Explorations of a self-dual Yang-Mills hierarchy

Diplomarbeit

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Oktober 2006

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1 Introduction

1.1 Overview

This thesis contributes to the theory of “integrable systems” , i.e., systems of “integrable” differential (or difference) equations. Roughly speaking, “integrability” should mean an exact (in contrast to numerical) method to solve differential equations. A given system of differential equations should certainly be called ”integrable”

- if the system can be transformed into a linear one. Linear equations are (at least in principle) always completely solvable.
- if there is a method to solve the initial data problem exactly.

For Hamiltonian systems with a finite number of degrees of freedom there is the notion of “Liouville integrability” [1]. The existence of a sufficient number of Poisson-commuting constants of motions implies the complete integrability (solvability) of the equations of motion. In the case of an infinite number of degrees of freedom, where we are dealing with partial differential (or difference) equations (PDEs), there is no analog of this Liouville theorem. Partially guided by results for systems with a finite number of degrees of freedom, several “integrability aspects” have been proposed which, however, do not necessarily imply a strong form of “integrability” :

- Lax pair or zero-curvature representation [2, 3] : The system is expressed as compatibility (integrability) condition of a *linear* system (see Sec.1.2).
- A formulation as an infinite-dimensional Hamiltonian system with an infinite number of Poisson-commuting conserved quantities. However, there is no analog of the Liouville theorem. A weaker condition would be :

  Existence of an infinite number of (independent) conservation laws.
- Bi-Hamiltonian (or multi-Hamiltonian) structure [4] : More than one Hamiltonian formulation for the dynamical system.
- Bäcklund transformations [5] : Solution-generating methods which produce a new solution from a given one.
- Inverse scattering theory [6, 7] : A method to construct the solution of a PDE for given initial data via calculation of scattering data of a Schrödinger-type operator (or a suitable generalization).
- Riemann-Hilbert-Problem (Riemann-Hilbert factorization, Birkhoff decomposition) [1].
- Formulation in terms of Hirota bilinear equations : “Bi-linearization” of a PDE [5, 8, 9].
- Painleve’ property [5, 10] : There is a conjecture that every (“complexified”) ordinary differential equation obtained from an “integrable” partial differential equation by a similarity reduction (typically determined by a symmetry group of the PDE) has the Painleve’ property that the only moving singularities (those which depend on the initial conditions) of its solutions are rational.

- Infinite number of symmetries [11, 12] in the sense of commuting flows : "hierarchy" of PDEs.

In this work we will concentrate on the hierarchy aspect (see also [13–15]). More precisely, we explore hierarchies associated with the (anti-) self-dual Yang-Mills (sdYM) equations.

The (anti-) sdYM equations play an important role in mathematics and physics. With Euclidean signature of the metric, special solutions of the (anti-) sdYM equations describe "instantons", i.e., minima of the Euclidean Yang-Mills action [16]. The (anti-) sdYM equations are also a central research object in the field of "integrable systems" . Many "integrable" systems, which include most of the known soliton equations, such as the Korteweg-de Vries (KdV) equation , N-wave system, nonlinear Schrödinger (NLS) equation , Kadomtsev-Petviashvili (KP) equation , have been obtained by reductions, i.e., special choices of the Yang-Mills potentials. It has been conjectured that actually (almost) *all* "integrable" systems can be obtained in this way [17].

The self-dual Yang-Mills hierarchy is a system which consists of an infinite number of commuting flows in which the sdYM equations are embedded. However, as a matter of fact, it is not clear in which sense this really defines a hierarchy (see Sec.2.2). Only by a suitable reduction, we actually obtain a hierarchy (see Sec.3). The purpose of this thesis is to explore examples of reductions and to find out under which conditions we indeed obtain hierarchies.

We consider more generally hierarchies with dependent variables in any (possibly noncommutative) associative algebra (see also [18–20], in particular). By specialization to matrix algebras, this includes cases of coupled systems of equations. Matrix versions of integrable equations are also a possible source for new integrable equations. Another motivation is given by the *operator method* [21–23], which associates with a (scalar) nonlinear equation an operator version. By a suitable map, solutions of an operator valued equation lead to solutions of the scalar one. This requires, of course, a generalization of the respective equation with dependent variable in a noncommutative associative algebra. Due to these reasons, in the following the calculations are performed "noncommutatively" unless we make a special assumption.

In the following subsections, we explain a bit more some points this thesis is based on. This includes the notions of a "Lax pair", "zero-curvature condition", "linear system", "symmetry of a PDE" and a short review of the (anti-) sdYM equations. In Sec.2.1, we give a definition of a sdYM hierarchy and build a relationship with the sdYM equations. We consider in detail a subsystem of the sdYM hierarchy (which we call pre-sdYM hierarchy). In Sec.3, we give a general overview of reductions and consider examples of reductions in order to show how to reduce the pre-sdYM hierarchy to a hierarchy associated with well-known integrable PDEs such as the N-wave system, KdV equation. In Sec.4 we consider an infinite system introduced in [13] which leads to 2+1-dimensional integrable systems. Examples of reductions of the system are discussed in Sec.4.2.

1.2 Lax pair, zero-curvature condition, linear system

The notion of Lax pair [2] as well as zero-curvature (or Zakharov-Shabat) condition is at the basics of the theory of integrable systems. These two notions are equivalent to the compatibility (integrability) condition of a linear system which is a good startpoint to consider integrable systems.
Firstly, we consider the Lax equation :

$$L_t = [\, P \,,\, L \,]. \tag{1.1}$$

where B, L are linear operators, This is the compatibility condition of the following linear system

$$\begin{aligned} (\partial_t - P)\Psi &= 0, \\ (L - \lambda)\Psi &= 0, \end{aligned} \tag{1.2}$$

where λ is a constant. If the equation (1.1) is equivalent to the considered PDE, we call P, L a "Lax pair" associated with the considered PDE. By choosing P, L,

$$\begin{aligned} L &:= \partial_x^2 - u, \\ P &:= \partial_x^3 - \frac{3}{2}u\partial_x - \frac{3}{4}u_x, \end{aligned} \tag{1.3}$$

the Lax equation (1.1) is equivalent to the KdV equation

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{4}(u^2)_x \tag{1.4}$$

where ∂_x is a differential operator with respect to x and u is a function of x, t.

Next we consider a zero-curvature (or Zakharov-Shabat) condition

$$P_x - Q_t + [\,P\,,\,Q\,] = 0 \tag{1.5}$$

where P, Q are linear operators. (1.5) is equivalent to the compatibility condition of the following linear system

$$\begin{aligned} (\partial_t - P)\Psi &= 0, \\ (\partial_x - Q)\Psi &= 0, \end{aligned} \tag{1.6}$$

If we choose

$$\begin{aligned} P &= \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \\ Q &= \begin{pmatrix} -u_x & 2u + 4\lambda \\ -u_{xx} - 2\lambda u + 4\lambda^2 - 2u^2 & u_x \end{pmatrix}, \end{aligned} \tag{1.7}$$

the equation (1.5) is equivalent to the KdV equation

$$u_t = u_{xxx} + 3(u^2)_x. \tag{1.8}$$

Only very few PDEs possess a "good" (see Appendix $A.3.2$) Lax pair or a zero curvature formulation and it is difficult to find one. Moreover, the two formalisms do not cover with each other. However, both are special cases of the compatibility condition of the following linear system

$$\begin{aligned} X[u]\Psi &= 0, \\ Y[u]\Psi &= 0, \end{aligned} \tag{1.9}$$

where the operators $X[u]$, $Y[u]$ are functionals of u. This means that the considered nonlinear system (e.g., the KdV equation) is expressed as

$$[\,X[u]\,,\,Y[u]\,] = 0. \tag{1.10}$$

1.3 Symmetries of PDEs

Let us consider the KdV equation

$$u_t = u_{xxx} + uu_x, \tag{1.11}$$

here with dependent variable in a commutative algebra. We look for solutions which depend differentiably on a parameter τ,

$$u = u(t, y, \tau). \tag{1.12}$$

A necessary condition for u in (1.12) to solve (1.11) is that u_τ has to satisfy the *linearized* KdV equation:

$$(u_\tau)_t = (u_\tau)_{xxx} + u_\tau u_x + u(u_\tau)_x \tag{1.13}$$

We may understand such a 1-parameter family of solutions as the result of the action of a 1-dimensional Lie group on a particular solution. Then u_τ should be viewed as the generator of this symmetry.

