Master thesis

Superstring and M Theory on a Manifold with G_2 Holonomy

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Chapter 1

Introduction

1.1 Introduction

String theory [1, 2] was first introduced as a model for explaining the exotic features of the strong interaction such as quark confinement, Regge slope, dual resonances, and so on. However, it was found that string theory is not successful and QCD is the correct description for strong interactions. Therefore, string theory seemed to die. However, after many turns and twists, string theory is now thought to be a strong candidate for unifying all four interactions. The main reasons are its self-consistency, existence of gravity, no UV divergences, and no free parameters. There are five string theories which have space-time supersymmetries, type I, type IIA, type IIB, heterotic string with $SO(32)$ gauge group, and heterotic string with $E_8 \times E_8$ gauge group and they all live in ten dimensions. At the perturbative level, they are totally different from each other. Therefore, in the past, it was conjectured that one of five theories was the truely unified theory and others were not so interesting. And that one was the $E_8 \times E_8$ heterotic string. The reason was the similarity to the gauge group of GUT. And, as we will explain in the next section, heterotic string on Calabi-Yau manifolds provides us with $N = 1$ supersymmetry in four dimensions. Therefore, heterotic strings on Calabi-Yau manifolds lead to $N = 1$ supersymmetric theories with realistic gauge groups and correct dimension. For these reasons, they have been studied vigorously. However, this point of view was changed dramatically by the emergence of D-branes. D-branes are the solitons of the string theory and inform us of non-perturbative aspects of string theory. And it was found that there is an unique eleven dimensional theory called "M-theory" which is the truly unified theory. From this perspective, string theories are thought to describe the physics around some particular vacuum. Under these circumstances, manifolds with G_2 holonomy have received much attention. The main reason for this is that if we compactify M-theory on such a manifold, we can get a four dimensional theory with 4 supercharges, which is of phenomenological importance. They also play fundamental roles in realizing the $N = 1$ supersymmetric gauge theories. In fact, Atiyah, Maldacena and Vafa proposed a new duality by using the M theory on G_2 manifolds [43]. This work is based on the previous work by Vafa and his coraborators [40, 41, 42]. These are the origin of the recent work by Dijkgraaf and Vafa

[56] and their work is the realization of the possibility that the non-perturbative physics can be understood by the perturbative calculations, which was first pointed out in [27]. Therefore, manifolds with G_2 holonomy are important from both phenomenological and theoretical point of view. With this in mind, we review the fundamental properties of superstring and M-theory on G_2 manifolds.

The organization is as follows. We will first recall some of the basic facts about string theory and geometry in chapter 1. Then, we will discuss the Witten's index in chapter 2. And we will go to the main chapter, M-theory on G_2 manifolds. In this chapter, we will review the work of Atiyah, Maldacena and Vafa and the fundamentals for understanding their work. Or more concretely, we will review conifold transitions, $N = 2$ superconformal field theories, open and closed topological strings, relations to superstring, and their duality properties. In chapter 4, we will discuss CFT description of G_2 manifolds. Chapter 5 is conclusion and summary.

1.2 Basic facts

1.2.1 Why superstings require 10 dimensions?

Let us first review briefly what determines the dimensions. This chapter only contains basic facts which we assume the reader to be familiar with. To avoid technical difficulties, we first consider bosonic string. The action is

$$S = -\frac{1}{4\pi\alpha'}\int_M d\sigma d\tau (-\gamma)^{\frac{1}{2}}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \tag{1.1}$$

where γ_{ab} is a world sheet metric (Lorentz signature$(-,+)$) and γ is determinant of it. μ runs 0 to $D-1$. (D is the dimension of spacetime.)
This action has the following symmetries.
1. D-dimensional Poincaré invariance

$$X'^\mu(\tau,\sigma) = \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu \tag{1.2}$$

$$\gamma'_{ab}(\tau,\sigma) = \gamma_{ab}(\tau,\sigma) \tag{1.3}$$

2. Two dimensional diff invariance

$$X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma) \tag{1.4}$$

$$\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma) \tag{1.5}$$

3. Two dimensional Weyl invariance

$$X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma) \tag{1.6}$$

$$\gamma'_{ab}(\tau,\sigma) = \exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma) \tag{1.7}$$

This action of course describes a string propagating in D dimensional spacetime. But, we can also regard this as a two dimensional field theory. If we see from this point of view, X^μ are two dimensional bosons, interacting with two dimensional gravitons γ^{ab}. And this theory has a large gauge symmetry which we must fix; i.e. local diff$\times$Weyl symmetry. Let us fix this gauge symmetry.

We first replace the world sheet metric with a Euclidean one and denote this new metric as g_{ab}. If we forget the global structures of the world sheet, it is easy to show that we can make this metric δ_{ab} by combining the diff and Weyl gauge symmetry. Therefore, we can fix the gauge by restricting to the flat metric. By using a well-known Fadeev-Popov method, the action becomes

$$S = \frac{1}{2\pi\alpha'} \int dz d\bar{z} \left(\partial X^\mu \bar{\partial} X_\mu + b \bar{\partial} c + \bar{b} \partial \bar{c} \right) \tag{1.8}$$

after the gauge fixing. Here we change the world sheet coordinates to $z = \sigma^1 + i\sigma^2 = \sigma + i(i\tau)$ and its complex cojugate $\bar{z}$. If we change coordinates in such a way that z' is a holomorphic function of z

$$z' = f(z) \tag{1.9}$$

and combine this with a Weyl transformation, the metric becomes

$$ds'^2 = \exp(2\omega) |\partial_z f|^{-2} dz' d\bar{z}'. \tag{1.10}$$

If we take $\omega = \ln |\partial_z f|$ here, we have the same metric as before. This means that locally, we have an extra gauge symmetry (this is a local conformal symmetry) even after fixing the metric. Therefore, after fixing the gauge, we have two dimensional conformal field theories which consist of matter CFT (D free bonsons X^μ) and ghost CFT. As we know, a free bonson has central charge 1 and the ghost has central charge -26. For the theory to be consistent, the diff$\times$Weyl invariance must not be amomalous. It can be shown that the theory can be regularized in a diff invariant way. So, we can concentrate only on the Weyl anomaly. Recall that the Weyl invariance makes the energy-momentum tensor traceless; i.e. $T^a{}_a = 0$. The only possible form of the anomaly is known to be $T^a{}_a = a_1 R$, where a_1 is some constant and R is a scalar curvature of the world sheet. This can be rewritten in complex coordinates of the form

$$T_{z\bar{z}} = \frac{a_1}{2} g_{z\bar{z}} R. \tag{1.11}$$

Let us calculate the constant a_1. Taking the covariant derivative of both sides, we have

$$\nabla^{\bar{z}} T_{z\bar{z}} = \frac{a_1}{2} \partial_z R. \tag{1.12}$$

By using the conservation of T_{ab}, this can be written as

$$\nabla^z T_{zz} = -\nabla^{\bar{z}} T_{z\bar{z}} = -\frac{a_1}{2} \partial_z R. \tag{1.13}$$

The Weyl transformation of the right hand side is

$$a_1\partial_z\nabla^2\delta\omega \approx 4a_1\partial_z^2\partial_{\bar{z}}\delta\omega \tag{1.14}$$

where we expand around a flat world sheet. We next calculate the Weyl transformation of the left hand side. The conformal transformation of the T_{zz} is

$$\delta T_{zz}(z) = -\frac{c}{12}\partial_z^3 v^z(z) - 2\partial_z v^z(z)T_{zz} - v^z(z)\partial_z T_{zz}(z) \tag{1.15}$$

where c is the central charge of the theory. As we saw, conformal transformation consists of two pieces; a coodinate transformation $\delta z = v$ and a Wely transformation $2\delta\omega = \partial v + (\partial v)^*$. The last two terms are the coordinate transformation of the tensor. Therefore the Weyl transformation of T_{zz} can, to leading order around the flat metric, be read as

$$\delta_W T_{zz} = -\frac{c}{6}\partial_z^2\delta\omega. \tag{1.16}$$

We have the result that a_1 is proportional to the central charge. Therefore, the total central charge of the theory must vanish. In our case, we must have $D + (-26) = 0$. This result give the answer to our question; the central charge of the ghost CFT determines the dimensions. If we consider the superstring theory, we have the superconformal symmetry. And we have D bosons, D fermions, and conformal and superconformal ghosts. A free fermion has central charge 1/2 and superconformal ghost has central charge 11. As above, we have $(1 + \frac{1}{2})D + (-26 + 11) = 0$. This shows that the superstring must be in 10 dimensions.

1.2.2 Emergence of the eleventh dimension

As a low energy effective field theory of the string theory we have a supergravity theory. Supergravity theories are constructed in dimensions less than 11. But superstring theory is in 10 dimensions. What does this mean? Why is there 11 dimensional theory? Is the supergravity in eleven dimensions only a theoretical possibility which is not relevant to the actual world? Today's our knowledge is that we have an eleven dimensional M-theory as a truely unified theory and the elevendimensional supergravity is the low energy effective field theory of it. Superstring theory is thought to describe a particular point of M-theory moduli space. (description around some particular vacuum of the M-theory). Let us explain this with one example. Recall that type IIA supergravity can be obtained by dimensional reduction of the 11-dimensional supergravity on S^1. So, it is natural to consider the type IIA superstring theory. Type IIA theory has 1-form A_μ, 3-form $A_{\mu\nu\lambda}$ in the ground state of RR sector. So, there are D0-brane, D2-brane, D4-brane, D6-brane and D8-brane in the type IIA. Let us concentrate on D0-branes. A D0-brane is a BPS state and it has a following mass

$$\tau_0 = \frac{1}{g\alpha'^{\frac{1}{2}}}. \tag{1.17}$$

The D0 brane is very heavy at weak coupling but becomes light in the strong coupling limit. We also know that for any number n, there is an ultrashort multiplet of bound sates of n D0 branes with mass

$$n\tau_0 = \frac{n}{g\alpha'^{\frac{1}{2}}}. \tag{1.18}$$

Because of the BPS property, this is exact and valid even if the coupling is strong. The spectrum becomes continuous as $g \to \infty$. Such a continuous spectrum is a characteristic of a system that is becoming noncompact. The above spectrum coincides with the spectrum of Kaluza-Klein states for a circle of radius

$$R_{10} = g\alpha'^{\frac{1}{2}} \tag{1.19}$$

As $g \to \infty$, the eleventh dimension appears. The reason that perturbative string theory can not detect the eleventh dimension is evident. A perturbation theory is an expansion around $g = 0$. But, this is the limit in which the eleventh dimension disappears.

1.2.3 The condition for $N = 1$ spacetime supersymmetry

So far, we have seen that superstrings require 10 dimensions and M-theory 11 dimensions. But we all know we live in 4 dimensions. Therefore it is natural to think that the other 6 or 7 dimensions are extremely small compared with our 4 dimensional world. Therefore, the problem of compactification down to 4 dimensions should be taken seriously. Then, what should we require for compactification? Compactification on general manifolds leads to a theory without supersymmetry. This causes the serious problems. One is a technical one. A theory without supersymmetry has too complicated quantum corrections and we want to avoid them. The other is a phenomenological one. There is a problem of "naturalness" or "gauge hierarchy" problem. We know we can avoid this problem by introducing supersymmetry. It was in fact one of the motivation of studying supersymmetry. So, we require that there be unbroken supersymmetries. However too much supersymmetry is also troublesome. For example, $N = 2$ supersymmetry in 4-dimensions have no chiral fermions. Therefore, it is reasonable to require $N = 1$ spacetime supersymmetry. Then, how can we have such models? In other words, what determines the number of unbroken supercharges when we compactify the superstring theory or M-theory? The answer turns out to be related to the holonomy group of the manifold. Let us explain this in the heterotic string case. We assume the four-dimensional Poincaré invariance in the following discussion. Then, the metric must be of the form

$$\begin{pmatrix} f(y)\eta_{\mu\nu} & 0 \\ 0 & G_{mn}(y) \end{pmatrix}. \tag{1.20}$$

We denote the noncompact coordinates by x^μ with $\mu, \nu = 0, 1, 2, 3$ and the compact coordinates by y^m with $m, n = 4, 5, ..., 9$. The other potential nonvanishing fields are $\Phi(y)$(dilaton), $H_{mnp}(y)$(three form field-strength) and $F_{mn}(y)$(field-strength). The condition for the unbroken supersymmetry is that the variations of the fermi fields are zero.

(This comes from $\langle\Omega|\{Q,...\}|\Omega\rangle = 0$) For the 10-dimensional $N = 1$ supergravity of the heterotic string these variations are

$$\delta\psi_\mu = \nabla_\mu \varepsilon \tag{1.21}$$

$$\delta\psi_m = \left(\partial_m + \frac{1}{4}\Omega^-_{mnp}\Gamma^{np}\right)\varepsilon \tag{1.22}$$

$$\delta\chi = \left(\Gamma^m \partial_m \Phi - \frac{1}{12}\Gamma^{mnp}H_{mnp}\right)\varepsilon \tag{1.23}$$

$$\delta\lambda = F_{mn}\Gamma^{mn}\varepsilon \tag{1.24}$$

where

$$\Omega^{\pm}_{MNP} = \omega_{MNP} \pm \frac{1}{2}H_{MNP} \tag{1.25}$$

and ω is the spin connection. These are the variations of the gravitino, dilatino, and gaugino respectively. Under the decomposition $SO(9,1) \to SO(3,1) \times SO(6)$, the **16** decomposes as

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \tag{1.26}$$

Thus, a Majorana-Weyl supersymmetry parameter can be written as

$$\varepsilon(y) \to \varepsilon_{\alpha\beta} + \varepsilon^*_{\alpha\beta}(y) \tag{1.27}$$

The indices on $\varepsilon_{\alpha\beta}$ transform respectively as $(\mathbf{2}, \mathbf{4})$. If there is any unbroken supersymmetry, we can generate further supersymmetry by $SO(3,1)$ rotations and reach the following form

$$\varepsilon_{\alpha\beta} = u_\alpha \zeta_\beta(y) \tag{1.28}$$

for an arbitrary Weyl spinor u. Therefore each internal spinor $\zeta_\beta(y)$ for which $\delta(\text{fermions}) = 0$ gives one copy of the minimal four-dimensional supersymmetry algebra. Let us look for the solutions. We make an additional assumption that the antisymmetric tensor field strength vanishes

$$H_{mnp} = 0. \tag{1.29}$$

In this case, from the vanishing of $\delta\chi$ we can deduce that the dilaton must be constant

$$\partial_m \Phi = 0. \tag{1.30}$$

Then the vanishing of $\delta\psi_\mu$ forbids a y-dependent scale factor of the metric; i.e. $f(y) = 1$. And finally the vanishing of $\delta\psi_m$ gives

$$\nabla_m \zeta = 0. \tag{1.31}$$

This shows that the internal space must admit the existence of a covariantly constant spinor. This is a strong condition and restricts a holonomy group of the internal space. When the internal space is given a metric and oriented, the holonomy group is generally $SO(d)$ where d is the dimension of the internal space. Covariant constant spinors are invariant under the action of the holonomy group. So, having a covariant constant spinor means that the holonomy group is the subgroup G of $SO(d)$ and under $G \subset SO(d)$, the spinor of the $SO(d)$ must decompose like

$$\text{spinor of } SO(d) \to (\text{something}) + \mathbf{1} \tag{1.32}$$

For example, if we want to compactify the heterotic string down to four dimensions with $N = 1$ supersymmetry, the holonomy of the internal six dimensional mainifold must be $SU(3) \subset SO(6)$. This is because under $SO(6) \to SU(3)$, the spinor decomposes $\mathbf{4} \to \mathbf{3} + \mathbf{1}$ and we can use $\mathbf{1}$ to define the supersymmetry. The number of remaining supercharges are $16 \times \frac{1}{4} = 4$, the correct number for $N = 1$ SUSY in 4 dimensions. The other possibilities are listed in the table.

Holonomy group	Dimension	Name	Remaining supercharges
$SU(2)$	4	K3 surface	$\frac{1}{2}$
$SU(3)$	6	Calabi-Yau	$\frac{1}{4}$
G_2	7	G_2	$\frac{1}{8}$
$Spin(7)$	8	$Spin(7)$	$\frac{1}{16}$

From this table, we see that superstrings on Calabi-Yau manifolds and M-theory on G_2 are the most interesting cases. Understanding the physics of superstring and M-theory on G_2 is the aim of this review. But also, studying the $K3$ surface is interesting and teaches us a lot about the non-perturbative aspects of the string theories [9].

1.2.4 Calabi-Yau manifolds

Let us recall some of the basic facts about complex geometry. We say that M is a complex manifold if and only if the following conditions are satisfied.

1. M is a topological space.
2. M is provided with a family of pairs $\{(U_i, \varphi_i)\}$
3. $\{U_i\}$ is a family of open sets which covers M. And φ_i is a homeomorphism from U_i to an open subset U'_i of $\mathbf{C}^m$. In particular, M is even dimensional.
4. For given U_i and U_j such that $U_i \cap U_j \neq \varnothing$, the map $\psi_{ji} = \varphi_j \varphi_i^{-1}$ from $\varphi_i(U_i \cap U_j)$ to $\varphi(U_i \cap U_j)$ is holomorphic.

The term "analytic" is meaningful only on complex manifolds. Let $z^i, z^{\bar{j}}$ be a local coordinate on some complex manifold. We can define the following form

$$K = -ig_{i\bar{j}} dz^i \wedge dz^{\bar{j}}, \tag{1.33}$$

which is called a Kähler form. A Kähler form is not generally a closed form. But in some cases, it is closed. A complex manifold with closed Kähler form has many good properties. Therefore let us introduce a new term for such objects. A complex manifold is called a Kähler manifold if and only if the Kähler form is closed; $dK = 0$. Therefore, in Kähler manifolds, a Kähler form is in $H^{1,1}(M)$. Let us call the class to which Kähler form belongs a Kähler class. A Kähler class is related to the volume of the manifold. Some properties of the Kähler manifold are listed below.

1. Existence of the Kähler potential
   A Kähler metric can be expressed locally as $g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \mathcal{K}$. Where $\mathcal{K}$ is a function called Kähler potential.

2. Torsionless
   A Kähler metric is torsionless.

3. Special Holonomy
   The nonvanishing connection coefficients are of the form Γ^i_{jk} or $\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}$ This implies that if we parallel transport the vectors of the form $\frac{\partial}{\partial z^i}$ along some closed path, the vectors are rotated only among holomorphic part. This means that the holonomy group is at most $U(n)$, where n is a complex dimension of the manifold.

4. Equvalence of various Laplacians
   In general complex manifolds, we can define three Laplacians based on d, ∂, and $\bar{\partial}$. For example, we can define the Laplacian from d as $dd^* + d^*d$. Let us denote these Laplacians by Δ_d, Δ_∂, and $\Delta_{\bar{\partial}}$ respectively. In a general complex manifold, these objects are not related at all. But in a Kähler manifold, the following relations are satisfied

   $$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \tag{1.34}$$

   In particular, $H^n_{DR}(M) = \oplus_{p+q=n} H^{p,q}(M)$.

Some famous examples of Kähler manifolds are complex Euclid space $\mathbf{C}^n$, complex projective space $\mathbf{CP}^n$, Riemann surfaces, and so on.

A Calabi-Yau manifold is a Kähler manifold with vanishing first Chern class. Long ago, Calabi conjectured that the Kähler manifold with vanishing Chern class has an unique Ricci flat metric for fixed Kähler class. Later, Yau showed that the Calabi's conjecture is true. Therefore, the vanishing first Chern class asserts the existence of the Ricci flat metric. Some important properties of Calabi-Yau manifolds are

1. Special holonomy
   Calabi-Yau manifolds have smaller holonomy $SU(n)$.

2. Holomorphic n-form
   The condition that Kähler manifolds with vanishing first Chern class is equivalent to the existence of a nowhere vanishing holomorphic $(n, 0)$ form Ω. This $(n, 0)$ form is covariant constant in the Ricci-flat metric.

3. Simple Hodge diamond

We consider the case in which the Calabi-Yau manifold is 6-dimensional. By using the above holomorphic form, it can be shown that

$$h^{p,0} = h^{3-p,0}. \tag{1.35}$$

Also, it can be shown that for a Calabi-Yau whose holonomy is exactly $SU(3)$

$$h^{1,0} = h^{0,1} = 0. \tag{1.36}$$

Combining these facts with the property of the Kähler manifold

$$h^{p,q} = h^{q,p}, \quad h^{n-p,n-q} = h^{p,q} \tag{1.37}$$

and usual

$$h^{0,0} = 1, \tag{1.38}$$

we have the following Hodge diamond

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1. \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{1.39}$$

From this diamond, we see that the Euler number is given by

$$\chi = 2(h^{1,1} - h^{2,1}). \tag{1.40}$$

4. Mirror symmetry

To describe the string theory on Calabi-Yau manifolds, we use $(2,2)$ superconformal field theories with $c = 9$. There are two types of marginal operators in such theories. These correspond to the two geometrical ways of deforming Calabi-Yau manifolds, one is Kähler moduli and the other is complex moduli. It is evident that the number of Kähler moduli is given by $h^{1,1}$. On the other hand, the complex moduli are given by the fields g_{ij}. This is because a coordinate change is needed to bring the metric back to the Hermitian form. However, this is not a form. Even so, we can make the following form

$$g_{i\bar{l}\bar{m}} = g_{ij} G^{j\bar{k}} \Omega_{\bar{k}\bar{l}\bar{m}}. \tag{1.41}$$

where G is the metric. Then, this shows that the the number of complex structure moduli is $h^{2,1}$. These two kinds of moduli are totally different objects from a geometrical point of view. However, there is an asymmetry between the abstract conformal field theory description and the geometrical realization. From the CFT viewpoint, the two kinds of operators differ only in the conventional sign of a $U(1)$ charge. This suggests that the Calabi-Yau manifolds exist with pairs and the Hodge number of each pair is mirror to each other. In other words, if we have a Calabi-Yau manifold M with the Hodge number $(h^{1,1}, h^{2,1})$, then there is an another Calabi-Yau manifold W satisfying

$$(h^{1,1}, h^{2,1})_M = (h^{2,1}, h^{1,1})_W. \tag{1.42}$$

There are now vast evidences for this conjecture.

1.2.5 Manifolds with G_2 holonomy

We saw above that the holonomy group of the internal space determines the remaining supercharges. And we noted that the M theory on a manifold having G_2 holonomy is one of the interesting cases. Then, how can we characterize the G_2 manifolds? Let us explain this. As we know, the group G_2 is one of the exceptional Lie groups. But we give another definition here. We can define the group G_2 as a subgroup of $SO(7)$ whose action on $\mathbf{R}^7$ preserves the following form

$$\begin{aligned}\varphi = dx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge dx^4 \wedge dx^5 + dx^1 \wedge dx^6 \wedge dx^7 + dx^2 \wedge dx^4 \wedge dx^6 \\ - dx^2 \wedge dx^5 \wedge dx^7 - dx^3 \wedge dx^4 \wedge dx^7 - dx^3 \wedge dx^5 \wedge dx^6.\end{aligned} \tag{1.43}$$

$SO(7)$ has rank 3 and dimension 21. The important representations are the vector $\mathbf{7}$, the spinor $\mathbf{8}$, and the adjoint $\mathbf{21}$. On the other hand, G_2 has rank 2 and dimension 14. Its important representations are the fundamental $\mathbf{7}$ and the adjoint $\mathbf{14}$. Under $G_2 \subset SO(7)$ the branching rules of these representations are

$$\begin{aligned}\mathbf{21} &= \mathbf{14} \oplus \mathbf{7} \\ \mathbf{7} &= \mathbf{7} \\ \mathbf{8} &= \mathbf{7} \oplus \mathbf{1}\end{aligned} \tag{1.44}$$

The decomposition $\mathbf{8} = \mathbf{7} \oplus \mathbf{1}$ means that G_2 holonomy manifolds admit precisely one covariant constant spinor, and remaining supercharges become $\frac{1}{8}$. Note that when we consider the superstring on G_2 holonomy manifolds, the above differential form that is invariant under the G_2 action becomes one of the generators of the G_2 CFT.

We mainly interested in M theory on G_2 manifolds. However, we still do not understand what M theory is. So, we must use indirect approaches and there are two main tools. One is based on supergravity and the other is based on CFT. We review from both point of view.

Chapter 2

Witten index

Consider any supersymmetric theory. If we want to know whether supersymmetry is spontaneously broken or not, we have only to check the energy of the vacuum. However, it is not easy to determine the energy of the vacuum. Even if the vacuum energy appears to be zero in some approximation, tiny corrections which have been neglected may cause the energy to be small but non-zero. Witten partially solved this problem by introducing Witten's index. In this chapter, we will review his work following [13].

2.1 Witten index

2.1.1 Definitions and physical implications

It is useful to consider the supersymmetric theories formulated in a finite spacial volume. In a finite volume, the spectrum is discrete and can be counted in a well-defined way. Since translations are part of the supersymmetry algebra, we must impose the boundary conditions that preserve translational invariance. This leads us to use periodic boundary conditions both for bosons and fermions. In general, it is not meaningful to ask whether an internal symmetry is spontaneously broken in finite volume because usually an internal symmetry is unbroken in a finite volume. However, supersymmetry can be spontaneously broken in a finite volume. This is because supersymmetry breaking just means that the ground state energy is positive, which is possible in a finite volume or even in a finite number of degrees of freedom.

Given a theory defined in a volume V with a Hilbert space $\mathcal{H}$, our main concern is the existence of zero-energy states in $\mathcal{H}$. In supersymmetric theories, the energy E is equal to or greater than the momentum $|\mathbf{P}|$. Therefore, the zero-energy states must have $\mathbf{P} = 0$ and we can restrict our attention to $\mathbf{P} = 0$ without losing any information. In the subspace of zero momentum, the supersymmetry algebra takes the following simple form

$$Q_1^2 = Q_2^2 = \dots = Q_K^2 = H \tag{2.1}$$

$$Q_i Q_j + Q_j Q_i = 0, \quad \text{for } i \neq j. \tag{2.2}$$

Let us first work with any one of the Q_i's that we denote as Q.

Let $|b\rangle$ be a bosonic state satisfying $\exp(2\pi i J_z)|b\rangle = |b\rangle$. And let $|f\rangle$ be a fermionic state with $\exp(2\pi i J_z)|f\rangle = -|f\rangle$. We define the following operator

$$(-1)^F = \exp(2\pi i J_z). \tag{2.3}$$

As we know, the states of non-zero energy are paired by the action of Q. If $|b\rangle$ is state of non-zero energy E, we can define a normalized fermionic state $|f\rangle = (1/\sqrt{E})Q|b\rangle$. Then, we have

$$Q|b\rangle = \sqrt{E}|f\rangle, \quad Q|f\rangle = \sqrt{E}|b\rangle, \tag{2.4}$$

where the second equation is chosen to satisfy $Q^2 = H$. All states of non-zero energy are paired in two dimensional supermultiplets with this structure. On the other hand, the zero-energy states are not paired in this way. By using $Q^2 = H$ and the hermitian property of Q, each state annihilated by H is also annihilated by Q. Therefore, they form trivial one dimensional supersymmetry multiplets. Therefore, in the general situations, there are paired states of positive energy and states of zero energy which are not necessarily paired. From this simple observation, we can deduce the following important facts. Suppose we vary the parameters of the theory. Then, the states of non-zero enegy move around in energy in Bose-Fermi pairs. They can move down to $E = 0$. If this happens, the number of bosonic states of zero energy $n_B^{E=0}$ and the number of fermionic states of zero energy $n_F^{E=0}$ increase in the same way. On the other hand, some states of zero energy may gain non-zero energy. Of course it is not possible for a single zero-energy state to acquire a non-zero energy. What can occur is that pairs of states can move from $E = 0$ to some non-zero energy state. In this case, $n_B^{E=0}$ and $n_F^{E=0}$ decrease by the same number. In either case, the difference $n_B^{E=0} - n_F^{E=0}$ is invariant under the change of parameters. Therefore, this quantity can be calculated reliably. It can be calculated in a convenient limit such as small volume, large mass, and weak coupling. And when such calculation is done, following two cases are possible.

(1) $n_B^{E=0} - n_F^{E=0} \neq 0$

In this case, supersymmetry is not broken. This is because we have either $n_B^{E=0} \neq 0$ or $n_F^{E=0} \neq 0$ and there are some states of zero energy.

(2) $n_B^{E=0} - n_F^{E=0} = 0$

In this case, there are two possibility.

(A) $n_B^{E=0} = n_F^{E=0} = 0$. Supersymmetry is broken and there is a massless Goldstone fermion in the infinite volume theory.

(B) $n_B^{E=0} = n_F^{E=0} \neq 0$. Supersymmetry is not broken. There is no Goldstone fermion but there are zero-energy fermionic states which are interpreted as evidence that the infinite volume theory has a massless fermion. In either case, we conclude that if $n_B^{E=0} - n_F^{E=0} = 0$, the infinite volume theory has a massless fermion.

The quantity $n_B^{E=0} - n_F^{E=0}$ can be regarded as the trace of the operator $(-1)^F$ since states of non-zero energy do not contribute to the trace of $(-1)^F$ because of the pairing. Thus, we have

$$\mathrm{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0}, \tag{2.5}$$

which is called the Witten's index. This formula should be regarded as merely a useful definition because the infinite sum over all states in Hilbert space required to define $\mathrm{Tr}(-1)^F$ is ill-defined. We could regularize $\mathrm{Tr}(-1)^F$ by writing $\mathrm{Tr}(-1)^F \exp(-\beta H)$ for arbitrary positive β.
The name "index" comes from the fact that the quantity $\mathrm{Tr}(-1)^F$ is an example of a standard mathematical concept; the index of an operator. Since the supercharge Q maps bosons into fermions and vice versa, it takes the following form

$$\begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix} \tag{2.6}$$

if the states are arranged in the form

$$\begin{pmatrix} B \\ F \end{pmatrix}. \tag{2.7}$$

Note that because Q is Hermitian, $M^\dagger$ is the adjoint of M. Since $H = Q^2$, the zero-energy states are precisely those annihilated by Q. Bosonic states annihilated by Q are states ψ which satisfy $M\psi = 0$. In the same manner, fermionic states annihilated by Q are states ψ that satisfy $M^\dagger\psi = 0$. Therefore, the quantity $\mathrm{Tr}(-1)^F$ is equal to the number of solutions of $M\psi = 0$ minus the number of solutions of $M^\dagger\psi = 0$. By definition, the latter quantity is the index of the operator M. Then, the fact that the Witten's index is independent of the parameters of the theory is a special case of the fact that the index of some operator is invariant under small deformations.

2.1.2 Forbidden changes in parameters

We have seen that the Witten's index is invariant under the change of parameters. However, there are a few subtleties. The most serious problem comes from the behavior of the potential energy for large field strenghs. For instance, consider the following potential

$$V(\phi) = (m\phi - g\phi^2)^2. \tag{2.8}$$

At $g = 0$, low energy states correspond to $\phi \sim 0$ but for $g \neq 0$, low energy states may correspond to $\phi \sim m/g$. Therefore, an arbitrarily small but non-zero g causes the existence of extra low energy states at $\phi \sim m/g$ that have no counterpart in the $g = 0$ case. In such a case, the Witten's index at $g = 0$ has a different value from that at $g \neq 0$. This is related to the change in asymptotic behavior of V. At $g = 0$, $V \sim \phi^2$ for large ϕ, but for non-zero g, $V \sim \phi^4$. In this way, the change in asymptotic behavior can change the Witten's index discontinuously. The general rule is that $\mathrm{Tr}(-1)^F$ is invariant under any change in parameters in which the Hamiltonian changes by terms no bigger than the terms already present in the large field limit. This is the crucial ingredient in showing the constancy of $\mathrm{Tr}(-1)^F$.

2.2 Some generalizations

In the last section, we have seen that the difference $n_B^{E=0} - n_F^{E=0}$ is invariant under reasonable changes in the parameters of the theory. In this section, we will see that the two numbers $n_B^{E=0}$ and $n_F^{E=0}$ are separately invariant under a smaller but still interesting class of changes in parameters. Though we worked with a single supersymmetry charge, we must make use of two supercharges in this section. We denote these two supercharges by Q_1 and Q_2. If we define $Q_\pm = \sqrt{\frac{1}{2}}(Q_1 \pm iQ_2)$, the supersymmetry algebra in the zero momentum sector takes the following simple form

$$Q_+^2 = Q_-^2 = 0, \quad Q_+Q_- + Q_-Q_+ = H. \tag{2.9}$$

Any operator Q_+ with $Q_+^2 = 0$ can be put in Jordan canonical form by a linear transformation. The Jordan canonical form of Q_+ would be of the form

$$\begin{bmatrix} 0 & 1 & & & & & & & \\ 0 & 0 & & & & & & & \\ & & 0 & 1 & & & & & \\ & & 0 & 0 & & & & & \\ & & & & 0 & 1 & & & \\ & & & & 0 & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix}. \tag{2.10}$$

The structure $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ corresponds to the two dimensional supermultiplets. In addition, there are an arbitrary number of unpaired zeros. They are the supersymmetric zero-energy states.

It is evident that $Q_+\chi = 0$ if χ can be written as $\chi = Q_+\psi$. Then, let us ask the converse question. Given any state χ with $Q_+\chi = 0$, when can we write $\chi = Q_+\psi$ for some ψ? The answer turns out to be that we can write χ in the form $\chi = Q_+\psi$ if and only if $E \neq 0$. The proof is as follows.

(1) The case $E \neq 0$

Define $\psi = (1/E)Q_-\chi$. Then, $Q_+\psi = (1/E)(Q_+Q_- + Q_-Q_+)\chi = \chi$. In this way, we have explicitly found a state ψ.

(2) The case $E = 0$

Suppose there exists some ψ such that $\chi = Q_+\psi$. Since Q_+ commutes with the Hamiltonian, the equation $Q_+\psi = \chi$ implies that ψ has the same energy as χ. By assumption, $E_\chi = 0$. Therefore, ψ must have zero energy. However, since the Q_i annihilate all states of zero energy, it follows $Q_+\psi = 0$, which is a contradiction.

