G_{AB} = \vec{\lambda}_A \cdot \vec{\lambda}_B$ transforms linearly under $SL(d+1, \mathbf{Z})$. In this sense, it is possible to claim that $SL(d+1, \mathbf{Z})$ is a part of the U-duality symmetry and G_{AB} gives the parameter which transforms covariantly under $SL(d+1, \mathbf{Z})$. The closure of these parameters under $SL(d+1, \mathbf{Z})$ was discussed in the literature, for example, [103].

Towards the whole of U-duality in Dp-branes' cases

The parameters obtained from $\vec{\lambda}_A$, however, do not describe the full parameter space to implement U-duality. In the following, we compare it with the dimensions of the parameter space. As we see, for $d = 1$, it correctly reproduces the moduli. The discrepancy of the number of parameters starts from $d > 1$. We will explain some part of the missing parameters is given as the deformation of Lie 3-algebra (7.68).

D3-brane $(d = 1)$: It corresponds to M-theory compactified on T^2. The parameter space in this case is $\big(SL(2)/U(1)\big) \times \mathbf{R}$ which gives 3 scalars. They correspond to G_{00}, G_{01} and g, in other words, g_{YM}^2, $C_{\hat{1}}$ and g_{11}, all of which appear in the D3-brane action (7.117).

D4-branes $(d = 2)$: It corresponds to M-theory compactified on T^3. The parameter space in this case is $\big(SL(3)/SO(3)\big) \times \big(SL(2)/U(1)\big)$ which gives 7 parameters. They correspond to G_{ab}, B_{ab}, Φ and $C_{\hat{a}}$ which transform in the $\mathbf{3}+\mathbf{1}+\mathbf{1}+\mathbf{2}$ representations of $SL(2)$. Φ and $C_{\hat{a}}$ is dilaton and R-R 1-form (or $p-3$ form) field which are the same in the above discussion.

B_{ab} is NS-NS 2-form field which we have not discussed so far. As we commented in the footnote 4, such parameters were introduced in eq. (7.58) as the deformation of the Lie 3-algebra, $[u_0, u_a, u_b] = B_{ab}T^0_{\vec{0}}, \cdots$. It describes the noncommutativity on the torus along the line of [93, 104]. We have not used this generalized algebra for the simplicity of the computation but can be straightforwardly included in the L-BLG model. It is interesting that some part of moduli are described as dynamical variable ("Higgs VEV") while the other part comes from the modification of Lie 3-algebra which underlies the L-BLG model.

D5-branes $(d=3)$: It corresponds to M-theory compactified on T^4. The parameter space in this case is $SL(5)/SO(5)$ which gives 14 parameters. They correspond to G_{ab}, B_{ab}, Φ, $C_{\hat{a}}$ and $C_{\hat{a}\hat{b}\hat{c}}$ which transform in the $\mathbf{6}+\mathbf{3}+\mathbf{1}+\mathbf{3}+\mathbf{1}$ representations of $SL(3)$.

$C_{\hat{a}\hat{b}\hat{c}}$ is R-R 0-form (or $p-5$ form) field which causes the interaction like as $\epsilon^{\mu\nu\lambda}C_{\hat{a}\hat{b}\hat{c}}$ $\times F_{\mu\nu}F_{\lambda a}F_{bc}$ or $\epsilon^{\mu\nu\lambda}C_{\hat{a}\hat{b}\hat{c}}F_{\mu a}F_{\nu b}F_{\lambda c}$. In the context of Lie 3-algebra, there is a room to include such coupling. It is related to the Lie 3-algebra associated with Nambu-Poisson bracket. As shown in Part II, the worldvolume theory becomes not the super Yang-Mills but instead described by self-dual 2-form field which describes the M5-brane.[5] The precise statement on the moduli becomes obscure in this sense.

To see U-duality, we must also consider the transformation of B_{ab} and $C_{\hat{a}\hat{b}\hat{c}}$. Especially, the interchange $B_{ab} \leftrightarrow C_{\hat{a}}$ and $C_{\hat{a}\hat{b}\hat{c}} \leftrightarrow \Phi$ means S-duality.

D6-branes $(d=4)$: It corresponds to M-theory compactified on T^5. The parameter space in this case is $SO(5,5)/\big(SO(5)\times SO(5)\big)$ which gives 25 scalars. They correspond to G_{ab}, B_{ab}, Φ, $C_{\hat{a}}$ and $C_{\hat{a}\hat{b}\hat{c}}$ which transform in the $\mathbf{10}+\mathbf{6}+\mathbf{1}+\mathbf{4}+\mathbf{4}$ representation of $SL(4)$. To see U-duality, we must also consider the transformation of B_{ab} and $C_{\hat{a}\hat{b}\hat{c}}$.