For example, the KdV equation (1.11) has the symmetry

$$u_\tau = \frac{1}{6}u_{xxxxx} + \frac{5}{18}uu_{xxx} + \frac{5}{9}u_x u_{xx} + \frac{5}{36}u^2 u_x. \tag{1.14}$$

Although it is straightforward to verify that u_τ satisfies the linearized KdV equation (1.13), this is a lengthy calculation and it is therefore helpful to use computer algebra. In Appendix $C.4.1$ a corresponding FORM program is attached.

If an equation possesses an infinite number of symmetries which commute with each other, then we have a hierarchy associated with this equation [1]. We refer to [11, 12] for further details on symmetries of PDEs.

[1] There are symmetries which do not belong to a hierarchy. Such symmetries are called additional symmetries. See [24], Chapter 7.

1.4 (anti-) self-dual Yang-Mills (sdYM) equations

In the following we recall the definition of (anti-) self-dual Yang-Mills equation [13, 25] and its properties. We use Cartesian coordinates $x = (x^\mu), \mu = 0, 1, 2, 3$ on Euclidean space $\mathbb{R}^4$ with the Euclidean metric [2]

$$ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{1.15}$$

The Yang-Mills field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu], \tag{1.16}$$

where A_μ has values in a matrix Lie algebra $\mathcal{G}$, and $\partial_\mu = \frac{\partial}{\partial x_\mu}$.

$F_{\mu\nu}$ is obviously antisymmetric and invariant under the *gauge-transformation*

$$A_\mu \mapsto \tilde{A}_\mu = g^{-1} A_\mu g - g^{-1} \partial_\mu g \tag{1.17}$$

where g is an element of a Lie group G with Lie algebra $\mathcal{G}$. Now we define the duality operation (Hodge star operator)

$$*F_{\mu\nu} = \frac{1}{2} \sum_{\kappa,\lambda=0}^{3} \epsilon_{\mu\nu\kappa\lambda} F_{\kappa\lambda}, \tag{1.18}$$

where $\epsilon_{\mu\nu\kappa\lambda}$ is the totally antisymmetric Levi-Civita pseudo-tensor with $\epsilon_{0123} = 1$ in the coordinates chosen above. The (anti-) self-dual Yang-Mills equations are defined by

$$*F_{\mu\nu} = \pm F_{\mu\nu} \tag{1.19}$$

where the sign $\pm$ selects the self-dual (+) and the anti-self-dual case (-), respectively. From (1.18) we obtain for the self-dual case

$$F_{01} = F_{23}, \qquad F_{02} = F_{31}, \qquad F_{03} = F_{12}. \tag{1.20}$$

These are the sdYM equations in the Cartesian coordinates on $\mathbb{R}^4$. For the anti-self-dual case, we obtain (see [16], for example)

$$F_{01} = -F_{23}, \qquad F_{02} = -F_{31}, \qquad F_{03} = -F_{12}. \tag{1.21}$$

In the following, let us concentrate on the self-dual case.

With a different choice of the complex coordinates, the sdYM equations (1.20) are rewritten in a different way. In terms of the complex coordinates

$$\begin{aligned} y = x^1 + ix^2, \quad \overline{y} = x^1 - ix^2, \\ z = x^0 + ix^3, \quad \overline{z} = x^0 - ix^3, \end{aligned}$$

(1.20) reads

$$F_{yz} = 0, \qquad F_{\overline{y}\,\overline{z}} = 0, \qquad F_{y\overline{y}} + F_{z\overline{z}} = 0. \tag{1.22}$$

[2] Alternatively, one can use the metric with signature (+ + - -), see [26], for example.

These equations are the compatibility condition of the linear system

$$D_1\Psi = A_1\Psi, \qquad D_2\Psi = A_2\Psi, \tag{1.23}$$

where

$$\begin{aligned} D_1 &:= \partial_y + \lambda\partial_{\bar z}, \quad & D_2 &:= \partial_z - \lambda\partial_{\bar y}, \\ A_1 &:= A_y + \lambda A_{\bar z}, \quad & A_2 &:= A_z - \lambda A_{\bar y}, \end{aligned}$$

and λ is a (spectral) parameter.

Remark. In particular,

$$0 = F_{\bar y\bar z} = \partial_{\bar y}A_{\bar z} - \partial_{\bar z}A_{\bar y} - [A_{\bar y}, A_{\bar z}] \tag{1.24}$$

is a zero curvature condition in the $\bar y\bar z$-plane. This implies that $A_{\bar y}, A_{\bar z}$ are locally "pure gauge". Then there is a gauge transformation which transforms $A_{\bar y}, A_{\bar z}$ to *commuting constant* matrices such that

$$[A_{\bar y}, A_{\bar z}] = 0 \tag{1.25}$$

(see Lemma 1 in [13]). The sdYM equations (1.22) are then reduced to

$$\partial_{\bar y}A_y + \partial_{\bar z}A_z = 0, \qquad F_{yz} = 0, \qquad . \tag{1.26}$$

1.5 Potential forms of the (anti-) sdYM equations

As we mentioned in the last subsection, by a gauge transformation we can reach (1.26).
a) The last equation in (1.26) is the integrability condition of the linear system

$$(\partial_y - A_y)J = 0 = (\partial_z - A_z)J \ . \tag{1.27}$$

If J is an invertible matrix field, this is equivalent to

$$A_y = J_y\,J^{-1}\,, \qquad A_z = J_z\,J^{-1} \tag{1.28}$$

and the remaining sdYM equation becomes

$$\partial_{\bar z}(J_z\,J^{-1}) + \partial_{\bar y}(J_y\,J^{-1}) = 0 \ . \tag{1.29}$$

b) Instead, we can solve the first of equations (1.26) locally by introducing a potential K such that

$$A_y = \partial_{\bar z}K\,, \qquad A_z = -\partial_{\bar y}K \ . \tag{1.30}$$

The remaining sdYM equation then takes the form

$$(\partial_z\partial_{\bar z} + \partial_y\partial_{\bar y})K - [\partial_{\bar z}K, \partial_{\bar y}K] = 0 \ . \tag{1.31}$$

For more details, see [16], for example.

2 Self-dual Yang-Mills hierarchies

In this section we consider the sdYM hierarchy introduced in [27], starting from an infinite linear system. We show that the linear system (1.23) in Sec.1.4 can be generalized to a infinite linear system and this leads to the sdYM hierarchy considered in [27]. Furthermore, we can see that this sdYM hierarchy contains several subsystems such as the sdYM hierarchy introduced in [13] and also other versions of sdYM hierarchies (see [14–16, 26, 28, 29]) by a suitable restriction on it.

2.1 From a linear system to a hierarchy

In Sec.1.4, we have seen that the compatibility condition of the linear system (1.23) is equivalent to the sdYM equations . As a generalization of (1.23), we consider the following linear system studied in [27] [3]

$$
\begin{aligned}
(\lambda\partial_{x_{k-1}} - \partial_{x_k})\Psi &= (\lambda A_{k-1} - D_k)\Psi, && (2.1)\\
(\lambda\partial_{t_{l-1}} - \partial_{t_l})\Psi &= (\lambda C_{l-1} - B_l)\Psi, \qquad k, l = 1, 2, \ldots && (2.2)
\end{aligned}
$$

where x_k, t_l are independent variables and A_k, B_k, C_k, D_k are functions which take values in $\mathcal{A}$, a (Lie) algebra of $N \times N$ matrices. Obviously, the linear system (2.1), (2.2) is gauge covariant under the gauge transformation

$$
\begin{aligned}
\Psi \mapsto \tilde{\Psi} &= g^{-1}\Psi,\\
A_\mu \mapsto \tilde{A}_\mu &= g^{-1}A_\mu g - g^{-1}\partial_\mu, \qquad A_\mu,\ g \in \mathcal{A}. && (2.3)
\end{aligned}
$$

If there is a nontrivial Ψ which solves (2.1) and (2.2) for k, $l = 1, 2, \ldots$, the compatibility (integrability) condition must be satisfied, so that

$$
\begin{aligned}
&[\lambda\partial_{x_{k-1}} - \partial_{x_k} - (\lambda A_{k-1} - D_k), \lambda\partial_{t_{l-1}} - \partial_{t_l} - (\lambda C_{l-1} - B_l)] = 0, && (2.4)\\
&\qquad k, l = 1, 2,\\
&[\lambda\partial_{x_{k-1}} - \partial_{x_k} - (\lambda A_{k-1} - D_k), \lambda\partial_{x_{k'-1}} - \partial_{x_{k'}} - (\lambda A_{k'-1} - D_{k'})] = 0, && (2.5)\\
&\qquad k, k' = 1, 2, \ldots, \qquad k \neq k\\
&[\lambda\partial_{t_{l-1}} - \partial_{t_l} - (\lambda C_{l-1} - B_l), \lambda\partial_{t_{l'-1}} - \partial_{t_{l'}} - (\lambda C_{l'-1} - B_{l'})] = 0, && (2.6)\\
&\qquad l, l' = 1, 2, \ldots, \qquad l \neq l'
\end{aligned}
$$

The first equation (2.4) is equivalent to the definition of the sdYM hierarchy introduced in [16, 26]. In the following, we assume that we can find such functions (typically matrix-valued functions) A_k, B_k, C_k, D_k. First we show that the equations (2.1) and (2.2) are indeed a generalization of (1.23).