Therefore, if we define $N = n_B^{E=0} + n_F^{E=0}$, we have

$$N = \dim(\mathrm{Ker}(Q_+)/\mathrm{Im}(Q_+)). \tag{2.11}$$

We would like to find the conditions under which N is invariant under changes in parameters of the theory. Of course, the invarinace of N doesn't follow from the supersymmetry alone. To see this, let us consider the following representation of the supersymmetry algebra.

$$Q_+ = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad Q_- = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}. \tag{2.12}$$

For $\lambda \neq 0$, no state has zero energy $N = 0$. But for $\lambda = 0$, there are two zero-energy states $N = 2$. In this way, the number of zero-energy states can change when the parameters are changed. However, there is a restricted class of changes in parameters which does not change the total number of zero-energy states N. Consider the following new operators

$$\tilde{Q}_+ = M^{-1} Q_+ M \tag{2.13}$$

$$\tilde{Q}_- = M^\dagger Q_- {M^\dagger}^{-1} \tag{2.14}$$

$$\tilde{H} = \tilde{Q}_+ \tilde{Q}_- + \tilde{Q}_- \tilde{Q}_+ \tag{2.15}$$

where M is an arbitrary invertible linear opertor. If M is unitary, $M^\dagger = M^{-1}$, the new operators differ from the original ones only a change of basis of the Hilbert space. Therefore, the theory described by the new operators is inequivalent to the original one if and only if M is not unitary. The important thing is that the number of zero-energy states for $\tilde{H}$ is always equal to the number of zero-energy states of H. This can be seen as follows. Let $\chi \in \mathcal{H}$ be the state which satisfies $Q_+\chi = 0$ but not $\chi = Q_+\psi$ for some ψ. Then let us define $\tilde{\chi} = M^{-1}\chi$, which satisfies $\tilde{Q}_+\tilde{\chi} = 0$ by definition. If we suppose $\tilde{\chi} = \tilde{Q}_+\tilde{\psi}$, then this leads to the contradiction because in this case $\chi = Q_+(M\tilde{\psi})$. This completes the proof.

Then, which parameters correspond to such desirable changes? There are mainly following parameters.
(1) The usual mass terms, self-coupling of scalars, and Yukawa couplings. These can be written in the form of superpotentials.
(2) Gauge couplings
(3) The θ term.
(4) The Fayet-Iliopoulos D term.
We first show that parameters in the group (1) can be changed by the above method. Here, we work with γ^0 eigenstates rather than a Majorana basis. As γ^0 is imaginary in the Majorana basis, the $\gamma^0 = +1$ supercharges are a complex doublet Q_α and the $\gamma^0 = -1$ charges are complex conjugates Q^*_β.

Consider the simplest Wess-Zumino model. It has a single complex scalar field ϕ and a single spinor field ψ. The parameters of this theory are mass m and the coupling g. Q_α for parameters (m_1, g_1) is related to $\tilde{Q}_\alpha$ for parameters (m_2, g_2) by the following way

$$\tilde{Q}_\alpha(m_2, g_2) = M^{-1} Q_\alpha(m_1, g_1) M \tag{2.16}$$

where

$$M = \exp\left(2\mathrm{Re} \int d^3x((m_2 - m_1)\frac{1}{2}\phi^2 + (g_2 - g_1)\frac{1}{3}\phi^3)\right). \tag{2.17}$$

Standard definition of Q_α shows that the terms in Q_α which do not commute with M are those involving the time derivative $\dot{\phi}$ or $\dot{\phi}^*$. These terms are

$$Q_\alpha^{N.C} = \int d^3x(\dot{\phi}^*\psi_\alpha + \dot{\phi}\epsilon_{\alpha\beta}\psi^{*\beta}) \tag{2.18}$$

where ψ_α are the spinor components with $\gamma^0 = +1$. Then, under conjugation by M, the Q_α becomes

$$\begin{aligned} Q_\alpha \to Q_\alpha + \int d^3x[&\{(m_2 - m_1)\phi^* + (g_2 - g_1)\phi^{*2}\}\psi_\alpha \\ &+ \{(m_2 - m_1)\phi + (g_2 - g_1)\phi^2\}\epsilon_{\alpha\beta}\psi^{*\beta}]. \end{aligned} \tag{2.19}$$

Then, the comparison with the standard formula for Q_α shows that the last term is precisely the change in Q_α under $(m_1, g_1) \to (m_2, g_2)$. This can be extended to arbitrary renormalizable theories with spin 0 and spin $\frac{1}{2}$ fields only. The superpotential W_1 can be changed to another potential W_2 by $Q_\alpha \to M^{-1}Q_\alpha M$ where

$$M = \exp\left(2\mathrm{Re}\int d^3x(W_2(\phi_i(x)) - W_1(\phi_i(x)))\right). \tag{2.20}$$

Note that we must be careful in two points. One is again the behavior for large field limit. For M to be a well-defined operator in the Hilbert space of our theory it is necessary that when M acts on an energy eigenstate, it must give a normalizable finite energy state. If $\Delta W = W_2 - W_1$ increases too rapidly for large ϕ, M produces an unnormalizable state that diverges expontentially for large ϕ. The necessary requirement is that the change in W under conjugation must grow no more rapidly than W itself for large ϕ.

The other is the ultraviolet divergences. When we compute $\mathrm{Tr}(-1)^F$, we do not have to worry the effects of UV divergences. This is because introducing cut-off affects only the highly excited states while $\mathrm{Tr}(-1)^F$ involves only the low-lying states. On the other hand, the arguments in this section must be checked if they are consistent with the renomalization because each ground states are sensitive to the tiny changes of the theory. If we consider the theory with spin 0 and spin $\frac{1}{2}$ fields only, there is a simple way to avoid the problem. We can regularize the theory in a supersymmetrically invariant way while preserving the fact that changing the parameters by conjugation is possible. In the interaction terms, we replace all superfields $\Phi_i(x)$ by $\tilde{\Phi}_i(x,t) = \int d^3yG(x,y)\Phi_i(y,t)$, where $G(x,t)$ is some kernel. The superspace interaction terms are now written as $\int d^4xd^2\theta W(\tilde{\Phi}_i(x,t))$. This eleminates the UV divergences and W can be still be changed by conjugation in the same way as before. Then, after ending all the calculations, we take $G(x,y) \to \delta^3(x-y)$. However, when we introduce the gauge fields, this procedure conflicts with the gauge invariances. Though there is no proof in this case, the conjugation process is believed to be possible.

Similar analysis leads us to the following results [13].

1. Superpotential can be changed by conjugation.

2. Abelian gauge counpling can be changed.

3. Non-abelian counpling can be changed if the theory has no θ dependece.

4. θ cannot be changed unless physics is independent of θ.

5. The D term cannot be changed.

2.3 A simple example

Consider the supersymmetric version of a particle moving in a line in one dimension. The supercharges are given by

$$Q_1 = \frac{1}{2}(\sigma_1 p + \sigma_2 W(x)) \tag{2.21}$$

$$Q_2 = \frac{1}{2}(\sigma_2 p - \sigma_1 W(x)) \tag{2.22}$$

where σ_i are Pauli matices and $p = -i\frac{d}{dx}$. Then, the Hamiltonian is

$$H = \frac{1}{2}\left(p^2 + W^2 + \sigma_3 \frac{dW}{dx}\right). \tag{2.23}$$

From these definitions, we have

$$Q_+ = \frac{1}{\sqrt{2}}(Q_1 + iQ_2) = -\frac{i}{2\sqrt{2}}(\sigma_1 + i\sigma_2)\left(\frac{d}{dx} + W(x)\right). \tag{2.24}$$

Let us consider the change of superpotential by conjugation. The relation between $Q_+(W)$ and $\tilde{Q}_+(\tilde{W})$ is given by

$$\tilde{Q}_+(\tilde{W}) = \exp(-F(x))Q_+(W)\exp(F(x)) \tag{2.25}$$

where $F(x)$ is a function which satisfies

$$\frac{dF}{dx} = \tilde{W}(x) - W(x). \tag{2.26}$$

Let us study the special case $W(x) = x^2 + a^2$, where a is some constant. For $a^2 > 0$, supersymmetry is spontaneously broken at the tree level because of the potential energy $V(x) = W^2(x)$. For $a^2 < 0$, supersymmetry is not broken in perturbation theory. However, it turns out that the dynamical supersymmetry breaking occurs in this case. To see this, note the fact that the sign of a^2 can be changed by the following conjugation

$$Q_+(-a^2) = \exp(2a^2 x)Q_+(a^2)\exp(-2a^2 x). \tag{2.27}$$

Therefore, the total number of zero-energy must be independent of the sign of a^2. We saw that for $a^2 > 0$ there are no zero-energy states. Combining these facts, we can conclude that the supersymmetry is also broken for negative a^2 even if the there appear to be two zero-energy states in perturbation theory. This is consistent with the usual analysis [12].

2.4 Non-linear sigma models

In this section, we will calculate $\mathrm{Tr}(-1)^F$ for supersymmetric non-linear sigma models in two dimensions. We denote the target space as X. The non-linear sigma model is defined by the action

$$S = \int d^2x \left(\frac{1}{2} g_{ij} \partial_\mu \phi^i \partial_\mu \phi^j + \frac{1}{2} \bar{\psi}_j i \gamma^k D_k \psi_j + \frac{1}{8} R_{ijkl}(\phi) \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l \right) \tag{2.28}$$

where g_{ij} is the metric of the manifold X, R_{ijkl} is a curvature tensor of X and D_k is a covariant derivative. From this definition we find it difficult to calculate $\mathrm{Tr}(-1)^F$ because the classical vacuum possesses a continuous degeneracy. Any configuration of the form $\phi = \mathrm{const}$ has zero-energy at the classical level. However, we can add following terms to the action to lift the degeneracy

$$\Delta S = \frac{1}{2} \int d^2x d^2\theta h(\Phi) \tag{2.29}$$

or in terms of components

$$\Delta S = \int d^2x \left(-\frac{1}{2} g^{ij} \frac{\partial h}{\partial \phi^i} \frac{\partial h}{\partial \phi^j} - \frac{1}{2} \frac{\partial^2 h}{\partial \phi^i \partial \phi^j} \bar{\psi}^i \psi^j \right) \tag{2.30}$$

where $h(\phi^i)$ is an arbitrary function defined on X. Then, the scalar potential and the fermion mass matrix become

$$V(\phi^i) = \frac{1}{2} |\nabla h|^2, \quad m_{ij}^2 = \frac{\partial^2 h}{\partial \phi^i \partial \phi^j}. \tag{2.31}$$

It is evident that the potential energy lift the vacuum degeneracy. The classical vacua are now those satisfying

$$\frac{\partial h}{\partial \phi^i} = 0. \tag{2.32}$$

We assume that there are only isolated points in X at which $\partial h / \partial \phi^i = 0$ is satisfied. Let us denote such points as $p^a, a = 1, 2, ..., k$. We also impose the following condition on h. The condition is that the matrix $\partial^2 h / \partial \phi^i \partial \phi^j$ has no zero eigenvalue. This condition means that there are no massless particles at the level of perturbation theory. Therefore, if we expand around any one of the points p^a, we have only one zero-energy state. Then the calculation of Witten index is reduced to the problem of determining the statistics of each vacuum. In order to do so, we must define $(-1)^F$ in two dimensions. In four dimensions we defined it by using the angular momentum. However, we have now only one spacial dimension. Therefore the term "angular momentum" has no meaning here. We adopt the following definition. The operator $(-1)^F$ is defined to be the operator which satisfies

$$(-1)^F \phi = \phi (-1)^F, \quad (-1)^F \psi = -\psi (-1)^F. \tag{2.33}$$

where ϕ and ψ are any elementary Bose and Fermi fields. Note that the overall sign cannot be fixed. This means that we must define any one of the k vacua to be bosonic (or fermionic) and determine whether others are bosonic or fermionic. This seems difficult. However, Witten found that this problem is related to the Morse theory and we can compute $\mathrm{Tr}(-1)^F$ in a simple way. To explain this, let us first consider the easy example. Our example is a free Majorana fermion with mass m. The action is given by

$$S = \frac{1}{2} \int d^2x (\bar{\psi} i \partial\!\!\!/ \psi - m \bar{\psi} \psi). \tag{2.34}$$

Note that by the chiral symmetry, physics is independent of the sign of m. Then, we define the zero modes of the fermion as

$$\sigma_1 = \frac{1}{\sqrt{L}} \int dx \psi_1(x), \quad \sigma_2 = \frac{1}{\sqrt{L}} \int dx \psi_2(x). \tag{2.35}$$

σ_1 and σ_2 form the following algebra

$$\sigma_1^2 = \sigma_2^2 = 1, \quad \sigma_1\sigma_2 + \sigma_2\sigma_1 = 0. \tag{2.36}$$

Then, the Hamiltonian for these zero modes is simple because the zero modes have no kinetic energy. The explicit form is

$$H = -im\sigma_1\sigma_2 = m\sigma_3 \tag{2.37}$$

where we defined $\sigma_3 = -i\sigma_1\sigma_2$, which is the number operator of the zero mode. We see from this form of Hamiltonian that the sign of m determines whether the zero mode is filled or empty, that is, bosonic or fermionic. In this way, we can determine the contribution to $\mathrm{Tr}(-1)^F$ from each vacuum. The general rule is that if the number of negative eigenvalues of the fermion mass matrix $\partial^2 h/\partial\phi^i\phi^j$ evaluated at some vacuum is even, that vacuum is bosonic (or fermionic depending on the choice of the sign of $(-1)^F$). Summing up all the contributions, we have

$$\mathrm{Tr}(-1)^F = \sum_a (-1)^{n^a} \tag{2.38}$$

where n^a is the number of negative eigenvalues of the fermion mass matrix evaluated at the a-th vacuum. At first sight, this result seems to be strange. This is because the result appears to depend on the particular choice of the function h. However, from one of the main theorem of Morse theory, the right-hand side of the result is equal to the Euler number of the target space X. Therefore, the result does not depend on h. In this way, we have

$$\mathrm{Tr}(-1)^F = \chi(X). \tag{2.39}$$

Chapter 3

M theory on G_2

In this chapter, we review the duality proposed by Atiyah, Maldacena and Vafa. To understand this duality, we need two tools; a flop and the topological string. Let us first discuss a flop following [15].

3.1 Flop

3.1.1 Preliminaries

We use linear sigma model approach. In order to do so, we need $N=2$ supersymmetric gauge theories as a world sheet theory. We mainly consider the case in which the target space is Calabi-Yau.

Consider the usual $N=1$ supersymmetric abelian gauge theory in four dimensions. We have two kinds of superfield. One is chiral superfield which obeys $\bar{D}_{\dot{\alpha}}\Phi = 0$ and can be expanded

$$\Phi(x,\theta) = \phi(y) + \sqrt{2}\theta^{\alpha}\psi_{\alpha}(y) + \theta^{\alpha}\theta_{\alpha}F(y). \tag{3.1}$$

And the other is a vector superfield which is real and can be expanded

$$V = -\theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}v_m + i\theta^{\alpha}\theta_{\alpha}\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} - i\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^{\alpha}\lambda_{\alpha} + \frac{1}{2}\theta^{\alpha}\theta_{\alpha}\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}D \tag{3.2}$$

in Wess-Zumino gauge, where $y^m = x^m + i\theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$ as usual. We have the usual gauge symmetry $\Phi \to \exp(iQa)\Phi, v \to v - da$. We will denote the lowest component of the superfields as ϕ_i for the superfield Φ_i.

Now let us make the dimensional reduction to two dimensions. In doing so, we take the fields to be independent of x^1 and x^2. The components v_1 and v_2 of the gauge field in the x^1 and x^2 directions, along with all the other fields, are functions of x^0 and x^3 only. Set $\sigma = (v_1 - iv_2)/\sqrt{2}$ and $\bar{\sigma} = (v_1 + iv_2)/\sqrt{2}$. And let y^0 and y^1 to be x^0 and x^3 respectively. After dimensional reduction, we label the fermion components as $(\psi^1,\psi^2) = (\psi^-,\psi^+)$, $(\psi_1,\psi_2) = (\psi_-,\psi_+)$ and similarly for dotted spinors. In two dimensions, in addition to chiral superfields, obeying $\bar{D}_+\Phi = \bar{D}_-\Phi = 0$, it is possible to have twisted chiral superfields which obeys $\bar{D}_+\Sigma = D_-\Sigma = 0$. And in two dimensions, the basic gauge invariant field strength of the superspace gauge field is a twisted chiral superfield. This quantity is

$$\Sigma = \frac{1}{2\sqrt{2}}\{\bar{\mathcal{D}}_+, \mathcal{D}_-\} \tag{3.3}$$

where $\mathcal{D}$ are gauge covariant derivatives. In the abelian case, we have

$$\begin{aligned} \Sigma = \sigma - i\sqrt{2}\theta^+\bar{\lambda}_+ - i\sqrt{2}\theta^+\bar{\theta}^-(D - iv_{01}) \\ - i\bar{\theta}^-\theta^-(\partial_0 - \partial_1)\sigma - i\theta^+\bar{\theta}^+(\partial_0 + \partial_1)\sigma + \sqrt{2}\bar{\theta}^-\theta^+\theta^-(\partial_0 - \partial_1)\bar{\lambda}_+ \\ + \sqrt{2}\theta^+\bar{\theta}^-\bar{\theta}^+(\partial_0 + \partial_1)\lambda_- - \theta^+\bar{\theta}^-\theta^-\bar{\theta}^+({\partial_0}^2 - {\partial_1}^2)\sigma \end{aligned} \tag{3.4}$$

In the presence of gauge fields, a chiral superfield in a given representation is a superfield obeying $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. If we write

$$\Phi = e^V\Phi_0 \tag{3.5}$$

then Φ_0 obeys $\bar{D}_{\dot{\alpha}}\Phi_0 = 0$.

With above notations, the Lagrangian for the theory with gauge group $U(1)^s$ and s vector superfields V_a, $a = 1, 2, ..., s$, and k chiral superfields Φ_i, $i = 1, 2, ..., k$ which are charged under each gauge particles with charge $Q_{i,a}$ is of the form

$$L = L_{\text{kin}} + L_W + L_{\text{gauge}} + L_{D,\theta} \tag{3.6}$$

where

$$\begin{aligned} L_{\text{kin}} &= \int d^4\theta \sum_i \bar{\Phi}_i\Phi_i = \int d^4\theta \sum_i \bar{\Phi}_{0,i}e^{2\sum_a Q_{i,a}V_a}\Phi_{0,i} \\ &= \sum_i (-D_\mu\bar{\phi}_iD^\mu\phi_i + i\bar{\psi}_{-,i}(D_0 + D_1)\psi_{-,i} + i\bar{\psi}_{+,i}(D_0 - D_1)\psi_{+,i} + |F_i|^2 \\ &- 2\sum_a \bar{\sigma}_a\sigma_aQ_{i,a}^2\bar{\phi}_i\phi_i - \sqrt{2}\sum_a Q_{i,a}(\bar{\sigma}_a\bar{\psi}_{+i}\psi_{+i} + \sigma_a\bar{\psi}_{-i}\psi_{+i}) + \sum_a D_aQ_{i,a}\bar{\phi}_i\phi_i \\ &- \sum_a i\sqrt{2}Q_{i,a}\bar{\phi}_i(\psi_{-i}\lambda_{+a} - \psi_{+i}\lambda_{-a}) - \sum_a i\sqrt{2}Q_{i,a}\phi_i(\bar{\lambda}_{-a}\bar{\psi}_{+i} - \bar{\lambda}_{+a}\bar{\psi}_{-i})) \end{aligned} \tag{3.7}$$

$$\begin{aligned} L_W &= -\int d\theta^+d\theta^-W(\Phi_i)|_{\bar{\theta}^+=\bar{\theta}^-=0} - h.c \\ &= -\left(F_i\frac{\partial W}{\partial\phi_i} + \frac{\partial^2W}{\partial\phi_i\partial\phi_j}\psi_{-i}\psi_{+j}\right) - h.c. \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
L_{\text{gauge}} &= -\sum_a \frac{1}{4{e_a}^2}\int d^4\theta \bar{\Sigma}_a \Sigma_a \\
&= -\sum_a \frac{1}{e_a^2}\bigg(\frac{1}{2}v_{01,a}^2 + \frac{1}{2}D_a^2 + i\bar{\lambda}_{+a}(\partial_0 - \partial_1)\lambda_{+a} \\
&+ i\bar{\lambda}_{-a}(\partial_0 + \partial_1)\lambda_{-a} - |\partial_\mu \sigma_a|^2\bigg)
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
L_{D,\theta} &= \frac{it'}{2\sqrt{2}}\int d\theta^+ d\bar{\theta}^- \Sigma|_{\theta^-=\bar{\theta}^+=0} - \frac{i\bar{t}'}{2\sqrt{2}}\int d\theta^- d\bar{\theta}^+ \bar{\Sigma}|_{\theta^+=\bar{\theta}^-=0} \\
&= -rD + \frac{\theta}{2\pi}v_{01}
\end{aligned} \tag{3.10}
$$

where t' is defined by

$$
t' = ir + \frac{\theta}{2\pi}. \tag{3.11}
$$

This action has the following symmetries.

1. $N = 2$ supersymmetries in two dimensions

Two left-moving supersymmetries and two right moving supersymmetries acting on $(\theta^-, \bar{\theta}^-)$ and $(\theta^+, \bar{\theta}^+)$ respectively.

2. If we ignore the superpotential L_W, we have two additional R symmetries.

Right moving one acts on $(\theta^+, \psi_{+i}, F_i, \sigma_a, \lambda_{-a})$ with charges $(1, -1, -1, 1, 1)$ and their complex conjugate with opposite charges. Left moving one acts on $(\theta^-, \psi_{-i}, F_i, \sigma_a, \lambda_{+a})$ with charges $(1, -1, -1, 1, 1)$ and their complex conjugates with opposite. Other fields are neutral. Let us denote the currents associated with these symmetries as J_R and J_L respectively. The existence of these symmetries seems natural from the four dimensional point of view. Many four dimensional supersymmetric theories have a single R symmetry. In addition, in the process of dimensional reduction, we have a chance to have a second $U(1)$ symmetry corresponding to rotations of the two extra dimensions. However, the R symmetries are sometimes anomalous. The condition for these to be non anomalous is given by

$$
\sum_i Q_{i,a} = 0 \quad \text{for} \quad a = 1, 2, ..., s. \tag{3.12}
$$

Then, what happens when we include superpotential terms? The superpotential term is of the form $\int d^2\theta W$ and by definition, have charge 1 under J_R. To preserve this, we should add J_R of the form $\Phi_i \to \exp(i\alpha k_i)\Phi_i$ under which $W \to e^{-i\alpha}W$. If this is possible, the superpotential is said to be quasi-homogeneous.

Later, this charge condition turns out to be equivalent to the target space Calabi-Yau condition. Usually, the sigma model apporach is a semiclassical approximation. So we can get correct information when the target space is large compared with the string scale.

3.1.2 Calabi-Yau/Landau-Ginzburg correspondence

Consider the model with gauge group $U(1)$ and n chiral fields S_i of charge 1 and one chiral field P of charge $-n$. We take the superpotential to be

$$W = PG(S_1, S_2, ..., S_n) \tag{3.13}$$

where G is a homogeneous polynomial of degree n. Note that this superpotential is quasi-homogeneous. If we take k_i to be -1 for P and 0 for S_i, we can have the desired symmetry. In fact, the superpotentials we will consider in this section are all quasi-homogeneous.

We impose the following condition on $G(S_1, ..., S_n)$:

$$\frac{\partial G}{\partial S_1} = ... = \frac{\partial G}{\partial S_n} = 0 \tag{3.14}$$

has no solution except $S_1 = ... = S_n = 0$. This means that the hypersurface defined by $G = 0$ in $\mathbf{CP}^{n-1}$ is smooth. We call this a transverse condition. By solving equations of motion for auxiliary fields D and F_i, we have

$$D = -e^2 \left(\sum_i Q_i |\phi_i|^2 - r \right) \tag{3.15}$$

$$F_i = \frac{\partial W}{\partial \phi_i}. \tag{3.16}$$

Then, the potential energy for the scalar fields s_i, p, σ is easily read from the Lagrangian to be

$$U = |G(s_i)|^2 + |p|^2 \sum_i \left| \frac{\partial G}{\partial s_i} \right|^2 + \frac{1}{2e^2} D^2 + 2|\sigma|^2 \left(\sum_i |s_i|^2 + n^2 |p|^2 \right) \tag{3.17}$$

where

$$D = -e^2 \left(\sum_i \bar{s}_i s_i - n\bar{p}p - r \right). \tag{3.18}$$

We want to discuss the low energy physics. When $r \gg 0$, for D to vanish, we must have some nonzero s_i's. Then vanishing of the term $|p|^2 \sum |\partial_i G|^2$ in the potential requires that $p = 0$.Then vanishing of D gives us

$$\sum_i \bar{s}_i s_i = r. \tag{3.19}$$

And vanishing of the remaining terms gives $G = 0$ and $\sigma = 0$. Let us look at the classical vacua from a geometrical point of view. We can regard s_i's to be coordinates of $\mathbf{C}^n$. Then, imposing the condition $\sum \bar{s}_i s_i = r$ and dividing by the gauge group $U(1)$ means that we have $(n-1)$ dimensional complex projective space $\mathbf{CP}^{n-1}$ with Kähler class proportional to r (Gauge symmetry is spontaneously broken but this is due to the expectation values of s_i's). In addition, we have $G = 0$. So, the moduli space of classical vacua is isomorphic to the hypersurface in projective space defined by $G = 0$. Recall that hypersurfaces defined by homogeneous polynomials of degree n in $\mathbf{CP}^{n-1}$ are Calabi-Yau manifolds. Note that this condition comes from the charge neutrality condition. Therefore, the Calabi-Yau condition is equivalent to the non-anomalous R symmetry. The meaning of this is that non-anomalous R invariance gives us $U(1)$ piece of the $N = 2$ superconformal algebra and the world sheet theory becomes a superconformal theory. Recall that strings in curved back ground gives conformal anomaly proportional to $R_{\mu\nu}$ in one loop level and this vanishes in Calabi-Yau manifold.

All modes other than oscillations tangent to X are massive. So, the low energy theory is a sigma model with target space Calabi-Yau manifold and Kähler class proportional to r.

Let us turn to the case when $r \ll 0$. In this case, vanishing of D requires p to be nonzero. Then, vanishing of $|p|^2 \sum |\partial_i G|^2$ requires all s_i to be 0. From vanishing of D, we have $|p| = \sqrt{-r/n}$. By the gauge transformation, we can fix the argument of p to 0. The theory has a unique classical vacuum. If you expand around this vaccum, the s_i's are all massless (for $n > 2$). By setting p to its expectation value, we have a effective field theory for s_i's with superpotential $\tilde{W} = \sqrt{-r}G(s_i)$. This effective action has by definition a degenerate critical point at the origin for $n > 2$. It vanishes up to order n. Therefore the low energy theory is a Landau-Ginzburg theory. (To be precise, this is a Landau-Ginzburg orbifold because the vacuum expectation value of p does not break the gauge symmetry completely and $\mathbf{Z}_n$ subgroup remains unbroken which acts on s_i as $s_i \to \zeta s_i$ where ζ is an n-th root of 1).

It seems that the Calabi-Yau and Ginzburg-Landau theories are the different phases of the same theory with phase transition at $r = 0$. But surprisingly, it turns out that there is no phase transition between them and they are connected smoothly.

Let us discuss this point. Recall that phase transitions occur in infinite degrees of freedom. In our model, we are interested in the compact worldsheet theory. So, singularities arise when the target space loses its effective compactness and gives a continuous spectrum. Then, where does the compactness of the target space break down? To see this, let us forget the σ field. Then the region in which the potential energy is less than some given value is always compact. So, there is no way for singularities to arise. But including the σ field changes the situation. If we set $s_i = p = 0$, the potential energy is $e^2r^2/2$ and independent of σ. This means that the region where the energy is less than some value can be noncompact. The critical value is, in the semiclassical approximation, $U_{cr} = e^2r^2/2$. But, near the region $r \sim 0$, the approximation breaks down. So we must be more presice though the idea is the same. There are three corrections to be considered. One is from fluctuations of s_i and p. But these only gives corrections of negative powers of $|\sigma|$ (note that the masses of these fields are proportional to $|\sigma|$). So, in the region where we are considering now (large σ) we can ignore them. The second one is from ultraviolet behavior. But in our case this is not important. For example, the one loop correction to the expectation value of $-D/e^2$ is of the form

$$\sum_i Q_i \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + 2|\sigma|^2}. \tag{3.20}$$

This vanishes because of the Calabi-Yau condition. The case in which this condition is not obeyed is discussed in [15]. So, the remaining issue is non-perturbative effects. This is the last one we should consider. From the above arguments, we can simply throw away the supermassive chiral superfields. And we have the effective theory of the massless gauge multiplet with Lagrangian $L_{\text{gauge}} + L_{D,\theta}$. This effective theory is a free theory and supersymmetry ensures the cancelation of zero point energy corrections to the ground state energy from σ and λ. The gauge field, on the other hand, has an important correction. The action for the gauge field is

$$S = \int d^2y \left(\frac{1}{2e^2} v_{01}^2 + \frac{\theta}{2\pi} v_{01} \right). \tag{3.21}$$

As we know, the θ term in abelian gauge theory in two dimensions induces the constant electric field in the vacuum, which is equal to the electric field made by a charge of strength $\theta/2\pi$. So this gives a contribution to the vacuum energy $(e^2/2)(\theta/2\pi)^2$ (We take the range $|\theta| \le \pi$). In sum, we have an exact, quantum corrected critical value, which is,

$$U_{cr} = \frac{e^2}{2} \left(r^2 + \left(\frac{\theta}{2\pi} \right)^2 \right). \tag{3.22}$$

So, in the presence of the θ term, this cannot vanish. This means that only singularity is at $r = \theta = 0$ and by swiching on the θ term, we can smoothly connect the two phases; Calabi-Yau and Ginzburg-Landau. This is the Calabi-Yau/Ginzburg-Landau correspondence from a sigma model point of view.

3.1.3 Symplectic quotient

Let us introduce some geometrical aspects. These turn out to be useful later. Consider the manifold $Y = \mathbf{C}^{n+1}$ with coordinates $s_1, s_2, ..., s_n$ and p with $\mathbf{C}^*$ action defined by

$$\begin{aligned} s_i &\to \lambda s_i \\ p &\to \lambda^{-n} p \end{aligned} \tag{3.23}$$

Suppose we want to form a quotient of Y by this action. As C^* acts freely on $(Y - O)$ (O is the origin), one might think one could form a reasonable quotient $(Y - O)/\mathbf{C}^*$. However, because C^* action forms a noncompact group, this is not so easy. Let P be a point in Y with $p = 0$, and P' be a point with $s_i = 0$ for all i. Then, P can be brought arbitrarily close to the origin by the C^* action. And similarly for P'. This means that the quotient $(Y - O)/\mathbf{C}^*$ with its natural topology is not a Hausdorff space. How can we save the situation? One obvious answer is the following one. Let Y_1 be the subset of Y with $p = 0$ and the s_i not all 0. Let Y_2 be $s_i = 0$ and $p \neq 0$. And set $\tilde{Y} = Y - (Y_1 \cup Y_2 \cup O)$. Since $\tilde{Y}$ contains only good $\mathbf{C}^*$ orbits, the quotient $\tilde{Y}/\mathbf{C}^*$ becomes a manifold. However deleting Y_1,Y_2, and O is too much. And there is a systematic way to include some of the ill-behaved orbits. This is the symplectic quotient and is directly related to the physical process explained above.

Y is endowed with the Kähler metric $ds^2 = \sum_i |ds_i|^2 + |dp|^2$. This metric is not invariant under the action of $\mathbf{C}^*$. Therefore, $\mathbf{C}^*$ is not a symmetry group of our physical models (Recall that in $N = 2$ theories in two dimensions the target space is Kähler). However, under the maximal compact subgroup $U(1)$, the metric is invariant ($U(1)$ action is defined by those with $|\lambda| = 1$). This was a gauge group in the previous section. Then consider the following function

$$\tilde{D} = \sum_i |s_i|^2 - n|p|^2 - r. \tag{3.24}$$

It is easy to see that this function generates the $U(1)$ action on Y by Poisson brackets. In other words, this is a Hamiltonian function. In the previous section, this was a part of the potential energy and what we did was to set $D = 0$ and divide by the gauge group $U(1)$. The operation of setting $D = 0$ and dividing by the gauge group is called the symplectic quotient. In our example, we say that we do the simplectic quotient of Y by $U(1)$. We denote this as $Y//U(1)$. Of course, this operation depends on r.

Let us study this operation in some detail. We first consider $\tilde{Y}//U(1)$. The $\tilde{D}$ function restricted to some $\mathbf{C}^*$ orbit is

$$|\lambda|^2 \sum_i |s_i|^2 - n|\lambda|^{-2n}|p|^2 - r. \tag{3.25}$$

As we are in $\tilde{Y}$, this is a monotonically increasing function of $|\lambda|$, and goes to $+\infty$ as $\lambda \to +\infty$ and to $-\infty$ as $\lambda \to 0$. So, $\tilde{D} = 0$ has a unique solution for $|\lambda|$. Since $|\lambda|$ is uniquely determined by $\tilde{D} = 0$ and the argument of λ is absorbed in the $U(1)$ action, the given $\mathbf{C}^*$ orbit contributes precisely one point to $\tilde{Y}//U(1)$. This shows that $\tilde{Y}//U(1)$ is naturally identified with $\tilde{Y}/\mathbf{C}^*$. Next, we consider $Y//U(1)$.

(1) $r > 0$

Setting $\tilde{D} = 0$ is possible only in $\tilde{Y} \cup Y_1$. $Y_1//U(1)$ is a copy of $\mathbf{CP}^{n-1}$ with Kähler form proportional to r. So, $Y//U(1)$ is the union of $\tilde{Y}/\mathbf{C}^*$ and $\mathbf{CP}^{n-1}$.

(2) $r = 0$

The origin is the only bad point. Its symplectic quotient is of course a point. So, $Y//U(1)$ is the union of $\tilde{Y}/\mathbf{C}^*$ and that point.

(3) $r < 0$

Setting $\tilde{D} = 0$ is possible only in Y_2. The symplectic quotient $Y_2//U(1)$ is a single point. So, $Y//U(1)$ is the union of $\tilde{Y}/\mathbf{C}^*$ and that point.

This generalize the simple procedure explained first. Indeed these include some of the bad points, which is precisely what we wanted to do. Note that the $Y//U(1)$ for positive, negative, or zero are all equivalent on dense open sets because on a dense open set, they are all coincide with $\tilde{Y}/\mathbf{C}^*$. This means that they are all birationally equivalent (Not topologically in general). This birational equivalence lies behind the Calabi-Yau/Ginzburg-Landau correspondence.