D7-branes $(d=5)$: It corresponds to M-theory compactified on T^6. The parameter space in this case is $E_6/USp(8)$ which gives 42 scalars. They correspond to G_{ab}, B_{ab}, Φ, $C_{\hat{a}}$, $C_{\hat{a}\hat{b}\hat{c}}$ and $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ which transform in the $\mathbf{15}+\mathbf{10}+\mathbf{1}+\mathbf{5}+\mathbf{10}+\mathbf{1}$ representations of $SL(5)$.

$C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ is R-R 0-form (or $p-7$ form) field which causes the interaction like as $\epsilon^{\mu\nu\lambda}C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ $\times F_{\mu\nu}F_{\lambda a}F_{bc}F_{de}$ and so on. Note that $C_{\hat{a}}$ in this case must be the self-dual 4-form field.

To see U-duality, we must also consider the transformation of B_{ab}, $C_{\hat{a}\hat{b}\hat{c}}$ and $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$. Especially, the interchange $B_{ab} \leftrightarrow C_{\hat{a}\hat{b}\hat{c}}$ and $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} \leftrightarrow \Phi$ means S-duality. However we don't know the way to introduce the field $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ at this moment in time, so this discussion may be difficult.

[5] In order to satisfy the fundamental identity, Nambu-Poisson bracket must be equipped on a 3-dim manifold. So, in this case, we must choose the specific T^3 where Nambu-Poisson bracket is defined from the whole compactified torus T^4.

D8-branes $(d = 6)$: It corresponds to M-theory compactified on T^7. The parameter space in this case is $E_7/SU(8)$ which gives 70 scalars. They correspond to G_{ab}, B_{ab}, Φ, $C_{\hat{a}}$, $C_{\hat{a}\hat{b}\hat{c}}$ and $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ which transform in the $\mathbf{21}+\mathbf{15}+\mathbf{1}+\mathbf{6}+\mathbf{20}+\mathbf{6}$ representations of $SL(6)$, plus one additional scalar B_{abcdef} which is the dual of NS-NS 2-form $*B_{(2)}$. To see U-duality, we must consider the transformation of all these fields.

D9-branes $(d = 7)$: It corresponds to M-theory compactified on T^8. The parameter space in this case is $E_8/SO(16)$ which gives 128 scalars. They correspond to G_{ab}, B_{ab}, Φ, $C_{\hat{a}}$, $C_{\hat{a}\hat{b}\hat{c}}$, $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ and $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}\hat{g}}$ which transform in the $\mathbf{28}+\mathbf{21}+\mathbf{1}+\mathbf{7}+\mathbf{35}+\mathbf{21}+\mathbf{1}$ representations of $SL(7)$, plus 14 additional scalars B_{abcdef} and $C_{\mu a}$. This $C_{\mu a}$ is R-R 2-form field which has legs belong to one of worldvolume coordinates x^μ and one of torus coordinates y^a.

To see U-duality, we must consider the transformation of all these fields. However we don't know the way to introduce the field $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}$ and $C_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}\hat{g}}$ at this moment in time, so this discussion may be very difficult.

7.5 Summary and remarks

In this chapter, we considered some general Lorentzian Lie 3-algebras and studied the BLG models based on the symmetry. In the examples we studied, we naturally obtain the M/string theory compactification on the torus. The mass term generated by the Higgs VEV's of ghost fields can be identified with the Kaluza-Klein mass in the toroidal compactification. The dimension of the torus can be identified with the number of negative-norm generators of the Lie 3-algebra.

We also argued that one may use our technique to consider the D-branes' system where its gauge symmetry is described by infinite-dimensional loop algebras. In particular, we presented a detailed derivation of Dp-brane action on a torus T^d $(d = p - 2)$ from BLG model. In this case, the Higgs VEV's of ghost fields $\vec{\lambda}_A$ give the moduli of torus g_{ab}, the coupling constants g_{YM} of super Yang-Mills and the R-R $(p-3)$-form field $C_{\hat{a}}$ through the 'metric' $G_{AB} = \vec{\lambda}_A \cdot \vec{\lambda}_B$.

For D3-branes $(d = 1)$, the parameters thus obtained are enough to realize full Montonen-Olive duality group $SL(2, \mathbf{Z})$ through the linear transformation on $\vec{\lambda}_A$. Moreover, some part of the symmetry is actually the automorphism of Lie 3-algebra. For higher dimensional case $d > 1$ (Dp-branes with $p > 3$), these parameters are enough to implement a subgroup of U-duality transformation, $SL(d + 1, \mathbf{Z})$, which acts linearly on $\vec{\lambda}_A$. The transformations of various parameters can be determined through the linear transformation of the metric G_{AB}. In order to realize the full U-duality group, however, they are not enough.