Theorem 2.1 *The compatibility condition of an arbitrary pair of equations from* (2.1) *and* (2.2) *yields the sdYM equations (or a restriction of it).*

Proof. Obviously, the compatibility condition of (2.1) and (2.2) for fixed k, l yields the sdYM equations by choosing

[3] One can also consider the case where the indices k,l extend to negative integers. We restrict our considers to positive indices, however.

$$x_{k-1} = \bar{z}, \qquad x_k = -y, \qquad t_{l-1} = -\bar{y}, \qquad t_l = -z\ ,$$
$$A_{x-1} = A_{\bar{z}}, \quad B_l = -A_z, \quad C_{l-1} = -A_{\bar{y}}, \quad D_k = -A_y\ .$$

Next we consider the following linear system, which consists of two different equations of (2.1),

$$\begin{aligned}(\lambda\partial_{x_{k-1}} - \partial_{x_k})\Psi &= (\lambda A_{k-1} - D_k)\Psi,\\ (\lambda\partial_{x_{k'-1}} - \partial_{x_{k'}})\Psi &= (\lambda A_{k'-1} - D_{k'})\Psi, \qquad k \neq k', k'-1.\end{aligned} \tag{2.7}$$

For $k \neq k'$, the compatibility condition of (2.7) also yields the sdYM equations under the identification

$$x_{k-1} = \bar{z}, \qquad x_k = -y, \qquad x_{k'-1} = -\bar{y}, \qquad x_{k'} = -z\ ,$$
$$A_{x-1} = A_{\bar{z}}, \quad D_{k'} = -A_z, \quad A_{k'-1} = -A_{\bar{y}}, \quad D_k = -A_y\ .$$

For $k = k'-1$, the linear system (2.7) reads

$$\begin{aligned}(\lambda\partial_{x_{k-1}} - \partial_{x_k})\Psi &= (\lambda A_{k-1} - D_k)\Psi,\\ (\lambda\partial_{x_k} - \partial_{x_{k+1}})\Psi &= (\lambda A_k - D_{k+1})\Psi,\end{aligned} \tag{2.8}$$

whose compatibility condition leads to

$$\begin{aligned}&\partial_{x_k} A_{k-1} - \partial_{x_{k-1}} A_k + [\, A_{k-1}\, ,\, A_k\,] = 0,\\ &\partial_{x_{k+1}} D_k - \partial_{x_k} D_{k+1} + [\, D_k\, ,\, D_{k+1}\,] = 0,\\ &\partial_{x_k} A_k + \partial_{x_{k-1}} D_{k+1} - \partial_{x_{k+1}} A_{k-1} - \partial_{x_k} D_k - [\, A_{k-1}\, ,\, D_{k+1}\,] - [\, D_k\, ,\, A_k\,] = 0.\end{aligned} \tag{2.9}$$

Under the identification

$$x_{k-1} = \bar{z}, \qquad x_k = -y, \qquad x_{k+1} = -z,$$
$$A_{x-1} = A_{\bar{z}}, \quad A_k = -A_{\bar{y}}, \quad D_k = -A_y, \quad D_{k+1} = -A_z,$$

we find that (2.9) yields

$$\begin{aligned}&\partial_{\bar{z}} A_{\bar{y}} - \partial_y A_{\bar{z}} - [\, A_{\bar{z}}\, ,\, A_{\bar{y}}\,] = 0,\\ &\partial_z A_y - \partial_y A_z - [\, A_z\, ,\, A_y\,] = 0,\\ &\partial_y A_{\bar{y}} + \partial_y A_{\bar{y}} - \partial_{\bar{z}} A_z - \partial_y A_y + [\, A_{\bar{z}}\, ,\, A_z\,] - [\, A_y\, ,\, A_{\bar{y}}\,] = 0.\end{aligned} \tag{2.10}$$

This is the sdYM equation with $y = \bar{y}$. In the same way, we can see that the compatibility condition of two different equations of (2.2) leads to the sdYM equations (or the sdYM equations with the condition $y = \bar{y}$). ■

Next we write the linear system (2.1), (2.2) in a different form. For this purpose, we eliminate the derivative with respect to the variables $x_1, \ldots, x_{m-1}$, $m = 1, 2, \ldots$ and $t_1, \ldots, t_{n-1}$, $n = 1, 2, \ldots$ as follows.

$$\left.\begin{array}{rcll}(\lambda\partial_{x_0} - \partial_{x_1})\Psi &=& (\lambda A_0 - D_1)\Psi & \times\ \lambda^{m-1}\\ (\lambda\partial_{x_1} - \partial_{x_2})\Psi &=& (\lambda A_1 - D_2)\Psi & \times\ \lambda^{m-2}\\ &\vdots&&\\ (\lambda\partial_{x_{m-1}} - \partial_{x_m})\Psi &=& (\lambda A_{m-1} - D_m)\Psi & \times\ \lambda^{0}\end{array}\right\}\ \text{adding the first } m \text{ equations}$$

$$\vdots$$

$$\left.\begin{array}{rcll}(\lambda\partial_{t_0}-\partial_{t_1})\Psi &=& (\lambda C_0-B_1)\Psi & \times\ \lambda^{n-1}\\ (\lambda\partial_{t_1}-\partial_{t_2})\Psi &=& (\lambda C_1-B_2)\Psi & \times\ \lambda^{n-2}\\ &\vdots&&\\ (\lambda\partial_{t_{n-1}}-\partial_{t_n})\Psi &=& (\lambda C_{n-1}-B_n)\Psi & \times\ \ \lambda^{0}\end{array}\right\}\text{adding the first } n \text{ equations}$$

$$\vdots$$

$$\Downarrow$$

$$\begin{aligned}(\lambda^m\partial_{x_0}-\partial_{x_m})\Psi &= (\sum_{i=1}^{m}A_{m-i}\lambda^i-\sum_{i=0}^{m-1}D_{m-i}\lambda^i)\Psi,\\ (\lambda^n\partial_{t_0}-\partial_{t_n})\Psi &= (\sum_{i=1}^{n}C_{n-i}\lambda^i-\sum_{i=0}^{n-1}B_{n-i}\lambda^i)\Psi\end{aligned}\tag{2.11}$$

Setting $A_i=C_i=0,\ i=0,1,\ldots$, (2.11) yields the "positive" part of the linear system considered by Takasaki [4] [29] (see also [14, 15]).

We define the sdYM hierarchy as the compatibility condition (Zakharov-Shabat equations) of the linear system (2.1), (2.2), respectively (2.11). By choosing a fixed m and n, $m,n=1,2,\ldots$, we obtain several subsystems which are equivalent to other versions of sdYM hierarchies. We should consider the compatibility condition of the linear system (2.11) as a large "framework", in which we can find "smaller" sdYM hierarchies, which appeared in the literature, as subsystems. In the following, we give two examples of such subsystems for fixed m. For convenience, in the following we use the identifications

$$\begin{array}{llll}x_0=\bar{z}, & x_m=-y,\ i=1,2,\ldots & t_0=-\bar{y}, & t_1=-z\ ,\\ A_0=A_{\bar{z}}, & B_1=-A_z, & C_0=-A_{\bar{y}}, & D_1=-A_y.\end{array}$$

m = 1 Setting $C_i=0,\ i=1,2,\ldots$, (2.11) reads

$$\begin{aligned}(\partial_y+\lambda\partial_{\bar{z}})\Psi &= (A_y+\lambda A_{\bar{z}})\Psi,\\ (\partial_{t_n}+\lambda^n\partial_{\bar{y}})\Psi &= (\sum_{i=1}^{n-1}B_{n-i}\lambda^i+A_{\bar{y}}\lambda^n)\Psi.\end{aligned}\tag{2.12}$$

whose compatibility condition of the system (2.11) yields

$$[\,\partial_y+\lambda\partial_{\bar{z}}-A_y-\lambda A_{\bar{z}}\,,\,\partial_{t_n}+\lambda^n\partial_{\bar{y}}-\sum_{i=1}^{n-1}B_{n-i}\lambda^i-A_{\bar{y}}\lambda^n\,]=0 \qquad n=1,2,\ldots \tag{2.13}$$

[4]Note that Takasaki considers also a sdYM hierarchy for negative indices m, n.