3.1.4 A flop

Now we change the model. Here, we consider a $U(1)$ gauge theory with two chiral superfields A_i of charge 1 and two chiral superfields B_j of charge -1. We regard them as coordinates on $V = \mathbf{C}^4$. We take the superpotential to be zero. Then, the potential energy for bosonic fields at $\sigma = 0$ is

$$U(a_i, b_j) = \frac{e^2}{2}\left(\sum_i |a_i|^2 - \sum_j |b_j|^2 - r\right)^2. \tag{3.26}$$

The $U(1)$ action is defined by

$$\begin{aligned} a_i &\to \lambda a_i \\ b_j &\to \lambda^{-1} b_j, \end{aligned} \tag{3.27}$$

which can be extended to a $\mathbf{C}^*$ action. The low energy effective theory is obtained as before by $V//U(1)$. As in the previous section, denote the region where the a_i are not both 0 and the b_j are not both 0 as $\tilde{V}$. Also, define V_1 to be the region in which a_i are not both 0 and b_j are both 0. And similarly for V_2 by exchanging a_i and b_j. There are three possible symplectic quotient corresponding to $r > 0$, $r = 0$ and $r < 0$. We call them Z_+, Z_0 and Z_- respectively.
(1) Z_+
For $r > 0$, setting the potential U to be zero is possible in $\tilde{V} \cup V_1$ and each $\mathbf{C}^*$ orbit contributes to precisely one point to Z_+. Therefore, $Z_+ = (\tilde{V} \cup V_1)/\mathbf{C}^*$. $\tilde{V} \cup V_1$ is the region where the a_i are not both 0. The values of the a_i, up to scaling by $\mathbf{C}^*$, determine a point in a copy of $\mathbf{CP}^1$ that we call $\mathbf{CP}^1_a$. Z_+ is fibered over $\mathbf{CP}^1_a$ by forgetting the values of b_j. Since the values of the b_j are arbitrary, the fiber is a copy of $\mathbf{C}^2$. The zero section of $Z_+ \to \mathbf{CP}^1_a$ gives an embedding $\mathbf{CP}^1_a \subsetneqq Z_+$. Hence, we can identify $\mathbf{CP}^1_a$ with the image of this embedding. For $b_j = 0$, the vanishing of V gives $\sum |a_i|^2 = r$. Therefore the Kähler form of $\mathbf{CP}^1_a$ is proportional to r.
(2) Z_0
The only ill-behaved orbit is O and it contributes a single point to Z_0. This point turns out to be singular and we will study this singularity later.
(3) Z_-
This is similar to (1). $Z_- = (\tilde{V} \cup V_2)/\mathbf{C}^*$. $\tilde{V} \cup V_2$ is the region in which the both b_j are not 0. So just as in the case (1), there is a fibration $Z_- \to \mathbf{CP}^1_b$ whose zero section gives an embedding $\mathbf{CP}^1_b \subsetneqq Z_-$. As above, we identify $\mathbf{CP}^1_b$ with this image. $\mathbf{CP}^1_b$ has Kähler form proportional to $-r$.

Z_+, Z_0, Z_- are all non-compact Calabi-Yau manifolds. To see this, consider the holomorphic 4-form $\Theta = da_1 \wedge da_2 \wedge db_1 \wedge db_2$ on V. As $\sum_i Q_i = 0$, this is invariant under the $\mathbf{C}^*$ action. Contracting this form with the vector field generating the $\mathbf{C}^*$ action gives an everywhere non-zero holomorphic 3-form whose restriction to the suitable region of V is the pullback of a holomorphic volume form on Z_+, Z_0, or Z_-.

As we said, Z_+ and Z_- are birationally equivalent but not topologically. What does this mean physically? In string theory, Fayet-Iliopoulos parameters are like all operators in the world-sheet Lagrangian interpreted as expectation values of some fields in space-time. So, they are dynamical variables and free to change in time. Suppose we are in the region $r > 0$. Also, suppose that the Fayet-Iliopoulos term r somehow decreases as time goes. The size of $\mathbf{CP}^1_a$ becomes smaller and smaller, shrinking to 0. Then a change of topology occurs and $\mathbf{CP}^1_a$ is replaced by $\mathbf{CP}^1_b$ for $r < 0$. As we explained, the interpolation is smooth. In this way, we have achieved the smooth change of topology in string theory.

Next, we want to study the singularity at the origin more closely. Let x, y, z, t be coordinates on a copy of $\mathbf{C}^4$ that we call W. Consider the following $\mathbf{C}^*$ invariant map $V \to W$.

$$\begin{aligned} x &= a_1 b_1 \\ y &= a_2 b_2 \\ z &= a_1 b_2 \\ t &= a_2 b_1. \end{aligned} \tag{3.28}$$

Let us call this formula $(*)$. By restricting these formulas to $\tilde{V} \cup O$ and dividing by $\mathbf{C}^*$ we get a map $Z_0 \to W$. It is obvious that the image of Z_0 in W lies in the Q defined by

$$xy - zt = 0. \tag{3.29}$$

In fact the map is an isomorphism between Z_0 and Q.
Proof
(1) Surjectivity
If $x = y = z = t = 0$, we take $a_i = b_j = 0$. If, for instance, $x \neq 0$ we pick $a_1 = 1$ and then iteratively solve $(*)$. In this way, we see that the map is surjective.
(2) Injectivity
If $x = y = z = t = 0$, from $(*)$, we must have either the a_i or b_j are both zero. In that case, for a_i and b_j to define a point in Z_0, they must all be zero. Therefore, $x = y = z = t = 0$ is the image only of the point $O \in Z_0$. If, say, $x \neq 0$, then $(*)$ requires $a_1 \neq 0$. By using the $\mathbf{C}^*$ action, we can set $a_1 = 1$. Then $(*)$ uniquely determines a_2, b_1, b_2. This completes the proof.

Therefore we must study the singularity of the affine quadric Q. This is an example of a conical singularity. More precisely, this is a cone on $SU(2) \times SU(2)/U(1)$. A conical singularity on an n-dimensional manifold X_n is a point (we will label it as $r = 0$) near which the metric can be put in the form

$$h_{mn} dx^m dx^n = dr^2 + r^2 g_{ij} dx^i dx^j. \tag{3.30}$$

Here, g_{ij} is a metric on an $n-1$ dimensional manifold Y_{n-1}. The point $r=0$ is singular unless Y_{n-1} is a round sphere. The conelike property is due to the fact that there is a group of diffeomorphisims of X_n that rescale the metric; i.e. the group $r \to tr$ with $t>0$. This is isomorphic to $\mathbf{R}_+^*$. We call X_n a cone over Y_{n-1}. To identify Y_{n-1} for given X_n is easy. Just omit the singularity at the origin and divide by $\mathbf{R}_+^*$.

Let us apply this to our case. After change of coordinates, we can describe Q as

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0. \tag{3.31}$$

The z_i transform in the four-dimensional representation of $SO(4)$. This is a cone because the equation defining it transforms with definite weight under $z_i \to tz_i$. To identify Y_5, we note that after omitting the singularity at the origin, dividing by $\mathbf{R}_+^*$ is equivalent to intersecting it with the unit sphere

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1. \tag{3.32}$$

The group $SO(4)$ acts transitively on this intersection. Any given point on the intersection, such as $P = (1/\sqrt{2}, i/\sqrt{2}, 0, 0)$, is invariant under only a single $U(1) \subset SO(4)$ (For example, P is invariant under the subgroup $U(1) = SO(2)$ that rotates z_3 and z_4). Therefore Q is a cone on $SO(4)/U(1) = SU(2) \times SU(2)/U(1)$. Moreover, from the work of Hopf, we know that S^3 is a $U(1)$ bundle over S^2. Since the singularity is a local one, we can say Q is a cone on $S^3 \times S^2$. This singularity can be resolved by blowing up, but that would ruin the Calabi-Yau condition. There are mainly three ways to resolve this singularity while preserving the Calabi-Yau condition. The two of them are to replace the singular point with a copy of $\mathbf{CP}^1$. These are called small resolutions and the two ways exactly correspond to $\mathbf{CP}^1_a$ and $\mathbf{CP}^1_b$. Thus Z_+ and Z_- are precisely the small resolutions of Q. The topology changing transition we found above is a transition between these two small resolutions. This is called a flop. The third way to resolve the singularity instead of changing FI term or Kähler parameter is to deform the complex structure. This can be done as follows. The defining equation is $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$. We change the equation to $z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu$, where μ is a real number. Then it is evident that there is now no singularity. But what is this geometrically? To see this, let us write the complex coordinates z_j in terms of real ones $z_j = x_j + iy_j$. The equation becomes

$$\sum_i (x_i^2 - y_i^2) = \mu \tag{3.33}$$

$$\sum_i x_i y_i = 0. \tag{3.34}$$

If we set $y_i = 0$, we have $\sum_i x_i^2 = \mu$. By differentiating this equation with respect to x_i, we have $\sum_i x_i dx_i = 0$. From this observation, it is easy to see that the resulting geometry is T^*S^3, the cotangent bundle of S^3. Since μ is a radius of the S^3, we can say that we have deleted the singularity by replacing the singular point by S^3. Unlike the transition between the small resolutions, the transition between the resolved conifold and the deformed conifold is singular in string theory.

3.2 Topological closed strings

We next study topological strings. We will concentrate on $N = 2$ topological strings. $N = 2$ topological field theory can be constructed from $N = 2$ superconformal field theory by twisting. By coupling it to gravity, we have the topological strings.

3.2.1 $N = 2$ superconformal field theory

General aspects

There are four defining operators in $N = 2$ superconformal field theory [20]. Energy-momentum $T(z)$, its superpartners $G^+(z)$,$G^-(z)$, and $U(1)$ current $J(z)$. Their OPEs are

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \tag{3.35}$$

$$T(z)G^{\pm}(w) \sim \frac{3/2}{(z-w)^2} G^{\pm}(w) + \frac{\partial_w G^{\pm}(w)}{z-w} \tag{3.36}$$

$$T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} \tag{3.37}$$

$$G^+(z)G^-(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial_w J(w)}{z-w} \tag{3.38}$$

$$J(z)G^{\pm}(w) \sim \pm \frac{G^{\pm}(w)}{z-w} \tag{3.39}$$

$$J(z)J(w) \sim \frac{c/3}{(z-w)^2}. \tag{3.40}$$

We expand them in modes

$$T(z) = \sum_n L_n z^{-n-2} \tag{3.41}$$

$$J(z) = \sum_n J_n z^{-n-1} \tag{3.42}$$

$$G^{\pm}(z) = \sum_n G^{\pm}_{n\pm a} z^{-(n\pm a)-3/2}, \tag{3.43}$$

where we impose the boundary condition to be

$$G^{\pm}(e^{2\pi i} z) = -e^{\mp 2\pi i a} G^{\pm}(z). \tag{3.44}$$

Then the commutation relations become as follows

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{3.45}$$

$$[L_n, J_m] = -mJ_{m+n} \tag{3.46}$$

$$[L_n, G^{\pm}_{m\pm a}] = \left(\frac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a} \tag{3.47}$$

$$[J_m, J_n] = \frac{c}{3}m\delta J_{m+n,0} \tag{3.48}$$

$$[J_n, G^{\pm}_{m\pm a}] = \pm G^{\pm}_{m+n\pm a} \tag{3.49}$$

$$\{G^{+}_{m+a}, G^{-}_{m-a}\} = 2L_{m+n} + (n-m+2a)J_{m+n} + \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n,0}, \tag{3.50}$$

or equivalently

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{3.51}$$

$$[L_n, J_m] = -mJ_{m+n} \tag{3.52}$$

$$[L_n, G^{\pm}_{r}] = \left(\frac{n}{2} - r\right) G^{\pm}_{n+r} \tag{3.53}$$

$$[J_m, J_n] = \frac{c}{3}m\delta_{m+n,0} \tag{3.54}$$

$$[J_n, G^{\pm}_{r}] = \pm G^{\pm}_{n+r} \tag{3.55}$$

$$\{G^{\pm}_{r}, G^{\mp}_{s}\} = 2L_{r+s} - (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \tag{3.56}$$

Note that each state can be labeled by the $U(1)$ charge (the eigenvalue of J_0) and the conformal weight. Left-chiral states are defined to be the states in the NS Hilbert space which satisfy

$$G^{+}_{-1/2}|\phi\rangle = 0 \tag{3.57}$$

and anti-chiral states are similarly defined by replacing G^+ with G^-. And similarly for right-chiral states, replacing G with $\bar{G}$. We will concentrate on the left-movers. Primary chiral states are defined to be the states satisfying

$$G^{+}_{-1/2}|\phi\rangle = G^{-}_{n+1/2}|\phi\rangle = G^{+}_{n+1/2}|\phi\rangle = 0 \quad \text{for} \quad n \geq 0. \tag{3.58}$$

By using the commutation relations and the above definitions, we deduce that for such states

$$\{G^{-}_{1/2}, G^{+}_{-1/2}\}|\phi\rangle = (2L_0 - J_0)|\phi\rangle = 0. \tag{3.59}$$

Therefore, for a primary chiral state the dimension h is one-half of its charge q; i.e. $h = \frac{q}{2}$. By replacing the primary chiral state with a general one, and taking its expectation value, we deduce

$$h \geq \frac{|q|}{2} \tag{3.60}$$

for an unitary theory. This inequality is saturated prcisely for the primary chiral and anti-chiral states. One can also go in the opposite direction and show that the states with $h = \frac{q}{2}$ are both chiral and primary. To prove this, suppose $|\phi\rangle$ saturates the inequality. Then, we have

$$\langle\phi|\{G^-_{1/2}, G^+_{-1/2}\}|\phi\rangle = 0 = |G^-_{1/2}|\phi\rangle|^2 + |G^+_{-1/2}|\phi\rangle|^2. \tag{3.61}$$

By unitarity, we deduce

$$G^+_{-1/2}|\phi\rangle = G^-_{1/2}|\phi\rangle = 0. \tag{3.62}$$

Any operator which lowers the L_0 eigenvalue but does not change the $U(1)$ charge must annihilate $|\phi\rangle$ because of the bound $h \geq |q/2|$. In particular

$$J_n|\phi\rangle = 0 \quad \text{for} \quad n > 0. \tag{3.63}$$

By combining the commutation relation

$$[J_n, G^\pm_r] = \pm G^\pm_{n+r} \tag{3.64}$$

with (3.62), we easily see that $|\phi\rangle$ is a primary chiral state.

Next, we want to show that for a primary chiral state

$$h \leq c/6. \tag{3.65}$$

To see this, use the $N = 2$ algebra

$$\{G^-_{3/2}, G^+_{-3/2}\} = 2L_0 - 3J_0 + 2c/3 \tag{3.66}$$

and take the expectation value of this positive operator for any chiral primary state. Taking into consideration the fact that $h = q/2$, we get the result. This fact is very important. Since the dimension of primary chiral fields is always less than or equal to some given value $c/6$, it follows that in non-degenerate $N = 2$ conformal theories, for which the spectrum of L_0 is discrete, there is only a finite number of primary chiral operators.

Now consider the operator algebra of primary chiral fields. In a general conformal field theory, we have to worry about how to define the composite operators. However, for chiral primary fields ϕ and χ we can choose the naive product

$$(\phi\chi)(z) = \lim_{z' \to z} \phi(z')\chi(z). \tag{3.67}$$

This is non-singular because $U(1)$ charges of the fields are additive and the conformal weights satisfy

$$h_{\phi\chi} \geq \frac{1}{2}(q_\phi + q_\chi) = h_\phi + h_\chi. \tag{3.68}$$

When we take the limit $z' \to z$ of the right hand side of (3.67), the fields except the primary chiral fields vanishes because of (3.68). Therefore, the primary chiral fields form a finite ring known as "chiral ring". To be more prcise, there are four rings that we can obtain in this way, depending on whether the left- and right-moving states are chiral or anti-chiral primary states. These four rings are pairwise conjugate.

We now turn our attention to the Ramond sector. By using the commutation relation $\{G_0^-, G_0^+\}$, we deduce that $h \geq c/24$. Equality is hold if and only if the state is annihilated by both G_0^+ and G_0^-. These are precisely the states which contribute to Witten's index $\mathrm{Tr}(-1)^F$. For theories satisfying

$$q_L - q_R \in \mathbf{Z} \tag{3.69}$$

for left- and right-moving $U(1)$ charges (q_L, q_R), the operator $(-1)^F$ where $F = F_L + F_R$ and F_L,F_R are left- and right-moving fermion numbers can be defined in terms of the $U(1)$ current as

$$(-1)^F = \exp\{i\pi(J_0 - \bar{J}_0)\}. \tag{3.70}$$

As is well-known, we can continuously connect the NS sector and the R sector by spectral flow. For a twist parameter θ we consider the Hilbert space H_θ of states which differ from the original one H_0 only in that their $U(1)$ charges are shifted by $-\frac{c}{3}\theta$. We denote the corresponding flow operator as U_θ

$$U_\theta : H_0 \longrightarrow H_\theta. \tag{3.71}$$

Under the spectral flow, the $N = 2$ algebra flows to an isomorphic algebra:

$$\begin{aligned} U_\theta L_n U_\theta^{-1} &= L_n + \theta J_n + \frac{c}{6}\theta^2 \delta_{n,0} \\ U_\theta G_r^+ U_\theta^{-1} &= G_{r+\theta}^+ \\ U_\theta G_r^- U_\theta^{-1} &= G_{r-\theta}^- \\ U_\theta J_n U_\theta^{-1} &= J_n + \frac{c}{3}\theta \delta_{n,0} \end{aligned} \tag{3.72}$$

and similarly for right movers. Note that for $\theta \in \mathbf{Z} + \frac{1}{2}$, it interpolates between the NS and R sectors and for $\theta \in \mathbf{Z}$ it takes the NS to NS and R to R. We first examine the effect of flow with $\theta = 1/2$. If we concentrate on chiral states, we see

$$U_{1/2} G_{-1/2}^+ U_{1/2}^{-1} U_{1/2} |\phi\rangle = G_0^+ |\tilde{\phi}\rangle = 0 \tag{3.73}$$

where $|\tilde{\phi}\rangle = U_{1/2}|\phi\rangle$. If $|\phi\rangle$ is also primary, it follows that

$$G_n^- |\tilde{\phi}\rangle = G_{n+1}^+ |\tilde{\phi}\rangle = 0 \quad \text{for} \quad n \geq 0. \tag{3.74}$$

Combining these we see that $|\tilde{\phi}\rangle$ is in the ground state of the R sector. Therefore, under the flow by $\theta = 1/2$, the chiral primary states flow to the ground states of the Ramond sector. Under the left-right symmetric spectral flow, $q_L - q_R$ is invariant. Therefore, the index

$$\mathrm{Tr}(-1)^F = \mathrm{Tr}_R[(-1)^{J_0-\bar{J}_0} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}] \tag{3.75}$$

can be calculated in terms of chiral primary states of the NS sector by

$$\mathrm{Tr}(-1)^F = \sum_{\mathrm{chiralring}} \exp(i\pi(q_L - q_R)). \tag{3.76}$$

In fact, we can do better. The difference between the charges of the chiral primary fields and the charges of the ground states of the Ramond sector is the shifted value $c/6$. Thus, we can relate the $U(1)$ character valued degeneracy of the ground states of the R sector to the character valued sum over the chiral ring in the NS sector:

$$\mathrm{Tr}_{G_0^\pm=\bar{G}_0^\pm=0}[t^{J_0}\bar{t}^{\bar{J}_0}] = (t\bar{t})^{-c/6}\mathrm{Tr}_{\mathrm{chiralring}}[t^{J_0}\bar{t}^{\bar{J}_0}] = (t\bar{t})^{-c/6}P(t,\bar{t}) \tag{3.77}$$

where we define $P(t,\bar{t}) = \mathrm{Tr}_{\mathrm{chiralring}}[t^{J_0}\bar{t}^{\bar{J}_0}]$. Due to the charge conjugation invariance of the Ramond sector, $P(t,\bar{t})$ satisfies the following duality property

$$P(t,\bar{t}) = (t\bar{t})^{c/3}P(1/t, 1/\bar{t}). \tag{3.78}$$

We call $P(t,\bar{t})$ the Poincaré polynomial. This property implies that there exists an unique primary chiral field with the highest possible left and right charges $q_R = q_L = c/3$ and dimensions $h_L = h_R = c/6$. This state is the Poincaré dual of the vacuum.

Now, we consider the flow with flow parameter $\theta = 1$. Under this flow, (chiral,chiral) primary states are mapped to the (anti-chiral,anti-chiral) primary states. In particular, the vacuum, which is primary and is chiral as well as anti-chiral (annihilated by $G^\pm_{-1/2}$) flows to an anti-chiral state $|\bar{\rho}\rangle = U_1|0\rangle$, which satisfies

$$G^+_{n+1/2}|\bar{\rho}\rangle = G^-_{n-3/2}|\bar{\rho}\rangle = 0 \quad \text{for} \quad n \geq 0. \tag{3.79}$$

By the commutation relation of $\{G^-_{-3/2}, G^+_{3/2}\}$ we see that

$$(2L_0 + 3J_0 + 2c/3)|\bar{\rho}\rangle = 0. \tag{3.80}$$

Since $|\bar{\rho}\rangle$ is anti-chiral and primary, we have

$$h_{\bar{\rho}} = \frac{c}{6}. \tag{3.81}$$

So, the vacuum flows to the conjugate of the chiral primary state with the highest charge mentioned above.

Let us bosonize the $U(1)$ current as

$$J(z) = i\sqrt{c/3}\,\partial\phi(z). \tag{3.82}$$

The normalization is fixed by the commutation relations. Then, a state with charge (q_L, q_R) can be represented by

$$O_{q_L,q_R} = \exp[i\sqrt{3/c}(q_L\phi_L - q_R\phi_R)]\chi \tag{3.83}$$

where χ is a neutral operator. And the spectral flow operator is expressed as

$$U_\theta = \exp(-i\theta\sqrt{c/3}(\phi_L - \phi_R)). \tag{3.84}$$

Aspects of strings

Now, we discuss the relations to spacetime supersymmetry. The necessary condition to have spacetime supersymetry is that the left and right $U(1)$ charges should be integers. If this is satisfied, we can define the $(-1)^{F_L}$ and $(-1)^{F_R}$ separately in terms of the currents as

$$(-1)^{F_L} = \exp[i\pi J_0] \tag{3.85}$$

$$(-1)^{F_R} = \exp[-i\pi \bar{J}_0]. \tag{3.86}$$

In such a case, we can perform spectral flow for left and right movers independently. For instance, set $\theta_R = 0$ and $\theta_L = \frac{1}{2}$. The (R,R) sector flows to the (NS,R) sector. The corresponding operator can be represented by

$$\exp\left[-\frac{i}{2}\sqrt{c/3}\phi_L\right] \tag{3.87}$$

To make this a well-defined in the entire theory, this must be augmented by similar flow operators in the ghost and noncompact sectors. In this way, we can construct the space-time supersymmetry operator from the spectral flow operator if the charge integral condition is imposed. In fact, Gepner constructed some models for Calabi-Yau compactification by using $N = 2$ minimal models. He projected to the states with odd integral charges to have space-time supersymmetry [81].

An easy example

The example is the simplest type of superstring compactification, namely, on a two dimensional torus. Let us examine the chiral ring. This theory has one complex boson, and one complex fermion. So, the central charge is 3. The $N = 2$ algebra is realized by

$$\begin{aligned} G^+(z) &= \psi^* \partial x \\ G^-(z) &= \psi \partial x^* \\ J(z) &= \psi \psi^* \end{aligned} \tag{3.88}$$

and similary for right movers. Left primary chiral states are

$$|0\rangle, \quad \psi^*_{-1/2}|0\rangle. \tag{3.89}$$

The easiest way to achieve this result is to note the fact $h \leq \frac{1}{2}$.

Therefore the (chiral,chiral) primary states are

$$|0\rangle, \quad \psi^*_{-1/2}|0\rangle, \quad \bar{\psi}^*_{-1/2}|0\rangle \quad \psi^*_{-1/2}\bar{\psi}^*_{-1/2}|0\rangle \tag{3.90}$$

The corresponding (c, c) ring is

$$\{1, \psi^*, \bar{\psi}^*, \psi^*\bar{\psi}^*\}. \tag{3.91}$$

The Poincaré polynomial is

$$P(t, \bar{t}) = 1 + t + \bar{t} + t\bar{t}. \tag{3.92}$$

It is easy to check the duality property.

Chiral rings and cohomology

Look at the results obtained in the previous paragraph. Under the identification

$$\begin{aligned} 1 &\rightarrow 1 \\ \psi^* &\rightarrow dz \\ \bar{\psi}^* &\rightarrow d\bar{z} \\ \psi^* \bar{\psi}^* &\rightarrow dz \wedge d\bar{z} \end{aligned} \tag{3.93}$$

the ring structure for the superconformal model on the torus is isomorphic to the ring structure of the Dolbeault cohomology groups of the torus. The duality property of the Poincaré polynomial is precisely translated to the Poincaré duality of the cohomology groups. The fact that dim $H^{d,d}$ = dim $H^{0,0} = 1$ corresponds to the uniqueness of the vacuum and of the field with highest charge. This can be understood as follows. If we are considering a supersymmetric non-linear σ-model, we know from the work of Witten [13] that there is a one to one correspondence between the cohomology classes of the manifold and the ground states of the Ramond sector. Since, we know that the ground states of the Ramond sector are related to the chiral primary fields by spectral flow, there is a one-to-one correspondence between the harmonic forms that represent the Dolbeault cohomology and the elements of the chiral ring. Note that the $U(1)$ charge corresponds to the grade of the differential forms, and the isomorphism preserves the left-right grading.

But it should be noted that this is only a large radius approximation. When we decrease the radii, the semi-classical approximation is no longer justifiable. Indeed, there are many examples that invalidate the isomorphisms of the two rings. The chiral rings get deformed, and these deformed cohomology rings are called the quantum cohomology ring.

3.2.2 Topological twists

Here, we twist the superconformal theory studied above [21, 22, 23].

Let us begin with an easy example. Consider the level-1 $N = 2$ superconformal theory in two dimensions whose central charge is 1. There exists a free field realization, which is [24]

$$\begin{aligned} T(z) &= -\frac{1}{2} : \partial\phi\partial\phi : \\ G^+(z) &= \sqrt{\frac{2}{3}} : e^{i\sqrt{3}\phi(z)} : \\ G^-(z) &= \sqrt{\frac{2}{3}} : e^{-i\sqrt{3}\phi(z)} : \\ J(z) &= \frac{i}{\sqrt{3}} \partial\phi(z). \end{aligned} \tag{3.94}$$

Here, $\phi(z)$ is a free boson with OPE $\partial\phi(z)\partial\phi(w) \sim -\frac{1}{(z-w)^2}$.

Now the important step. Re-define the energy-momentum by

$$T'(z) = T(z) + \frac{1}{2}\partial J(z) \tag{3.95}$$

It may seem that nothing important has occured. To see what really occurs, let us calculate the new central charge

$$\begin{aligned} T'(z)T'(w) &= \left(T(z) + \frac{1}{2}\partial J(z)\right)\left(T(w) + \frac{1}{2}\partial J(w)\right) \\ &= \frac{c/2}{(z-w)^4} + \frac{1}{4}\frac{c}{3}\frac{-2\times 3}{(z-w)^4} + \text{less singular terms} \\ &= O\left(\frac{1}{(z-w)^2}\right). \end{aligned} \tag{3.96}$$

So, the central charge vanishes. Similar caluculation yields the following important relation

$$\{\int dz G^+(z), G^-(w)\} = 2\left(T(w) + \frac{1}{2}\partial J(w)\right) = 2T'(w). \tag{3.97}$$

The meaning of these facts are as follows. First note that changing the energy-momentum tensor induces the change of conformal weight $h' = h - \frac{q}{2}$. From the algebra of $N = 2$, we see that G^+ has originally a conformal weight $3/2$, and charge 1. Therefore, after twisting, the G^+ has a conformal dimension 1. So, $Q = \int dz G^+(z)$ is a scalar. In fact, from the algebra of G_0^+, we easily see $Q^2 = 0$. Therefore, We can regard the Q as the BRST operator. Then, the above fact shows that the energy-momentum tensor is not only BRST closed but also BRST exact. Since the energy-momentum tensor is an operator of an evolution of space and time, it follows that the correlation functions are independent of where operators are. Also, from the definition of the energy-momentum tensor the correlation functions are independent of the two dimensional metric. Therefore, what we have is a topological field theory! Moreover, by using the free field representation above, the new energy momentum tensor can be expressed as

$$T'(z) = -\frac{1}{2}(\partial\phi)^2 + \frac{i}{2\sqrt{3}}\partial^2\phi(z). \tag{3.98}$$

On the other hand, the unitary discrete series of the Virasoro algebra has the following free field representation

$$T(z) = -\frac{1}{2}(\partial\phi)^2 + i\alpha_0\partial^2\phi(z) \tag{3.99}$$

where $\alpha_0 = \frac{1}{\sqrt{2(l+2)(l+3)}}$ and the central charge is $c = 1 - 12\alpha_0^2$. We easily see that the above twisted theory is equivalent to the Virasoro case of $l = 0$.

Though we explained in a paricular example, the above facts are generally true. But, there are in fact two ways to twist the theory. What we have done is called chiral twist. The reason for this name is the fact that the BRST closed observables are the chiral primary fields (Note that chiral primary fields are by definition those satisfying $\oint_w dz G^+(z)\phi(w) = 0$). On the other hand, modifying the energy-momentum tensor as

$$T'(z) = T(z) - \frac{1}{2}\partial J(z) \tag{3.100}$$

gives so-called anti-chiral twist. In that case, the new conformal weights are $h' = h + \frac{q}{2}$ and the roles of G^+ and G^- are exchanged.

3.2.3 Topological sigma models

We will discuss the topological sigma models or so called A-model and B-model. As we studied above, there are two ways to twist the $N = 2$ theory. So, starting from the $N = 2$ theory, we have two topological field theories. Therefore, taking into consideration the fact that there are actually left- and right-movers, one might think that there are four possible twists. But this is not the case. The (chiral, chiral) twist is equivalent to the (antichiral, antichiral) twist by redefinition. Similarly, (chiral, antichiral) and (antichiral, chiral) are the same. So, there are two independent ways. We call A-model in the first case, and B-model in the latter. These names come from the relation to the type IIA and type IIB compactifications. We assume that the target space of the sigma models are always Calabi-Yau.

Preliminaries

We consider the $N = 2$ supersymmetric non-linear sigma model. The action is

$$S = 2t\int_\Sigma d^2z\left(\frac{1}{2}g_{IJ}\partial_z\phi^I\partial_{\bar z}\phi^J + i\psi_-^{\bar\imath}D_z\psi_-^i g_{\bar\imath i} + i\psi_+^{\bar\imath}D_{\bar z}\psi_+^i g_{\bar\imath i} + R_{i\bar\imath j\bar\jmath}\psi_+^i\psi_+^{\bar\imath}\psi_-^j\psi_-^{\bar\jmath}\right) \tag{3.101}$$

where $R_{i\bar\imath j\bar\jmath}$ is the Riemann tensor of the target space and the covariant derivative $D_{\bar z}$ are given by

$$D_{\bar z}\psi_+^i = \partial_{\bar z}\psi_+^i + \partial_{\bar z}\phi^j\Gamma^i_{jk}\psi_+^k. \tag{3.102}$$

And similarly for D_z. The transformation laws are

$$\begin{aligned}
\delta\phi^i &= i\alpha_-\psi_+^i + i\alpha_+\psi_-^i\\
\delta\phi^{\bar\imath} &= i\tilde\alpha_-\psi_+^{\bar\imath} + i\tilde\alpha_+\psi_-^{\bar\imath}\\
\delta\psi_+^i &= -\tilde\alpha_-\partial_z\phi^i - i\alpha_+\psi_-^j\Gamma^i_{jm}\psi_+^m\\
\delta\psi_+^{\bar\imath} &= -\alpha_-\partial_z\phi^{\bar\imath} - i\tilde\alpha_+\psi_-^{\bar\jmath}\Gamma^{\bar\imath}_{\bar\jmath\bar m}\psi_+^{\bar m}\\
\delta\psi_-^i &= -\tilde\alpha_+\partial_{\bar z}\phi^i - i\alpha_-\psi_+^j\Gamma^i_{jm}\psi_-^m\\
\delta\psi_-^{\bar\imath} &= -\alpha_+\partial_{\bar z}\phi^{\bar\imath} - i\tilde\alpha_-\psi_+^{\bar\jmath}\Gamma^{\bar\imath}_{\bar\jmath\bar m}\psi_+^{\bar m}.
\end{aligned} \tag{3.103}$$

Our assignments of the quantum numbers (h_L, q_L, h_R, q_R) are

$$
\begin{aligned}
\phi &\to (0,0,0,0) \\
\psi_+^i &\to (\frac{1}{2},\frac{1}{2},0,0) \\
\psi_+^{\bar{\imath}} &\to (\frac{1}{2},-\frac{1}{2},0,0) \\
\psi_-^i &\to (0,0,\frac{1}{2},-\frac{1}{2}) \\
\psi_-^{\bar{\imath}} &\to (0,0,\frac{1}{2},\frac{1}{2}) \\
\alpha_- &\to (-\frac{1}{2},-\frac{1}{2},0,0) \\
\tilde{\alpha}_- &\to (-\frac{1}{2},\frac{1}{2},0,0) \\
\alpha_+ &\to (0,0,-\frac{1}{2},\frac{1}{2}) \\
\tilde{\alpha}_+ &\to (0,0,-\frac{1}{2},-\frac{1}{2}).
\end{aligned}
\tag{3.104}
$$

Note that these are not fixed canonically. If we adopt other possible assignments, the chiral twist and anti-chiral twist can be exchanged. This is the nature of the mirror symmetry, as will be gradually clear. We denote the Riemann surface by Σ and the target space by X.

A model

Apply the (chiral,chiral) twist to the above sigma model. After this procedure, ψ_+^i and $\psi_-^{\bar{\imath}}$ become scalars, $\psi_+^{\bar{\imath}}$ a (1,0) form, and ψ_-^i a (0,1) form on the Riemann surface Σ. We set $\chi^i = \psi_+^i$, $\chi^{\bar{\imath}} = \psi_-^{\bar{\imath}}$, $\psi_z^{\bar{\imath}} = \psi_+^{\bar{\imath}}$, and $\psi_{\bar{z}}^i = \psi_-^i$. We next want to consider the topological transformation laws. Since α_- and $\tilde{\alpha}_+$ become scalars we can set them to constants, which we call α and $\tilde{\alpha}$. By doing so and setting $\alpha_+ = \tilde{\alpha}_- = 0$, we have

$$
\begin{aligned}
\delta\phi^i &= i\alpha\chi^i \\
\delta\phi^{\bar{\imath}} &= i\tilde{\alpha}\chi^{\bar{\imath}} \\
\delta\chi^i &= \delta\chi^{\bar{\imath}} = 0 \\
\delta\psi_z^{\bar{\imath}} &= -\alpha\partial_z\phi^{\bar{\imath}} - i\tilde{\alpha}\chi^{\bar{\jmath}}\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{m}}\psi_z^{\bar{m}} \\
\delta\psi_{\bar{z}}^i &= -\tilde{\alpha}\partial_{\bar{z}}\phi^i - i\alpha\chi^j\Gamma^i_{jm}\psi_{\bar{z}}^m.
\end{aligned}
\tag{3.105}
$$

The supersymmetry algebra of the original model collapses for these topological transformation laws to $\delta^2 = 0$, which holds modulo the equations of motion.