We argued that one of the missed parameters, NS-NS 2-form background, can be introduced through the deformation of the Lie 3-algebra. For higher d, however, we need extra R-R background which we could not succeed to explain in the context of BLG model so far. To achieve this, we may have to consider more general structures of algebra.

7.5.1 Relation with ABJM model

It is interesting to derive the U-duality symmetry from ABJM model, which we mentioned in §1.5. While some work have been done in [28] for D3-branes, in their study, the coupling constants of the super Yang-Mills depend on one real parameter and the proof of the S-duality is limited. In particular, it may be interesting how to incorporate the loop algebras in ABJM context which would help us to go beyond D3-branes' case. As we explained, the loop algebra is suitable symmetry to describe the Kaluza-Klein modes.

7.5.2 Toward lower dimensional branes' theory

It is also interesting to see the U-duality in lower dimensional branes from BLG model. It is rather trivial to Dp-branes' system ($p \leq 1$), since we obtain it essentially by the double dimensional reduction of D2-branes' worldvolume. Therefore, if we want to have nontrivial discussion, we may have to derive the multiple F1-strings' system from BLG model, and discuss the U-duality relation with M2-branes and Dp-branes.

However, to obtain the multiple F1-strings is a challenging subject, since in particular we have no idea how to eliminate the Chern-Simons gauge field in BLG model without breaking supersymmetry. On the other hand, if BLG model actually describes the multiple M2-branes, the multiple F1-strings' system must be derived. Then this is also an important future work.

Conclusion and Discussion

In this Ph.D. thesis, we discuss how BLG model can describe the various M-branes' and D-branes' system, and U-duality between M2-branes and D-branes.

BLG model is originally proposed as the multiple M2-branes' worldvolume theory [1–4]. This breakthrough is very meaningful, since it gave the researchers the first opportunity to discuss multiple M2-branes' system. This model is the Chern-Simons matter system which has $\mathcal{N} = 8$ supersymmetry in $(2+1)$-dim spacetime. Their action is distinctive in that the gauge symmetry is based on a new mathematical framework, Lie 3-algebra.

The Lie 3-algebra which can be used in BLG model must satisfy the fundamental identity and invariant metric condition, in order that the model properly has gauge symmetry and supersymmetry. However, it was soon realized that these constraints are too restrictive that the only allowed Lie 3-algebra is so-called $\mathcal{A}_4$ algebra

$$[T^a, T^b, T^c] = i\epsilon^{abcd} T^d, \quad \langle T^a, T^b \rangle = \delta^{ab}\,, \tag{7.140}$$

where $a, b, \cdots = 1, \cdots, 4$, if we consider the finite-dimensional representation of Lie 3-algebra with positive-definite metric. BLG model in this case describes the two M2-branes' system [11, 12, 24].

Then, in order to obtain the description of larger number of M2-branes, many studies have been made to generalize the BLG framework. Concretely, we consider the *infinite*-dimensional representation of Lie 3-algebra or Lie 3-algebra with *negative-norm* generators. Regrettably, however, nobody finds how to obtain the more than two (and finite number of) M2-branes' system, up to now. Instead of this, we could show that the BLG model offers the *broader* framework, in that it can describe the other systems than M2-branes' one.

In Part II of the thesis, we discuss BLG model with Nambu-Poisson bracket [5, 6, 13]

$$\{f^a, f^b, f^c\} = \epsilon^{\mu\nu\rho} \frac{\partial f^a}{\partial y^\mu} \frac{\partial f^b}{\partial y^\nu} \frac{\partial f^c}{\partial y^\rho}, \quad \langle f^a, f^b \rangle = \int_{\mathcal{N}} d^3y\, f^a f^b\,, \tag{7.141}$$

as an infinite-dimensional representation of Lie 3-algebra. Here $\mathcal{N}$ is a 3-dim manifold on which the Nambu-Poisson bracket is defined, y^μ ($\mu = 1, 2, 3$) is coordinates on $\mathcal{N}$, and $f^a = f^a(y^\mu)$ ($a = 1, \cdots, \infty$) is the functions on $\mathcal{N}$. BLG model in this case describes the infinite number of M2-branes' system and realizes the worldvolume theory of a single M5-brane in the C-field background on the 3-dim manifold.

Moreover, we show that when we use the truncation version of Nambu-Poisson bracket, we can discuss the celebrated $N^{\frac{3}{2}}$ law for the degrees of freedom of the multiple M2-branes' system in BLG model.