We mainly consider this system (in the following, we call this system "pre-sdYM hierarchy") in this thesis (note that we name $t_n = -z_{n+1}, B_n = -L_n, n = 1, 2, \dots$ and $L_0 = -A_{\bar{y}}$ in the following sections). However, this is just a part of the compatibility condition of (2.11). We must also consider the compatibility condition of (2.11) for $n, n' = 1, 2, \dots, n \neq n'$, so that

$$[\, \partial_{t_n} + \lambda^n \partial_{\bar{y}} - \sum_{i=0}^{n-1} B_{n-i}\lambda^i - A_{\bar{y}}\lambda^n \,,\, \partial_{t_{n'}} + \lambda^{n'} \partial_{\bar{y}} - \sum_{i=0}^{n'-1} B_{n'-i}\lambda^i - A_{\bar{y}}\lambda^{n'} \,] = 0, \qquad n, n' = 1, 2, \dots \qquad n \neq n' \tag{2.14}$$

We will treat this system in more detail in Sec.2.2 and Sec.2.3. Furthermore, we consider its reductions in Sec.3.

Remark. The reduced linear system (2.12) is *not* gauge covariant anymore (note that the linear system (2.11) is gauge covariant) since we impose $C_i = 0$, $i = 1, 2, \dots$ Rewriting (2.12) in the following form,

$$(\mathcal{D}_1 - P)\Psi = 0, \tag{2.15}$$
$$(\mathcal{D}_n - Q)\Psi = 0. \tag{2.16}$$

where

$$\mathcal{D}_1 := \partial_y + \lambda \partial_{\bar{z}}, \qquad \mathcal{D}_n := \partial_{t_n} + \lambda^n \partial_{\bar{y}},$$
$$P := A_y + A_{\bar{z}}\lambda, \qquad Q := \sum_{i=1}^{n-1} B_{n-i}\lambda^i + A_{\bar{y}}\lambda^n.$$

(2.15) is gauge covariant under the gauge transformation (2.3),

$$(\mathcal{D}_1 - P)\Psi \mapsto (\mathcal{D}_1 - \tilde{P})\tilde{\Psi} = g^{-1}(\mathcal{D}_1 - P)\Psi. \tag{2.17}$$

On the contrary, (2.16) is transformed as follows

$$(\mathcal{D}_n - Q)\Psi \mapsto (\mathcal{D}_n - \tilde{Q})\tilde{\Psi} = g^{-1}(\mathcal{D}_n - Q - \sum_{i=1}^{n-1} \lambda^i g_{t_{n-i}} g^{-1})\Psi. \tag{2.18}$$

This means that the compatibility condition of (2.15), (2.16) is gauge dependent.

m = 2 Setting $x_2 = x_1 = -y$, $A_1 = 0$, $C_i = 0$, $i = 1, 2, \dots$, and $D_2 = 0$, we obtain

$$[\, \lambda^2 \partial_{\bar{z}} + \partial_y - \lambda A_y - \lambda^2 A_{\bar{z}} \,,\, \partial_{t_n} + \lambda^n \partial_{\bar{y}} - \lambda^n A_{\bar{y}} - \sum_{i=0}^{n-1} B_{n-i}\lambda^i \,] = 0, \qquad n = 1, 2, \dots \tag{2.19}$$

and

$$[\, \partial_{t_n} + \lambda^n \partial_{\bar{y}} - \lambda^n A_{\bar{y}} - \sum_{i=0}^{n-1} B_{n-i}\lambda^i \,,\, \partial_{t_{n'}} + \lambda^{n'} \partial_{\bar{y}} - \lambda^{n'} A_{\bar{y}} - \sum_{i=0}^{n'-1} B_{n'-i}\lambda^i \,] = 0, \qquad n, n' = 1, 2, \dots \qquad n \neq n'. \tag{2.20}$$

which lead to a (2+1)-dimensional generalization of the Derivative Nonlinear Schrödinger (DNLS) hierarchy studied by Strachan (see [27]). Obviously, we can obtain a large number of other systems by fixing the m and n. As mentioned above, in this thesis we will especially concentrate on the system (2.13) and (2.14).

2.2 A pre-sdYM hierarchy

In this subsection we turn to a study of the system (2.13). In the following, $y, z = z_2, z_k$, $k = 3, 4, \ldots$, are complex variables, $\mathcal{G}$ a Lie algebra, and $\hat{\mathcal{G}}$ the Lie algebra of objects which are polynomials in an indeterminate λ and formal power series in its inverse λ^{-1} with coefficients in $\mathcal{G}$. Furthermore, ad denotes the adjoint action of the Lie group G which is generated by $\mathcal{G}$,

$$\mathrm{ad}A(B) = [A, B], \qquad A, B \in \mathcal{G}.$$

$\hat{G}_-$ names the formal group generated by $\hat{\mathcal{G}}_-$. In the following, partial derivatives of elements of $\mathcal{G}$ or $\hat{\mathcal{G}}$ are sometimes expressed in index notation, e.g., $A_{1,y} := \partial_y A_1$.

Moreover we name $t_n = -z_{n+1}, n = 1, 2, \ldots$, following [13] and in (2.13), (2.14), we write

$$\sum_{i=0}^{n-1} B_{n-i}\lambda^i + \lambda A_{\bar{y}} = -\sum_{i=0}^{n} L_{n-i}\lambda^i, \qquad L_0 := -A_{\bar{y}} \tag{2.21}$$

Furthermore, we rename the index

$$n \to k-1, \qquad k = 2, 3, \ldots \tag{2.22}$$

and introduce the differential operators

$$D_1 = \partial_y + \lambda\,\partial_{\bar{z}} \tag{2.23}$$
$$D_k = \partial_{z_k} - \lambda^{k-1}\,\partial_{\bar{y}} \qquad k = 2, 3, \ldots \tag{2.24}$$

Then we find that (2.13) is written by

$$[D_1 - A_1, D_k - A_k] = 0 \qquad k = 2, 3, \ldots \tag{2.25}$$

with

$$A_1 = A_y + \lambda\, A_{\bar{z}} \tag{2.26}$$
$$A_k = \sum_{j=0}^{k-1} L_{k-1-j}\,\lambda^j \qquad k = 2, 3, \ldots \tag{2.27}$$

Using the above definitions of D_1, D_k, (2.25) becomes

$$A_{1,z_k} - A_{k,y} + [A_1, A_k] - \lambda\,(A_{k,\bar{z}} - [A_{\bar{z}}, A_k]) - \lambda^{k-1}\,A_{1,\bar{y}} = 0\ . \tag{2.28}$$

Then we obtain

$$A_{y,z_k} = (\partial_y - \mathrm{ad}A_y)L_{k-1}, \qquad k = 2, 3, \ldots \tag{2.29}$$
$$(\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}})L_0 = -A_{\bar{z},\bar{y}} \tag{2.30}$$
$$(\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}})L_1 = -(\partial_y - \mathrm{ad}A_y)L_0 - A_{y,\bar{y}} \tag{2.31}$$
$$(\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}})L_i = -(\partial_y - \mathrm{ad}A_y)L_{i-1} \qquad i = 2, \ldots, k-2 \tag{2.32}$$
$$(\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}})L_{k-1} = -(\partial_y - \mathrm{ad}A_y)L_{k-2} + A_{\bar{z},z_k}\ . \tag{2.33}$$

Imposing the condition

$$A_{\bar{z},z_k} = 0 \qquad k = 2, 3, \dots \tag{2.34}$$

this reduces to the "evolution equations"

$$A_{y,z_k} \;\; = \;\; (\partial_y - \mathrm{ad}A_y)L_{k-1} \qquad k = 2, 3, \dots \tag{2.35}$$

and the "recursion formula"

$$(\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}})L_i \;\; = \;\; -(\partial_y - \mathrm{ad}A_y)L_{i-1} - A_{y,\bar{y}}\,\delta_{i,1} - A_{\bar{z},\bar{y}}\,\delta_{i,0} \qquad i = 0, 1, 2, \dots \tag{2.36}$$

(where $L_{-1} = 0$). One easily verifies that in terms of

$$L = \sum_{i=0}^{\infty} L_i\,\lambda^{-i} \tag{2.37}$$

the recursion relations (2.36) can be written equivalently as

$$D_1 L = [A_1, L] - \partial_{\bar{y}} A_1 \ . \tag{2.38}$$

Remark. The equation (2.36) simply becomes a recursion formula if the operator $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ possesses an inverse, which however is rarely the case in reductions. Moreover, the authors of [13] showed that (2.13) implies indeed (2.14) if the operator $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is invertible (see sec.2.3). However, in several examples of reductions we obtain hierarchies associated with the considered system from (2.13), though the operator is *not* invertible (See Sec.3). But as a matter of fact, the system (2.13) is *not* automatically a hierarchy and we should rather call this system as "pre-sdYM hierarchy" (where we do not assume that the operator $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is invertible) . We sketch the relations in the following, assuming that a reduction has been chosen.