From now on, we generally set $\alpha = \tilde{\alpha}$ for simplicity. In this case, the first two lines of the above transformation laws combine to $\delta\Phi = i\alpha\chi^I$. Also, we define the BRST operator Q such that $\delta W = -i\alpha\{Q, W\}$ for any field W. In terms of these variables, the action is given by

$$S = 2t \int_\Sigma d^2z \left(\frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i\psi_z^{\bar{\imath}} D_{\bar{z}} \chi^i g_{\bar{\imath}i} + i\psi_z^i D_z \chi^{\bar{\imath}} g_{\bar{\imath}i} - R_{i\bar{\imath}j\bar{\jmath}} \psi_{\bar{z}}^i \psi_z^{\bar{\imath}} \chi^j \chi^{\bar{\jmath}} \right). \quad (3.106)$$

We note the important fact that this can be written modulo terms that vanish by the equation of motion as

$$S = it \int_\Sigma d^2z \{Q, V\} + t \int_\Sigma \Phi^*(K) \quad (3.107)$$

where

$$V = g_{i\bar{\jmath}} \left(\psi_z^{\bar{\imath}} \partial_{\bar{z}} \phi^j + \partial_z \phi^{\bar{\imath}} \psi_{\bar{z}}^j \right). \quad (3.108)$$

And

$$\int_\Sigma \Phi^*(K) = \int_\Sigma d^2z \left(\partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{\jmath}} g_{i\bar{\jmath}} - \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{\jmath}} g_{i\bar{\jmath}} \right) \quad (3.109)$$

is the integral of the pullback of the Kähler form $K = -ig_{i\bar{\jmath}} dz^i \wedge dz^{\bar{\jmath}}$. Therefore $\int \Phi^*(K)$ depends only on the cohomology class of K and the homotopy class of the map Φ. If $H^2(X, \mathbf{Z}) \cong \mathbf{Z}$ and the metric is normalized so that the periods of K are integer multiples of 2π, then

$$\int \Phi^*(K) = 2\pi n. \quad (3.110)$$

We adopt this terminology for simplicity.

Suppose we wish to compute the path integral for fields of degree n. With insertions of BRST invariant operators O_a, we want to compute

$$\langle \prod O_a \rangle_n = e^{-2\pi nt} \int_{B_n} D\phi D\chi D\psi e^{-it\int\{Q,V\}} \prod O_a \quad (3.111)$$

Here B_n is the component of the field space for maps of degree n. In fact, this expression is independent of t (we need Re $t > 0$ for convergence) except the factor of $e^{-2\pi nt}$. The reason for this is that differentiating the other t dependent factor with respect to t gives only the irrelevant factors of the form $\{Q, ...\}$. Thus, we can compute the path integral by taking the limit of large Re t. This is the conventional weak coupling limit. Looking at the original form of the Lagrangian, we see that the bosonic part of S is minimized for holomorphic maps; that is

$$\partial_{\bar{z}} \phi^i = \partial_z \phi^{\bar{\imath}} = 0. \quad (3.112)$$

Therefore the weak coupling limit involves a reduction to the moduli space $\mathcal{M}_n$ of holomorphic maps of degree n.

Note also that in the A model the correlation functions are independent of the complex structure of Σ and X, and depend only on the cohomology class of the Kähler form K. This is certainly true for $\int_\Sigma \Phi^*(K)$. For the rest, all dependence of the Lagrangian on the complex structure is in V, which appears only in the form $\{Q, V\}$. Therefore, varying the path integral with respect to the complex structure will therefore bring only irrelevant factors of the form $\{Q, ...\}$.

The action has a "ghost number" conservation law at the classical level. χ has ghost number 1, ψ has -1, and ϕ has 0. The BRST operator has ghost number 1. The ghost number is not a symmetry at the quantum level because of the anomaly. Let a_n be the number of χ zero modes, or the dimension of the space of solutions of $D_{\bar{z}}\chi^i = D_z\chi^{\bar{i}} = 0$, and similarly, b_n the number of ψ zero modes, solutions of $D_{\bar{z}}\psi_z^{\bar{i}} = D_z\psi_{\bar{z}}^i = 0$. Then the index theorem gives a formula for the difference $w_n = a_n - b_n$. If X is a Calabi-Yau manifold of complex dimensison d, and Σ has genus g, then $w_n = 2d(1-g)$, which is independent of n. Therefore, the expression $\langle \prod O_a \rangle$ vanishes unless the sum of the ghost numbers of the O_a is equal to w_n. The number w_n is often called the "virtual dimension" of $\mathcal{M}_n$. The reason for this is that in a sufficiently generic situation, one would expect that if $w_n > 0$, then $b_n = 0$ hence $w_n = a_n$.

Next, we discuss the observables of the A model. The BRST cohomology of the A model, in the space of local operators, can be represented by those that are functions of ϕ and χ only. They have the following simple construction. Let $W = W_{I_1 I_2 ... I_n}(\phi) d\phi^{I_1} \wedge d\phi^{I_2} \wedge ... \wedge d\phi^{I_n}$ be an n-form on X. Define the corresponding local operator as

$$O_W(P) = W_{I_1 I_2 ... I_n}(\phi)\chi^{I_1}\chi^{I_2}...\chi^{I_n}(P). \tag{3.113}$$

The ghost number of O_W is n. A simple calculation shows

$$\{Q, O_W\} = -O_{dW} \tag{3.114}$$

Therefore, taking $W \to O_W$ gives a map from the de Rham cohomology of X to the BRST cohomology of the quantum field theory. If we restrict to local operators, this map is an isomorphism. There are following convenient representatives of the cohomology. Let H be a homology cycle. The Poincaré dual of H is a cohomology class that counts intersections with H. It can be represented by a differential form $W(H)$ that has delta function support on H. We refer to $O_{W(H)}$ as O_H. The ghost number of O_H is the codimension of H.

Now let us evaluate the path integral. We pick some homology cycles H_a, $a = 1, 2, ..., s$ of codimension q_a. We want to compute the quantity

$$\langle O_{H_1}(P_1)...O_{H_s}(P_s)\rangle_n = e^{-2\pi nt} \int D\phi D\chi D\psi e^{it\int\{Q,V\}} \prod O_{H_a}(P_a). \tag{3.115}$$

As we saw, the path integral reduces to an integral over the moduli space $\mathcal{M}_n$ of instantons. Moreover, since we have picked $O_{H_a}(P_a)$ to have delta function support for instantons Φ such that

$$\Phi(P_a) \in H_a, \tag{3.116}$$

the path integral actually reduces to an integral over the moduli space $\tilde{\mathcal{M}}_n$ of instantons obeying the above condition. In a generic situation, the dimension a_n of $\mathcal{M}_n$ coincides with the virtual dimension w_n. And, requiring $\Phi(P_a) \in H_a$ imposes q_a conditions. Therefore the dimension of $\tilde{\mathcal{M}}_n$ is $w_n - \sum_a q_a = 0$. In such cases, $\tilde{\mathcal{M}}_n$ consists of a finite set of poins.. We denote the number of such points by $\#\tilde{\mathcal{M}}_n$ We take Re $t \to \infty$. The computation reduces to evaluation of a ratio of boson and fermion determinants and this ratio is simply 1 because of the BRST symmetry which ensures cancellation between bose and fermi modes. Therefore, in a generic situation

$$\langle \prod O_{H_a}(P_a) \rangle_n = e^{-2\pi n t} \#\tilde{\mathcal{M}}_n. \tag{3.117}$$

Summing over n, we have

$$\langle \prod O_{H_a}(P_a) \rangle = \sum_n e^{-2\pi n t} \#\tilde{\mathcal{M}}_n. \tag{3.118}$$

In complex geometry, the term "generic" doesn't mean that other cases are unimportant. Therefore we should consider the case $\tilde{\mathcal{M}}_n$ have components of positive dimensions. Let s be the dimension of $\tilde{\mathcal{M}}_n$. In this case, the space V of ψ zero modes is s dimensional and varies as the fibers of a vector bundle $\mathcal{V}$ of dimension s over $\tilde{\mathcal{M}}_n$. The known result is that we simply replace

$$\#\tilde{\mathcal{M}}_n \to \int_{\tilde{\mathcal{M}}_n} \chi(\mathcal{V}) \tag{3.119}$$

where $\chi(\mathcal{V})$ is the Euler class of the bundle $\mathcal{V}$. See [25] for detail. Thus, the generalization is

$$\langle \prod O_{H_a}(P_a) \rangle = \sum_n e^{-2\pi n t} \int_{\tilde{\mathcal{M}}_n} \chi(\mathcal{V}). \tag{3.120}$$

B model

We study the B model in a similar way. Apply the (anti-chiral, chiral) twist to the nonlinear sigma model. Then, $\psi_\pm^{\bar{\imath}}$ are scalars, ψ_+^i is a (1,0) form, and ψ_-^i is a (0,1) form. We set

$$\eta^{\bar{\imath}} = \psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}} \tag{3.121}$$

$$\theta_i = g_{i\bar{\imath}} \left(\psi_+^{\bar{\imath}} - \psi_-^{\bar{\imath}} \right). \tag{3.122}$$

Also, we combine $\psi_\pm^i$ into a one form ρ. Thus, the (1,0) part of ρ is $\rho_z^i = \psi_+^i$ and the (0,1) part is $\rho_{\bar{z}}^i = \psi_-^i$. As for the transformation laws, we set $\alpha_\pm = 0$ and $\tilde{\alpha}_+ = \tilde{\alpha}_- = \alpha$ as in the A model. Here, α is a constant. Then, the transformation laws are

$$\begin{aligned} \delta\phi^i &= 0 \\ \delta\phi^{\bar{\imath}} &= i\alpha\eta^{\bar{\imath}} \\ \delta\eta^{\bar{\imath}} &= \delta\theta_i = 0 \\ \delta\rho^i &= -\alpha d\phi^i. \end{aligned} \tag{3.123}$$

The BRST operator is defined by $\delta(...) = -i\alpha\{Q, ...\}$ and obeys $Q^2 = 0$ modulo the equations of motion.
The action is

$$S = t\int_\Sigma d^2z(g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + i\eta^{\bar{\imath}}(D_z\rho^i_{\bar{z}} + D_{\bar{z}}\rho^i_z)g_{i\bar{\imath}} \\ +i\theta_i(D_{\bar{z}}\rho^i_z - D_z\rho^i_{\bar{z}}) + R_{i\bar{\imath}j\bar{\jmath}}\rho^i_z\rho^j_{\bar{z}}\eta^{\bar{\imath}}\theta_k g^{k\bar{\jmath}}) \tag{3.124}$$

As in the A model, this can be rewritten

$$S = it\int\{Q, V\} + tW \tag{3.125}$$

where

$$V = g_{i\bar{\jmath}}\left(\rho^i_z\partial_{\bar{z}}\phi^{\bar{\jmath}} + \rho^i_{\bar{z}}\partial_z\phi^{\bar{\jmath}}\right) \tag{3.126}$$

and

$$W = \int_\Sigma\left(-\theta_i D\rho^i - \frac{i}{2}R_{i\bar{\imath}j\bar{\jmath}}\rho^i\wedge\rho^j\eta^{\bar{\imath}}\theta_k g^{k\bar{\jmath}}\right). \tag{3.127}$$

Here D is the exterior derivative on Σ extended to act on forms with values in $\Phi^*(T^{1,0}X)$ by using the pullback of the Levi-Civita connection of X. We can see that the B model is independent of the complex structure of Σ and the Kähler metric of X. Under a change of complex structure of Σ or Kähler metric of X, the action changes by irrelevant terms of the form $\{Q, ...\}$ only. This is obvious for the $\{Q, V\}$ term. As for W, it is independent of the complex structure of Σ, because it is written in terms of differential forms which depend only on the differentiable structure. It is less obvious, but can be shown that under change of Kähler metric of X, W changes by $\{Q, ...\}$. These are mirror to our earlier result that the A model is independent of the complex structure of Σ and X but depends on the Kähler class of the metric of X. In fact, as mirror symmetry reverses one of the $U(1)$ quantum numbers in the $N = 2$ algebra, it can be taken to exchange the twists of ψ_+ while leaving ψ_- alone. Therefore, mirror symmetry exchanges A model and B model. The B model is independent of the coupling constant t except for a trivial factor which will appear shortly (as long as Re $t > 0$). Under a change of t, the $t\{W, V\}$ term changes by $\{Q, ...\}$. As for the t in tW this can be removed by redefining $\theta \to \theta/t$. This is possible because V is independent of θ and W is homogeneous of degree one. Hence, the B model is independent of t except for factors that come from the θ dependence of the observables. If O_a are BRST invariant operators that are homogeneous in t of degree k_a, the t dependence of $\langle\prod O_a\rangle$ is a factor of $t^{-\sum_a k_a}$. This arises from the rescaling of θ. This trivial t dependence should be contrasted with the complicated t dependence of the A model, coming from the instanton sum.

As the t dependence of the B model is known, all calculations can be performed in the limit Re $t \to \infty$, that is, in the weak coupling limit. In this limit, we expand around minima of the bosonic part of the Lagrangian. These are just the constant maps; $d\phi = 0$.

Since the space of such maps is a copy of X, the path integral reduces to an integral over X.

The fermion determinant of the A model is real and positive as the χ^i,$\psi_z^{\bar{\imath}}$ determinant is the complex conjugate of the $\chi^{\bar{\imath}}$,$\psi_{\bar{z}}^i$ determinant. In particular, there is no problem in defining this determinant as a function, and the A model makes sense as a quantum field theory for any complex manifold X. The B model is very different. Since the 0-forms $\eta^{\bar{\imath}}$, $g^{\bar{\imath}i}\theta_i$ are sections of $T^{0,1}X$ and the 1-forms η^i are sections of $T^{1,0}X$, the fermion determinant in the B model is complex. The B model does not make any sense as a quantum field theory without an anomaly cancellation condition that makes it possible to define the fermion determinants as functions. The relevant condition is $c_1(X) = 0$, that is, X must be a Calabi-Yau manifold. Thus in the B model, the Calabi-Yau condition plays an even more fundamental role than it does in the untwisted model, where it is merely necessary for conformal invariance. Like the A model, the B model has an important ghost number. The ghost number is 1 for η and θ, -1 for ρ, and 0 for ϕ. If X is a Calabi-Yau manifold of complex dimension d, and O_a are BRST invariant operators of ghost number w_a, $\langle O_a \rangle$ vanishes in genus g unless

$$\sum_a w_a = 2d(1-g). \tag{3.128}$$

There is actually a more refined $\mathbf{Z} \times \mathbf{Z}$ grading, which we have obscured by setting $\tilde{\alpha}_+ = \tilde{\alpha}_-$ and combining $\psi_\pm^i$ into ρ.

Now we consider the observables of the B model. Instead of the cohomology of X, we consider $(0,p)$ forms on X with values in $\wedge^q T^{1,0}X$, the q-th exterior power of the holomorphic tangent bundle of X. We can write such an object as

$$V = d\bar{z}^{i_1} d\bar{z}^{i_2} ... d\bar{z}^{i_p} V_{\bar{\imath}_1\bar{\imath}_2...\bar{\imath}_p}{}^{j_1 j_2 ... j_q} \frac{\partial}{\partial z_{j_1}} \frac{\partial}{\partial z_{j_2}} ... \frac{\partial}{\partial z_{j_q}}, \tag{3.129}$$

where V is antisymmetric in the j's as well as in the $\bar{\imath}$'s.

For every V and $P \in \Sigma$, we can make the operator

$$O_V = \eta^{\bar{\imath}_1} ... \eta^{\bar{\imath}_p} V_{\bar{\imath}_1...\bar{\imath}_p}{}^{j_1...j_q} \theta_{j_1} ... \theta_{j_q}. \tag{3.130}$$

Simple calculation gives

$$\{Q, O_V\} = -O_{\bar{\partial} V}. \tag{3.131}$$

Consequently O_V is BRST closed if $\bar{\partial} V = 0$ and exact if $V = \bar{\partial} S$ for some S. Thus, $V \to O_V$ gives a natural map from $\oplus_{p,q} H^p(X, \wedge^q T^{1,0}X)$ to the BRST cohomology of the B model. As long as local operators are concerned, this is in fact an isomorphism.

Next, we want to compute

$$\langle \prod O_{V_a}(P_a) \rangle \tag{3.132}$$

where $P_a \in \Sigma$ are points and $V_a \in H^{p_a}(X, \wedge^{q_a} T^{1,0}X)$. We consider only the case of genus zero. Then, by considering the ghost number (this is related to a more prcise left-right-moving ghostnumbers), this correlation function vanishes unless

$$\sum_a p_a = \sum_a q_q = d. \tag{3.133}$$

Taking Re $t \to \infty$ we reduce the path integral to an integral over the space of constant maps. In addition to the bose zero modes which are the displacements of the constant map, there are fermi zero modes, which are the constant modes of η and θ. The nonzero bose and fermi modes are independent of the particular constant map about which one is expanding. So, these go into the definition of the string coupling constant. Therefore, we reduce to a computation involving the zero modes only. This means that the correlation functions of the B model reduce to classical expressions. Once we restrict to the space of zero modes, a function of ϕ,η and θ which is of p-th order in η and q-th order in θ can be interpreted as a $(0,p)$ form on X with values in $\wedge^d T^{1,0}X$. In multiplying such functions, we automatically antisymmetrizes on the appropriate indices because of the fermi statistics. Therefore, $\prod O_{V_a}$ can be interpreted as a d form with values in $\wedge^d T^{1,0}X$ (the number d comes from the anomalous ghost number conservation). The map

$$\otimes_a H^{p_a}(X, \wedge^{q_a} T^{1,0}X) \to H^d(X, \wedge^d T^{1,0}X) \tag{3.134}$$

is the classical wedge product. What remains to be done is to integrate over X the element of $H^d(X, \wedge^d T^{1,0}X)$ obtained in this way. The Calabi-Yau condition ensures that $H^d(X, \wedge^d T^{1,0}X)$ is non-zero and one dimensional. The space of linear forms on this space is thus likewise one-dimensional; any such non-zero form gives a method of integration unique up to a constant multiple. Of course, the path integral of the B model gives formally a method of evaluating the correlation functions, this procedure formally is unique up to a multiplicative constant (a correction to the string coupling constant). We noted that the B model is anomalous except for Calabi-Yau manifolds. The restriction to Calabi-Yau manifolds amounts to the fact that what can be integrated naturally are top forms, elements of $H^d(X, \Omega^d X)$. In general the relation between $\Omega^d X$ and $\wedge^d T^{1,0}X$ is that they are inverses each other but in Calabi-Yau cases, they are both trivial and hence isomorphic. Indeed, multiplication by the square of a holomorphic d form gives a map from $\wedge^d T^{1,0}X$ to $\Omega^d X$.

Reduction from another viewpoint

In the A model, we reduce the path integral to integrals over moduli spaces of holomorphic maps, while in the B model we reduce to integrals over spaces of constant maps. We explain this in an alternative way.

Consider an arbitrary quantum field theory, with some function space $\mathcal{E}$ over which we want to integrate. Let F be a group of symmetries of the theory. We first assume that F acts on $\mathcal{E}$ freely. Then, we have a fibration $\mathcal{E} \to \mathcal{E}/F$. If we first integrate over the fibers, we have

$$\int_{\mathcal{E}} e^{-S} O = \mathrm{vol}(F) \int_{\mathcal{E}/F} e^{-S} O. \tag{3.135}$$

where we assume Os are invariant under the F action. Now we want to apply this to the case in which F is generated by the BRST operator Q. In this case, the volume of the group F is zero, because for a fermionic variable θ,

$$\int d\theta 1 = 0. \tag{3.136}$$

This tells us that if Q acts freely, the expectation value of any BRST closed operator vanishes. Therefore, this result is a localization formula expressing the path integral as an integral on a fixed point locus $\mathcal{E}_0$. In other words, the contribution comes entirely from the fixed points of Q.

In the A model, requiring $\delta\psi^I = 0$ gives $\chi^I = 0$ and setting $\delta\psi = 0$, we have

$$\partial_{\bar{z}} \phi^i = \partial_z \phi^{\bar{\imath}} = 0. \tag{3.137}$$

In the B model $\delta\rho^i = 0$ gives us

$$d\phi^i = 0. \tag{3.138}$$

These are precisely the conditions that we got before.

Relations to physical models

So far, we have discussed the A model and the B model. These are of course not equivalent to the standard untwisted theories. However, in some cases, the calculations based on these topological sigma models coincides with those of "physical" untwisted ones. In general, computations in the twisted models are easier than in the untwisted ones. So, we can compute certain quantities which are difficult to compute in the ordinary models by replacing them with the twisted models. Understanding the precise relations is the theme of this section. In order to do so, it is convenient to use some mathematical terms.

Let K and $\bar{K}$ be the canonical and anti-canonical bundles over Σ. These are the bundles of 1-forms of types $(1,0)$ and $(0,1)$. And let $K^{1/2}$ and $\bar{K}^{1/2}$ be square roots of these. The index denotes the conformal weight. For example, in these terms, the field ψ^i_+ is a section of $K^{1/2} \otimes \Phi^*(T^{1,0}X)$ and after the chiral twisting, it is a section of $\Phi^*(T^{1,0}X)$. In constructing the twisted models, we twisted various fields by $K^{1/2}$ and $\bar{K}^{1/2}$. If K is trivial and we choose $K^{1/2}$ and $\bar{K}^{1/2}$ to be trivial, the models coincide because in this case, the twisting does nothing. The important example is the case in which Σ is the Riemann surface of genus zero with two points deleted. Such a surface can be described as a cylinder with a complete, flat metric $ds^2 = d\tau^2 + d\sigma^2$, $-\infty < \tau < \infty$, $0 \le \sigma \le 2\pi$. When we compute the path integrals on such a surface, we must pick initial and final states. Let the initial state be $|i\rangle$ and final state $|f\rangle$. Picking also points $P_a \in \Sigma$ and BRST invariant operators O_a, we are interested in the objects

$$\langle f| \prod_a O_a(P_a)|i\rangle. \tag{3.139}$$

As K is trivial, we can pick $K^{1/2}$ and $\bar{K}^{1/2}$ to be trivial. If we do so, the twist does nothing. Therefore the above quantity is equivalent to some matrix element in the untwisted model. Of course, we can get only a few of them in this way because the untwisted model has a lot of matrix elements. Let us first see from the viewpoint of untwisted model. The operators O_a corespond to the chiral rings or standard vertex operators of massless bosons. Let us call the such vertex operators B_a. Then, what about the initial and final states? The equivalence between the untwisted and twisted models is a consequence of choosing K and their square roots to be trivial. The fact that the $K^{1/2}$ is trivial means that we impose the boundary condition along the σ-direction to be periodic. So, the initial and final states are in the Ramond sector, i.e. fermions. Let us denote these by f_1 and f_2. From the point of view of the untwisted model, the above quantity can be written

$$\langle f_2| \prod_a B_a|f_1\rangle \tag{3.140}$$

Note that all we have done is just to change the notation. This is a coupling of two ground state fermions $|f_1\rangle$ and $|f_2\rangle$ to an arbitrary number of ground state bosons B_a that are all from the same chiral ring. These are particularly important because they determine the superpotential.

On the other hand, the above expression can be interpreted in another way by compactifying Σ - adding points P and P' and conformally rescaling the metric to bring them to a finite distance. Then the states $|i\rangle$ and $|f\rangle$ will correspond to BRST invariant operators $O_i(P)$ and $O_f(P')$. The matrix element is then equivalent to a correlation function

$$\langle O_f(P')O_i(P) \prod_a O_a(P_a)\rangle \tag{3.141}$$

of the twisted model. In this way, particular matrix elements of the untwisted model can be identified with observables of the twisted model.

Some comments

We have seen that the A model is sensible to the Kähler class of the Calabi-Yau manifold and B model to the complex structure. They are exchanged when we redefine the charge correctly. This means that the mirror symmetry exchanges the Kähler and complex structure moduli. Let us consider the most easiest case, the torus. The shape of the torus is characterized by R_1/R_2 and the volume is $R_1 R_2$. Here, R_1, R_2 are radii of the torus. Then, in this case, exchanging the Kähler moduli and complex moduli is equal to

$$R_2 \leftrightarrow \frac{1}{R_2}. \tag{3.142}$$

This is nothing but a T-duality.

3.2.4 Topological strings; coupling to gravity

So far, we concentrated on the sigma models. These are the correct descriptions when the target space is large enough. To have topological strings, we have to couple to gravity, that is, including the integration on moduli space of the Riemann surface. In order to do so it is convenient to use the analogy with the bosonic string theory. We concentrate on the A model case. Recall that the bosonic string theory has a scalar supercharge $Q_{BRST} = Q + \bar{Q}$, anti-ghosts, $b, \bar{b}$ of spin (2,0) and (0,2) with the property

$$Q^2 = b_0^2 = 0 \tag{3.143}$$

$$\{Q, b_0\} = L_0 \tag{3.144}$$

and two $U(1)$'s, $G, \bar{G}$ corresponding to the left and right ghost numbers. We identify

$$\begin{aligned} 2j_{BRST} &\leftrightarrow G^+ \\ b &\leftrightarrow G^- \\ bc &\leftrightarrow J \end{aligned} \tag{3.145}$$

and similarly for right-movers. Then, the notion of a physical state in the bosonic string becomes exactly the same as that of a chiral primary state in the twisted theory. Therefore we can define coupling of twisted $N = 2$ theory to gravity by integrating correlation functions of chiral primary fields over moduli space of Riemann surface with the insertion of G^-'s folded with $3g - 3$ Beltrami differentials. In particular, the partition function of the twisted $N = 2$ theory coupled to gravity at genus $g > 1$ that we denote by F_g is defined by

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{k=1}^{3g-3} \int_{\Sigma_g} G^- \mu_k \int_{\Sigma_g} \bar{G}^- \bar{\mu}_k \rangle \tag{3.146}$$

where $\mu_i \sim g^{-1} \frac{\partial g}{\partial \tau_k}$ denote the Beltrami differentials, τ_k the complex moduli of the Riemann surfaces, and $\mathcal{M}_g$ denotes the moduli space of genus g Riemann surfaces. For F_1 the answer can be written using the corresponding analysis of the bosonic string case. In bosonic string, one inserts $bc\bar{b}\bar{c}$ to absorb the ghost zero modes. This is translated in the twisted theory to the insertion of left and right fermion number currents. Also, to fix the normalization it is best to write the answer in the operator formulation, which is

$$F_1 = \frac{1}{2} \int \frac{d^2\tau}{\tau_2} \mathrm{Tr}[(-1)^F F_L F_R q^{L_0} \bar{q}^{\bar{L}_0}] \tag{3.147}$$

where the factor 1/2 takes care of the fact that there is a $\mathbf{Z}_2$ reflection symmetry for al tori. For genus 0, the 0,1, and 2 point funtions are zero, as is the case in bosonic string, and the three point functions can be written as

$$\langle \phi_i \phi_j \phi_k \rangle = C_{ijk} \tag{3.148}$$

where we define C_{ijk} by the coefficient of the OPE $\phi_i \phi_j = C_{ij}^k \phi_k$ with lowering the indices by $\eta_{ij} = \langle j|i\rangle$.

It is a rather good property of twisted unitary $N = 2$ theories that F_g is finite and thus well-defined. The only potential divergence would have come from the regions near the boundary of moduli space. But in such cases, the fact that the propagator on a long tube is given by $G_0^- \frac{1}{L_0+\bar{L}_0} \bar{G}_0^-$ and that this annihilates the massless modes imply that only the massive modes propagate and the integrand in F_g is exponentially small.

There are two notable differences between bosonic string and twisted $N = 2$ theories. One is that the ghost number violation in bosonic string at genus g is universal and is given by $3g-3$, while for twisted $N = 2$ theories it is given by $\hat{c}(g-1)$, where we defined $\hat{c} = c_{\text{untwisted}}/3$. In particular, we see that $\hat{c} = 3$ is a critical case in that it gives the same degree of charge violation as bosonic string. This means that only for $\hat{c} = 3$ the F_g has a chance to be non-zero for $g > 1$. Therefore, the most interesting case is again the Calabi-Yau 3-fold. The other difference between the two even if we choose $\hat{c} = 3$ is that in the case of topological strings, the G^- cohomology is generically non-trivial whereas absolute b-cohomology is always trivial in the bosonic strings. This leads to the "holomorphic anomaly". Viewing chiral fields as the first component of a superfields, we can modify the action by perturbing with them as

$$t_i \int d^2z d^2\theta^+ \phi_i + \bar{t}_i \int d^2z d^2\theta^- \bar{\phi}_i = t^i \int d^2z \phi_i^{(2)} + \bar{t}_i \int d^2z \bar{\phi}_i^{(2)} \tag{3.149}$$

where ϕ's are chiral primary, $\bar{\phi}_i$' are anti-chiral primary, $\phi_i^{(2)} = \{G^-, [\bar{G}^-, \phi_i]\}$, and t_i are complex parameters. Since, in the topological string based on the A model, insertion of anti-chiral fields in the action modifies the theory by Q-trivial terms, we expect that the amplitude is independent of $\bar{t}_i$ therefore holomorphic in t_i. However, this is correct only in the tree level. "The holomorphic anomaly" emerges in the string loop level from the boundary of the moduli space [28]. So, in general the topological string amplitudes depend not only on t_i but also on $\bar{t}_i$. Note that from the above form of the perturbation, we see that by differentiating the genus 0 amplitude F_0 with respect to t_i's, we have the following equation

$$C_{ijk} = \frac{\partial^3 F_0}{\partial t_i \partial t_j \partial t_k}. \tag{3.150}$$

Therefore, the genus 0 partition function is the prepotential and the Yukawa coupling can be derived from it.

Relations to superstrings

We have defined the topological string theory. We want to discuss here the relations to the ordinary superstring theories [27]. One of the main motivations to study $N = 2$ superconformal field theories comes from the fact that they serve as building blocks for string vacua. In this connection, particular objects which have natural interpretations for the $N = 2$ SCFT's turm out to also have some interesting phenomenological implications in string models. One such object is the Yukawa coupling. If we consider heterotic strings compactified on a Calabi-Yau 3-fold with gauge connection identified with the spin connection of Calabi-Yau, then, the chiral primary fields of charge 1 give rise to massless generations and the chiral ring coefficients C_{ijk} give the Yukawa couplings between the different generations. The fact that Yukawa couplings are simply the three point function of topological gravity leads us to expect that all the other computations of twisted theories also have similar physical significance for an appropriate string theory. We discuss the significance of F_g in connection with standard superstring theories in the case $g \geq 1$.

We first give an idea. The massless fields in general end up as lowest component of chiral superfields from the spacetime point of view. Usually, we have superpotential terms or F-terms which involve only chiral superfields. This means that such terms depend holomorphically on the massless fields or moduli. So, if we forget the holomorphic anomaly, we expect that F_g contributes to the superpotential terms. We show this below.

We must start by asking which string theory the F_g should be related with? By considering the fact that it is left-right symmetric, and is related to the twisting of a supersymmetric sigma model for closed string theory, we are naturally led to consider the type II strings compactified on a $\hat{c} = 3$ internal theory (A-model is related to type IIA and B-model to type IIB). Therefore we are searching for low energy effective field theory terms which F_g is computing. When we compactify the type II theory on Calabi-Yau 3-folds, we have $N = 2$ supergravity theory in four dimensions. In the multiplet having graviton in it, there is a $U(1)$ gauge field called graviphoton. We denote the field strength of this field by F. This field comes from the RR sector of type II string. In the limit of vanishing momentum $k \to 0$, the vertex operator for this field is proportional to

$$V_F^{\pm\pm} = k_{\pm\pm} S^{\pm} \bar{S}^{\pm} \sigma \bar{\sigma} e^{-\phi/2} \tag{3.151}$$

here ϕ is part of the bosonized β, γ field and S denotes the left-moving 4-dimensional spinor vertex operator and σ denotes the vertex operator for the left-moving unique charge $3/2$ Ramond ground vacuum state for the internal $N = 2$ theory. The $\pm$ denote the index of the spinor. Note that σ, S and $e^{-\phi/2}$ together with their right-moving countarparts generate the spectral flow from the NS sector to the R sector. Recall that the ordinary type II strings have more fields than the twisted ones and also that ordinary strings have the fermionic diffeomorphism ghosts (b, c), the bosonic super-diffeomorphism ghosts (β, γ), the space-time fields, which we take to be two complex bosons X^i, two complex fermions ψ^i, and their conjugates $\chi_{\bar{\imath}}$, and their right-moving part. If we could twist the $1/2$ integral spin fields by half a unit, then their spins would be the same as the integral spin fields but with opposite statistics. So they would tend to cancel out of the partition function. In addition, we would need to twist the internal $N = 2$ theory which is also the same as shifting the $1/2$ integral fermion spins of the internal theory. In fact, both of these can be accomplished by insertion of $(2g - 2)$ vertex operators for gravi-photon V_F^{++}. The way to see this is that the spin content of fields can be changed by adding the following term to the action

$$\frac{1}{2} \int R\varphi \tag{3.152}$$

where φ denotes the bosonized version of the fields. We can choose the curvature R to have delta-function like support at $2g-2$ points (Recall the Gauss-Bonnet theorem). From the above arguments, this is equivalent to the insertion of V_F^{++}. To write it in a conformally meaningful way, we have to integrate it over the Riemann surface (Note that V_F^{++} has dimension (1,1) and the above procedure is equivalent to choosing the delta-function support for R by averaging over all points). We thus have

$$\left\langle \left[\int V_F^{++}\right]^{2g-2} ... \right\rangle_{\text{untwisted}} = \langle ... \rangle_{\text{twisted}} . \tag{3.153}$$

This is not the whole story. We have to take the zero modes of extra fields into consideration. There are zero modes for b, β and the ψ, χ system that have to be absorbed in order for the partition functions not to vanish.