In Part III, we discuss BLG model with Lorentzian Lie 3-algebra [15, 60, 61], which is the Lie 3-algebra with a pair of Lorentzian metric generators u, v and arbitrary Lie algebra generators T^i, such that

$$[u, T^i, T^j] = i f^{ij}{}_k T^k\,, \quad [T^i, T^j, T^k] = -i f^{ijk} v\,, \quad [v, *, *] = 0\,,$$
$$\langle u, v\rangle = 1\,, \quad \langle T^i, T^j\rangle = h^{ij}\,, \quad \text{otherwise} = 0\,, \tag{7.142}$$

where $f^{ij}{}_k$ and h^{ij} are structure constants and metric (or Killing form) of the Lie algebra $\mathcal{G}$. While the components associated with the generators u, v become ghosts, they can be removed by a new kind of Higgs mechanism proposed by [25]. This mechanism is unusual procedure, and in the future research, we must discuss closely whether it can be justified from the viewpoint of quantum field theory. However, at this moment, it is regarded as the very useful procedure for obtaining unitary theories from BLG model with Lorentzian Lie 3-algebra. In fact, surprisingly enough, this mechanism keeps all the gauge symmetry and $\mathcal{N} = 8$ supersymmetry. After the ghost is removed in such a way, when we choose $U(N)$ as Lie algebra $\mathcal{G}$, the Chern-Simons matter system is reduced to the ordinary super Yang-Mills system which describes multiple D2-branes' system [7, 26, 105–113].

This framework is also generalized by including more Lorentzian metric generators [7, 26, 113]. In particular, we show that when we use the central extension of Kac-Moody algebra or multiple loop algebra, BLG model describes the multiple Dp-branes' system on $(p-2)$-dim torus T^{p-2}. In this framework, we can discuss *U-duality* by comparing BLG model in this case and the original BLG model which describe Dp-branes and M2-branes, respectively. As a result, we can show that BLG model can properly realize (a part of) U-duality.

Until now, while we cannot get many pieces of information on the multiple M2-branes' system itself, we can make sure that BLG model has nontrivial information on M-theory, by showing that it can also describe M5-brane and D-branes and realize U-duality. Finally, for future directions, we want to discuss the following subjects:

Whole U-duality As we mentioned in Chapter 7, in order to see whole U-duality, we need to improve the Lie 3-algebra so that the algebra includes all the degrees of freedom of parameter space of U-duality group.

Multiple F1-strings' system This is also mentioned in Chapter 7. It is easy to do the double dimensional reduction of M2-branes' worldvolume in BLG model, but we have no idea how to eliminate the Chern-Simons gauge field without breaking supersymmetry. We have to find a good way to do it, since there must be no gauge field on F1-strings' worldsheet.

Covariant formulation of multiple M2-branes As we saw in Chapter 1, the covariant action of a single M2-brane is DBI-type action. Thus it is natural that the covariant action of multiple M2-branes is also of DBI-type. However, it is difficult to put the Chern-Simons gauge field in the action of such a form.

More than two M2-branes' system This may be one of the ultimate goals for BLG model. In this moment, we have no idea. One way is to improve the truncated Nambu-Poisson bracket which we used in Chapter 5, so that we can analyze BLG model in the Lagrangian level. Or, perhaps this cannot be done by Lie 3-algebra, and we need to consider more general Lie n-algebra ($n \geq 3$).

Relation to ABJM model and BFSS/BMN matrix model As we mentioned in Chapter 1, these models also describe the multiple M2-branes' system. Therefore it is interesting to study whether or not there are agreements and contradictions among BLG model and these models.

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Bibliography

[1] J. Bagger and N. Lambert, “Modeling multiple M2’s,” Phys. Rev. **D75** (2007) 045020 [arXiv:hep-th/0611108].

[2] J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” Phys. Rev. **D77** (2008) 065008 [arXiv:0711.0955 [hep-th]].

[3] J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” JHEP **0802** (2008) 105 [arXiv:0712.3738 [hep-th]].

[4] A. Gustavsson, “Algebraic structures on parallel M2-branes,” Nucl. Phys. **B811** (2009) 66 [arXiv:0709.1260 [hep-th]].

[5] P.-M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, “M5-brane in three-form flux and multiple M2-branes,” JHEP **0808** (2008) 014 [arXiv:0805.2898 [hep-th]].

[6] C.-S. Chu, P.-M. Ho, Y. Matsuo and S. Shiba, “Truncated Nambu-Poisson Bracket and Entropy Formula for Multiple Membranes,” JHEP **0808** (2008) 076 [arXiv:0807.0812 [hep-th]].

[7] P.-M. Ho, Y. Matsuo and S. Shiba, “Lorentzian Lie (3-)algebra and toroidal compactification of M/string theory,” JHEP **0903** (2009) 045 [arXiv:0901.2003 [hep-th]].