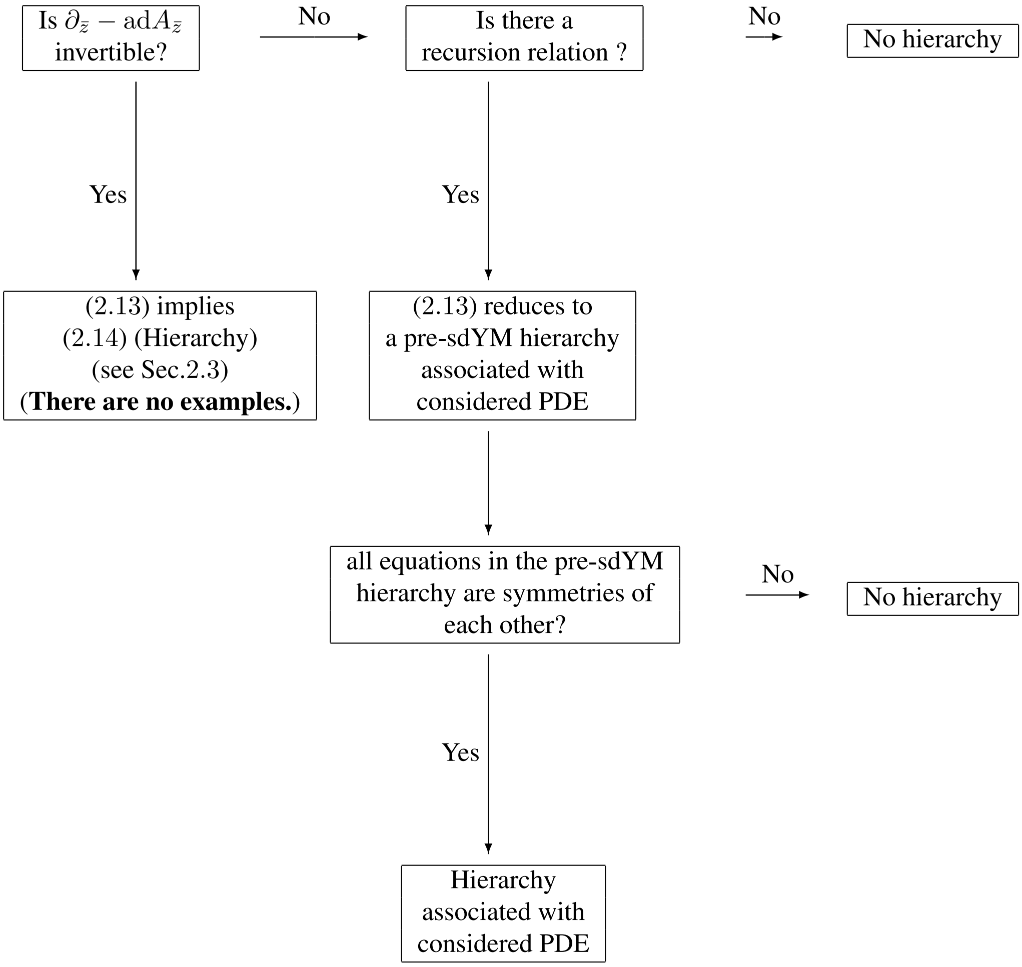

Figure 1 : Pre-sdYM hierarchy

Remark. In general, the equation (2.35) and (2.36) are not genuinly the "evolution equation" and "recursion relation". Depending on the reduction ansatz, (2.35) can include conditions which are necessary to determine L_i (see Sec.3.3.1).

2.3 More about the pre-sdYM hierarchy

In this subsection we show that we can reach (2.14) starting from the pre-sdYM hierarchy (2.13) under the particular assumption that the operator $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is invertible. In the following, we restrict to the gauge where $L_0 = -A_{\bar{y}}$ and $A_{\bar{z}}$ are constant commuting matrices .

Lemma 2.1 *Let $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ be invertible. There is an element $W \in \hat{\mathcal{G}}_-$ such that*

$$D_1 - A_1 = \hat{W}(D_1 - \lambda\, A_{\bar{z}})\hat{W}^{-1} \tag{2.39}$$

where $\hat{W}$ denotes the multiplication operator associated with W.

Proof: Rewritting (2.39) in the form

$$D_1(W) = A_1 W - \lambda W A_{\bar{z}}. \tag{2.40}$$

We look for W in the form

$$W = \mathbf{1} + \sum_{i=1}^{\infty} W_i \lambda^{-i}, \qquad W_i \in \mathcal{G}. \tag{2.41}$$

Inserting this in (2.40), we get

$$A_y + \sum_{i=1}^{\infty} (A_y W_i - W_{i,y}) \lambda^{-k} = \sum_{i=1}^{\infty} (\partial_{\bar{z}} - \mathrm{ad} A_{\bar{z}}) W_i \lambda^{-i+1}, \tag{2.42}$$

which yields the recursion relation

$$(\partial_{\bar{z}} - \mathrm{ad} A_{\bar{z}}) W_i = -(\partial_y - A_y) W_{i-1}, \qquad i = 1, 2, \ldots \tag{2.43}$$

for W_i with the initial condition

$$(\partial_{\bar{z}} - \mathrm{ad} A_{\bar{z}}) W_1 = -A_y. \tag{2.44}$$

Since $\partial_{\bar{z}} - \mathrm{ad} A_{\bar{z}}$ is assumed to be invertible, the W_i, $i = 1, 2, \ldots$ are determined recursively by (2.43). ■

Next we show that

$$L = W\,(\partial_{\bar{y}} - A_{\bar{y}})\,W^{-1}\ . \tag{2.45}$$

Lemma 2.2 *Let W satisfy (2.39) and let $A_{\bar{y}}, A_{\bar{z}}$ be constant and commuting. Then L defined by (2.45) satisfies*

$$D_1 L = [A_1, L] - \partial_{\bar{y}} A_1 \tag{2.46}$$

(which is (2.38)).

Proof: (See also Lemma 3 in [13].) First we rewrite (2.45) in the form

$$\partial_{\bar{y}} + \hat{L} = \hat{W}(\partial_{\bar{y}} - A_{\bar{y}})\hat{W^{-1}}\ .$$

Using also (2.39), we find

$$[D_1 - A_1, \partial_{\bar{y}} + \hat{L}] = \hat{W}\,[D_1 - \lambda\, A_{\bar{z}}, \partial_{\bar{y}} - A_{\bar{y}}]\,\hat{W}^{-1} = 0$$

which is (2.46). ■

Note that using (2.37), we can write

$$A_k = (\lambda^{k-1}\, L)_+\ . \tag{2.47}$$

Proposition 2.1 *If $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is invertible, the system (2.13) implies the* Wilson-Sato (WS) equations *[24] for $W \in \hat{\mathcal{G}}_-$,*

$$\partial_{z_k} W = -(\lambda^{k-1}\, L)_-\, W, \qquad k = 2, 3, \dots\ . \tag{2.48}$$

Proof:

$$\begin{aligned} 0 &= \hat{W}^{-1}[D_1 - A_1, D_k - A_k]\, \hat{W} \\ &= [\hat{W}^{-1}\,(D_1 - A_1)\, \hat{W}, \hat{W}^{-1}(D_k - A_k)\, \hat{W}] \\ &= [D_1 - \lambda\, A_{\bar{z}}, \hat{W}^{-1}(D_k - A_k)\, \hat{W}] \end{aligned}$$

where we applied the previous lemma. First, multiplying λ^{k-1} in (2.45) we find

$$\lambda^{k-1} W_{\bar{y}} = -\lambda^{k-1}(LW + WA_{\bar{y}}). \tag{2.49}$$

With the help of

$$\begin{aligned} (D_k - A_k)\, \hat{W} &= W_{z_k} - \underbrace{\lambda^{k-1} W_{\bar{y}}}_{(2.49)} - A_k W + W(\partial_{z_k} - \lambda^{k-1}\partial_{\bar{y}}) \\ &= W_{z_k} + (\lambda^{k-1}L)_- W + W(D_k + \lambda^{k-1}A_{\bar{y}}) \end{aligned} \tag{2.50}$$

and using that $A_{\bar{y}}$ and $A_{\bar{z}}$ are constant and commuting, this yields

$$\begin{aligned} 0 &= [D_1 - \lambda\, A_{\bar{z}}, W^{-1}(W_{z_k} + (\lambda^{k-1}L)_-\, W)] \\ &= (\partial_y + \lambda\,(\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}))W^{-1}(W_{z_k} + (\lambda^{k-1}L)_-\, W) \end{aligned}$$

and thus (2.48) if $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is invertible. ■

We already noted in section 2.2 that inserting the expansion (2.37) in (2.46) reproduces (2.36) (in fact without any restrictions on $A_{\bar{y}}, A_{\bar{z}}$). This shows that L defined in (2.45) indeed coincides with what we previously named by the same symbol (under the conditions stated there).