The b zero modes give rise to the measure over moduli space. We have to insert in the superstring measure a factor of the form

$$|b(\mu_1)...b(\mu_{3g-3})|^2 \tag{3.154}$$

to absorb the b zero modes. For the β zero modes we usually have to insert $2g-2$ factors of $\delta(\beta)G$ where G is the $N=2$ supersymmetry current for the full theory. But this is true for the partition function with no operators inserted. In our case inserting $2g-2$ vertex operators V_F^{++} which are in the $-1/2$ picture means that we need to insert $3g-3$ factors of $\delta(\beta)$. Moreover the fact that β, γ is effectively twisted means that we can choose the same basis for the Beltrami differentials to fold with them. Moreover by charge conservation for the internal theory only the G^- component of the internal theory gives non-vanishing amplitude. Thus we have

$$|\delta(\beta(\mu_1))...\delta(\beta(\mu_{3g-3}))|^2 |G^-(\mu_1)...G^-(\mu_{3g-3})|^2. \tag{3.155}$$

With this choice, the zero modes of b and $\delta(\beta)$ give opposite contributions and thus b, c and β, γ completely drop out of the picture, leaving us with the twisted $N=2$ theory with $3g-3$ insertions of G^-. This is presicely the prescription we had for computing F_g of the twisted string coupled to gravity. However we still have to consider the space-time fermion zero modes. There are g of ψ^i and one of $\chi_{\bar{\imath}}$ zero modes for each i. To absorb the χ zero mode and one ψ zero mode we insert the operator

$$\epsilon_{ij}\epsilon_{i'j'}\epsilon_{\bar{\imath}\bar{\jmath}}\epsilon_{\bar{\imath}'\bar{\jmath}'} \int \psi^i \chi_{\bar{\imath}} \bar{\psi}^{i'} \bar{\chi}_{\bar{\imath}'} \int \psi^j \chi_{\bar{\jmath}} \bar{\psi}^{j'} \bar{\chi}_{\bar{\jmath}'}. \tag{3.156}$$

This is equal to the insertion of two graviton vertex operators up to factors of momentum. We are left to absorb $g-1$ extra zero modes of ψ^i. Taking into account that after twisting ψ^i has spin 1, one is tempted to introduce $g-1$ operators of the form $\int \psi^i \bar{\psi}^j$ but this does not have a well-defined meaning as a vertex operator for the untwisted theory. Instead, we insert $g-1$ operators of the form $\psi^1\psi^2\bar{\psi}^1\bar{\psi}^2$ at $g-1$ points where we have taken the delta function curvature singularity. This choice have the property of absorbing the unwanted

ψ zero modes without getting an operator which does not make sense in the untwisted theory. This is because choosing this position for the $g-1$ curvature singularities will convert $g-1$ of V_F^{++} to V_F^{--} which is the vertex for gravi-photon field with opposite self-duality property. In this way we have for $g \geq 1$

$$F_g = \left\langle \left[\int V_F^{++}\right]^{g-1} \left[\int V_F^{--}\right]^{g-1} \epsilon_{ij}\epsilon_{i'j'}\epsilon_{\bar{\imath}\bar{\jmath}}\epsilon_{\bar{\imath}'\bar{\jmath}'} \int \psi^i \chi_{\bar{\imath}} \bar{\psi}^{i'} \bar{\chi}_{\bar{\imath}'} \int \psi^j \chi_{\bar{\jmath}} \bar{\psi}^{j'} \bar{\chi}_{\bar{\jmath}'} \right\rangle. \tag{3.157}$$

Putting the momentum factors, we see, from the above expression, that F_g is the coefficient in the low energy effective action for a term of the form

$$R_+^2 (F_+^2)^{g-1}. \tag{3.158}$$

Here, $+$ means the self-dual part of the field-strength. We should expect a term in the superpotential which gives rise to the effective action of the above form. In fact, we can find an F-term which gives rise to such a term:

$$\int d^4\theta \mathcal{W}^{2g}. \tag{3.159}$$

Here $\mathcal{W}_{\mu\nu} = F_{\mu\nu}^+ - R_{\mu\nu\lambda\rho}^+ \theta\sigma^{\lambda\rho}\theta + \ldots$ is a Weyl superfield and $\mathcal{W}^2 = \mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}$. Note that the above superpotential terms receive contributions only from genus g amplitudes in the physical untwisted string theory. Moreover there are no non-perturbative string corrections. This can be seen as follows. We have considered type IIA theory compactified on a Calabi-Yau manifold to 4 dimensions. We obtained an $N=2$ supersymmetric theory. The massless moduli is in the vector or hypermultiplets. The Kähler moduli are in the vector multiplets and the complex moduli in the hypermultiplets. In particular the dilaton is in the hypermultiplet. And the term $F_g R_+^2 F_+^{2g-2}$ involves only Kähler moduli. Then, combining these, we can say that the above superpotential terms receive contributions only from genus g amplitudes in the physical string theory because the dilaton cannot mix with the vector multiplet moduli. Therefore the partition function of topological strings computes certain exact quantities in the full, physical string theory.

Some known results

The general form (or the leading term for large volume limit) of the partition functions for $g > 1$ is calculated in [27] to be

$$F_g = \frac{1}{2}\chi(X) g_s^{2g-2} \int_{\mathcal{M}_g} c_{g-1}^3 + O(\exp(-A)). \tag{3.160}$$

Here $\chi(X)$ denotes the Euler characteristic of the target space X and c_{g-1} denotes the $(g-1)$-th Chern class of the Hodge bundle over $\mathcal{M}_g$. The Hodge bundle is a holomorphic vector bundle over $\mathcal{M}_g$ whose fiber is spanned by g holomorphic 1-forms on the Riemann surface Σ_g. The leading term comes from the constant map. The constant map is a map that sends the whole Riemann surface to a point in the target Calabi-Yau space. And the other terms come from the worldsheet instantons. The worldsheet instanton is a map that maps the Riemann surface to the nontrivial curve in the Calabi-Yau manifold X. The leading term was computed by mathematicians to be [31]

$$\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{B_g}{2g(2g-2)}\frac{B_{g-1}}{(2g-2)!} = (-1)^{g-1}\chi_g \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}} \tag{3.161}$$

where $\chi_g = (-1)^{g-1}\frac{B_g}{2g(2g-2)}$ is the Euler characteristic of $\mathcal{M}_g$, and the B_g are the Bernoulli numbers. Bernoulli numbers are defined by

$$\frac{x}{e^x-1} + \frac{x}{2} - 1 = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{(2n)!}x^{2n}. \tag{3.162}$$

On the other hand, the genus 0 amplitude for Calabi-Yau target space with only one Kähler moduli is calculated in [26] to be

$$F_0 = \frac{1}{g_s^2}\left[-\frac{\chi(X)}{2}\zeta(3) - \frac{\pi^2}{6}c_2 t + i\pi a t^2 - \frac{C}{3!}t^3 + \sum_{m,n} d_m \frac{1}{n^3}e^{-nmt}\right] \tag{3.163}$$

where $\chi(X)$ is the Euler number, c_2 is a second Chern class of X (moreprecisely $c_2 t = \int K \wedge c_2$), $Ct^3 = \int_X K \wedge K \wedge K$ is a self intersection number with respect to the Kähler class, and d_m is the number of primitive holomorphic curve of degree m. Here, a is predicted for Calabi-Yau manifolds with one Kähler moduli as $a = \frac{C}{2}$ mod $\mathbf{Z}$. And genus 1 partition function for Calabi-Yau with only genus 0 holomorphic curve was computed in [28] to be

$$F_1 = -\frac{c_2}{24}t + \frac{1}{12}\sum_m d_m \ln(1-e^{-mt}). \tag{3.164}$$

These are calculated by path integral or other direct calculational method. Such computations are of course very important. However, if we want to calculate the more complicated higher genus partition functions, these methods are not useful because of the difficulty in integrating over the moduli space of the Riemann surfaces. Therefore it seems that in practice we can't know the higher genus answers. However, Gopakumar and Vafa succeeded in computing all genus partition functions from a totally different point of view. Our next task is to explain this.

3.2.5 Deriving all genus answers

Schwinger's computations

Let us go to totally different subject here, the effects of electron-positron pair creation on the Maxwell's classical action. The relation will become clear later.

Our set up is as follows. There are fermions minimally coupled to the classical constant external electric field. The interaction part of the Lagrangian is

$$\mathcal{L}_I(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \tag{3.165}$$

Our aim is to compute the probability amplitude of emitting no electron-positron pair, which we denote by S_0. Usual quantum field theory tells us that this can be computed as

$$\begin{aligned} S_0(A) &= \langle 0|S|0\rangle \\ &= \sum_{n=0}^{\infty}\frac{(-ie)^n}{n!}\int dx_1...dx_n\langle 0|T[\bar{\psi}(x_1)\not{A}(x_1)\psi(x_1)...\bar{\psi}(x_n)\not{A}(x_n)\psi(x_n)]|0\rangle \end{aligned} \tag{3.166}$$

where S is the S matrix which is given by

$$S = T\exp\left[-ie\int d^4x\bar{\psi}_{\rm int}(x)\gamma^\mu\psi_{\rm int}(x)A_\mu(x)\right]. \tag{3.167}$$

According to the Wick's theorem, each term is a sum of products of the form

$$\langle 0|T\not{A}(x_k)\psi(x_k)\bar{\psi}(x_l)|0\rangle. \tag{3.168}$$

Define the 4×4 matrix by

$$C(\alpha_k, x_k; \alpha_l, x_l) = -ie\sum_\alpha\langle 0|T[\not{A}_{\alpha_k\alpha}(x_k)\psi_\alpha(x_k)\bar{\psi}_{\alpha_l}(x_l)]|0\rangle \tag{3.169}$$

for given x_k, x_l. In terms of these, S_0 can be written

$$S_0(A) = \sum\frac{(-1)^n}{n!}\int dx_1...dx_n\sum_P\epsilon_P\sum_{\alpha_1...\alpha_n}C(\alpha_1, x_1; \alpha_{P_1}, x_{P_1})...C(\alpha_n, x_n; \alpha_{P_n}, x_{P_n}). \tag{3.170}$$

Let us regard the discrete indices α_i and the continuous variables x_i as the same footing and denote them in a bracket notation $|x,\alpha\rangle$. It is convenient to introduce the matrix Γ, which is defined to be

$$\langle x,\alpha|\Gamma|y,\beta\rangle = C(x,\alpha;y,\beta). \tag{3.171}$$

In terms of these, S_0 can be expressed as

$$\begin{aligned} S_0(A) = \mathrm{Det}(I-\Gamma) &= \exp[\mathrm{Tr}\ \ln(I-\Gamma)] \\ &= \mathrm{Det}\left[I - e\not{A}(x)\frac{1}{\not{P}-m+i\epsilon}\right] \\ &= \mathrm{Det}\left\{[\not{P}-e\not{A}(x)-m+i\epsilon]\frac{1}{\not{P}-m+i\epsilon}\right\} \\ &= \exp\left\{\mathrm{Tr}\ \ln\left[(\not{P}-e\not{A}(x)-m+i\epsilon)\frac{1}{\not{P}-m+i\epsilon}\right]\right\}. \end{aligned} \tag{3.172}$$

Therefore all we have to do is to calculate

$$\ln(S_0) = \text{Tr } \ln\left\{[\not{P} - e\not{A}(x) - m + i\epsilon]\frac{1}{\not{P} - m + i\epsilon}\right\} \tag{3.173}$$

By inserting the charge conjugation matrix C and using the fact $C\gamma_\mu C^{-1} = -\gamma_\mu^T$, we can write the above expression as

$$\ln(S_0) = \text{Tr } \ln\left\{[\not{P} - e\not{A}(x) + m - i\epsilon]\frac{1}{\not{P} + m - i\epsilon}\right\}. \tag{3.174}$$

Summing the above two equations, we have

$$2\ln(S_0) = \text{Tr } \ln\left(\{[\not{P} - e\not{A}]^2 - m^2 + i\epsilon\}\frac{1}{P^2 - m^2 + i\epsilon}\right). \tag{3.175}$$

By using the identity,

$$\ln\frac{a}{b} = \int_0^\infty \frac{ds}{s}(e^{is(b+i\epsilon)} - e^{is(a+i\epsilon)}), \tag{3.176}$$

we have

$$\begin{aligned} w(x) &= \text{Re}\int_0^\infty \frac{ds}{s} e^{-is(m^2 - i\epsilon)} \text{tr } (\langle x|e^{is(\not{P} - e\not{A}(x))^2}|x\rangle - \langle x|e^{isP^2}|x\rangle) \\ &= \text{Re tr}\int_0^\infty \frac{ds}{s} e^{-is(m^2 - i\epsilon)}\langle x|\left(\exp\left\{is\left[(P - eA(x))^2 + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}(x)\right]\right\} - e^{isP^2}\right)|x\rangle \end{aligned} \tag{3.177}$$

where we defined $w(x)$ by

$$|S_0|^2 = \exp\left[-\int d^4x w(x)\right]. \tag{3.178}$$

The meaning of $w(x)$ is clear. It is a probability density for pair creation.

For a constant background field, this cannot depend on x. Moreover, $\sigma_{\mu\nu}F^{\mu\nu}$ commutes with all the other operators and we can compute its exponential. Also, note that the spin dependence comes entirely from here and the generalization to the particle with arbitrary spin is easy. From now on, we assume that the electro-magnetic field is purely electric and that the electric field is along the z axis. In addition, we choose a gauge such that only $A^3 = -Et$ is nonzero. In this case, it is easy to see

$$\text{tr } e^{ise\sigma_{\mu\nu}F^{\mu\nu}/2} = 4\cosh(seE). \tag{3.179}$$

Using the commutation relation $[X_0, P_0] = -i$, we have

$$\begin{aligned} (P - eA)^2 &= P_0^2 - \mathbf{P}_T^2 - (P^3 + eEX^0)^2 \\ &= e^{-iP^0P^3/eE}(P_0^2 - \mathbf{P}_T^2 - e^2E^2X^{02})e^{iP^0P^3/eE}. \end{aligned} \tag{3.180}$$

Therefore,

$$\begin{aligned}
&\mathrm{tr}\langle x|e^{is[(P-eA)^2+e\sigma_{\mu\nu}F^{\mu\nu}/2]}|x\rangle \\
&\quad = 4\mathrm{cosh}(seE)\int \frac{d^3p}{(2\pi)^4}d\omega d\omega' e^{i(\omega'-\omega)(t+p^3/eE)-isp_T^2}\langle\omega|e^{is(P_0^2-e^2E^2X_0^2)}|\omega'\rangle \\
&\quad = \frac{2eE}{(2\pi)^2 is}\mathrm{cosh}(eEs)\int_{-\infty}^{\infty} d\omega\langle\omega|e^{is(P_0^2-e^2E^2X_0^2)}|\omega\rangle.
\end{aligned} \tag{3.181}$$

The last integral can be regarded as the trace of the evolution operator of the harmonic oscillator via the correspondence $P_0 \to P, -X_0 \to Q, 2ieE \to \omega_0, 1/2 \to m_0$. By using the result of ordinary harmonic oscillators, we have

$$\begin{aligned}
\mathrm{Tr}\ \exp\left[is\left(\frac{P^2}{2m_0}+\frac{m_0\omega_0^2}{2}Q^2\right)\right] &= \sum_n \exp\left[is\left(n+\frac{1}{2}\right)\omega_0\right] \\
&= \frac{i}{2\mathrm{sin}\ s\omega_0/2}.
\end{aligned} \tag{3.182}$$

Collecting all the terms, we finally get

$$w = -\frac{1}{(2\pi)^2}\int_0^{\infty}\frac{ds}{s^2}\left[eE\mathrm{cosh}(eEs)-\frac{1}{s}\right]\mathrm{Re}\ (ie^{-is(m^2-i\epsilon)}). \tag{3.183}$$

The term $1/s$ corresponds to the subtraction at $e=0$. The effective action is of course given by

$$S_0(A) = \exp\left[i\int d^4x\delta L\right]. \tag{3.184}$$

Note that in the above calculation, we did not use any approximations. Therefore the result is exact and provides us with both perturbative and non-perturbarive information. As we said, the generalization to arbitrary spins is easy. Just replace $\sigma_{\mu\nu} \to 4J_{\mu\nu}$ and take care of the sign of statistics. Therefore, the generalization is (after analytic continuation)

$$\begin{aligned}
&\int_{\epsilon}^{\infty}\frac{ds}{s}\mathrm{Tr}(-1)^F e^{-s(\Delta+m^2+2eJ_{\mu\nu}F^{\mu\nu})} \\
=&\frac{1}{4}\int_{\epsilon}^{\infty}\frac{ds}{s}\frac{1}{\sinh^2(seF/2)}e^{-sm^2}\mathrm{Tr}(-1)^F e^{-2seJ_{\mu\nu}F^{\mu\nu}}.
\end{aligned} \tag{3.185}$$

Reduction to Schwinger's computation

What we wanted to do is to calculate the partition functions. We want to reduce this problem to the calculation discussed above. Suppose we give an expectation value to the self-dual field strength $\langle F_+\rangle = F$. Then, let us ask what the coefficient of the effective action of the form

$$S_{\text{eff}} = \int R_+^2\left(\sum_g F^{2g-2}A_g\right) \tag{3.186}$$

is. The answer is of course given by the partition function; $A_g = F_g$. Let us see from the point of view of effective field theory. Consider the particles in the half hypermultiplet which have BPS charge Z. Particles in the half hypermultiplet have quantum numbers under $SO(4) = SU(2)_L \times SU(2)_R$ and transform as

$$\left[2\left(\frac{1}{2},0\right) \oplus (0,0)\right]. \tag{3.187}$$

Then, the contribution to the R_+^2 term of the effective action can be computed by integrating out the half hypermultiplets which are charged under the graviphoton. Therefore we must calculate the loop amplitudes with $2g-2$ graviphotons and 2 gravitons on the external lines. This seems difficult. But, the tremendous fact that Gopakumar and Vafa found is that the amplitude we want to calculate is equal to the more easier one. By using the result of the work of Antoniadis, Gava, Narain, and Taylor [30], they found that the amplitude is the loop amplitude with scalar particles in the loop and $2g-2$ graviphotons on the external lines [40]; i.e. that of Scwinger's! Moreover, they found that the contribution to $R_+^2 F_+^{2g-2}$ from a state with spin $[(\frac{1}{2},0) \oplus 2(0,0)] \otimes [(j_1,j_2)]$ is equal to that to F_+^{2g-2} from a particle with spin $[(j_1,j_2)]$ in the Schwinger's case. Therefore what we have to do is to generarize the calculation discussed before and substituting the correct charge and mass to the result. In the next section, we will compute the amplitudes in this way but from a different point of view.

Calculation from M theory point of view

Let us turn our viewpoint to the case when the string coupling constant becomes large. In this case, it is suitable to think that we consider the M theory compactified on $CY_3 \times S^1$. The lightest objects are, in the type IIA terms, the D0 branes, their bound states, and D2 branes wrappped over the 2-cycles in the Calabi-Yau manifold. Let us set $\alpha' = \frac{1}{4\pi^2}$.

What we want to do is to show that the particles in the loop are these lightest D branes. The bound states of D0 branes always give the half hypermultiplets and therefore we can reduce to the scalar case. But the D2 branes are in general not so and we must consider the particles with higher spin. Though there is no problem in such cases, we want to avoid them for simplicity. Therefore we assume that the 2-cycle over which we wrap the D2 brane is topologically S^2. This condition assures that there is no bound states of multiple D2 branes wrapped over 2-cycles and there is only bound states of a single D2 brane with a number of D0 branes. This means that not only the bound states of D0 branes but also the bound state of a wrapped D2 brane with D0 branes gives the half hypermultiplet and the computation is reduced to the scalar case of Schwinger's. If this assumption is not obeyed, we have to use the Schwinger's result in the higher spin case. This point is discussed in [40].

Let us first consider the D0 branes. The bound states of the D0 branes are BPS states and the charge Z is given by

$$Z = \frac{2\pi i n}{g_s}. \tag{3.188}$$

The Schwinger's result for the scalar particle with BPS charge Z in a constant self-dual field F is

$$\mathcal{F}(Z) = \int_\epsilon^\infty \frac{ds}{s^3}\left(\frac{s/2}{\sinh s/2}\right)^2 e^{-\frac{sZ}{F}} \tag{3.189}$$

as we noted before. Here we define $\mathcal{F}(Z)$. This has a perturbative expansion of the form

$$\mathcal{F}(Z) = \sum_g F'_g F^{2g-2} + O(e^{-\frac{Z}{F}}) \tag{3.190}$$

where F'_g is equal to the partition function if we choose the particles correctly. Direct computation shows that $F'_g = -\chi_g Z^{2-2g}$. where χ_g is the Euler characteristic of the Riemann surface. We claim that the bound states of D0 branes and the bound states of a D2 brane with many D0 branes give the correct answer. To see this, let us substitute $Z = \frac{2\pi i n}{g_s}$ for F'_g and sum over n.

$$F'^{(0)}_g = -\chi_g \sum_{n\neq 0\in \mathbf{Z}} \left(\frac{2\pi i n}{g_s}\right)^{2-2g}. \tag{3.191}$$

This is equal to

$$-\left[(-1)^{g-1}\chi_g \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}}\right] g_s^{2g-2} \tag{3.192}$$

where we used $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$. This is the result for a single hypermultiplet. We must take into account all the multiplets for a fixed Kaluza-Klein momentum around the circle (for fixed n). The net contribution is known to be $h^{2,1}(X) - h^{1,1}(X) = -\chi(X)/2$ times the contribution of a single hypermultiplet. The derivation of this fact is not so easy, see [40]. Therefore the final answer for the contribution from the D0 branes is

$$F'^{(0)}_g = \frac{\chi(X)}{2}\left[(-1)^{g-1}\chi_g \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}}\right] g_s^{2g-2}. \tag{3.193}$$

Let us compare this result with the contribution from the constant map, the equation (3.160). We easily see that these are exactly the same. Therefore we can reinterpret the contribution from the constant map as the one from the bound states of the D0 branes.

We next discuss the contribution from the D2 brane. The BPS charge of the bound states of a D2 brane with n D0 branes is also determined by BPS property to be

$$Z = \frac{2\pi(A + in)}{g_s}, \tag{3.194}$$

where A denotes the area of the surface over which the D2 brane is wrapped. Note again that from our assumption, we have only one D2 brane and no multiwrapped D2 brane.

Following the D0 brane case, we have

$$
\begin{aligned}
F_g^{\prime(2)} &= -\frac{\chi_g g_s^{2g-2}}{(2\pi)^{2g-2}} \sum_{m\in\mathbf{Z}} (A+im)^{2g-2} \\
&= -\frac{\chi_g g_s^{2g-2}}{(2\pi)^{2g-2}(2g-3)!} \left(\frac{d}{dA}\right)^{2g-2} \ln(1-e^{2\pi A}) \\
&= \chi_g g_s^{2g-2} \frac{1}{(2g-3)!} \sum_m m^{2g-3} e^{-2\pi m A},
\end{aligned} \tag{3.195}
$$

where we use the following product formula

$$
\frac{\sinh(\pi x)}{\pi x} = \prod_{n=1}^{\infty} \left(1+\frac{x^2}{n^2}\right). \tag{3.196}
$$

For the genus 0 and 1, this expression should be interpreted in the regularized form $\chi_g/(2g-3)! = 1, -1/12$ respectively. This expression is known to agree for $g = 1, 2$ with the contribution from degenerate genus zero instantons and generalizes it to an arbitrary genus amiplitude [32]. In this way, not only we rederive the known results for topological string amplitudes, but also we can get the all genus answer in a simple way. In summary, we have

$$
F_0 = \frac{1}{g_s^2}\left[-\frac{\chi(X)}{2}\zeta(3) - \frac{\pi^2}{6}c_2 t + i\pi a t^2 - C\frac{t^3}{3!} + \sum_{n,m} d_m \frac{1}{n^3}\exp(-nmt)\right] \tag{3.197}
$$

$$
F_1 = -\frac{c_2}{24}t + \frac{1}{12}\sum_m d_m \ln(1-e^{-mt}) \tag{3.198}
$$

$$
F_g = g_s^{2g-2}\left[(-1)^{g-1}\chi_g \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}} - \frac{\chi_g}{(2g-3)!}\sum_n n^{2g-3}e^{-nt}\right]. \tag{3.199}
$$

3.3 Topological open strings

We have discussed the topological closed string. We started from $N = 2$ SCFT, twisted it to the topological field theory, and coupled it to gravity. Then we have found the relation to untwisted models, and by using that property we could get all genus answers. In this section, we would like to discuss the topological open string. The A model and the B model again play fundamental roles. But in the open strings, the world sheet has nontrivial boundaries. Therefore we must impose suitable boundary conditions. Then we will couple these systems to space-time gauge fields in the sense of string theory (we assume the coupling to two dimensional gravity implicitly). Again we will concentrate on the case in which the target space is a Calabi-Yau 3-fold. After investigating the models, we will find the interesting phenomena again. The main goal of this section is to demonstrate that under certain conditions, the A model with Calabi-Yau target space is equivalent to the Chern-Simons theory on the certain 3-dimensional submanifold in the Calabi-Yau. This is the work of Witten [61].

3.3.1 Boundary conditions

As we said above, we want to study the A model and the B model on Riemann surfaces Σ with boundaries with emphasis on the A model. The action and the transformation laws are the same as before. Therefore, we have only to consider the boundary conditions. Let each component of $\partial\Sigma$ be C_i. And let M_i be a special Lagrangian submanifold of the Calabi-Yau manifold. The definition of the special Lagrangian submanifold is as follows. A 3-dimensional submanifold M in the Calabi-Yau 3-fold X is a special Lagrangian submanifold if and only if the following conditions are satisfied

$$i^*(\Omega) = \Omega|_M, \quad i^*(K) = 0 \tag{3.200}$$

where $i : M \hookrightarrow X$ is an inclusion, Ω is a holomorphic 3-form on the Calabi-Yau and K is a Kähler form. And we denote the tangent bundle and normal bundles to M in X by TM and NM respectively. We regard them as the real and imaginary subbundles of $TX|_M$.

In the A model, regarding C_i as a real locus in the complex Riemann surface, we impose the following boundary conditions.

1. $\Phi|_{C_i}$ maps C_i to M, that is $\Phi|_{C_i}$ is real.

2. The normal derivative of Φ at C_i is imaginary, or it takes values in $\Phi^*(NM)$.

3. χ and the pullback of ψ to C_i are real, that is they take values in $\Phi^*(TM)$.

These are the boundary conditions for the A model. Above boundary conditions mean that we impose the Dirichlet boundary conditions because the boundary of the Riemann surface must be mapped to certain fixed manifold. Therefore, we have "D-branes" on M. As is well-known, we cannot put D-branes in an arbitrary place. The known result is that we must impose the SUSY cycle condition to put the D-branes. In our case, this condition requires M to be special Lagrangian submanifold. This is the reason we introduce this object. If we couple this model to gauge fields by introducing $U(N)$ Chan-Paton factors, we can interpret this as being N D-branes which are wrapped over M.

In the B model, we pick boundary conditions that do not require anything similar to the choice of the M. For the B model, we impose the following conditions.

1. The normal derivative of Φ is zero on $\partial\Sigma$.

2. θ vanishes on $\partial\Sigma$.

3. The pullback to $\partial\Sigma$ of $\star\rho$ vanishes. Here $\star$ is the Hodge star operator.

3.3.2 Coupling to gauge fields

We would like to couple the above systems to gauge fields on X. This can be done by introducing Chan-Paton factors. We will concentrate on the oriented Riemann surfaces.

Therefore we take the gauge group to be $U(N)$. This condition turns out to be essential later.

Before coupling to gauge fields, what we are interested in is the path integral

$$\int D\Phi_i \exp(-S(\Phi_i)) \tag{3.201}$$

where Φ_i are the various fields and S is the action. Then, what happens when we couple this to gauge fields? To see this, let $A = A_I d\phi^I$ be a gauge field of the gauge group $U(N)$. Note that the worldsheet Σ is oriented and the orientation of Σ induces the orientation of C_i. Therefore for a given $\Phi : \Sigma \to X$, we can define the following object (the holonomy of $\Phi^*(A)$ around C_i)

$$\mathrm{Tr} P \exp \oint_{C_i} \Phi^*(A) \tag{3.202}$$

where the trace is taken in the fundamental N dimensional representation of $U(N)$. Then, the coupling to gauge fields can be done by replacing the above path integral by

$$\int D\Phi_i \exp(-L(\Phi_i)) \prod_i \mathrm{Tr} P \exp \oint_{C_i} \Phi^*(A). \tag{3.203}$$

To be true, this expression is for the A model. The B model requires some modification, which we will discuss later. Therefore we first discuss the A model case. We must check whether this procedure preserves the fermionic symmetry of the theory. In general, the variation of the trace of the holonomy is given by

$$\delta \mathrm{Tr} P \exp \oint_C \Phi^*(A) = \mathrm{Tr} \oint_C \delta\phi^I \frac{d\phi^J}{d\tau} F_{IJ}(\tau) d\tau \cdot P \exp \oint_{C;\tau} \Phi^*(A). \tag{3.204}$$

Here, τ is a coordinate on C and $\exp \oint_{C;\tau} \Phi^*(A)$ is the holonomy of $\Phi^*(A)$ around C starting and ending at τ. F_{IJ} are of course the pullback of the space-time field strength $F_{IJ} = dA + A \wedge A$ by Φ. The transformation law of the A model is $\delta\phi^I = i\alpha\chi^I$. Therefore, the above expression is invariant under the fermionic symmetry if and only if the spacetime field strength vanishes. So, in the A model, it is possible to couple only to flat connections on M. We will later see that the target space physics is equivalent to the Chern-Simons gauge theory on M. The classical solutions of the Chern-Simons gauge theory are the flat connections. (The action of the Chern-Simons gauge theory is defined by $S = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$). This corresponds to the fact that in the ordinary string theory, the background fields that can be incorporated in the worldsheet theory are always classical solutions of the spacetime theory.

As for the B model, we can discuss in a similar way. But, as we mentioned, we must modify some expressions. Substituting the transformation law $\delta\phi^i = 0, \delta\phi^{\bar{\imath}} = i\alpha\eta^{\bar{\imath}}$ for the equation (3.203), we find that the holonomy is invariant under the symmetry if and only if the (1,1) and (0,2) parts of the field strength is zero. The vanishing of the (1,1) part of the field strength turns out to be too restrictive and we want to relax it. This can be done as follows. We replace the $\Phi^*(A)$ with the following new connection

$$\tilde{A} = \Phi^*(A) - i\eta^{\bar{i}} F_{\bar{i}j}\rho^j. \tag{3.205}$$

In the same way as before, we can show that for any circle C, the trace of the holonomy

$$\mathrm{Tr}P \exp \oint_C \tilde{A} \tag{3.206}$$

is invariant under the fermionic symmetry if and only if the (0,2) part of the field strength of A vanishes. Therefore, the coupling to the gauge fields should be modified in the B model to

$$\int D\Phi_i \exp(-S(\Phi_i)) \cdot \prod_i \mathrm{Tr}P \exp \oint_{C_i} \tilde{A}. \tag{3.207}$$

On the other hand, the fact that the (0,2) part of the field strength vanishes means that the (0,1) part of the gauge field A defines a holomorphic structure on the bundle E. Here E is a rank N complex vector bundle over X with structure group (or gauge group) $U(N)$ on which the connection A is defined.

3.3.3 Large t limit

We discuss here the large t limit or equivalently the large radius limit. We discuss in terms of the Hamiltonian description. Let us take Σ to be an infinite strip with coordinate σ, τ $(0 \leq \sigma \leq \pi, -\infty \leq \tau \leq \infty)$ and metric $ds^2 = d\sigma^2 + d\tau^2$.

A model

Recall that there are scalars ϕ, χ and the 1-form ψ. If we write $\psi = \psi_\sigma d\sigma + \psi_\tau d\tau$, then the self-duality condition relates ψ_σ and ψ_τ. Therefore we can regard χ and ψ_τ as the independent fermi fields. The canonical commutation relations are

$$\left[\frac{d\phi^I}{d\tau}(\sigma), \phi^J(\sigma')\right] = -\frac{i}{t} g^{IJ}\delta(\sigma - \sigma') \tag{3.208}$$

$$\{\psi_\tau(\sigma), \chi(\sigma')\} = \frac{1}{t}\delta(\sigma - \sigma'). \tag{3.209}$$

The Hilbert space consists of functionals $\mathcal{A}(\Phi, ...)$ where now Φ is a map from the interval $I = [0, \pi]$ to X whose restriction to ∂I is a map from ∂I to M. The Hamiltonian is given by

$$L_0 = \int_0^\pi d\sigma T_{00}. \tag{3.210}$$

From the canonical commutation relations, we can write $d\phi/d\tau$ in terms of $\delta/\delta\phi$. Then the Hamiltonian can be written as

$$L_0 = \frac{1}{2}\int_0^\pi \left(-\frac{1}{t}g^{IJ}\frac{\delta^2}{\delta\phi^I\delta\phi^J(\sigma)} + tg_{IJ}\frac{d\phi^I}{d\sigma}\frac{d\phi^J}{d\sigma}\right) + \text{terms with fermions.} \tag{3.211}$$

Recall that in the topological theories, the energy-momentum tensor can be written as

$$T_{\alpha\beta} = \{Q, b_{\alpha\beta}\} \tag{3.212}$$

for some field $b_{\alpha\beta}$. If we define the zero mode of the b field by

$$b_0 = \int_0^\pi d\sigma b_{00}, \tag{3.213}$$

then we have

$$L_0 = \{Q, b_0\}. \tag{3.214}$$

This implies that the Q cohomology can be computed in the subspace of the Hilbert space annihilated by L_0. Since the physics is independent of t, it is suffice to study the kernel of L_0 for large t.

By looking at the Hamiltonian, we see that such objects are localized near $d\phi^I/d\sigma = 0$ or the constant maps. As our boundary condition is such that ∂I is mapped to M, the constant map is in fact a map from I to M. Since the contribution to L_0 from the non-zero fermion modes is of order 1, the low-lying eigenfunctions of L_0 is described by the functional of the bose and fermi zero modes only. The other modes are in their Fock vacuum and the energy of this Fock vacuum is zero from the supersymmetry of the untwisted model.

From the above arguments, the bose and fermi zero modes are all tangent to M. Let us denote them as $q^a, \chi^a, {\psi_\tau}^a$, $a = 1, 2, 3$, where q^a are coordinates on M. The canonical commutation relations enable us to represent ${\psi_\tau}^a$ as $\partial/\partial\chi^a$. Therefore, the functional $\mathcal{A}$ reduces to a function $\mathcal{A}(q^a, \chi^a)$. If we expand in powers of χ^a, we have

$$\mathcal{A} = c(q) + \chi^a A_a(q) + \chi^a\chi^b B_{ab}(q) + ... \tag{3.215}$$

The successive terms can be interpreted as p-forms on M because of the fermi statistics. When we introduce the Chan-Paton factors, these differential forms become matrices that is, the differential forms with values in End(E), where End(E) consists of the endomorphisms of the flat vector bundle over M.

Now look at the commutation relations

$$[Q, \phi^I] = -\chi^I, \quad \{Q, \chi^J\} = 0. \tag{3.216}$$

If we interpret χ^I as $-d\phi^I$, we can identify Q with the exterior derivative d on M. Therefore, the Q cohomology is equal to $H^*(M, \mathrm{End}(E))$. Moreover, the form of the Hamiltonian shows that when acting on the differential forms on M, the L_0 reduces to

$$L_0 = \frac{\pi}{2t}\Delta \tag{3.217}$$

where $\Delta = dd^* + d^*d$ is the Laplacian. From the relation $L_0 = \{Q, b_0\}$ we can see that b_0 reduces to

$$b_0 = \frac{\pi}{2t} d^*. \tag{3.218}$$

B model

In the B model there are scalars ϕ, θ, η and 1-form ρ. From the similar argument in the A model, the functional reduces to a function of zero modes of ϕ^I and $\eta^{\bar{\imath}}$ (the zero modes of ρ is represented as $\partial/\partial\eta$). As in the A model, we expand

$$\mathcal{A}(\phi^I, \eta^{\bar{\imath}}) = c(\phi^I) + \eta^{\bar{\imath}} A_{\bar{\imath}}(\phi^I) + \eta^{\bar{\imath}}\eta^{\bar{\jmath}} B_{\bar{\imath}\bar{\jmath}}(\phi^I) + ... \tag{3.219}$$

The successive terms are again interpreted as $(0, q)$ forms on X with values in $\mathrm{End}(E)$. Where now E is a holomorphic vector bundle over X. From the commutation relations

$$\begin{aligned} [Q, \phi^i] &= 0 \\ [Q, \phi^{\bar{\imath}}] &= -\eta^{\bar{\imath}} \\ \{Q, \eta^{\bar{\imath}}\} &= 0, \end{aligned} \tag{3.220}$$

we can identify Q with the $\bar{\partial}$ operator if we regard $\eta^{\bar{\imath}}$ as $-\bar{\partial}\phi^{\bar{\imath}}$. Therefore the Q cohomology is $H^{0,*}(X, \mathrm{End}(E))$. L_0 and b_0 reduce to

$$L_0 = \frac{\pi}{2t}(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) \tag{3.221}$$

$$b_0 = \frac{\pi}{2t}\bar{\partial}^*. \tag{3.222}$$

One might wonder the absence of θ_i. But, θ_i do not play any role in the above discussion since $\delta\phi^i = 0$.