[8] T. Kobo, Y. Matsuo and S. Shiba, “Aspects of U-duality in BLG models with Lorentzian metric 3-algebras,” JHEP **0906** (2009) 053 [arXiv:0905.1445 [hep-th]].

[9] M. Kaku, “Introduction to Superstring and M-theory,” Springer-Verlag New York, Inc., 1999.

[10] N. Ohta, “Superstring theory, branes and M-theory” (in Japanese), Springer-Verlag Tokyo, 2002.

[11] G. Papadopoulos, “M2-branes, 3-Lie Algebras and Plucker relations,” JHEP **0805** (2008) 054 [arXiv:0804.2662 [hep-th]].

[12] P.-M. Ho, R.-C. Hou and Y. Matsuo, “Lie 3-Algebra and Multiple M2-branes,” JHEP **0806** (2008) 020 [arXiv:0804.2110 [hep-th]].

[13] P.-M. Ho and Y. Matsuo, “M5 from M2,” JHEP **0806** (2008) 105 [arXiv:0804.3629 [hep-th]].

[14] I. R. Klebanov and A. A. Tseytlin, “Entropy of Near-Extremal Black p-branes,” Nucl. Phys. **B475** (1996) 164 [arXiv:hep-th/9604089].

[15] P.-M. Ho, Y. Imamura and Y. Matsuo, “M2 to D2 revisited,” JHEP **0807** (2008) 003 [arXiv:0805.1202 [hep-th]].

[16] S. Coleman and J. Mandula, “All Possible Symmetries of the S Matrix,” Phys. Rev. **159** (1967) 1251.

[17] R. Haag, J. Łopuszánski and M. Sohnius, “All Possible Generators of Supersymmetries of the S Matrix,” Nucl. Phys. **B88** (1975) 257.

[18] E. Cremmer and B. Julia, “The $\mathcal{N} = 8$ Supergravity Theory – 1. The Lagrangian,” Phys. Lett. **B80** (1978) 48.

[19] E. Cremmer and B. Julia, “The SO(8) Supergravity,” Nucl. Phys. **B159** (1979) 141.

[20] B. E. Bergshoeff, E. Sezgin and P. K. Townsend, “Supermembranes and Eleven-Dimensional Supergravity,” Phys. Lett. **B189** (1987) 75.

[21] B. E. Bergshoeff, E. Sezgin and P. K. Townsend, “Properties of the Eleven-Dimensional Super Membrane Theory,” Ann. Phys. **185** (1988) 300.

[22] P. Pasti, D. P. Sorokin and M. Tonin,” “Covariant action for a $D = 11$ five-brane with the chiral field,” Phys. Lett. **B398** (1997) 41 [arXiv:hep-th/9701037].

[23] M. Green and J. H. Schwarz, Phys. Lett. **136B** (1984) 367.

[24] J. P. Gauntlett and J. B. Gutowski, “Constraining Maximally Supersymmetric Membrane Actions,” JHEP **0806** (2008) 053 [arXiv:0804.3078 [hep-th]].

[25] S. Mukhi and C. Papageorgakis, “M2 to D2,” JHEP **0805** (2008) 085 [arXiv:0803.3218 [hep-th]].

[26] P. de Medeiros, J. Figueroa-O’Farrill and E. Méndez-Escobar, “Metric Lie 3-algebras in Bagger-Lambert theory,” JHEP **0808** (2008) 045 [arXiv:0806.3242 [hep-th]].

[27] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP **0810** (2008) 091 [arXiv:0806.1218 [hep-th]].

[28] K. Hashimoto, T.-S. Tai and S. Terashima, “Toward a Proof of Montonen-Olive Duality via Multiple M2-branes,” JHEP **0904** (2009) 025 [arXiv:0809.2137 [hep-th]].

[29] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M Theory As A Matrix Model: A Conjecture," Phys. Rev. **D55** (1997) 5112 [arXiv:hep-th/9610043].

[30] D. Berenstein, J. M. Maldacena and H. Nastase, "Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills," JHEP **0204** (2002) 013 [arXiv:hep-th/0202021].

[31] M. Ali-Akbari, "3d CFT and Multi M2-brane Theory on $R \times S^2$," JHEP **0903** (2009) 148 [arXiv:0902.2869 [hep-th]].

[32] M. Li and T. Wang, "M2-branes Coupled to Antisymmetric Fluxes," JHEP **0807** (2008) 093 [arXiv:0805.3427 [hep-th]].

[33] D. S. Berman, "M-theory branes and their interactions," Phys. Rept. **456** (2008) 89 [arXiv:0710.1707 [hep-th]].

[34] J. H. Schwarz, "Superconformal Chern-Simons theories¡" JHEP **0411** (2004) 078 [arXiv:hep-th/0411077].