Note that (2.48) can be written in the following form,

$$D_k - A_k = \hat{W}\,(D_k + \lambda^{k-1}\, A_{\bar{y}})\, \hat{W}^{-1} \qquad k = 2, 3, \dots\ . \tag{2.51}$$

Together with (2.39) this expresses the pre-sdYM hierarchy in terms of a dressing by the intertwiner W.

Considering the following commutator

$$[D_k - A_k, \partial_{\bar{y}} + \hat{L}] = \hat{W}[D_k + \lambda^{k-1}A_{\bar{y}}, \partial_{\bar{y}} - A_{\bar{y}}]\widehat{W^{-1}} = 0, \tag{2.52}$$

which implies the generalized Lax equations

$$\partial_{z_k} L = -[(\lambda^{k-1}\, L)_-, L] + \partial_{\bar{y}}(\lambda^{k-1}\, L)_- \tag{2.53}$$

and thus

$$D_k L = [A_k, L] - \partial_{\bar{y}} A_k \qquad k = 2, 3, \dots \tag{2.54}$$

Note that (2.53) alone does not lead us back to the pre-sdYM hierarchy. We also need (2.46). In fact, the compatibility conditions of (2.46) and (2.54) are

$$(\partial_{\bar{y}} + \mathrm{ad}L)\,(D_k A_1 - D_1 A_k + [A_1, A_k]) = 0 \tag{2.55}$$

by use of the Jacobi identity. If $\partial_{\bar{y}} + \mathrm{ad}L$ is invertible, this implies $D_k A_1 - D_1 A_k + [A_1, A_k] = 0$ which is (2.25).

Theorem 2.2 *The flows given by the Wilson-Sato equations (2.48) commute, so that we have a hierarchy.*

Proof:

$$W_{z_k z_l} - W_{z_l z_k} \;=\; \Big((\lambda^{l-1}L)_{-,z_k} - (\lambda^{k-1}L)_{-,z_l} + [(\lambda^{k-1}L)_-, (\lambda^{l-1})_-]\Big)W. \tag{2.56}$$

Multiplying equations (2.53) for z_k and z_l by λ^{l-1} and λ^{k-1} respectively, subtracting and finally projecting the difference onto $\hat{\mathcal{G}}_-$, we obtain

$$\begin{aligned}(\lambda^{l-1}L)_{-,z_k} - (\lambda^{k-1}L)_{-,z_l} &= 2[(\lambda^{l-1}L)_-, (\lambda^{k-1}L)_-] \\ &\quad +[(\lambda^{l-1}L)_-, (\lambda^{k-1}L)_+]_- + [(\lambda^{l-1}L)_+, (\lambda^{k-1}L)_-]_- \\ &= \underbrace{[\lambda^{l-1}L, \lambda^{k-1}L]_-}_{=0} + [(\lambda^{l-1}L)_-, (\lambda^{k-1}L)_-] \\ &= [(\lambda^{l-1}L)_-, (\lambda^{k-1}L)_-].\end{aligned} \tag{2.57}$$

Then we obtain

$$W_{z_k z_l} - W_{z_l z_k} = 0. \tag{2.58}$$

■

Theorem 2.3 *As a consequence of (2.54) we have*

$$[D_k - A_k, D_l - A_l] = 0 \qquad k, l = 2, 3, \ldots\ . \tag{2.59}$$

Proof: First we note that (2.54) implies

$$D_k(A_l) = \Big([A_k, \lambda^{l-1}L] - \lambda^{l-1}\,\partial_{\bar{y}} A_k\Big)_+ .$$

Using this we find

$$\begin{aligned}&[D_l - A_l, D_k - A_k] = D_k(A_l) - D_l(A_k) - [A_k, A_l] \\ =\;& ([A_k, \lambda^{l-1}L] - [A_l, \lambda^{k-1}L] - [A_k, A_l])_+ + (\lambda^{k-1}\,\partial_{\bar{y}} A_l - \lambda^{l-1}\,\partial_{\bar{y}} A_k)_+ \\ =\;& ([A_k, (\lambda^{l-1}L)_-] + [(\lambda^{l-1}L)_-, \lambda^{k-1}L])_+ + (-\lambda^{k-1}\,\partial_{\bar{y}}(\lambda^{l-1}L)_- + \lambda^{l-1}\,\partial_{\bar{y}}(\lambda^{k-1}L)_-)_+ = 0\ .\end{aligned}$$

■

Without the last term, respectively when all fields are independent of the coordinate $\bar{y}$, equation (2.53) for L given by (2.37) is a well-known setting for (generalized) AKNS hierarchies. In this case the (reduced) pre-sdYM hierarchy is the extended AKNS hierarchy

$$\partial_{z_k} L = [(\lambda^{k-1}L)_+, L]\,, \qquad (\partial_y + \lambda\,\partial_{\bar{z}})L = [A_y + \lambda\, A_{\bar{z}}, L] \tag{2.60}$$

(where we assume that (2.34) holds).

3 Reductions of the pre-sdYM hierarchy

In this section, we show how the pre-sdYM hierarchy considered in Sec.2.2 leads to hierarchies associated with well-known integrable systems by reductions.

3.1 What is a reduction?

In general, a reduction means a special choice of the gauge fields. Typical ingredients of a reduction are

- the choice of the gauge group.
- the choice of concrete form of gauge fields in a certain gauge.
- restrictions on the dependence of the gauge fields on the coordinates.

See also [26] for a more restricted notion of a reduction. The following diagram shows schematically the relation between the sdYM equations, the pre-sdYM hierarchy, and a reduction map.

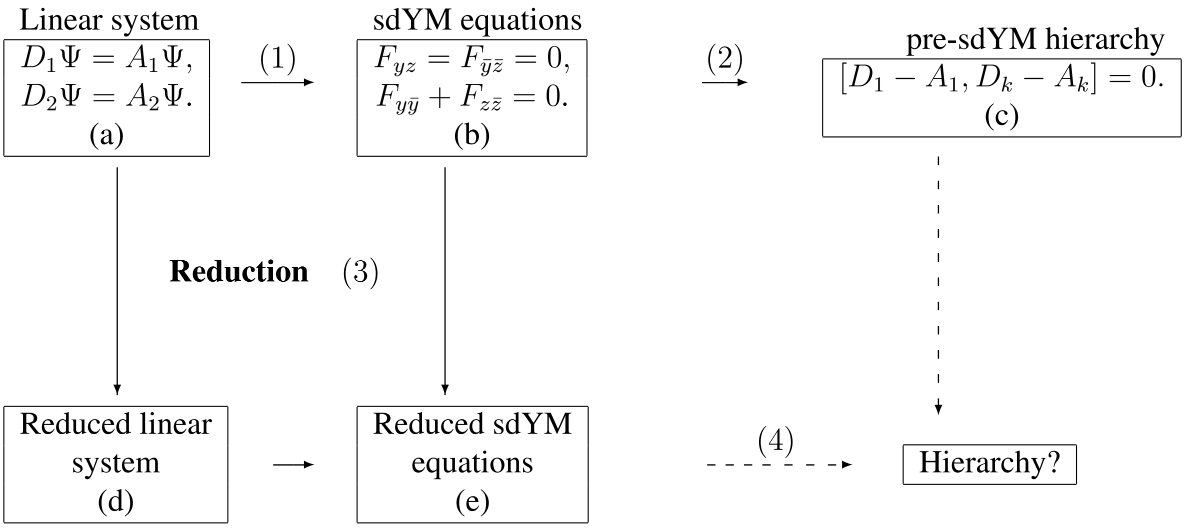

Figure 2. A reduction map

The relations in the above diagram are as follows.