3.3.4 Space-time physics

In this section, we study the space-time physics of the models costructed above by using the open string field theory. We will find that the A model with target space T^*M is equivalent to the Chern-Simons gauge theory on M.

Open string field theory

Let us first construct the open string field theory. We have string fields $\mathcal{A}$ with ghost number 1, a multiplication $\star$ and the BRST operator Q with ghost number 1 and $Q^2 = 0$. There is also a linear functional $\int$ of ghost number -3. This means that $\int b$ is zero unless b has ghost number 3. The conditions imposed on $\int$ and $\star$ are as follows

$$\int a \star b = (-1)^{\deg(a)\deg(b)} \int b \star a \tag{3.223}$$

$$\int Qb = 0 \tag{3.224}$$

where deg(a) is a ghost number of a. Then the action for the open string field theory is

$$S = \frac{1}{2}\int \left(\mathcal{A} \star Q\mathcal{A} + \frac{2}{3}\mathcal{A} \star \mathcal{A} \star \mathcal{A}\right). \tag{3.225}$$

This action is invariant under the gauge tranformation defined by

$$\delta\mathcal{A} = Q\epsilon - \epsilon \star \mathcal{A} + \mathcal{A} \star \epsilon. \tag{3.226}$$

We can also introduce the Chan-Paton factors. We have only to replace the string fields with the fields having values in matrtices and $\int$ with $\int \otimes$Tr. Here Tr is the usual trace on matrices. If a suitable reality condition is imposed, $\mathcal{A}$ take values in $N \times N$ hermitian matrices which is nothing but the Lie algebra of $U(N)$. The fact that integration law is of ghost number -3 comes from the fact that the Euler characteristic of a disk is -1 and the ghost number of the vacuum is $-3\chi(\Sigma)$ in the critical string theory. We take a topological sigma model with a Calabi-Yau target space X of dimension 6 as a world sheet theory. And we identify the multiplication and integration with standard gluing operations, and Q with the BRST operator of the topological field theory. Let us first study the A model.

Space-time physics of the A model

The boundary conditions are the same as before. When we studied a particular surface, we introduced a separate special Lagrangian submanifold M_i to which each component C_i is mapped. In string field theory, one generates all possible Σ's through Feynmann diagram expansions. Therefore the M_i's must be built in universally at the outset. We will do this by picking a single M once and for all.

Let us summarize what we will show here. A neighborhood of M in X is known to be equivalent topologically to a neighborhood of M in its cotangent bundle T^*M. Roughly speaking, the topological string theory consists of two pieces. One is the instantons with target space X and boundary values in M. And the other is the Chern-Simons theory with target space M. The instantons can be suppressed if we replace X with T^*M as will be proved below. Therefore, we will concentrate on the case $X = T^*M$.

We will show the vanishing thoerem explained just now. The claim is that the instantons mapping Σ to T^*M with $\partial\Sigma$ to M are constant. T^*M has the symplectic structure ω which can be written as $\omega = \sum dp_a \wedge dq^a$ with p_a linear coordinates in the fibers that vanish on M. This can be written as $\omega = d\rho$ where $\rho = \sum p_a dq^a$ vanishes on M. From this, we can read $g_{i\bar{j}} = g_{\bar{j}i} = -i\omega_{i\bar{j}}$. Of course, $g_{ij} = g_{\bar{i}\bar{j}} = 0$ as usual. An instanton is defined to be a map $\Phi : \Sigma \to X$ with $\bar{\partial}\phi^i = 0$. The bosonic part of the action is

$$\begin{aligned} I &= i\int_\Sigma dz \wedge d\bar{z} g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J. \\ &= 2i\int_\Sigma dz \wedge d\bar{z} g_{\bar{i}j}\partial_z\phi^{\bar{i}}\partial_{\bar{z}}\phi^j - i\int_\Sigma dz \wedge d\bar{z} g_{\bar{i}j}(\partial_z\phi^{\bar{i}}\partial_{\bar{z}}\phi^j - \partial_z\phi^j\partial_{\bar{z}}\phi^{\bar{i}}) \\ &= 2i\int_\Sigma dz \wedge d\bar{z} g_{\bar{i}j}\partial_z\phi^{\bar{i}}\partial_{\bar{z}}\phi^j + \int_\Sigma \Phi^*(\omega). \end{aligned} \tag{3.227}$$

The first term of the last form vanishes for instantons. Therefore for instantons, the action reduces to the second one. But this term is also zero in our special case. This is because

$$\int_\Sigma \Phi^*(\omega) = \int_{\partial\Sigma} \Phi^*(\rho) = 0 \tag{3.228}$$

where we used the fact that $\Phi(\partial\Sigma) \in M$. From the original form of I, we see that it vanishes only for constant maps, which completes the proof.

Now we study the spacetime physics. Recall that the low lying modes can be described as functions of the form $\mathcal{A}(q^a, \chi^b)$. In the string case, we need the fields with ghost number 1. Therefore the fields must be linear in χ. Then we have the expansion

$$\mathcal{A} = \chi^a A_a(q). \tag{3.229}$$

Recall also that in the large t limit, Q can be identified with the exterior derivative d. Therefore, in the limit $t \to \infty$, the first part of the string field action $\frac{1}{2}\int \mathcal{A} \star Q\mathcal{A}$ reduces to

$$\frac{1}{2}\int_M \mathrm{Tr} A \wedge dA. \tag{3.230}$$

We next consider the cubic part of the action. Let $A^{(j)}$ be a mode of the gauge field A. The corresponding vertex operator is given by $V^{(j)} = \chi^a A_a^{(j)}(q)$. Let us evaluate the coupling of three these modes on the disk by inserting three vertex operators which are put on the boundary of the disk. By using the $SL(2,\mathbf{R})$ symmetry, we have only to consider $\langle V^{(1)}(0)V^{(2)}(1)V^{(3)}(\infty)\rangle$. As we saw, the path integral reduces to an integral over zero modes in the large t limit. Therefore the expression is equal to

$$\int dq^1 dq^2 dq^3 d\chi^1 d\chi^2 d\chi^3 \mathrm{Tr}\chi^a A_a^{(1)}(q)\chi^b A_b^{(2)}(q)\chi_c^{(3)}(q) = \int_M \mathrm{Tr} A^{(1)} \wedge A^{(2)} \wedge A^{(3)}. \tag{3.231}$$

In summary, we have

$$\begin{aligned} S &= \frac{1}{2}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \\ &= \frac{1}{2}\int_M d^3q \mathrm{Tr}\left(\epsilon^{\mu\nu\lambda}(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda)\right). \end{aligned} \tag{3.232}$$

This is precisely the action for the Chern-Simons gauge theory. In the ordinary string theory, the above reduction is only an approximation because of the corrections of order α'. But in the topological string case, the above reduction is exact. The corrections corresponding to the $O(\alpha')$ corrections are perturbative corrections in $1/t$. But, as we saw, the topological string is independent of t. Therefore there are no such terms. Combining with the fact that the instantons are also absent, we see that the reduction we have done is exact. The perturbative calculations of both sides are also discussed in [61].

Space-time physics of the B model

We will be brief because the essential points are the same as the A model. The low-lying modes are functions $\mathcal{A}(\phi^I, \eta^{\bar{\imath}})$ of the zero modes. For the field $\mathcal{A}$ to have ghost number zero, it must be linear in η. Therefore the expansion is

$$\mathcal{A} = \eta^{\bar{\imath}} A_{\bar{\imath}}(\phi^I). \tag{3.233}$$

So, the physical field is a 1-form of type $(0,1)$. The linearized gauge transformation $\delta\mathcal{A}$ reduces to $\delta A \bar{\partial}\epsilon$. Therefore, we should regard A as the $(0,1)$ part of a connection on E. Recall that for the B model to be able to be coupled to gauge fields, it is necessary for the (0,2) part of the field strength vanishes. Therefore, we expect that the space-time action has the classical solution with this property. In fact, arguments similar to the A model case shows that the action can be reduced to

$$S = \frac{1}{2}\int_X \Omega \wedge \mathrm{Tr}\left(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A\right). \tag{3.234}$$

where Ω is an everywhere non-zero holomorphic 3-form on X.

Some comments

We should note one more point. In the ordinary string theory, we must consider the closed strings, too. However the simplification also occurs in this respect. Witten also showed in his paper [61] that the on-shell coupling between the open string and the closed string is zero. Therefore, in the topological theory, we can consider the physics of the open string directly without taking any limits as was done by Maldacena in the AdS/CFT correspondence [35].

3.3.5 Relations to superstring

In the closed string section, we have seen that topological closed string tells us about certain amplitude of the untwisted theory. We discuss here the open string version of this. The natural candidate is of course the type I theory. This is indeed so. We concentrate on the genus 0 oriented case. The idea is the same as the closed string case. The twisting is done by the gaugino field, which we denote by $V_\Psi^\pm$. This operator is again the spectral flow operator in the internal $N=2$ theory combined with the twisting operator acting on superconformal ghosts and space-time fermionic fields. This operator is again inserted where the world sheet has curvature singularity. Consider a cylinder with two boundaries and $h-2$ slits cut on it. This gives curvature singularities at the two end points of each of the slits. Therefore, we should insert the operator V_Ψ at each such point. Then, the twisting is done. After taking into account the zero modes, we have

$$F_{0,h} = \left\langle \left[\oint_{S_i} V_\Psi^+ \oint_{S_i} V_\Psi^-\right]^{h-2} \oint_S V_F \oint_S V_F \right\rangle_{\text{untwisted}}, \tag{3.235}$$

where S_i are slits on the cylinder, S is the one of the boundaries of the cylinder and V_F is the vertex operator for a gauge field (up to momentum factors). This gives the term of the form

$$\mathrm{Tr}F^2[\mathrm{Tr}\Psi^2]^{h-2}. \tag{3.236}$$

This should be written as an F term. There is indeed such superpotential term, which is

$$\int d^2\theta[(\mathrm{Tr}W^2)]^{h-1} \tag{3.237}$$

where W is the chiral superfield with gaugino as its bottom component. Similarly the higher genus amplitudes calculate the terms of the form

$$\int d^2\theta \mathcal{W}^{2g}[\mathrm{Tr}W^2]^{h-1}. \tag{3.238}$$

3.4 Duality of topological strings

In this section we discuss the topological version of the duality between open and closed strings. A more precise statement is that the $SU(N)$ Chern-Simons theory on S^3 is exactly dual to the topological closed string on the resolved conifold. This can be considered as a duality between the topological open string and the topological closed string, because the Chern-Simons gauge theory on S^3 can be realized as an open string on T^*S^3. This duality was proposed by Gopakumar and Vafa [41].

3.4.1 Open/Closed duality

Suppose we want to calculate the free energy or effective action by large N expansion. Our notations are as follows. We denote the coupling constant by κ, and we assume that the action is of the form $S \sim \frac{1}{\kappa}\int L$. We should sum over all the diagrams which are so similar to the open string diagrams, that is, the Feynman diagrams with the double line notation. There are N fermions in each loop. Therefore the power of N is given by the number of loops, or number of holes h. From the normalization of the kinetic term, each edge gives a factor of κ and each vertex gives $1/\kappa$. Therefore free energy can be expanded as

$$F = \sum N^h \kappa^{E-V} f(h, E, V) \tag{3.239}$$

where f is some function. By using the formula $V - E + h = 2 - 2g$, we can rewrite this as

$$F = \sum_{g=0,h=0}^{\infty} C_{g,h} N^h \kappa^{2g-2+h}. \tag{3.240}$$

We usually consider the large N limit by fixing the 't Hooft coupling $\lambda = \kappa N$. In terms of this we can write as

$$F = \sum_{g=0,h=1}^{\infty} C_{g,h} N^{2-2g} \lambda^{2g-2+h}. \tag{3.241}$$

Because of the expansion in genus and holes, we can regard this expression as a perturbative expansion of the open string theory. Then, the fact that the Chern-Simons gauge theory is equivalent to the open topological string theory means that the coefficient $C_{g,h}$ is given by the partition function of the open string of the Riemann surface with g genus and h holes, that is, $C_{g,h} = F_{g,h}$.

When the 't Hooft coupling λ is small, the open string description is good because of the factor λ^{2g-2+h}. However, as the 't Hooft coupling λ becomes larger, this description is getting worse. We must sum over all holes. After this process, the resulting expression should be of the form

$$F = \sum_{g=0} N^{2-2g} F_g(\lambda). \tag{3.242}$$

It is natural to interpret this expression as a closed string expansion. Therefore, we expect that the open string theory has its closed string dual pair. One example of this realization is the AdS/CFT correspondence. The AdS/CFT correspondence states that the $\mathcal{N} = 4$ SYM on N D3 branes in the flat space $\mathbf{R}^{10}$ is dual to the closed string propagating on $AdS_5 \times S^5$. In the same spirit, we want to find the dual pair of the topological open string theory on T^*S^3. In the AdS/CFT case, the open string is in flat space, but the closed string is in the curved space. Therefore, we expect that the closed string we are looking for is in the different geometry from T^*S^3. Then, what is the geometry on which the closed string propagates? As we saw in the section on flop, the cotangent bundle T^*S^3 is a deformed conifold. Therefore natural candidate is the resolved conifold which corresponds to turning on the FI term r. Therefore, we claim that

Topological Open String on $T^*S^3 \leftrightarrow$ Topological Closed String on resolved conifold.

We show below that this is indeed the case.

3.4.2 Proof of duality

We will prove the duality stated above by demonstrating that the partition functions of both sides exactly coincide. The S^2 resolved conifold has only one S^2 as an element of the nontrivial homology. This means that the Schwinger's calculation can be applied to the closed string side without any generalizations. On the other hand, the exact partition function of the Chern-Simons gauge theory on S^3 is also known. Therefore, the checks of the claim can be done in a straightforward way. Let us first discuss the Chern-Simons side.

Partition functions of Chern-Simons

The partition function of the Chern-Simons theory on a three manifold M is defined by

$$Z[M,N,k] = \int DA \exp\left(\frac{ik}{4\pi}\int_M \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\right). \tag{3.243}$$

The exact partition function on S^3 with gauge group $SU(N)$ is known to be

$$Z[S^3,N,k] = e^{\frac{i\pi}{8}N(N-1)}\frac{1}{(N+k)^{N/2}}\sqrt{\frac{N+k}{N}}\prod_{j=1}^{N-1}\left\{2\sin\left(\frac{j\pi}{N+k}\right)\right\}^{N-j}. \tag{3.244}$$

The bare 't Hooft coupling is $\lambda_b = \frac{2\pi N}{k}$. However, it is known that there is a finite renormalization of the coupling constant. The renormalized 't Hooft coupling is $\lambda = \frac{2\pi N}{k+N}$. This is the true parameter by which we should expand. Then, the result is

$$\begin{aligned} F(S^3,N,\lambda) &= -\frac{N}{2}\ln N + \frac{N-1}{2}\ln\lambda \\ &\quad + \sum_{j=1}^{N-1}(N-j)\left[\ln j + \ln\left(\frac{\lambda}{2\pi N}\right) + \sum_{n=1}^{\infty}\ln\left(1-\frac{j^2\lambda^2}{4\pi^2 n^2 N^2}\right)\right] \end{aligned} \tag{3.245}$$

where we defined $Z = e^{-F}$. We focus on the last term $\tilde{F} = \sum_j (N-j)\sum_n \ln(1-\frac{j^2\lambda^2}{4\pi^2n^2N^2})$. By expanding the logarithm and summing over n, we have

$$\tilde{F} = \sum_{m=1}^{\infty}\frac{\zeta(2m)}{m}\left(\frac{\lambda}{2\pi N}\right)^{2m}\sum_{j=1}^{N-j}(N-j)j^{2m} \tag{3.246}$$

where we have used the definition $\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z}$. The sum over j can be carried out with the result

$$\sum_{j-1}^{N-j}(N-j)j^{2m} = \frac{N^{2m+2}}{(2m+1)(2m+2)} + 2m\sum_{g=1}^{m}\binom{2m-1}{2g-3}(-1)^g\frac{B_g}{2g(2g-2)}N^{2m+2-2g}. \tag{3.247}$$

Then, with the replacement $m = g-1+p$, the result becomes

$$\begin{aligned} \tilde{F} &= \sum_{p=2}^{\infty}N^2\frac{\zeta(2p-2)}{p-1}\frac{(\frac{\lambda}{2\pi})^{2p-2}}{2p(2p-1)} - \sum_{p=1}^{\infty}B_1\frac{\zeta(2p)}{2p}\left(\frac{\lambda}{2\pi}\right)^{2p} \\ &\quad + \sum_{g=2}^{\infty}N^{2-2g}\frac{(-1)^g B_g}{2g(2g-2)}\sum_{p=1}^{\infty}\zeta(2g-2+2p)\binom{2g-3+2p}{2p}\left(\frac{\lambda}{2\pi}\right)^{2g-2+2p}. \end{aligned} \tag{3.248}$$

In this way, we have the partition function for every genus. The result is

$$F_0(\lambda) = \frac{3}{4} - \frac{1}{2}\ln\lambda + \sum_{p=2}^{\infty}\frac{\zeta(2p-2)}{p-1}\frac{(\frac{\lambda}{2\pi})^{2p-2}}{2p(2p-1)} \tag{3.249}$$

$$F_1(\lambda) = -\sum_{p=1}^{\infty} B_1\frac{\zeta(2p)}{2p}\left(\frac{\lambda}{2\pi}\right)^{2p} \tag{3.250}$$

$$F_g(\lambda) = \chi_g\left[1 + 2\sum_{p=1}^{\infty}\zeta(2g-2+2p)\binom{2g-3+2p}{2p}\left(\frac{\lambda}{2\pi}\right)^{2g-2+2p}\right] \tag{3.251}$$

where we use $\chi_g = (-1)^{g-1}\frac{B_g}{2g(2g-2)}$. The sum over p is of course the sum over the holes. Therefore, we must calculate the summation over p.

Comparing with closed string amplitudes

Let us first compare the genus zero partition function. We rewrite the above expression as

$$F_0 = \frac{3}{4} - \frac{1}{2}\ln\lambda + \frac{1}{\lambda^2}\sum_{p=2}^{\infty}\frac{\zeta(2p-2)}{p-1}\frac{\lambda^{2p}}{(2\pi)^{2p-2}2p(2p-1)}. \tag{3.252}$$

Then the sum over p of the last term can be done by differentiating twice with respect to λ, rewriting $\zeta(2m) = \sum\frac{1}{n^{2m}}$ with the result

$$\begin{aligned}
\sum_{p=1}^{\infty}\frac{\zeta(2p)}{2p}\left(\frac{\lambda}{2\pi}\right)^{2p} &= \sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2p}\left(\frac{\lambda}{2\pi n}\right)^{2p} \\
&= \frac{1}{2}\left(\ln\left(\Gamma\left(1-\frac{\lambda}{2\pi}\right)\right) + \ln\left(\Gamma\left(1+\frac{\lambda}{2\pi}\right)\right)\right) \\
&= \frac{1}{2}\ln\left(\frac{1}{2}\lambda\mathrm{Csc}\left(\frac{\lambda}{2}\right)\right) \\
&= -\frac{1}{2}\ln\left(\frac{2}{\lambda}\sin\left(\frac{\lambda}{2}\right)\right) \\
&\sim -\frac{1}{2}\ln\left(2\sin\left(\frac{\lambda}{2}\right)\right) \\
&= -\frac{i\lambda}{4} - \frac{1}{2}\ln(1-e^{-i\lambda}) \\
&= -\frac{i\lambda}{4} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-in\lambda}}{n}.
\end{aligned} \tag{3.253}$$

Then, the final result is

$$N^2F_0 = -\left(\frac{N}{\lambda}\right)^2\left[-\zeta(3) + i\frac{\pi^2}{6}\lambda - i(m+\frac{1}{4})\pi\lambda^2 + \frac{i\lambda^3}{12} + \sum_{n=1}^{\infty}\frac{e^{-in\lambda}}{n^3}\right]. \tag{3.254}$$

The first term, for example, is determined by the fact that $\lim_{\lambda\to 0}\lambda^2 F_0 = 0$. And, m is an integer which is not fixed uniquely. Let us compare this with the closed string result. The genus 0 amplitude for Calabi-Yau X with one Kähler class is

$$F_0 = \frac{1}{g_s^2}\left[-\frac{\chi(X)}{2}\zeta(3) - \frac{\pi^2}{6}c_2 t + i\pi a t^2 - C\frac{t^3}{3!} + \sum_{n,m} d_m \frac{1}{n^3}\exp(-nmt)\right]. \tag{3.255}$$

The notations are as before. For the S^2 resolved conifold, we have

$$\begin{aligned} &d_1 = 1, \quad d_i = 0, \quad \text{for } i > 1 \\ &C = \frac{1}{2}, \quad c_2 = -1, \quad \chi = 2 \end{aligned} \tag{3.256}$$

If we identify the parameters by $g_s = \frac{i\lambda}{N}$ and $t = i\lambda$, we easily see that both expressions are the same.

Let us next compare the genus 1 amplitude. In the same way as genus 0 amplitude, the summation on p can be carried out with the result

$$F_1(\lambda) = i\frac{B_1}{4}\lambda + \frac{B_1}{2}\ln(1 - e^{-i\lambda}). \tag{3.257}$$

On the other hand, the closed string side is

$$F_1 = -\frac{c_2}{24}t + \frac{1}{12}\sum_m d_m \ln(1 - e^{-mt}). \tag{3.258}$$

These expressions again coincide (we used $B_1 = \frac{1}{6}$).

Finally, we compare the genus $g > 1$ partition function. We add the "closed string sector ($p = 0$)" to the Chern-Simons free energy and weigh terms with $2p$ holes with a factor of $(2\pi A)^{2p}$.

$$\begin{aligned} N^{2-2g}F_g &= N^{2-2g}\chi_g\left[\sum_{n=1}^{\infty}\left(\frac{\lambda}{2\pi n}\right)^{2g-2}\sum_{p=0}^{\infty}(-1)^{g-1+p}\frac{1}{n^{2p}}\binom{2g-3+2p}{2p}(2\pi A)^{2p}\right] \\ &= N^{2-2g}\chi_g\left(\frac{\lambda}{2\pi}\right)^{2g-2}\sum_{n=1}^{\infty}\left[\frac{1}{(A+in)^{2g-2}} + \frac{1}{(A-in)^{2g-2}}\right] \\ &= \left(\frac{N}{\lambda}\right)^{2-2g}(-1)^{g-1}\left[(-1)^{g-1}\chi_g\frac{2\zeta(2g-2)}{(2\pi)^{2g-2}} - \frac{\chi_g}{(2g-3)!}\sum_{n\in\mathbf{Z}}\frac{1}{(\lambda+2\pi n)^{2g-2}}\right] \\ &= g_s^{2g-2}\left[(-1)^{g-1}\chi_g\frac{2\zeta(2g-2)}{(2\pi)^{2g-2}} - \frac{\chi_g}{(2g-3)!}\sum_{n=1}^{\infty}n^{2g-3}e^{-nt}\right]. \end{aligned} \tag{3.259}$$

This is again equal to the closed string amplitude. In this way we have a strong evidence for the duality. This duality is proposed by Gopakumar and Vafa in [41]. They also give some more evidences for the duality such as the coupling via the gravitational Chern-Simons terms and the Wilson loops. We do not discuss here. Interested reader should consult their paper.

3.5 Duality of superstrings

In the previous section, we see that topological strings also have a duality property which is similar to the AdS/CFT correspondence. However, what we are truely interested in is untwisted physical superstring theory. As explained before, the topological string amplitude gives a certain exact expression for F-term or superpotential. Therefore we can regard the above duality as the duality between the superstrings whose evidence is given by the above calculation. Embedding the duality in the ordinary string theory is the purpose of this section. Our main claim is that type IIA string on the deformed conifold with N D6 branes wrapped over S^3 is dual to the type IIA theory on the resolved conifold without D-branes but with RR fluxes. This is the work of Vafa [42].

3.5.1 RR fluxes and topological string amplitudes

Consider the type IIA theory compactified on a Calabi-Yau 3-fold. The resulting physics is of course the $N = 2$ theory. Then, what happens when we turn on the RR-fluxes in the internal space? The answer is that the RR fluxes generate superpotential terms and partially break the supersymmetry to $N = 1$. The superpotential generated by the RR 2-form F, 4-form G and 6-form G_6 is given by [64, 65]

$$g_s W = \int F \wedge K \wedge K + i \int G \wedge K + \int G_6 \tag{3.260}$$

where g_s is the coupling constant and K is the complexified Kähler class. This seems to have nothing with the topological string amplitudes. However, it turns out that we can rewrite this expression in terms of the prepotential or the genus 0 topological string amplitude. This is the crucial step to embed the duality. Below we show this fact. Let us go to the dual description by using the mirror symmetry. And consider the 3-cycle. Choose a basis for $H^3(M, \mathbf{Z})$ such that $A \cap A = B \cap B = 0$ and $A \cap B = 1$. Here the $\cap$ means the intersection number. Consider the following objects

$$\Pi = \int_A \Omega, \quad \Pi' = \int_B \Omega. \tag{3.261}$$

We know that there is a prepotential F_0 such that

$$\Pi' = \frac{\partial F_0}{\partial \Pi}. \tag{3.262}$$

When we return to the original description, the Π becomes the Kähler parameter t or the volume of the 2-cycle. Then, the Π' becomes the dual of it or, the volume of the 4-cycle. This shows that in terms of the complexified area of the basic 2-cycle the volumes of the $0, 2, 4$ cycles are given by

$$1, t, \frac{\partial F_0}{\partial t}. \tag{3.263}$$

Then, if we consider the N units of the two form flux F through the basic 2-cycle, the generated superpotential can be written as

$$\int F \wedge K \wedge K = N\frac{\partial F_0}{\partial t}. \tag{3.264}$$

In the same way, the superpotential generated by the N, L, P units of F, G, G_6 fluxes can be written as

$$g_s W = N\frac{\partial F_0}{\partial t} + itL + P. \tag{3.265}$$

The above F_0 is of course the topological string amplitude with RR flux turned on. We assume that the topological string amplitudes are not modified at all even if the RR flux are switched on. The following arguments may be helpful. Let us rewrite the superpotential as

$$g_s W = \int (F + i * G) \wedge K \wedge K + \int G_6. \tag{3.266}$$

The vector superfield with bottom component t has an auxiliary field in the superspace of the form

$$t + \theta^2(F + *iG) + ... \tag{3.267}$$

where F and G are the R fluxes of the internal Calabi-Yau space. In the usual set up, they are set to zero. Let $F_0^{\text{off}}(t)$ be the genus 0 topological string amplitude without the flux. If we turn N units of 2-form flux on, the superpotential is modified as

$$\int d^4\theta F_0^{\text{off}}(t) \rightarrow \int d^4\theta F_0^{\text{off}}(t + \theta^2 F + ...) = \int d^2\theta N\frac{\partial F_0^{\text{off}}}{\partial t}, \tag{3.268}$$

which is exactly the expected answer if $F_0^{\text{off}} = F_0^{\text{on}}$. Of course, this is not a proof.

The higher genus amplitudes also give rise to superpotential terms when we turn the flux on. For N units of RR 2-form,

$$\int d^4\theta \mathcal{W}^{2g} F_g \rightarrow N\int d^2\theta \mathcal{W}^{2g}\frac{\partial F_g}{\partial t}. \tag{3.269}$$

In this way, we see that under the assumption that the topological string amplitudes are not modified by the RR fluxes, we can determine the terms generated by the fluxes. If we have also NS fluxes, the expression $g_s W = N\frac{\partial F_0}{\partial t} + ...$ is the same. But in this case, N, L, P also have imaginary pieces. We will denote N, L, P with keeping in mind that they can be complex numbers.

3.5.2 Embedding the duality

Now we will interpret the topological string duality in terms of ordinary superstring theory. Let us make clear what we should do. What we want to show is that type IIA theory on the deformed conifold with N D6-branes wrapped on the S^3 of T^*S^3 (open side) is dual to the type IIA theory on the resolved conifold with RR flux but no D-branes (closed side). The open topological string amplitudes compute the superpotential terms of the gauge theory realized on the D-branes the form of which is

$$\int d^2\theta F_{g,h}\mathcal{W}^{2g}NhS^{h-1} \tag{3.270}$$

where we defined $S = g_s \mathrm{Tr} W^2$. The coefficient of Nh comes as follows. We have to choose $h-1$ holes to put the S fields and this can be done in h ways and the trace over the hole without a field gives a factor of N. If we define $F_g^{\mathrm{open}}(r) = \sum_h F_{g,h}r^h$, the fixed genus amplitude computes the term of the form

$$N\int d^2\theta \mathcal{W}^{2g}\frac{\partial F_g^{\mathrm{open}}(S)}{\partial S}. \tag{3.271}$$

Let us concentrate on the genus 0 amplitude. As we saw just now, the genus 0 amplitude of the open string computes

$$g_s W = N\frac{\partial F_0^{\mathrm{open}}}{\partial S} + \alpha S + \beta \tag{3.272}$$

where we showed explicitly the contribution coming from $h = 2$ in the form αS. The closed string computes

$$g_s W = N\frac{\partial F_0}{\partial t} + itL + P. \tag{3.273}$$

If both theories are dual to each other, these must be the same with suitable identification. Let us assume that they are equivalent. Then, we should identify α, β with iL, P respectively. Since the vacuum of both sides must satisfy $W = dW = 0$ and the W has the same form, this will identify

$$\langle S\rangle = \langle g_s \mathrm{Tr} W^2\rangle = t, \quad F_0^{\mathrm{open}} = F_0. \tag{3.274}$$

The same argument for the general genus leads us to the condition

$$\langle S\rangle = t, \quad F_g^{\mathrm{open}} = F_g. \tag{3.275}$$

This is nothing but the condition for the duality between the topological strings explained before. In this way, the topological string duality can be seen as a nontrivial check of the duality of the superstring theory.

Physical implications

Let us discuss some physics of this duality. The important thing is the fact that $\langle S\rangle \neq 0$, that is, the gaugino condensation. We first look at this from the purely gauge theoretical point of view. We consider the small S limit. The leading term with the lowest number of derivatives is given by the superpotential of the form

$$\frac{1}{g_s}\int d^2\theta SY. \tag{3.276}$$

Y is the chiral superfield with its bottom component $iC+\mu$, where C is the vacuum expectation value of the 3-form gauge field of IIA normalized with periodicity 2π and μ is the volume of S^3. Note that if Y is not a dynamical field, this is just a coupling constant. In particular, the analog of the 't Hooft coupling for this system is given by

$$\frac{Y}{Ng_s}. \tag{3.277}$$

Therefore, we have a non-standard supersymmetric gauge theory with coupling constant as a dynamical field. It is known that there is a nonperturbative effect and the superpotential becomes

$$W=\int d^2\theta\left(\frac{1}{g_s}SY+iN^2\xi e^{-Y/N}\right) \tag{3.278}$$

where ξ is a constant. By the shift $Y\to Y+Y_0$, ξ can be identified with a shift in the bare coupling constant of S

$$\xi=e^{-Y_0/N}. \tag{3.279}$$

We can integrated out the field Y by requiring

$$\partial_Y W=0, \tag{3.280}$$

which gives

$$Y=\ln\left(\frac{S}{iN\xi g_s}\right)^{-N}. \tag{3.281}$$

Then we have

$$W_{\mathrm{eff}}(S)=\frac{1}{g_s}\left[S\ln\left(\frac{S}{iN\xi g_s}\right)^{-N}+NS\right]. \tag{3.282}$$

Setting $\partial_S W_{\mathrm{eff}}=0$ gives

$$\left[\frac{S}{iN\xi g_s}\right]^N=1\to S=iN\xi g_s e^{2\pi il/N}=iNg_s e^{(-Y_0+2\pi il)/N}. \tag{3.283}$$

If we compare this expression with the gaugino condensate for standard $N = 1$ Yang-Mills theory [84]

$$\mathrm{Tr}W^2 = iN\Lambda^3 e^{\frac{-1}{Ng_{YM}^2} + \frac{2\pi il}{N}}, \tag{3.284}$$

we have the identification that Λ corresponds to the string scale and $1/g_{YM}^2 = Y_0$. We can also get this result from the dual closed topological string amplitude in the limit $t \to 0$. In this limit the topological closed string amplitude is

$$F_0 \to -\frac{1}{2}t^2 \ln t + at^2 + bt + c \tag{3.285}$$

which gives

$$W(S) = \frac{1}{g_s}[N\partial_S F_0(S) + \alpha S + \beta] = \frac{1}{g_s}(S\ln(S^{-N}) + \mathrm{const} \cdot NS + \mathrm{const} \cdot N). \tag{3.286}$$

This is perfect agreement with the open string analysis. This simple calculation shows the possibility that we might understand the non-perturbative dynamics of the gauge theory in terms of the perturbative topological string amplitudes.

Solution for t

To be true, we must be more careful in one point in establishing the duality. We must pay attention to the existence of the gravitational solution. When we try to solve some system of equations, there are sometimes topological obstructions. Unless these obstructions are absent, we cannot solve the equations. These phenomena are common in mathematics and physics such as Mittag-Leffler problem, Calabi's conjecture, and so on. Then, what is the obstruction relevant here? The similar example with the same number of supercharges was already studied from this point of view with the result that the conditions for the existence of the gravitational solution are equivalent to the condition $W = dW = 0$ [66, 67]. This is the condition to preserve the $N = 1$ supersymmetry.

Let us solve the equations $W = dW = 0$. There are four parameters; t, N, L, P. But N is fixed by the number of D6-branes. We solve the equations

$$\partial_t W = \to L = iNF_0'' \tag{3.287}$$
$$W = 0 \to P = -NF_0' + NtF_0''. \tag{3.288}$$

From $g_sW = N\partial_t F_0 + itL + P$ and the identification $\alpha S = itL$, we see that a shift in L is related to a shift in the bare coupling constant of the gauge system. In order to agree with the bare coupling constant of the gauge theory, we have $iL = \mu/g_s$ where μ is the volume of the S^3. Then, the value of t can be expressed in terms of N, μ, g_s by using the explicit form of the partition function $F_0 = \frac{1}{6}t^3 - \sum_{n>0}\frac{e^{-nt}}{n^3} + \text{quadratic term}$. The result is

$$(e^t - 1)^N = a \cdot \exp(-\mu/g_s) \tag{3.289}$$

where a is a constant which depends on the ambiguity in the quadratic part of F_0. This expression can be read as follows. For large μ, $Ng_{YM}^2 \sim Ng_s/\mu \ll 1$ and $t \to 0$. Therefore, in this case, the wrapped D-brane description is good and the resolved description is bad. On the other hand, when $t \gg 0$, the resolved description is good. But the wrapped D-brane description is bad since $\mu \to -\infty$.