[35] S. Lee, "Superconformal field theories from crystal lattices," Phys. Rev. **D75** (2007) 101901 [arXiv:hep-th/0610204].

[36] A. Basu and J. A. Harvey, "The M2-M5 brane system and a generalized Nahm's equation," Nucl. Phys. **B713** (2005) 136 [arXiv:hep-th/0412310].

[37] P. S. Howe, N. D. Lambert and P. C. West, "The self-dual string soliton," Nucl. Phys. **B515** (1998) 203 [arXiv:hep-th/9709014].

[38] N. R. Constable, R. C. Myers and O. Tafjord, "Non-Abelian brane intersections," JHEP **0106** (2001) 023 [arXiv:hep-th/0102080].

[39] D.-E. Diaconescu, "D-branes, monopoles and Nahm equations," Nucl. Phys. **B503** (1997) 220 [arXiv:hep-th/9608163].

[40] A. Kapustin and S. Sethi, "The Higgs branch of impurity theories," Adv. Theor. Math. Phys. **2** (1998) 571 [arXiv:hep-th/9804027].

[41] D. Tsimpis, "Nahm equations and boundary conditions," Phys. Lett. **B433** (1998) 287 [arXiv:hep-th/9804081].

[42] D. S. Berman and N. B. Copland, "A note on the M2-M5 brane system and fuzzy spheres," Phys. Lett. **B639** (2006) 553 [arXiv:hep-th/0605086].

[43] M. M. Sheikh-Jabbari, "Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture," JHEP **0409** (2004) 017 [arXiv:hep-th/0406214].

[44] Z. Guralnik and S. Ramgoolam, "On the polarization of unstable D0-branes into non-commutative odd spheres," JHEP **0102** (2001) 032 [arXiv:hep-th/0101001].

[45] J. Bagger and N. Lambert, "Three-Algebras and $\mathcal{N} = 6$ Chern-Simons Gauge Theories," Phys. Rev. **D79** (2009) 025002 [arXiv:0807.0163 [hep-th]].

[46] V. T. Filippov, "n-Lie algebras," Sib. Mat. Zh. **26** (1985) 126.

[47] B. Pioline, "Comments on the topological open membrane," Phys. Rev. **D66** (2002) 025010 [arXiv:hep-th/0201257].

[48] Y. Kawamura, "Cubic matrix, generalized spin algebra and uncertainty relation," Prog. Theor. Phys. **110** (2003) 579 [arXiv:hep-th/0304149].

[49] A. Gustavsson, "Selfdual strings and loop space Nahm equations," JHEP **0804** (2008) 083 [arXiv:0802.3456 [hep-th]].

[50] H. Awata, M. Li, D. Minic and T. Yoneya, "On the quantization of Nambu brackets," JHEP **0102** (2001) 013 [arXiv:hep-th/9906248].

[51] Y. Nambu, "Generalized Hamiltonian dynamics," Phys. Rev. D **7**, 2405 (1973).

[52] Ph. Gautheron, "Some remarks concerning Nambu mechanics," Lett. in Math. Phys. **37** (1996) 103.

[53] D. Alekseevsky and P. Guha, "On Decomposability of Nambu-Poisson Tensor," Acta. Math. Univ. Commenianae **65** (1996) 1.

[54] R. Ibánez, M. de León, J. C. Marrero and D. M. de Diego, "Dynamics of generalized Poisson and Nambu-Poisson brackets," J. of Math. Physics **38** (1997) 2332.

[55] N. Nakanishi, "On Nambu-Poisson Manifolds," Reviews in Mathematical Physics **10** (1998) 499.

[56] G. Marmo, G. Vilasi, A. M. Vinogradov, "The local structure of n-Poisson and n-Jacobi manifolds," J. Geom. Physics **25** (1998) 141.

[57] I. Vaisman, "A survey on Nambu-Poisson brackets," Acta. Math. Univ. Comenianae **2** (1999) 213.

[58] L. Takhtajan, "On Foundation Of The Generalized Nambu Mechanics (Second Version)," Commun. Math. Phys. **160** (1994) 295 [arXiv:hep-th/9301111].

[59] J. -P. Dufour, N. T. Zung, "Linearization of Nambu structures," Compositio Mathematica **117** (1999) 83.

[60] J. Gomis, G. Milanesi and J. G. Russo, "Bagger-Lambert Theory for General Lie Algebras," JHEP **0806** (2008) 075 [arXiv:0805.1012 [hep-th]].

[61] S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, “$\mathcal{N} = 8$ superconformal gauge theories and M2 branes,” JHEP **0901** (2009) 078 [arXiv:0805.1087 [hep-th]].

[62] I. A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. P. Sorokin and M. Tonin, “Covariant action for the super-five-brane of M-theory,” Phys. Rev. Lett. **78** (1997) 4332 [arXiv:hep-th/9701149].