(1). The compatibility condition of the linear system (a) is equivalent to the sdYM equations (b).
(2). The sdYM equations constitute the first member of the pre-sdYM hierarchy (c), with the identification $A_{\bar{y}} = -L_0, A_z = L_1$.
(3). By a reduction the linear system (a) and the sdYM equations are reduced to (d), (e), respectively. The compatibility conditions of (d) should be equivalent to the considered nonlinear system.
(4). It is not guaranteed whether there exists a hierarchy associated with the considered system. Only for special reductions of the sdYM equations it is possible to obtain a well-defined recursion formula for the corresponding hierarchy directly from the pre-sdYM hierarchy, which does not in general possess a well-defined recursion formula.

In this section we consider several reductions which lead to well-known soliton equations such as N-wave system, KdV equation, NLS equation and Sine-Gordon equation. Furthermore we discuss whether the reduction also yields a hierarchy associated with these reduced systems. Moreover, as mentioned in Sec.1.1, we allow the entries of the gauge potential matrices to be elements of an arbitrary associative and typically noncomutative algebra, e.g., a matrix algebra. Additionally, in the calculations we drop constants of integration unless stated otherwise.

3.2 Reduction to the N-wave system

Let us assume that all fields do not depend on the coordinates $\bar{y}$, $\bar{z}$, and let us choose

$$A_{\bar{z}} = \text{diag}(a_1, \ldots, a_n) \tag{3.1}$$

with all eigenvalues different from each other. As a consequence, $\mathrm{ad}A_{\bar{z}}$ is invertible on off-diagonal matrices.[5]

Under the specified conditions, the pre-sdYM hierarchy equations reduce to

$$U_{z_k} = (\partial_y - \mathrm{ad}U)\, L_{k-1} \qquad k = 2, 3, \ldots \tag{3.2}$$

together with

$$(\mathrm{ad}A_{\bar{z}})L_0 = 0\,, \qquad (\mathrm{ad}A_{\bar{z}})L_k = (\partial_y - \mathrm{ad}U)\, L_{k-1} \qquad k = 1, 2, \ldots \tag{3.3}$$

where we write U instead of A_y. In particular, this requires that L_0 $(= -A_{\bar{y}})$ is also diagonal. We write

$$L_0 = -\text{diag}(b_1, \ldots, b_n)\ . \tag{3.4}$$

Splitting (3.3) into off-diagonal and diagonal parts, we obtain

$$(L_{k+1})_{\text{off}-\text{diag}} = (\mathrm{ad}A_{\bar{z}})^{-1}\, (\partial_y L_k - [U, L_k])_{\text{off}-\text{diag}} \tag{3.5}$$

$$(\partial_y L_k - [U, L_k])_{\text{diag}} = 0 \tag{3.6}$$

where $k = 0, 1, 2, \ldots$.

1. Calculation of the first hierarchy equation

Since L_0 is diagonal, $(\mathrm{ad}U)L_0$ is off-diagonal. For $k = 0$ the last equation thus requires

$$L_{0,y} = 0 \tag{3.7}$$

and we obtain

$$[A_{\bar{z}}, L_1] + [U, L_0] = 0 \tag{3.8}$$

[5]Note that the commutator of a diagonal and an off-diagonal matrix is off-diagonal.

which determines the off-diagonal part of L_1 ($= A_z$),

$$(L_1)_{ij} = \lambda_{ij}\, U_{ij} \qquad\qquad i \neq j \tag{3.9}$$

where

$$\lambda_{ij} := \frac{b_i - b_j}{a_i - a_j} = \lambda_{ji} \qquad\qquad i \neq j\ . \tag{3.10}$$

It is convenient to set $\lambda_{ii} := 0$. Noting that

$$(\mathrm{ad}U)\, L_1 = [U, L_1] = \Big(\sum_{l=1}^{n} (\lambda_{il} - \lambda_{lj})\, U_{il}\, U_{lj} \Big) \tag{3.11}$$

has vanishing diagonal part, the diagonal part equation (3.6) for $k = 1$ requires $\partial_y (L_1)_{\mathrm{diag}} = 0$. Let us assume vanishing diagonal part of U, which implies vanishing diagonal part of L_1. The first evolution equation then becomes

$$U_{ij,z} - \lambda_{ij}\, U_{ij,y} = \sum_{l=1}^{n} (\lambda_{il} - \lambda_{lj})\, U_{il}\, U_{lj} \qquad\qquad i \neq j \tag{3.12}$$

which is known as the N-wave interaction equation [7,30–33] . For $n = 2$ this is a linear equation. The first interesting case is therefore $n = 3$.

2. Calculation of the second hierarchy equation

Using (3.9), the off-diagonal part of L_2 is given by the following equation,

$$(a_i - a_j)\, L_{2ij} = \lambda_{ij}\, U_{ij,y} + \sum_{l=1}^{n} (\lambda_{il} - \lambda_{lj})\, U_{il}\, U_{lj} \qquad\qquad i \neq j\ . \tag{3.13}$$

The diagonal part of L_2 is determined by

$$\begin{aligned} L_{2ii,y} &= \sum_{j \neq i} (U_{ij}\, L_{2ji} - L_{2ij}\, U_{ji}) \\ &= -\sum_{j \neq i} \frac{\lambda_{ij}}{a_i - a_j} (U_{ij}\, U_{ji})_{,y} + \sum_{j \neq i} \sum_{l \neq i,j} \frac{\lambda_{il} - \lambda_{lj}}{a_i - a_j} (U_{ij}\, U_{jl}\, U_{li} - U_{il}\, U_{lj}\, U_{ji}) \\ &= -\sum_{j \neq i} \frac{\lambda_{ij}}{a_i - a_j} (U_{ij}\, U_{ji})_{,y} + \sum_{\substack{j,l \neq i \\ j \neq l}} \Big(\frac{\lambda_{il} - \lambda_{lj}}{a_i - a_j} - \frac{\lambda_{ij} - \lambda_{lj}}{a_i - a_l} \Big) U_{ij}\, U_{jl}\, U_{li}\ . \end{aligned} \tag{3.14}$$

where,

$$\begin{aligned} \frac{\lambda_{il} - \lambda_{lj}}{a_i - a_j} - \frac{\lambda_{ij} - \lambda_{lj}}{a_i - a_l} &= (b_j - b_l) \Big(\frac{1}{(a_i - a_l)(a_i - a_j)} + \frac{1}{(a_i - a_l)(a_j - a_l)} \\ &\quad + \frac{1}{(a_i - a_j)(a_l - a_j)} \Big) \\ &= 0. \end{aligned} \tag{3.15}$$

Thus we obtain

$$L_{2ii} = -\sum_{k\neq i} \frac{\lambda_{ik}}{a_i - a_k}\, U_{ik}\, U_{ki}. \tag{3.16}$$

The next hierarchy equation then reads

$$\begin{aligned}
U_{ij,z_3} &= L_{2ij,y} - \sum_{k\neq j} U_{ik}\, L_{2kj} - U_{ij}\, L_{2jj} + \sum_{k\neq i} L_{2ik}\, U_{kj} + L_{2ii}\, U_{ij} \\
&= \frac{1}{a_i - a_j}\Big(\lambda_{ij}\, U_{ij,yy} + \sum_{k\neq i,j} (\lambda_{ik} - \lambda_{kj})\, (U_{ik}\, U_{kj})_{,y}\Big) \\
&\quad + \sum_{k\neq i,j} \Big(\frac{\lambda_{ik}}{a_i - a_k}\, U_{ik,y}\, U_{kj} - \frac{\lambda_{kj}}{a_k - a_j}\, U_{ik}\, U_{kj,y}\Big) \\
&\quad + \sum_{k\neq i} \frac{1}{a_i - a_k} \sum_{l=1}^{n} (\lambda_{il} - \lambda_{lk})\, U_{il} U_{lk} U_{kj} - \sum_{k\neq j} \frac{1}{a_j - a_k} \sum_{l=1}^{n} (\lambda_{jl} - \lambda_{lk})\, U_{ik} U_{kl} U_{lj} \\
&\quad + U_{ij} \sum_{k\neq j} \frac{\lambda_{jk}}{a_j - a_k}\, U_{jk}\, U_{kj} - \sum_{k\neq i} \frac{\lambda_{ik}}{a_i - a_k}\, U_{ik}\, U_{ki}\, U_{ij}\ .
\end{aligned} \tag{3.17}$$

Using FORM the commutativity of the first and second equation has been checked (see Sec.C.4.2).

Proceeding with these calculations, we can obtain the higher evolution equations. Hence, in this example of a reduction, the pre-sdYM hierarchy leads to a hierarchy for the reduced system (see also [13]).