3.6 M-theory flop and duality

Now we have come to one of the main themes. We have seen that the topological string theories also have a duality property which is similar to the AdS/CFT correspondence. By embedding the duality in the superstring, we have had a new duality. This duality is between the type IIA theories with and without D-branes. Moreover the D-branes which appear there are D6-branes. Therefore it is natural to ask, "How can we see this duality in terms of M-theory?". Atiyah, Maldacena, and Vafa [43] succeeded in deriving the duality from simple geometrical way by embedding type IIA in M-theory. Here we discuss their work.

3.6.1 Generalized flop

String case

Let us recall a flop. We considered a $(2,2)$ supersymmetric $U(1)$ gauge theory with four fields (a_1, a_2, b_1, b_2) and an FI term given by r and θ. The charges of the fields are $(1, 1, -1, -1)$ respectively. We set $t = r + i\theta$. The low energy vacuum is given by

$$V : |a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 = r. \tag{3.290}$$

The gauge inequivalent solutions are $V/U(1)$. This can be identified with $O(-1)+O(-1)$ bundle over $\mathbf{CP}^1$. When $r > 0$, the $\mathbf{CP}^1$ is identified as the locus $b_1 = b_2 = 0$ and the normal directions are b_1 and b_2. If we replace $t \to -t$, we have the same system with the role of the $(a_1, a_2) \leftrightarrow (b_1, b_2)$ exchanged. We also saw that the region $r > 0$ and $r < 0$ are completely different phase but connected smoothly if we turn the parameter θ on.

Here, we consider a modified theory where we mod out the a_i's and b_j's by some discrete group G that does not necessarily act symmetrically under the exchange $(a_1, a_2) \leftrightarrow (b_1, b_2)$. Let us denote the resulting theory as $Q_G[t]$. Because of the asymmetric action of G, $Q_G[-t] \neq Q_G[t]$. Instead, we have

$$Q_G[-t] = Q_{G'}[t] \tag{3.291}$$

where G' is related to G by conjugation with U

$$\begin{aligned} U : (a_1, a_2) &\leftrightarrow (b_1, b_2) \\ G' &= UGU^{-1}. \end{aligned} \tag{3.292}$$

Let us explain the effects of these processes in an example. Suppose G is generated by the element

$$(a_1, a_2, b_1, b_2) \to (\omega a_1, \omega^{-1} a_2, b_1, b_2) \tag{3.293}$$

where ω is an n-th root of unity. Geometrically this corresponds to the action on $O(-1)+O(-1)$ bundle over $\mathbf{CP}^1$

$$(\zeta_1, \zeta_2, z) \to (\omega\zeta_1, \omega\zeta_2, \omega^{-2} z) \tag{3.294}$$

where z is the coordinate of $\mathbf{CP}^1$ and ζ_i are the coordinates of the fiber. Note that modding by this discrete group gives two orbifold singularities; the north and south poles at the origin of ζ_i. On the other hand, the group G' is generated by

$$(\zeta_1, \zeta_2, z) \to (\omega\zeta_1, \omega^{-1}\zeta_2, z). \tag{3.295}$$

This action produces totally different singularities. It leads to an A_{N-1} singularity over $\mathbf{CP}^1$. Therefore, modding by the discrete group gives a totally new dual theory. This is just a warming up for the next subsection.

M-theory case

We first want to discuss the duality which can be obtained by the technique introduced just now, but here we are interested in M-theory not in string theory. What we finally want to have is the 4-dimensional $N = 1$ supersymmetric theory. Therefore, we must consider the M-theory on G_2 manifolds. We consider the following G_2 manifold. Let u_i, v_i, $i = 1, 2, 3, 4$ be real and impose the following condition.

$$(u_1^2 + u_2^2 + u_3^2 + u_4^2) - (v_1^2 + v_2^2 + v_3^2 + v_4^2) = V. \tag{3.296}$$

The topology of this manifold is $\mathbf{R}^4 \times S^3$. For $V > 0$, the S^3 is identified with the locus $v_i = 0$ and v_i correspond to the $\mathbf{R}^4$ normal directions over S^3. We denote the S^3 defined by $v_i = 0$ as $\tilde{S}^3$. When $V < 0$, the roles of the u_i's and v_i's are exchanged. And we denote the S^3 given by $u_i = 0$ as $\hat{S}^3$. A G_2 holonomy metric can be defined on this manifold by

$$ds^2 = \alpha^2 dr^2 + \gamma^2 (\tilde{w}^a)^2 + \beta^2 (\hat{w}^a - \frac{1}{2}\tilde{w}^a)^2 \tag{3.297}$$

with

$$\alpha^{-2} = 1 - \frac{a^3}{r^3}, \quad \beta^2 = \frac{r^2}{9}\left(1 - \frac{a^3}{r^3}\right), \quad \gamma^2 = \frac{r^2}{12} \tag{3.298}$$

and $\tilde{w}^a$ and $\hat{w}^a$ are left invariant one forms on two three-spheres which we defined as $\tilde{S}^3$ and $\hat{S}^3$. The variable r fills one of the S^3's depending on the sign of V. In the above form, $r \geq a$ and it fills the $\hat{S}^3$ associated with the 1-form $\hat{w}^a$. And $\tilde{S}^3$ is not filled and topologically non-trivial. We can regard the S^3's as the $SU(2)$ group manifolds. The following formulas are useful

$$g = e^{i\psi/2\sigma^3} e^{i\theta/2\sigma^1} e^{i\phi/2\sigma^3} \tag{3.299}$$

$$\frac{i}{2} w_R^a \sigma^a = dg g^{-1}, \quad \frac{i}{2} w_L^a \sigma^a = g^{-1} dg \tag{3.300}$$

$$w_R^1 + i w_R^2 = e^{-i\psi}(d\theta + i\mathrm{sin}\theta d\phi), \quad w_R^3 = d\psi + \mathrm{cos}\theta d\phi \tag{3.301}$$

$$w_L^1 + i w_L^2 = e^{i\phi}(d\theta - i\mathrm{sin}\theta d\psi), \quad w_L^3 = d\phi + \mathrm{cos}\theta d\psi \tag{3.302}$$

$$dw_R^a = -\frac{1}{2}\epsilon^{abc} w_R^b w_R^c, \quad dw_L^a = \frac{1}{2}\epsilon^{abc} w_L^b w_L^c. \tag{3.303}$$

From these definitions, we can see that the forms w_L^a are invariant under left multiplications $g \to h_L g$ and transform in the adjoint representation of $SU(2)$ under right multiplication $g \to g h_R$. In terms of these forms, the metric of the unit three-sphere can be written as

$$ds^2 = \frac{1}{4}\sum_a (w_L^a)^2 = \frac{1}{4}\sum_a (w_R^a)^2. \tag{3.304}$$

The G_2 metric has $SU(2) \times SU(2) \times SU(2)$ isometry. Two $SU(2)$'s come from left multiplication in each of the S^3 while the third $SU(2)$ arises from right multiplication on both three-spheres by the same group element, or the diagonal subgroup of $SU(2)_R \times SU(2)_R$. The last fact can be seen by noticing that the index a transforms in the adjoint of $SU(2)$ and is contracted in an $SU(2)$ invariant way. Let us denote this isometry group as

$$SU(2)_L^1 \times SU(2)_L^2 \times [SU(2)_R^1 \times SU(2)_R^2]_D \tag{3.305}$$

where L, R mean the left and right multiplication and $1, 2$ denote which S^3 they act. For example, $SU(2)_L^1$ acts on u_i by left multiplications. We associate $SU(2)^1$ with the $\tilde{S}^3$ and $SU(2)^2$ with the contractible $\hat{S}^3$.

The explicit form of the 3-form which defines the G_2 structure is given by

$$\begin{aligned}\varphi &= \frac{a^3}{72}\epsilon_{abc}\tilde{w}^a\tilde{w}^b\tilde{w}^c - \frac{1}{18}(r^3 - a^3)\epsilon_{abc}(\tilde{w}^a\hat{w}^b\hat{w}^c - \tilde{w}^a\tilde{w}^b\hat{w}^c) + \frac{r^2}{3}dr\tilde{w}^a\hat{w}^a \\ &= \frac{a^3}{72}\epsilon_{abc}\tilde{w}^a\tilde{w}^b\tilde{w}^c + d\left(\frac{r^3 - a^3}{9}\tilde{w}^a\hat{w}^a\right).\end{aligned} \tag{3.306}$$

From this, we see that φ is invariant under the isometry group of the metric. This means that if we quotient the manifold by the subgroup of the isometry group, we still have a manifold with G_2 holonomy metric.

If we consider the limit $r \to \infty$, the metric becomes

$$ds^2 \sim dr^2 + \frac{r^2}{9}[(\tilde{w}^a)^2 + (\hat{w}^a)^2 - \tilde{w}^a\hat{w}^a]. \tag{3.307}$$

This is the cone on $\tilde{S}^3 \times \hat{S}^3$. Therefore, the space we are considering is asymptotic to a cone on $\tilde{S}^3 \times \hat{S}^3$. In other words, the space can be made by eliminating the singularity of the cone by giving a finite volume to one of the S^3's. If we give a finite volume to $\tilde{S}^3$, we have the above metric. On the other hand, if we give a finite volume to $\hat{S}^3$, we obtain the metric with $\tilde{w}^a \leftrightarrow \hat{w}^a$. These two manifolds are related by a flop.

In general, the number of moduli of the Ricci-flat metric of a compact manifold is given by the dimension of the third homology class or b_3. Therefore, the point on the moduli space of Ricci-flat metrics is characterized by the volume of some basis for 3-cycles. This corresponds physically to the lowest components of chiral fields in $N = 1$ multiplets. Thus, we expect that the complexification of the volumes occurs. In fact, we have a 3-form gauge field C in M-theory and its vacuum expectation values about each cycle give the imaginary part of the volume. Let us denote this combination as $V_M = V + iC$. Just as in the string propagating on the resolved confolds, the phase structure of M-theory as a function of V_M is expected to have a singularity only at the origin $V = C = 0$ and we can connect the two regions ($V > 0$ and $V < 0$) smoothly. Though there is no rigorous arguments, this can be seen as follows. The fact that V_M is the lowest component of a chiral field implies that the moduli space of M-theory compactified on G_2 manifolds are given by analytic expressions therefore singularities occur in complex codimension 1 or higher. For the theory we are considering, this means that the singularities can arise only at $V = C = 0$ in particular not at $V = 0, C \neq 0$. Therefore as in the resolved conifold case, two regions are connected smoothly. Let us denote the M-theory in the presence of the back ground V_M as $Q[V_M]$. It is evident that

$$Q[V_M] = Q[-V_M]. \tag{3.308}$$

As in the string case, we can consider modding out by the action of the subgroup of the isometry group. In this case, the duality property is expressed by

$$Q_G[-V_M] = Q_{G'}[V_M] \tag{3.309}$$

where $G = UG'U^{-1}$ and U is the $\mathbf{Z}_2$ outer automorphism which exchanges the u_i's and v_i's and acts on $SU(2)$'s as

$$U[SU(2)^{1,2}_{L,R}]U^{-1} = SU(2)^{2,1}_{L,R}. \tag{3.310}$$

We take G as a $\mathbf{Z}_N$ subgroup of $SU(2)^2_L$. To write down its action explicitly, it is convenient to introduce the complex notation like $z_1 = u_1 + iu_2$. In terms of these, our manifold is expressed as

$$(|z_1|^2 + |z_2|^2) - (|z_3|^2 + |z_4|^2) = V. \tag{3.311}$$

Note that the G_2 holonomy manifold is not a complex manifold. Therefore to write the equation in terms of the complex coordinates has no intrinsic meaning other than for simpler notations. In terms of these, the group action is defined by

$$g : (z_1, z_2, z_3, z_4) \to (z_1, z_2, \omega z_3, \omega z_4). \tag{3.312}$$

Then, G' is generated by

$$g' : (z_1, z_2, z_3, z_4) \to (\omega z_1, \omega z_2, z_3, z_4). \tag{3.313}$$

Let us first consider $Q_G[V_M]$ for $V_M \gg 0$. In this case, the elements $g \in G$ do not act freely. There are fixed points at $\tilde{S}^3$ defined by $z_3 = z_4 = 0$. Since the normal directions are $\mathbf{R}^4/\mathbf{Z}_N$ with the usual action of $\mathbf{Z}_N$ on $\mathbf{R}^4$, this singularity is the A_{N-1} singularity. This singularity gives rise to an $SU(N)$ gauge symmetry on the singular locus. Therefore, we have an $N = 1$ sypersymmetric $SU(N)$ Yang-Mills theory on $\tilde{S}^3$ times Minkowski space.

Let us next consider $Q_G[V_M]$ for $V_M \ll 0$. By the duality property, it is equal to $Q_{G'}[V_M]$ for $V_M \gg 0$. In this case, the elements $g' \in G'$ act freely. This is because the locus $z_1 = z_2 = 0$ is not on the manifold for $V_M \gg 0$. There is no sign of gauge symmetry. What does this mean? We discuss this in the next section.

3.6.2 Gauge theoretical interpretation

We see above that $Q_G[V_M]$ for $V_M \gg 0$ contains $N = 1$ Yang-Mills sector in four dimensions. Recall that the Yang-Mills coupling are given by

$$\frac{1}{g_{YM}^2} + i\theta = V_M \tag{3.314}$$

where g_{YM} should be regarded as the gauge coupling at the Planck scale. We know that the effective coupling constant depends on the energy scale we probe. Then, the above relation shows that V_M also runs and its value depends on the energy scale. Let us denote the running volume as $V_M(\mu_{\text{ren}})$. Then $V_M(\mu_{\text{ren}})$ decreases logarithmically as

$$V_M(\mu_{\text{ren}}) = V_M + \text{const} \cdot \ln\left(\frac{\mu_{\text{ren}}}{M_{\text{pl}}}\right). \tag{3.315}$$

Let us consider the region for small μ_{ren}. In this case, $V_M(\mu_{\text{ren}})$ is very small. Moreover, it seems that we would get a negative volume. From the purely field theoretical point of view, this cannot be allowed. However, we already know that this is a flop. Therefore, we are led to the idea that for large $V_M \gg 0$, we have an ordinary gauge theoretical description $Q_G[V_M]$ but as the energy scale becomes smaller and smaller until V_M becomes negative, we have an another description $Q_{G'}[V_M]$. Recall that in terms of $Q_{G'}[V_M]$, there is no singularity in geometry so we have $N = 1$ theory without any sign of gauge symmetry. This is exactly what we expect for a confining phase!

3.6.3 Type IIA description

We have seen that not only in string theory but also in M-theory we can use the idea of flop. And we saw that we have a candidate for describing the confining gauge theory. However, our main claim was how we can derive the superstring duality from M-theory. In order to do so, we must choose the eleventh dimension. There are many ways to do so. To derive the duality proposed by Vafa, we identify the 11-th direction with the fibers of $U(1)$ sitting in $SU(2)^2_L = \hat{S}^3$. The $\mathbf{Z}_N$ that we modded out is a subgroup of it, that is, $\mathbf{Z}_N \subset U(1) \subset SU(2)^2_L = \hat{S}^3$. Let us forget the modding out for a while and consider how the M-theory flop described in the previous section can be seen from the type IIA viewpoint. As we saw, the geometry for $Q[V_M], V_M \gg 0$ is a "resolved" conifold over $\tilde{S}^3 \times \hat{S}^3$, where the desingularization is done by replacing the singular point with $\tilde{S}^3$. With the identification of the eleventh dimension as above, this geometry can be described as an ordinary deformed conifold T^*S^3 in type IIA. On the other hand, $Q[V_M], V_M \ll 0$ is a "resolved" conifold over $\tilde{S}^3 \times \hat{S}^3$, where we replace the singular point with $\hat{S}^3$. Therefore, from a 10-dimensional viewpoint, this is an ordinary resolved confold. Let us next consider the effect of modding out. The geometry for $Q_G[V_M], V_M \gg 0$ has A_{N-1} type singularity. The A_{N-1} type singularity in M-theory corresponds to the N D6-branes in type IIA theory. Therefore $Q_G[V_M]$ for $V_M \gg 0$ corresponds to the deformed conifold T^*S^3 with N D6-branes wrapped over $\tilde{S}^3$. At large distances from D6-branes, we detect the presence of the branes by the presence of a two form field strength on the surrounding S^2. If we lift up to M-theory, this means that the eleventh circle is non-trivially fibered over the S^2. The total topology of this fibration is $S^3/\mathbf{Z}_N$. On the other hand, $Q_G[V_M]$ for $V_M \ll 0$ or $Q_{G'}[V_M]$ for $V_M \gg 0$ has no singularity. Therefore there is no D-branes. However, the eleventh circle is non-trivially fibered over S^2. This means that in the type IIA terms we have N units of RR 2-form flux through S^2. In this way, we can derive the duality from M-theory. Let us check whether the relations of parameters are consistent with the type IIA duality. From the relation between the M-theory parameters and type IIA ones, we deduce

$$V_M = V_{IIA}/g_s = \mu/g_s. \tag{3.316}$$

And the volume of the minimal S^2 (Kähler class) is given by

$$t = -V_M/N. \tag{3.317}$$

This relation is correct for large t where the supergravity approximation is good. From these two relations we have

$$t = \frac{-V_{IIA}}{Ng_s} = \frac{-\mu}{Ng_s}. \tag{3.318}$$

This coincides with the large t limit of the result obtained before

$$(e^t - 1)^N = a \cdot \exp(-\mu/g_s). \tag{3.319}$$

The latter expression contains the contribution of world sheet instantons which is neglected at the large volume limit.

Chapter 4

Superstring on G_2

In this chapter we discuss the CFT description of manifolds with G_2 holonomy.

4.1 Fundamental properties

4.1.1 G_2 CFT

Let us first construct the algebra that underlies sigma models with $N = 1$ superconformal symmetry on manifolds with G_2 holonomy. The idea is the same as in the Calabi-Yau case. In that case, we start from $N = 1$ superconformal algebra and we include the $U(1)$ current. This is because we have a priori a $U(n)$ symmetry when we consider the sigma model on a Kähler manifold and the holomomy of the Calabi-Yau manifold $SU(n)$ breaks part of the symmetry leaving the $U(n)/SU(n) = U(1)$ unbroken. Then the closure of the algebra requires $N = 2$ superconformal algebra. We will construct the G_2 CFT in the same way [69].

We start from the flat 7 dimensional space and construct the chiral operators. We assume that they continue to exist even after we perturb the metric to a non-trivial G_2 holonomy. The $N = 1$ superconformal has two operators, the energy-momentum tensor $T(z)$ and its superpartner $G(z)$. They are represented as

$$T(z) = T_b + T_f = \frac{1}{2}\sum_{i=1}^{7} :J^i J^i: -\frac{1}{2}\sum_{i=1}^{7} :\psi^i \partial\psi^i: \tag{4.1}$$

$$G(z) = \sum_{i=1}^{7} :J^i \psi^i: \tag{4.2}$$

where we defined $J^k = i\partial x^k$. Their OPE's are given by

$$J^k(z)J^l(w) \sim \frac{1}{(z-w)^2}\delta_{kl}, \quad \psi^k(z)\psi^l(w) \sim \frac{1}{z-w}\delta_{kl}. \tag{4.3}$$

Then, what is the operator which captures the G_2 structure? Recall that the G_2 structure is characterized by the 3-form φ which is invariant under the holonomy group. This suggests that we should add the following operator to the $N = 1$ generators

$$\begin{aligned}\Phi(z) =&\psi^1\psi^2\psi^5 + \psi^1\psi^3\psi^6 + \psi^1\psi^4\psi^7 - \\ &\psi^2\psi^3\psi^7 + \psi^2\psi^4\psi^6 - \psi^3\psi^4\psi^5 + \psi^5\psi^6\psi^7.\end{aligned} \tag{4.4}$$

The OPE of the algebra must close. Other operators are determined by this condition. If we calculate the OPE of Φ with itself, we have

$$\Phi(z)\Phi(w) \sim -\frac{7}{(z-w)^3} + \frac{6}{z-w}X(w) \tag{4.5}$$

where we have a new operator X with spin 2 defined by

$$\begin{aligned}X(z) =& -\psi^1\psi^2\psi^3\psi^4 + \psi^1\psi^2\psi^6\psi^7 - \psi^1\psi^3\psi^5\psi^7 + \psi^1\psi^4\psi^5\psi^6 - \\ &\psi^2\psi^3\psi^5\psi^6 - \psi^2\psi^4\psi^5\psi^7 - \psi^3\psi^4\psi^6\psi^7 - \frac{1}{2}\sum_i :\partial\psi^i\psi^i: \\ =& - *\Phi + T_f\end{aligned} \tag{4.6}$$

where $*\Phi$ is defined by the dual form $*\varphi$. Next we compute the OPE of the form $G(z)\Phi(w)$, and $G(z)X(w)$. The result is

$$G(z)\Phi(w) \sim \frac{1}{z-w}K(w) \tag{4.7}$$

$$G(z)X(w) \sim -\frac{1}{2}\frac{1}{(z-w)^2}G(w) + \frac{1}{z-w}M(w) \tag{4.8}$$

where we again obtain the new operators K, M defined by

$$\begin{aligned}K(z) =& J^1\psi^2\psi^5 + J^1\psi^3\psi^6 + J^1\psi^4\psi^7 - J^2\psi^1\psi^5 - J^2\psi^3\psi^7 + \\ &J^2\psi^4\psi^6 - J^3\psi^1\psi^6 + J^3\psi^2\psi^7 - J^3\psi^4\psi^5 - J^4\psi^1\psi^7 - J^4\psi^2\psi^6 + \\ &J^4\psi^3\psi^5 + J^5\psi^1\psi^2 - J^5\psi^3\psi^4 + J^5\psi^6\psi^7 + J^6\psi^1\psi^3 + J^6\psi^2\psi^4 - \\ &J^6\psi^5\psi^7 + J^7\psi^1\psi^4 - J^7\psi^2\psi^3 + J^7\psi^5\psi^6\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}M =& - J^1\psi^2\psi^3\psi^4 + J^1\psi^2\psi^6\psi^7 - J^1\psi^3\psi^5\psi^7 + J^1\psi^4\psi^5\psi^6 + J^2\psi^1\psi^3\psi^4 - \\ &J^2\psi^1\psi^6\psi^7 - J^2\psi^3\psi^5\psi^6 - J^2\psi^4\psi^5\psi^7 - J^3\psi^1\psi^2\psi^4 + J^3\psi^1\psi^5\psi^7 + \\ &J^3\psi^2\psi^5\psi^6 - J^3\psi^4\psi^6\psi^7 + J^4\psi^1\psi^2\psi^3 - J^4\psi^1\psi^5\psi^6 + J^4\psi^2\psi^5\psi^7 + \\ &J^4\psi^3\psi^6\psi^7 - J^5\psi^1\psi^3\psi^7 + J^5\psi^1\psi^4\psi^6 - J^5\psi^2\psi^3\psi^6 - J^5\psi^2\psi^4\psi^7 + \\ &J^6\psi^1\psi^2\psi^7 - J^6\psi^1\psi^4\psi^5 + J^6\psi^2\psi^3\psi^5 - J^6\psi^3\psi^4\psi^7 - J^7\psi^1\psi^2\psi^6 + \\ &J^7\psi^1\psi^3\psi^5 + J^7\psi^2\psi^4\psi^5 + J^7\psi^3\psi^4\psi^6 + \sum_i\left(\frac{1}{2}J^i\partial\psi^i - \frac{1}{2}\partial J^i\psi^i\right).\end{aligned} \tag{4.10}$$

The nontrivial fact is that the operator expansion algebra formed by the six operators T, G, Φ, X, K and M closes. The OPE's of these operators are given by

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \tag{4.11}$$

$$T(z)G(w) \sim \frac{3}{2}\frac{1}{(z-w)^2}G(w) + \frac{1}{z-w}\partial G(w) \tag{4.12}$$

$$G(z)G(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2}{z-w}T(w) \tag{4.13}$$

$$T(z)\Phi(w) \sim \frac{3/2}{(z-w)^2} + \frac{1}{z-w}\partial\Phi(w) \tag{4.14}$$

$$G(z)\Phi(w) \sim \frac{1}{z-w}K(w) \tag{4.15}$$

$$\Phi(z)\Phi(w) \sim -\frac{7}{(z-w)^3} + \frac{6}{z-w}X(w) \tag{4.16}$$

$$\Phi(z)X(w) \sim -\frac{2}{15}\frac{1}{(z-w)^2}\Phi(w) - \frac{5}{2}\frac{1}{z-w}\partial\Phi(w) \tag{4.17}$$

$$\Phi(z)K(w) \sim -\frac{3}{(z-w)^2}G(w) - \frac{3}{z-w}\left(M + \frac{1}{2}\partial G\right)(w) \tag{4.18}$$

$$\Phi(z)M(w) \sim \frac{9}{2}\frac{1}{(z-w)^2}K(w) - \frac{1}{z-w}\left(3 : G(w)\Phi(w) : -\frac{5}{2}\partial K(w)\right) \tag{4.19}$$

$$T(z)X(w) \sim -\frac{7}{4}\frac{1}{(z-w)^3} + \frac{2}{(z-w)^2}X(w) + \frac{1}{z-w}\partial X(w) \tag{4.20}$$

$$G(z)X(w) \sim -\frac{1}{2}\frac{1}{(z-w)^2}G(w) + \frac{1}{z-w}M(w) \tag{4.21}$$

$$X(z)X(w) \sim \frac{35}{4}\frac{1}{(z-w)^4} - \frac{10}{(z-w)^2}X(w) - \frac{5}{z-w}\partial X(w) \tag{4.22}$$

$$X(z)K(w) \sim -\frac{3}{(z-w)^2}K(w) + \frac{3}{z-w}\left(: G(w)\Phi(w) : -\partial K(w)\right) \tag{4.23}$$

$$\begin{aligned} X(z)M(w) \sim {} & -\frac{9}{2}\frac{1}{(z-w)^3}G(w) - \frac{1}{(z-w)^2}\left(5M + \frac{9}{4}\partial G\right)(w) + \\ & \frac{1}{z-w}\left(4 : G(w)X(w) : -\frac{7}{2}\partial M(w) - \frac{3}{4}\partial^2 G(w)\right) \end{aligned} \tag{4.24}$$

$$T(z)K(w) \sim \frac{2}{(z-w)^2}K(w) + \frac{1}{z-w}\partial K(w) \tag{4.25}$$

$$G(z)K(w) \sim \frac{4}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial\Phi(w) \tag{4.26}$$

$$K(z)K(w) \sim -\frac{21}{(z-w)^4} + \frac{6}{(z-w)^2}(X-T)(w) + \frac{3}{z-w}(\partial X - \partial T)(w) \tag{4.27}$$

$$K(z)M(w) \sim -\frac{15}{(z-w)^3}\Phi(w) - \frac{11}{2}\frac{1}{(z-w)^2}\partial\Phi(w) +$$

$$\frac{3}{z-w}(: G(w)K(w) : -2 : T(w)\Phi(w) :) \tag{4.28}$$

$$T(z)M(w) \sim -\frac{1}{2}\frac{1}{(z-w)^3}G(w) + \frac{5}{2}\frac{1}{(z-w)^2}M(w) + \frac{1}{z-w}\partial M(w) \tag{4.29}$$

$$G(z)M(w) \sim \frac{7}{2}\frac{1}{(z-w)^4} + \frac{1}{(z-w)^2}(T+4X)(w) + \frac{1}{z-w}\partial X(w) \tag{4.30}$$

$$M(z)M(w) \sim -\frac{35}{(z-w)^5} + \frac{1}{(z-w)^3}(20X - 9T)(w) + \frac{1}{(z-w)^2}\left(10\partial X - \frac{9}{2}\partial T\right)(w)$$
$$+ \frac{1}{z-w}\left(\frac{3}{2}\partial^2 X(w) - \frac{3}{2}\partial^2 T(w) - 4 : G(w)M(w) : +8 : T(w)X(w) :\right) \tag{4.31}$$

where the central charge is $c = \frac{21}{2}$. This is the definition of the G_2 CFT. An important fact is that above algebra contains two $N = 1$ superconformal subalgebras. One is given by the original $N = 1$ algebra generated by T and G. And the other is generated by $T_I = -\frac{1}{5}X$ and $G_I = \frac{i}{\sqrt{15}}\Phi$. The latter has central charge $c_I = \frac{7}{10}$. Therefore, the G_2 CFT contains the tri-critical Ising model. Moreover, there is a decomposition of the form

$$T_I(z)T_r(w) \sim 0, \quad T(z) = T_I(z) + T_r(z). \tag{4.32}$$

This fact allows us to classify the highest weight representation of our algebra by using two numbers: tri-critical Ising weight and the eigenvalue of the zero mode of the remaining part of the energy-momentum tensor. We use the notation $[\Delta_I, \Delta_r]$ for operators that correspond to the Virasoro highest weights $|\Delta_I, \Delta_r\rangle$. The first dimension is the dimension of the tri-critical Ising part and the second is the dimension of the remaining Virasoro algebra T_r.

The emergence of the tri-critical Ising model is not an accident. In the 7-dimensional manifolds, we have a priori $SO(7)$ symmetry or more precisely $SO(7)$ current algebra at level 1. However, we are left with only the residual symmetry $SO(7)/G_2$ because of the holonomy. $SO(7)/G_2$ is no longer a group but we can see physically that this is the coset model. $SO(7)$ at level 1 has central charge 7/2 and G_2 at level 1 has 14/5. Therefore, the central charge for the residual symmetry is

$$\frac{7}{2} - \frac{14}{5} = \frac{7}{10}, \tag{4.33}$$

which is precisely the central charge of the tri-critical Ising model. Therefore, the existence of the tricritical Ising model is a consequence of the G_2 holonomy.

4.1.2 Heighest weight states and a twist operator

From the fact that the total energy-momentum tensor is a sum of two commutative Virasoro generators one of which is tri-critical Ising, we see that unitary highest weight representations must have the following tri-critical Ising dimensions

$$[0]_{\rm Vir}, \quad \left[\frac{1}{10}\right]_{\rm Vir}, \quad \left[\frac{6}{10}\right]_{\rm Vir}, \quad \left[\frac{3}{2}\right]_{\rm Vir} \tag{4.34}$$

or in $N=1$ terms

$$\begin{aligned} {\rm NS}: [0], \quad & \left[\frac{1}{10}\right] \\ {\rm R}: \left[\frac{7}{16}\right], \quad & \left[\frac{3}{80}\right]. \end{aligned} \tag{4.35}$$

We first show that the Witten's index $(-1)^F$ for the full theory can be identified with the $(-1)^{F_I}$ which is the $\mathbf{Z}_2$ symmetry of the tri-critical Ising model viewed as an $N=1$ superconformal system. Our algebra has three bosonic generators T, X, K and three fer-minonic generators G, Φ, M. Let us expand these in modes as $A(z) = \sum_n A_n z^{-n-\Delta}$. The commutation relations of these modes are found in [69]. We use the following commutation relations.

$$[G_n, X_m] = -\frac{1}{2}\left(n+\frac{1}{2}\right) G_{n+m} + M_{n+m} \tag{4.36}$$

$$[X_n, K_m] = 3(m+1)K_{n+m} + 3 : G\Phi :_{n+m} \tag{4.37}$$

$$\begin{aligned} [X_n, M_m] = & \left[\frac{9}{4}(n+1)\left(m+\frac{3}{2}\right) - \frac{3}{4}\left(n+m+\frac{3}{2}\right)\left(n+m+\frac{5}{2}\right)\right] G_{n+m} \\ & - \left[5(n+1) - \frac{7}{2}\left(n+m+\frac{5}{2}\right)\right] M_{n+m} + 4 : GX :_{n+m} \end{aligned} \tag{4.38}$$

$$\{G_n, M_m\} = -\frac{7}{12}\left(n^2-\frac{1}{4}\right)\left(n-\frac{3}{2}\right)\delta_{n+m,0} + \left(n+\frac{1}{2}\right) L_{n+m} + (3n-m)X_{n+m} \tag{4.39}$$

$$\begin{aligned} \{M_n, M_m\} = & -\frac{35}{24}\left(n^2-\frac{1}{4}\right)\left(n^2-\frac{9}{4}\right)\delta_{n+m,0} + \left[\frac{3}{2}(n+m+2)(n+m+3)\right. \\ & \left. - 10\left(n+\frac{3}{2}\right)\left(m+\frac{3}{2}\right)\right] X_{n+m} + \left[\frac{9}{2}\left(n+\frac{3}{2}\right)\left(m+\frac{3}{2}\right)\right. \\ & \left. - \frac{3}{2}(n+m+2)(n+m+3)\right] L_{n+m} - 4 : GM :_{n+m} + 8 : LX :_{n+m} \end{aligned} \tag{4.40}$$

where we defined $: AB :_n = \sum_{p<-\Delta_A-1} A_p B_{n-p} + (-1)^{AB} \sum_{p>\Delta_A} B_{n-p} A_p$. The important property is that

$$\Phi_n^\dagger = -\Phi_{-n}, \quad K_n^\dagger = -K_{-n}, \quad M_n^\dagger = \frac{1}{2}G_{-n} - M_{-n}. \tag{4.41}$$

From this property and the commutation relations, we see that

$$|M_{-1/2}|0,0\rangle|^2 = \langle 0,0|M_{1/2}^\dagger M_{-1/2}|0,0\rangle = 0 \tag{4.42}$$

$$|M_{-3/2}|0,0\rangle|^2 = \langle 0,0|M_{3/2}^\dagger M_{-3/2}|0,0\rangle = 0. \tag{4.43}$$

On the other hand, the $\mathbf{Z}_2$ assignments for the tricritical is

$$[0]^+, \left[\frac{1}{10}\right]^-, \left[\frac{6}{10}\right]^+, \left[\frac{3}{2}\right]^- . \tag{4.44}$$

To prove that $(-1)^F = (-1)^{F_I}$, it is sufficient to derive the tri-critical Ising dimensions of our generators and see if the two $\mathbf{Z}_2$ assignments agree. To see this, let us calculate as follows

$$\begin{aligned} T_0^I G_{-3/2}|0,0\rangle &= -\frac{1}{5} X_0 G_{-3/2}|0,0\rangle \\ &= \frac{1}{10} G_{-3/2}|0,0\rangle - \frac{1}{5} M_{-3/2}|0,0\rangle \\ &= \frac{1}{10} G_{-3/2}|0,0\rangle. \end{aligned} \tag{4.45}$$

This shows

$$G_{-3/2}|0,0\rangle = \left|\frac{1}{10}, \frac{14}{10}\right\rangle^- \tag{4.46}$$

where $-$ denotes the $(-1)^{F_I}$. Then, the similar calculation gives us

$$\begin{aligned} &L_{-2}|0,0\rangle = |2,0\rangle^+ + |0,2\rangle^+, \quad X_{-2}|0,0\rangle = |2,0\rangle^+ \\ &K_{-2}|0,0\rangle = \left|\frac{6}{10}, \frac{14}{10}\right\rangle^+, M_{-5/2}|0,0\rangle = a\left|\frac{1}{10}, \frac{24}{10}\right\rangle^- + b\left|\frac{1}{10}+1, \frac{14}{10}\right\rangle^- . \end{aligned} \tag{4.47}$$

Then, we easily see that $(-1)^F = (-1)^{F_I}$. This simple fact implies that in the NS sector of the full theory only NS dimensions of tri-critical model show up and similarly for the R sector. We next discuss the analog of the spectral flow operator. Recall that supersymmetry requires that the Ramond vacuum for the whole theory has dimensions $\frac{c}{24} = \frac{D}{16} = \frac{7}{16}$. This lead to the following unitary highest weight representations of our theory in the Ramond ground state

$$\left|\frac{7}{16}, 0\right\rangle \quad \left|\frac{3}{80}, \frac{2}{5}\right\rangle. \tag{4.48}$$

Note that there exists a ground state in the R sector which entirely consists of the tri-critical Ising sector. The operator corresponding to the state $|\frac{7}{16}, 0\rangle$ plays the same role as the spectral flow operator in $N = 2$ theories which is also entirely constructed out of the $U(1)$ piece of $N = 2$. To have one spacetime supersymmetry we want to have a theory with only one Ramond ground state of the form $|\frac{7}{16}, 0\rangle$. To see that the operator is the spectral flow operator, let us study the action of $[\frac{7}{16}, 0]$ on various states. We have unique fusion rules for the operator $[\frac{7}{16}]$

$$\left[\frac{7}{16}\right]\left[\frac{7}{16}\right] = [0]_{\text{Vir}} + \left[\frac{3}{2}\right]_{\text{Vir}} = [0] \tag{4.49}$$

$$\left[\frac{7}{16}\right]\left[\frac{3}{80}\right] = \left[\frac{1}{10}\right]_{\text{Vir}} + \left[\frac{6}{10}\right]_{\text{Vir}} = \left[\frac{1}{10}\right]. \tag{4.50}$$

The existence of this operator enables us to predict the existence of certain states in the NS sector. This is because of the fact that it sits entirely in the tri-critical Ising part of the theory therefore its OPE with other fields depend only on the tri-critical Ising content of other state and thus by considering the OPE of the operator corresponding to $[\frac{7}{16}, 0]$ with the other states in the R sector we end up with certain special NS state. The Ising spin field $[\frac{7}{16}]$ maps Ramond ground state $|\frac{7}{16}, 0\rangle$ to NS vacuum $|0, 0\rangle$ and vice versa. Similarly, the spin field $[\frac{7}{16}, 0]$ maps the state $|\frac{3}{80}, \frac{2}{5}\rangle$ to a primary state in the NS sector $|\frac{1}{10}, \frac{2}{5}\rangle$. This procedure can be repeated in opposite direction; the spin field maps $|\frac{1}{10}, \frac{2}{5}\rangle$ to $|\frac{3}{80}, \frac{2}{5}\rangle$. In this way, the tri-critical fusion rule leads to the prediction of existence of the following NS sector

$$\mathrm{NS} : |0, 0\rangle, \quad \left|\frac{1}{10}, \frac{2}{5}\right\rangle. \tag{4.51}$$

Note that the state $|\frac{1}{10}, \frac{2}{5}\rangle$ has dimension $\frac{1}{2}$ and is primary. This means that $G_{-1/2}|\frac{1}{10}, \frac{2}{5}\rangle$ is of dimension 1 and preserves $N = 1$ supersymmetry and thus a candidate for the exactly marginal pertubation of the theory. This is indeed the case, which is proved in [69].