[63] M. Aganagic, J. Park, C. Popescu and J. H. Schwarz, “World-volume action of the M-theory five-brane,” Nucl. Phys. **B496** (1997) 191 [arXiv:hep-th/9701166].

[64] P. S. Howe, G. Sierra, P. K. Townsend, “Supersymmetry in six dimensions,” Nucl. Phys. **B221** (1983) 331.

[65] L. J. Romans, “Self-duality for interacting fields: Covariant field equations for six-dimensional chiral supergravities,” Nucl. Phys. **B276** (1986) 71.

[66] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, “Noncommutative Yang-Mills in IIB matrix model,” Nucl. Phys. **B565** (2000) 176 [arXiv:hep-th/9908141].

[67] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP **9909** (1999) 032 [arXiv:hep-th/9908142].

[68] J. A. Harvey and G. W. Moore, “Superpotentials and membrane instantons,” arXiv:hep-th/9907026.

[69] J. S. Park, “Topological open p-branes,” arXiv:hep-th/0012141.

[70] F. Lund and T. Regge, “Unified Approach to Strings and Vortices with Soliton Solutions,” Phys. Rev. **D14** (1976) 1524.

[71] E. Bergshoeff, D. S. Berman, J. P. van der Schaar and P. Sundell, “A noncommutative M-theory five-brane,” Nucl. Phys. B **590**(2000) 173 [arXiv:hep-th/0005026].

[72] S. Kawamoto and N. Sasakura, “Open membranes in a constant C-field background and noncommutative boundary strings,” JHEP **0007** (2000) 014 [arXiv:hep-th/0005123].

[73] N. Ikeda, “Deformation of BF theories, topological open membrane and a generalization of the star deformation,” JHEP **0107** (2001) 037 [arXiv:hep-th/0105286].

[74] D. S. Berman and B. Pioline, “Open membranes, ribbons and deformed Schild strings,” Phys. Rev. **D70** (2004) 045007 [arXiv:hep-th/0404049].

[75] Y. Matsuo and Y. Shibusa, “Volume preserving diffeomorphism and noncommutative branes,” JHEP **0102** (2001) 006 [arXiv:hep-th/0010040].

[76] P.-M. Ho and Y. Matsuo, "A toy model of open membrane field theory in constant 3-form flux," Gen. Rel. Grav. **39** (2007) 913 [arXiv:hep-th/0701130].

[77] J. Figueroa-O'Farrill, P. de Medeiros and E. Mendez-Escobar, "Lorentzian Lie 3-algebras and their Bagger-Lambert moduli space," JHEP **0807** (2008) 111 [arXiv:0805.4363 [hep-th]].

[78] U. Gran, B. E. W. Nilsson and C. Petersson, "On relating multiple M2 and D2-branes," JHEP **0810** (2008) 067 [arXiv:0804.1784 [hep-th]].

[79] N. B. Copland, "Aspects of M-Theory Brane Interactions and String Theory Symmetries," arXiv:0707.1317 [hep-th].

[80] S. Ramgoolam, "On spherical harmonics for fuzzy spheres in diverse dimensions," Nucl. Phys. **B610**, 461 (2001) [arXiv:hep-th/0105006].

[81] S. Ramgoolam, "Higher dimensional geometries related to fuzzy odd-dimensional spheres," JHEP **0210** (2002) 064 [arXiv:hep-th/0207111].

[82] G. Dito, M. Flato, D. Sternheimer and L. Takhtajan, "Deformation quantization and Nambu mechanics," Commun. Math. Phys. **183** (1997) 1 [arXiv:hep-th/9602016].

[83] T. Curtright and C. K. Zachos, "Classical and quantum Nambu mechanics," Phys. Rev. **D68** (2003) 085001 [arXiv:hep-th/0212267].

[84] Y. Kawamura, "Cubic matrix, Nambu mechanics and beyond," Prog. Theor. Phys. **109** (2003) 153 [arXiv:hep-th/0207054].

[85] M. A. Bandres, A. E. Lipstein and J. H. Schwarz, "Ghost-Free Superconformal Action for Multiple M2-Branes," JHEP **0807** (2008) 117 [arXiv:0806.0054 [hep-th]].

[86] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, "Supersymmetric Yang-Mills Theory From Lorentzian Three-Algebras," JHEP **0808** (2008) 094 [arXiv:0806.0738 [hep-th]].

[87] N. Lambert and D. Tong, "Membranes on an Orbifold," Phys. Rev. Lett. **101** (2008) 041602 [arXiv:0804.1114 [hep-th]].

[88] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, "M2-branes on M-folds," JHEP **0805** (2008) 038 [arXiv:0804.1256 [hep-th]].