4.1.3 Geometry and G_2 CFT

Let us recall some basic facts about Witten's index. If we consider the Hilbert space on a circle with periodic boundary conditions (Ramond sector) of a 2-dimensional supersymmetric sigma model with n-dimensional target space M, we know

$$\mathrm{Tr}(-1)^F \exp(-\beta H) = \chi(M) = \sum_{i=0}^{n} (-1)^i b_i = n_+ - n_- \tag{4.52}$$

where $\chi(M)$ is the Euler number of M, b_i are the Betti numbers of M, and n_+, n_- denote the total numbet of even and odd dimensional cohomologies. The important idea is that only the ground states $H = 0$ contribute to the index and in a suitable limit the ground states are related to the harmonic forms on M and $(-1)^F$ can be identified with the parity of the degree of the harmonic forms up to an overall sign. It is also known that the number of ground states of the theory are equal to the number of harmonic forms. This means

$$\mathrm{Tr}\exp(-\beta H)|_{\beta\to\infty} = n_+ + n_-. \tag{4.53}$$

We note here two important facts. One is that from the above two physical computations, we can determine only n_+ and n_- not all b_i. This implies the limitation of $N = 1$ SCFT as a probe of geometry. The other is that the sign of $(-1)^F$ cannot be fixed canonically. Therefore, we can only deduce n_+ and n_- up to the exchange $n_+ \leftrightarrow n_-$ from the physical computations. The failure in fixing the sign reminds us of the mirror symmetry. In fact, we can understand the mirror symmetry from this point of view. Suppose M is a Kähler manifold, the fermions are complex and so there is a $U(1)$ conserved charge corresponding to the fermion number F whose action on ground states can be identified with the number of holomorphic forms p minus the number of antiholomorphic forms q of the harmonic form

$$F = p - q. \tag{4.54}$$

By decomposing the ground state to eigenstates of F we can compute the number of cohomology elements with a given value of $p - q$. If in addition M is Calabi-Yau, there is the chiral fermion number F_A which is conserved non-perturbatively. The action of F_A on ground states can be identified with

$$F_A = p + q - d \tag{4.55}$$

where d is the complex dimension of M. From these, we can compute p, q as

$$p - \frac{d}{2} = \frac{1}{2}(F_A + F) = F_L \tag{4.56}$$

$$q - \frac{d}{2} = \frac{1}{2}(F_A - F) = F_R. \tag{4.57}$$

Just as in the previous calculation, there is an ambiguity in the identification of the sign of F_R, F_L. This means that we can determine the Hodge numbers $h^{p,q}$ only up to the ambiguity

$$h^{p,q} \leftrightarrow h^{d-p,q}. \tag{4.58}$$

As we explained, this is the mirror symmetry of the Calabi-Yau manifold. Then, this implies the generalization of the mirror symmetry to G_2 manifolds. In fact, Shatashvili and Vafa claim that the degree of ambiguity left by being unable to decipher all the topological aspects of the target manifold using the algebraic formulation of quantum field theory should be explained by having topologically inequivalent manifolds allowed by the ambiguity to lead to the same quantum field theory up to deformation in the moduli of the quantum field theory. They call this a generalized mirror conjecture.

We next discuss the dimension of moduli space. The dimension of moduli space of deformation of G_2 manifolds is b_3. However, the dimension of moduli space of the CFT is bigger than the geometrical one. This is because we can add the antisymmetric 2-form to the action and this has no geometrical analog. Therefore, we have

$$\dim(\text{moduli}_{\text{physical}}) = \dim(\text{moduli}_{\text{geometrical}}) + b_2. \tag{4.59}$$

As in the Calabi-Yau case, we are interested in manifolds with $b_1 = 0$ in order to realize the minimum number of covariantly constant spinor. In other words, for the manifold to have exactly G_2 holonomy, the b_1 must vanish. Therefore, in the case of G_2 holonomy, there are two independent betti numbers b_2, and b_4 because of the Poincaré duality $b_3 = b_4, b_5 = b_2$. As discussed, we can physically know only the sum $b_2 + b_4 = b_2 + b_3 = b_5 + b_3$, that is, we can compute only the dimension of moduli space.

4.1.4 Left-Right sector

Now we are ready to discuss the non-chiral sector. Our first claim is that there are only following states in the (R, R) ground states

$$(\mathrm{R},\mathrm{R}): \left|\left(\frac{7}{16},0\right)_L;\left(\frac{7}{16},0\right)_R;\pm\right\rangle, \quad \left|\left(\frac{3}{80},\frac{2}{5}\right)_L;\left(\frac{3}{80},\frac{2}{5}\right)_R;\pm\right\rangle. \tag{4.60}$$

The meaning of $\pm$ will be explained momentarily. In principle, we have two other possibilities of left-right combinations; $|(\frac{7}{16},0)_L;(\frac{3}{80},\frac{2}{5})_R;\pm\rangle$ and the same with $L\leftrightarrow R$. The reason for this is as follows. By using the fusion rules (4.49),(4.50), we see that primary field corresponding to first ground state of the above two would lead to the highest weight state $|(0,0)_L;(\frac{1}{10},\frac{2}{5})_R\rangle$ in the NS sector if it acts on the states $|(\frac{7}{16},0)_L;(\frac{3}{80},\frac{2}{5})_R;\pm\rangle$. But this operator has total dimension $\frac{1}{2}$ and is chiral. So, we get an additional chiral operator with spin $\frac{1}{2}$ which is not present in our original extended chiral algebra. Therefore, these additional states do not exist in general.

We next explain the meaning of $\pm$. If we act Φ_0 on the ground states we have $\{\Phi_0,\bar{\Phi}_0\}=0, \Phi_0^2=\bar{\Phi}_0^2=\frac{6}{15}$. Thus, they form a 2-dimensional representation. The $\pm$ signs reflect this fact and denote the two different $(-1)^F$ assignments. The fact that the states exist in pairs is a consequence of the fact that in odd dimensional manifolds, the dual of each cohomology is another cohomology with different degree mod 2. Therefore we can say that the $+$ states correspond to even cohomology elements and $-$ to the odd. So, we can concentrate on the even cohomology elements.

We next act the state $|(\frac{7}{16},0)_L;(\frac{7}{16},0)_R;+\rangle$ on all $+$ Ramond ground states. Then, we obtain the following (NS,NS) states

$$(\mathrm{NS},\mathrm{NS}): |(0,0)_L;(0,0)_R\rangle \quad \left|\left(\frac{1}{10},\frac{2}{5}\right)_L;\left(\frac{1}{10},\frac{2}{5}\right)_R\right\rangle. \tag{4.61}$$

The latter states have dimension $(\frac{1}{2},\frac{1}{2})$ and as mentioned before or proved in [69], these are the exactly marginal operators that preserve the G_2 structure. Therefore the number of such states are given by b_2+b_4. And it is evident that the number of these states are the same as the number of states $|(\frac{3}{80},\frac{2}{5})_l;(\frac{3}{80},\frac{2}{5})_R;+\rangle$. The total number of Ramond ground states is $b_0+b_2+b_4+b_6=1+b_2+b_4+b_1=1+b_2+b_4$. This shows the consistency between the geometrical and physical considerations. The holonomy is exatly G_2 requires $b_1=0$. On the other hand, we see above that the uniqueness of the state $|(\frac{7}{16})_L;(\frac{7}{16})_R;+\rangle$ leads us to the same conclusion.

Now we discuss the relations to compactification of string theory. If we compactify the string theory on G_2 manifolds, we have $N=2$ supersymmetry for type II and $N=1$ for heterotic strings. We construct the corresponding supersymmetry generators. There is a standard ansatz for target space supersymmetry current [82]. This is

$$J_{L,R}=e^{-\frac{\phi_{gh}}{2}}S_3^{\alpha}\sigma_{\frac{7}{16}}^{L,R} \tag{4.62}$$

where ϕ_{gh} is a bosonized 10-dimensional ghost field, S_3^{α} are 3-dimensional spinor and σ is the tri-critical Ising model spin field. This operator has dimension 1. The dimension of 10-dimensional ghost part is always equal to $\frac{3}{8}$, dimension of 3-dimensional spin field is $3\times\frac{1}{16}$ and the dimension of σ is $\frac{7}{16}$. These add up to 1. Note that σ has a unique OPE with the vacuum [0] in the right hand side and we can consider J as a chiral operator. The subscript L, R explains this. From this operator it is easy to construct the 3-dimensional supersymmetry generators. They are

$$Q_{L,R} = \oint J_{L,R}. \tag{4.63}$$

One can show that these generate the supersymmetry algebra.

4.1.5 Example of Joyce

Here we study one of the examples constructed by Joyce [71]. Let us consider T^7 modded out by $\mathbf{Z}_2{}^3$ and denote the generators of the $\mathbf{Z}_2$'s as α, β, γ. We represent each of them by a pair of row vectors. The holonomy part of these elements, which are simultaneously diagonal, are represented by a row of seven (± 1)'s. And they are accompanied by the translation on the torus, which is again written by another row vector. The seven coordinates x_i are taken to have period 1. In terms of these the action of α, β, γ is given by

$$\alpha = [(-1,-1,-1,-1,1,1,1);(0,0,0,0,0,0,0)] \tag{4.64}$$

$$\beta = [(-1,-1,1,1,-1,-1,1);(0,\frac{1}{2},0,0,0,0,0)] \tag{4.65}$$

$$\gamma = [(-1,1,-1,1,-1,1,-1);(\frac{1}{2},0,0,0,0,0,0)]. \tag{4.66}$$

Note that the above holonomies preserve φ. However, they do not sit in an $SU(3)$ group. Let us first look at the untwisted Ramond sector. They can be identified with the cohomology elements of the torus. We project out all the cohomologies except for the following ones. H^0, H^7 which are one dimensional and 7 in H^3 and 7 in H^4. The seven invariant elements precisely correspond to the seven monomials in the definition of φ and $*\varphi$.

We next consider the sectors which give rise to new ground states in the Ramond sector. For this to happen, there must be fixed points of the group action. Out of the seven non-trivial elements only three have fixed points. The three elements are α, β and γ. The fixed points of α consist of sixteen 3-tori and each of them has 8 cohomology elements, $(b_0, b_1, b_2, b_3) = (1,3,3,1)$. To get the final answer, we must project to invariant subsectors under the action of the full group. On this set, the β and γ act freely and leave us with four invariant combinations of the sixteen 3-tori. Therefore, we have four copies of $(1,3,3,1)$ added to the Ramond ground state from this sector. Similar analysis for β gives us again four copies of 3-tori. We get a contribution to the $b_2 + b_4 = 4\times 4\times 2 = 32$ and to the $b_3 + b_5 = 4\times 4\times 2 = 32$ from the α and β sector. On the other hand, the γ sector projected to its invariant fixed point set gives eight copies of $T^3/\mathbf{Z}_2$. The $\mathbf{Z}_2$ acts in the neighborhood of each of these T^3's by

$$(y_1, y_2, y_3, z_1, z_2) \to (\frac{1}{2} + y_1, -y_2, -y_3, z_1, -z_2) \tag{4.67}$$

where y_i denote the coordinates of the fixed T^3 and the z_i denote the orthogonal direction in complex notation. That is, the z_i's are those that goes to minus itself under the action of γ. Of the cohomologies of each of the 3-tori from the above $\mathbf{Z}_2$ action only two elements survive, two in odd and two in even. Therefore, we get the additional 16 elements to $b_2 + b_4$ and 16 to $b_3 + b_5$ from the total of eight 3-tori. In all, we have

$$b_0 = 1, b_2 + b_4 = 55, b_3 + b_5 = 55, b_7 = 1. \tag{4.68}$$

Therefore, we have a 55 dimensional moduli space. 7 of them come from the untwisted sector and correspond to the 7 radii of T^7. Other 48 come from blow up modes in the twisted sectors.

Now we discuss how these moduli fit with the CFT. We concentrate on untwisted moduli. The primary superconformal field of dimension $\frac{1}{2}$ which correspond to the untwisted moduli are ψ^i. From the $X\psi$ OPE, we see that ψ^i has dimension $-1/2$ under X_0, which means that ψ^i have dimension $(T_I)_0 = -X_0/5 = 1/10$ for the tri-critical part of the energy momentum tensor. This coincides with the prediction of the previous section. Moreover, $G_{-1/2}\psi^i = \partial X^i = J^i$ and they commute with $T_I = -X_0/5$. This means that the tri-critical dimension of the moduli is 0, which is necessary to preserve the G_2 structure. In this way, we see that the geometrical analysis is consistent with the CFT description.

As we explained, physically we can know the sum $b_2 + b_4$ but cannot identify b_2 and b_4 separately. The above example indeed implies this. However, this fact is also reflected mathematically. There are singularities in the manifold considered just now. To get a smooth manifold, we must resolve the singularities. However there are inequivalent ways to resolve them with different Betti numbers b_2, b_4 but the same $b_2 + b_4$. In fact, Joyce found that depending on how the singularities are resolved, the resulting smooth manifolds have following Betti numbers

$$b_2 = 8 + l, b_4 = 47 - l \tag{4.69}$$

where l runs from 0 to 8. The meaning of this is as follows. We know physically that the moduli space is smooth near the orbifold points. Therefore, the difference between these answers have to do with turning on different marginal operators. Thus, this example supports the conjecture that topologically distinct manifolds coming from the ambiguity of b_2, b_4 give rise to the same conformal field theory. However, whether this is generally correct is not understood yet.

4.1.6 Topological twist?

We have seen that G_2 compactifications are very similar to the $N = 2$ superconformal field theories which correspond to Calabi-Yau compactification. Both of them have

$N = 1$ space-time supersymmetry upon heterotic compactification and the supersymmetry is obtained from the spectral flow operator. Basically the spectral flow operator is responsible for the twisting. Therefore, it is natural to expect that the G_2 CFT has topological theories, too. As we will see, it seems that there is indeed a topological theory for G_2. However, whether the topological theory exists is far from well-understood.

Recall that the twisting is the same as the insertions of $2g-2$ of these spectral flow operators. If we consider on the sphere, we can define the twsited correlation functions by insertions of two spin fields $\sigma_{\frac{7}{16}}$

$$\langle V_1(z_1,\bar{z}_1)...V_n(z_n,\bar{z}_n)\rangle_{\text{twisted}} = \langle \sigma(0)V_1(z_1,\bar{z}_1)...V_n(z_n,\bar{z}_n)\sigma(\infty)\rangle_{\text{untwisted}}. \tag{4.70}$$

Let us check that this works well. The tri-critical Ising model is one of the minimal models. Such models have well-known free field representations. The bosonized tri-critical Ising stress tensor and supercurrent have the form

$$T_I = -\frac{1}{5}X = (\partial\varphi)^2 + \sqrt{2}\alpha_0\partial^2\varphi = (\partial\varphi)^2 + \frac{1}{2\sqrt{5}}\partial^2\varphi \tag{4.71}$$

$$G_I = \frac{i}{\sqrt{15}}\Phi = e^{\frac{6}{\sqrt{5}}\varphi} \tag{4.72}$$

where we defined $\varphi = \frac{i}{\sqrt{2}}\phi$ and the ϕ is a free boson with $\partial\phi(z)\partial\phi(w) \sim \frac{-1}{(z-w)^2}$. And $\alpha_0 = \frac{1}{2\sqrt{10}}$ is a background charge. The operator $[\frac{7}{16}]$ is represented as

$$\left[\frac{7}{16}\right] = e^{-\frac{5}{2\sqrt{5}}\varphi}. \tag{4.73}$$

Then, inserting the spin fields induces the change of background charge to $\tilde{\alpha}_0 = \frac{3}{\sqrt{10}}$. The change in the charge at infinity induces the change of the energy momentum tensor. The new energy momentum tensor is now given by

$$T_{\text{twisted}} = (\partial\varphi)^2 + \frac{3}{\sqrt{5}}\partial^2\varphi. \tag{4.74}$$

Therefore, the new central charge is $c_{\text{twisted}} = 1 - 12\tilde{\alpha_0}^2 = -\frac{98}{10}$. As we did not change the remaining part T_r, and the remaining part has central charge $c_r = \frac{98}{10}$, the new total central charge becomes

$$c_{\text{twisted}} + c_r = -\frac{98}{10} + \frac{98}{10} = 0. \tag{4.75}$$

This indeed implies the existence of a topological theory. After twisting, we get the special states in the NS sector that have total dimension 0. These are

$$\left|\frac{1}{10}, \frac{2}{5}\right\rangle \to \left|-\frac{2}{5}, \frac{2}{5}\right\rangle \tag{4.76}$$

$$\left|\frac{6}{10}, \frac{2}{5}\right\rangle \to \left|-\frac{2}{5}, \frac{2}{5}\right\rangle \tag{4.77}$$

$$\left|\frac{3}{2}, 0\right\rangle \to |0,0\rangle. \tag{4.78}$$

These states are of course those we would expect for topological observables. Another remarkable fact is that the operators forming a G_2 CFT algebra have integral dimensions after twisting. G has dimension 1, Φ dimension 0, M dimension 2, and K dimension 1. Thus G is a candidate for BRST current of the topological theory.

Chapter 5

Conclusion and summary

We have seen that the conifold transition provides us with interesting dualities from both string and field theoretical points of view. We first showed that the transition between small resolutions is smooth and we know that there can be a smooth change of topology in nature. Then, we performed topological twists, which gives rise to topological field theories. And then, we found the relations to untwisted theories. By making full use of such relations, we could know all genus partition functions of topological closed strings. We next considered topological open strings and demonstated that the topological open string with "D-branes" is equivalent to the Chern-Simons theory on the "D-branes". And we found the duality between topological open and closed strings. Then, we embeded the duality in superstring and finally we showed that this duality can be understood as an M theory flop. The duality indicates the possibility of understanding non-perturbative dynamics of the gauge theory such as instanton effects, gaugino condensation, quark confinement, and so on by using only the perturbative methods.

We also considered a CFT description of G_2 manifolds. We constructed the G_2 CFT and found that it contains the tri-critical Ising model and that this is the sign of the G_2 holonomy. Then we studied highest weight states and found that there is a unique operator which serves as a spectral flow operator. And we discussed the relations to geometry and knew that we can only know the dimension of moduli space from physical computations. And we considered an example of Joyce and it supported the generalized mirror conjecture. Then, we discussed a trial to topological twists. Although there are some indications that there are many interesting phenomena in CFT description of G_2 manifolds, these are not well understood.

Recently, Dijkgraaf and Vafa considered the B model version of the duality. The B model version of

$$\text{Chern-Simons} \leftrightarrow \text{Type IIA}$$

is

$$\text{Holomorphic Chern-Simons} \leftrightarrow \text{Type IIB}.$$

In [55], they showed that the B model topological strings on local Calabi-Yau threefolds are the large N duals of matrix models. By using these facts, they found that the superpotentials of $N = 1$ supersymmetric gauge theories can be computed by the *planar* diagrams of the matrix model [56]. This is the "T-dual" version of the duality that we have explained. One of the advantages of this work comes from the fact that computing the planar amplitudes of the matrix model is generally easier than the computation of the topological string amplitudes. This area has now received much attention.

On the other hand, CFT description of the G_2 manifolds is not well understood yet. Some non-trivial examples are constructed. In [74, 76] the G_2 manifolds of the form $(CY_3 \times S^1)/\mathbf{Z}_2$ are constructed by using the Gepner constructions, and in [78] CFT is realized by the coset construction. One of the main problems of the CFT approach is that except for some examples, the geometrical meanings of the models are unknown. The difficulty of the G_2 comes from the fact that unlike Calabi-Yau manifolds the G_2 manifolds are not complex manifolds.

We have discussed mainly new dualities. Understanding the non-perturbative physics of string theories and field theories is the main theme of current elementary particle physics. Though the duality we have discussed cannot give all what we want to know, it is certainly the cornerstone. And behind the interesting phenomena we have discussed in this review underlie the abundant structures of $N = 1$ supersymmetric theories and $N = 2$ superconformal field theories.

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References

[1] J.Polchinski, *String theory I,II*, Cambridge University Press (1998)

[2] M.B Green, J.H Schwarz, and E.Witten, *Superstring Theory*, vol.2 Cambridge University Press (1987)

[3] P.Griffiths and J.Harris, *Principles of Algebraic Geometry*, John Wiley & Sons (1978)

[4] P.D.Francesco, P.Mathieu and D.Senechal, *Conformal Field Theory*, Springer-Verlag, (1997)

[5] T.Hübsch, *Calabi-Yau Manifolds: A Bestiary for Physicists*, World Scientific (1992)

[6] D.Joyce, *Compact Manifolds With Special Holonomy*, Oxford University Press, (2000)

[7] B.R Greene, *String Theory On Calabi-Yau Manifolds*, hep-th/9702155

[8] S.Gubser, *TASI Lectures: Special Holonomy In String Theory And M-Theory*, hep-th/0201114

[9] P.S.Aspinwall, *K3 Surfaces And String Duality*, hep-th/9611137

[10] E.Cremmer, B.Julia and J.Scherk, *Supergravity Theory In 11 Dimensions*, Phys.Lett. **76B** (1978) 409

[11] E.Witten, *Search For A Realistic Kaluza-Klein Theory*, Nucl.Phys. **B186** (1981) 412

[12] E.Witten, *Dynamical Breaking of Supersymmetry*, Nucl.Phys. **B188** (1981) 513

[13] E.Witten, *Constraints On Supersymmetry Breaking*, Nucl.Phys. **202** (1982) 253

[14] E.Witten, *Supersymmetry and Morse Theory*, J.Diff.Geom. **17** (1982) 661

[15] E.Witten, *Phases Of $N=2$ Theories In Two Dimensions*, hep-th/9301042

[16] P.Aspinwall, B.Greene, and D.Morrison, *Multiple Mirror Manifolds And Topology Change In String Theory*, hep-th/9301043

[17] P.S.Aspinwall, B.R.Greene and D.R.Morrison, *Calabi-Yau Moduli Space, Mirror Manifolds And Space-Time Topology Change In String Theory*, hep-th/9309097

[18] A.Strominger, *Massless Black Holes And Conifolds In String Theory*, hep-th/9504090

[19] B.Greene, D.R.Morrison and A.Strominger, *Black Hole Condensation And The Unification Of String Vacua*, hep-th/9504145

[20] W.Lerche, C.Vafa, and N.Warner, *Chiral Rings In $N = 2$ Superconformal Theories*, Nucl. Phys. **B324** (1989) 427

[21] E.Witten, *Topological Sigma Models*, Commun.Math.Phys. **118** (1988) 411

[22] T.Eguchi, and S.K Yang, *$N = 2$ Superconformal Models As Topological Field Theories*, Mod.Phys.Lett. **A5** (1990) 1693

[23] E.Witten, *Mirror Manifolds And Topological Field Theory*, hep-th/9112056

[24] G.Waterson, *Bosonic Construction Of An $N = 2$ Extended Superconformal Theory In Two Dimensions*, Phys.Lett. **B171** (1986) 77

[25] E.Witten, *The N Matrix Model and Gauged WZW Models*, Nucl.Phys. **371** (1992) 191

[26] P.Candelas, X.C De La Ossa, P.S.Green, and L.Parkes, *A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nucl.Phys. **B359** (1991) 21

[27] M.Bershadsky, S.Cecotti, H.Ooguri, and C.Vafa, *Kodaira-Spencer Theory Of Gravity And Exact Results For Quantum String Amplitudes*, hep-th/9309140

[28] M.Bershadsky, S.Cecotti, H.Ooguri, and C.Vafa, *Holomorphic Anomalies in Topological Field Theories*, hep-th/9302103

[29] I.Antoniadis, E.Gava, K.S.Narain and T.R.Taylor, *Topological Amplitudes In String Theory*, hep-th/9307158

[30] I.Antoniadis, E.Gava, K.S.Narain and T.R.Taylor, *$N = 2$ Type II - Heterotic Duality and Higher Derivative F-terms*, (1995) hep-th/9507115

[31] S.Cecotti, P.Fendley, K.Intriligator and C.Vafa, *A New Supersymmetric Index*, Nucl.Phys.**B386** (1992) 405

[32] A.Klemm, P.Mayr and C.Vafa, *BPS States of Exceptional Non-Critical Strings*, hep-th/9607139

[33] J.Schwinger, *On Gauge Invariance And Vacuum Polarization*, Phys.Rev. **82** (1951) 664

[34] C.Itzykson, J.B.Zuber, *Quantum Field Theory*, McGraw-Hill (1980)

[35] J.M.Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, hep-th/9711200

[36] E.Witten, *Anti de Sitter Space and Holography*, hep-th/9802150

[37] S.S.Gubser, I.R.Klebanov and A.M.Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, hep-th/9802109

[38] I.R.Klebanov and E.Witten, *Superconformal Field Theory On Threebranes at a Calabi-Yau Singularity*, hep-th/9807080

[39] L.Susskind and E.Witten, *Holographic Bound In Anti-de Sitter Space*, hep-th/9805114

[40] R.Gopakumar and C.Vafa, *M-Theory And Topological Strings-I,II*, hep-th/9809187, hep-th/9812127

[41] R.Gopakumar and C.Vafa, *On the Gauge Theory/Geometry Correspondence*, hep-th/9811131

[42] C.Vafa, *Superstrings and Topological Strings at Large N*, hep-th/0008142

[43] M.Atiyah, J.M. Maldacena, and C.Vafa, *An M Theory Flop as a Large N Duality*, hep-th/0011256

[44] B.Acharya, *On Realising* $\mathcal{N}=1$ *Super Yang-Mills in M Theory*, hep-th/0011089

[45] M.Atiyah and E.Witten, *M-Theory Dynamics On A Manifold of* G_2 *Holonomy*, hep-th/0107177

[46] E.Witten, *Anomaly Cancellation On* G_2*-Manifolds*, hep-th/0108165

[47] B.Acharya and E.Witten, *Chiral Fermions from Manifolds Of* G_2 *Holonomy*, hep-th/0109152

[48] E.Witten, *Deconstruction,* G_2 *Holonomy, And Doublet Triplet Splitting*, hep-ph/0201018

[49] T.Friedmann and E.Witten, *Unification Scale, Proton Decay, And Manifolds Of* G_2 *Holonomy*, hep-th/0211269

[50] G.Curio, *Superpotentials For M-Theory On a* G_2 *Holonomy Manifold And Triality Symmetry*, hep-th/0212211

[51] H.Partouche and B.Pioline, *Rolling among* G_2 *vacua*, hep-th/0011256

[52] F.Cachazo, K.Intriligator and C.Vafa, *A Large N Duality via a Geometric Transition*, hep-th/0103067

[53] M.Aganagic and C.Vafa, *G_2 Manifolds, Mirror Symmetry and Geometric Engineering*, hep-th/0110171

[54] M.Aganagic and C.Vafa, *Mirror Symmetry and a G_2 Flop*, hep-th/0105225

[55] R.Dijkgraaf and C.Vafa, *Matrix Models, Topological Strings, and Supersymmetric Gauge Theories*, hep-th/0206255

[56] R.Dijkgraaf and C.Vafa, *A Perturbative Window into Non-Perturbative Physics*, hep-th/0208048

[57] Y.Tachikawa, *Derivation Of The Konishi Anomaly Relation From Dijkgraaf-Vafa With (Bi)-Fundamental Matters*, hep-th/0211189

[58] F.Cachazo, M.R.Douglas, N.Seiberg and E.Witten, *Chiral Rings And Anomalies In Supersymmetric Gauge Theory*, hep-th/0211170

[59] F.Cachazo, N.Seiberg and E.Witten, *Phases of $\mathcal{N} = 1$ Supersymmetric Gauge Theories and Matrices*, hep-th/0301006

[60] J.Gomis, *D-Branes, Holonomy, and M-Theory*, hep-th/0103115

[61] E.Witten, *Chern-Simons Gauge Theory as a String Theory*, hep-th/9207094

[62] E.Witten, *Non-Commutative Geometry And String Field Theory*, Nucl.Phys. **268** (1986) 253

[63] M.Bershadsky and C.Vafa, *D-Branes and Topological Field Theories*, hep-th/9511222

[64] T.R.Taylor and C.Vafa, *RR Flux on Calabi-Yau and Partial Supersymmetry Breaking*, hep-th/9912152

[65] P.Mayr, *On Supersymmetry Breaking In String Theory And Its Realization In Brane Worlds*, hep-th/0003198

[66] K.Becker and M.Becker, *M-Theory on Eight-Manifolds*, hep-th/9605053

[67] S.Gukov, C.Vafa, and E.Witten, *CFT's From Calabi-Yau Four-folds*, hep-th/9906070

[68] G.Papadopoulos and P.K.Townsend, *Compactification Of D=11 Supergravity On Spaces Of Exceptional Holonomy*, hep-th/9506150

[69] S.L.Shatashvili and C.Vafa, *Superstrings and Manifolds of Exceptional Holonomy*, hep-th/9407025

[70] J.M.Figueroa-O'Farrill, *A note on the extended superconformal algebras associated with manifolds of exceptional holonomy*, hep-th/9609113

[71] D.Joyce, *Compact 7-manifolds with holonomy G_2 I,II*, J.Diff.Geom. **43** (1996) 291, J.Diff.Geom. **43** (1996) 329

[72] R.Bryant and S.Salamon, *On The Construction Of Some Complete Metrics With Exceptional Holonomy*, Duke Math.J. **58** (1989) 829

[73] G.W.Gibbons, D.N.Page and C.N.Pope, *Einstein Metrics On S^3, R^3 and R^4 Bundles*, Commun.Math.Phys.**127** (1990) 529

[74] T.Eguchi and Y.Sugawara, *String Theory on G_2 Manifolds Based on Gepner Construction*, hep-th/0111012

[75] T.Eguchi and Y.Sugawara, *CFT Description of String Theory Compactified on Non-compact Manifolds with G_2 Holonomy*, hep-th/0108091

[76] R.Blumenhagen and V.Braun, *Superconformal Field Theories for Compact G_2 Manifolds*, hep-th/0110232

[77] K.Sugiyama and S.Yamaguchi, *Cascade of special holonomy manifolds and heterotic string theory*, hep-th/0108219

[78] K.Sugiyama and S.Yamaguchi, *Coset Construction of Noncompact Spin(7) and G_2 CFTs*, hep-th/0204213

[79] R.Roiban and J.Walcher, *Rational Conformal Field Theories With G_2 Holonomy*, hep-th/0110302

[80] D.Gepner, *Space-time Supersymmetry in Compactified String Theory and Superconformal Models*, Nucl.Phys. **B296** (1988) 757

[81] D.Gepner, *Exactly Solvable String Compactifications on Manifolds of SU(N) Holonomy*, Phys.Lett.**B199** (1987) 380

[82] T.Banks, L.Dixon, D.Friedan and E.Martinec, *Phenomenology And Conformal Field Theory or Can String Theory Predict The Weak Mixing Angle?*, Nucl.Phys.**B299** (1988) 613

[83] L.Dixon, J.A.Harvey, C.Vafa and E.Witten, *Strings on Orbifolds (I)(II)*, Nucl.Phys. **B261** (1985) 678, Nucl.Phys. **B274** (1986) 285

[84] G.Veneziano and S.Yankielowicz, *An Effective Lagrangian For The Pure $N = 1$ Supersymmetric Yang-Mills Theory*, Phys.Lett. **B113** (1982) 231

[85] M.Shifman and A.Vainshtein, *Instantons Versus Supersymmetry: Fifteen Years Later*, hep-th/9902018