[89] N. Ishibashi, "p-branes from (p-2)-branes in the bosonic string theory," Nucl. Phys. **B539** (1999) 107 [arXiv:hep-th/9804163].

[90] N. Ishibashi, "A relation between commutative and noncommutative descriptions of D-branes," arXiv:hep-th/9909176.

[91] L. Cornalba and R. Schiappa, "Matrix theory star products from the Born-Infeld action," Adv. Theor. Math. Phys. **4** (2000) 249 [arXiv:hep-th/9907211].

[92] L. Cornalba, "D-brane physics and noncommutative Yang-Mills theory," Adv. Theor. Math. Phys. **4** (2000) 271 [arXiv:hep-th/9909081].

[93] P.-M. Ho, Y.-Y. Wu and Y.-S. Wu, "Towards a noncommutative geometric approach to matrix compactification," Phys. Rev. **D58** (1998) 026006 [arXiv:hep-th/9712201].

[94] H. Lin, "Kac-Moody Extensions of 3-Algebras and M2-branes," JHEP **0807** (2008) 136 [arXiv:0805.4003 [hep-th]].

[95] W. Taylor, "D-brane field theory on compact spaces," Phys. Lett. **B394** (1997) 283 [arXiv:hep-th/9611042].

[96] P.-M. Ho "Twisted bundle on quantum torus and BPS states in matrix theory," Phys. Lett. **B434** (1998) 41 [arXiv:hep-th/9803166].

[97] D. Brace, B. Morariu and B. Zumino, "Dualities of the matrix model from T-duality of the type II string," Nucl. Phys. **B545** (1999) 192 [arXiv:hep-th/9810099].

[98] L. Dolan and M. J. Duff, "Kac-Moody Symmetries Of Kaluza-Klein Theories," Phys. Rev. Lett. **52** (1984) 14.

[99] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray and D. Stevenson, "Twisted K-theory and K-theory of bundle gerbes," Commun. Math. Phys. **228** (2002) 17 [arXiv:hep-th/0106194].

[100] A. Bergman and U. Varadarajan, "Loop groups, Kaluza-Klein reduction and M-theory," JHEP **0506** (2005) 043 [arXiv:hep-th/0406218].

[101] P. Bouwknegt and V. Mathai, "T-Duality as a Duality of Loop Group Bundles," J. Phys. **A42** (2009) 162001 [arXiv:0902.4341 [hep-th]].

[102] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," Nucl. Phys. **B438** (1995) 109 [arXiv:hep-th/9410167].

[103] C. M. Hull, "Matrix theory, U duality and toroidal compactifications of M theory," JHEP **9810** (1998) 011 [arXiv:hep-th/9711179].

[104] P.-M. Ho and Y.-S. Wu, "Noncommutative gauge theories in matrix theory," Phys. Rev. **D58** (1998) 066003 [arXiv:hep-th/9801147].

[105] B. Ezhuthachan, S. Mukhi and C. Papageorgakis, "D2 to D2," JHEP **0807** (2008) 041 [arXiv:0806.1639 [hep-th]].

[106] Y. Honma, S. Iso, Y. Sumitomo and S. Zhang, "Scaling limit of $\mathcal{N} = 6$ superconformal Chern-Simons theories and Lorentzian Bagger-Lambert theories," Phys. Rev. **D78** (2008) 105011 [arXiv:0806.3498 [hep-th]].

[107] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, "A Massive Study of M2-brane Proposals," JHEP **0809** (2008) 113 [arXiv:0807.1074 [hep-th]].

[108] H. Verlinde, "D2 or M2? A Note on Membrane Scattering," arXiv:0807.2121 [hep-th].

[109] S. Banerjee and A. Sen, "Interpreting the M2-brane Action," Mod. Phys. Lett. **A24** (2009) 721 [arXiv:0805.3930 [hep-th]].

[110] S. Cecotti and A. Sen, "Coulomb Branch of the Lorentzian Three Algebra Theory," arXiv:0806.1990 [hep-th].

[111] E. Antonyan and A. A. Tseytlin, "On 3d $\mathcal{N} = 8$ Lorentzian BLG theory as a scaling limit of 3d superconformal $\mathcal{N} = 6$ ABJM theory," arXiv:0811.1540 [hep-th].

[112] B. Ezhuthachan, S. Mukhi and C. Papageorgakis, "The Power of the Higgs Mechanism: Higher-Derivative BLG Theories," arXiv:0903.0003 [hep-th].

[113] P. de Medeiros, J. Figueroa-O'Farrill, E. Méndez-Escobar and P. Ritter, "Metric 3-Lie algebras for unitary Bagger-Lambert theories," JHEP **0904** (2009) 037 [arXiv:0902.4674 [hep-th]].