The equation (3.3) determines the diagonal and off-diagonal part of L_i, $i = 0, 1, \ldots$, i.e., L_i is determined sufficiently. From (3.2) we obtain the evolution equations for the off-diagonal entries of U. This is one of particular examples of reductions for which (3.2) implies “genuinely” the evolution equation and (3.3) implies the recursion formula which determines each L_k, $k = 0, 1, \ldots$, respectively. As we have discussed in Sec.2.2, in general (3.2) includes conditions that determine L_i (see Sec.3.3.1, for example).

3.2.1 Recursion formula of the N-wave hierarchy

Now we present a more complete treatment of the N-wave hierarchy by deriving a recursion formula.

Proposition 3.1 *The reduced pre-sdYM hierarchy is equivalent to*

$$U_{ij,z_k} = (a_i - a_j)(L_k)_{ij} \qquad i \neq j, \qquad k = 2, 3, \ldots \tag{3.18}$$

and

$$\begin{aligned}
(L_k)_{ij} &= \frac{1}{a_i - a_j}\Big((L_{k-1})_{ij,y} - \sum_{l} (U_{il} (L_{k-1})_{lj} - (L_{k-1})_{il} U_{lj})\Big) \qquad i \neq j, \\
(L_k)_{ii,y} &= \sum_{l} (U_{il} (L_k)_{li} - (L_k)_{il} U_{li})
\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&= \sum_l \frac{1}{a_l - a_i}\Big(U_{il}(L_{k-1})_{li,y} + (L_{k-1})_{il,y}U_{li}\Big) \\
&\quad + \sum_l \sum_m \Big[\{U_{im}(L_{k-1})_{ml} - (L_{k-1})_{im}U_{ml}\}U_{li} \\
&\quad - U_{li}\{U_{lm}(L_{k-1})_{mi} - (L_{k-1})_{lm}U_{mi}\}\Big], \qquad k = 2, 3, \ldots
\end{aligned} \tag{3.20}
$$

with

$$
(L_1)_{ij} = \lambda_{ij}U_{ij} \qquad i \neq j, \qquad (L_1)_{ii} = 0 \tag{3.21}
$$

(λ_{ij} is defined in (3.10)).

Proof. Inserting (3.3) in (3.2) we find for $k \geq 2$,

$$
U_{z_k} = \mathrm{ad}A_{\bar{z}}(L_k). \tag{3.22}
$$

The diagonal parts of these equations yield (3.18). Furthermore, splitting (3.3) into off-diagonal and diagonal parts, we obtain the recursion formula for the corresponding entries of L_k, (3.19) and (3.20) for $k \geq 2$. ■

Due to the reduction ansätze (3.1), the operator $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is invertible only on the off-diagonal part of arbitrary non-zero matrices. In this example, the weak invertibility of the operator leads to the recursion relation of the off-diagonal part of L_k. The missing diagonal part of L_k is obtained from the diagonal part of (3.2).

Applying (3.18), (3.19) and (3.20), the higher hierarchy equations are calculated recursively. For the first values of k we find

$$
\begin{aligned}
(L_2)_{ij} &= \frac{1}{a_i - a_j}\Big((L_1)_{ij,y} - \sum_l (U_{il}(L_1)_{lj} - (L_1)_{il}U_{lj})\Big) \\
&= \frac{1}{a_i - a_j}\Big(\lambda_{ij}U_{ij,y} + \sum_l (\lambda_{il} - \lambda_{lj})U_{il}U_{lj}\Big) \qquad i \neq j \\
(L_2)_{ii,y} &= \sum_l \frac{1}{a_l - a_i}\Big(U_{il}(L_1)_{li,y} + (L_1)_{il,y}U_{li}\Big) \\
&\quad + \sum_l \sum_m [\{U_{im}(L_1)_{ml} - (L_1)_{ml} - (L_1)_{im}U_{ml}\}U_{li} \\
&\quad - U_{li}\{\{U_{lm}(L_1)_{mi} - (L_1)_{lm}U_{mi}\}] \\
&= \sum_{l \neq i} \frac{\lambda_{il}}{a_i - a_j}(U_{il}U_{li})_y \\
(L_3)_{ij} &= \frac{1}{a_i - a_j}\Big((L_2)_{ij,y} - \sum_l (U_{il}(L_2)_{lj} - (L_2)_{il}U_{lj})\Big) \\
&= \frac{1}{a_i - a_j}\Big((L_2)_{ij,y} - U_{ij}(L_2)_{jj} + (L_2)_{ii}U_{ij} - \sum_{l \neq i,j}(U_{il}(L_2)_{lj} - (L_2)_{il}U_{lj})\Big) \\
&= \frac{1}{(a_i - a_j)^2}\Big(\lambda_{ij}U_{ij,yy} + \sum_{m \neq i,j}(\lambda_{im} - \lambda mj)(U_{im}U_{mj})_y\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{a_i-a_j}\Big(\sum_{l\neq i,j}(\frac{\lambda_{il}}{a_i-a_l}U_{il,y}U_{lj}-\frac{\lambda_{lj}}{a_l-a_j}U_{il}U_{lj,y})\\
&+\sum_{l\neq i}\frac{1}{a_i-a_l}\sum_m(\lambda_{im}-\lambda_{ml})U_{im}U_{ml}U_{lj}\\
&-\sum_{l\neq j}\frac{1}{a_l-a_j}\sum_m(\lambda_{lm}-\lambda_{mj})U_{il}U_{lm}U_{mj}\\
&U_{ij}\sum_{l\neq j}\frac{\lambda_{jl}}{a_j-a_l}U_{jl}U_{lj}-\sum_{l\neq i}\frac{\lambda_{il}}{a_i-a_l}U_{il}U_{li}U_{ij}\Big)
\end{aligned}
$$

$$
\begin{aligned}
(L_3)_{ii,y} \;=\; &\sum_{l\neq i}\frac{\lambda_{il}}{(a_i-a_l)^2}(U_{il}U_{li,y}-U_{il,y}U_{li})_y\\
&+\sum_{l\neq i}\sum_{m\neq i,l}\frac{(\lambda_{lm}-\lambda_{mi})}{(a_i-a_l)^2}\Big(U_{il}(U_{lm}U_{mi})_y+(U_{im}U_{mi})_yU_{li}\Big)\\
&+\sum_{l\neq i}\frac{1}{a_l-a_i}\sum_{p\neq i}\frac{\lambda_{ip}}{a_i-a_p}(U_{il}U_{li}U_{ip}U_{pi}-U_{ip}U_{pi}U_{il}U_{li})\\
&\sum_{l\neq i}\frac{1}{a_i-a_l}\sum_{p\neq l}\frac{\lambda_{lp}}{a_l-a_p}(U_{il}U_{lp}U_{pl}U_{li}-U_{il}U_{lp}U_{pl}U_{li})\\
&\sum_{l\neq i}\frac{1}{a_l-a_i}\sum_{n\neq i,l}\Big(\frac{\lambda_{ln}}{a_l-a_n}(U_{il}U_{ln,y}U_{ni}-U_{in}U_{nl,y}U_{li})\\
&+\frac{\lambda_{in}}{a_i-a_n}(U_{il}U_{ln}U_{ni,y}-U_{in,y}U_{nl}U_{li})\Big)\\
&\sum_{l\neq i}\frac{1}{a_l-a_i}\sum_{m\neq i,l}\sum_{n\neq i,l,m}\Big(\frac{\lambda_{ln}-\lambda_{nm}}{a_l-a_m}(U_{il}U_{ln}U_{nm}U_{mi}+U_{im}U_{mn}U_{nl}U_{li})\\
&-\frac{\lambda_{in}-\lambda_{nm}}{a_i-a_m}(U_{in}U_{nm}U_{ml}U_{li}+U_{il}U_{lm}U_{mn}U_{ni})\Big) \qquad (3.23)
\end{aligned}
$$

These reproduce the first member of the N-wave hierarchy

$$
\begin{aligned}
U_{ij,z_2} &= (a_i-a_j)(L_2)_{ij}\\
&= \lambda_{ij}U_{ij,y}+\sum_l(\lambda_{il}-\lambda_{lj})U_{il}U_{lj}
\end{aligned}
$$

$$
\begin{aligned}
U_{ij,z_3} &= (a_i-a_j)(L_3)_{ij}\\
&= \frac{1}{a_i-a_j}\Big(\lambda_{ij}U_{ij,yy}+\sum_l(\lambda_{il}-\lambda_{lj})(U_{il}U_{lj})_y\Big)\\
&\quad+\sum_{l\neq i,j}(\frac{\lambda_{il}}{a_i-a_l}U_{il,y}U_{lj}-\frac{\lambda_{lj}}{a_l-a_j}U_{il}U_{lj,y})\\
&\quad+\sum_{l\neq i}\frac{1}{a_i-a_l}\sum_m(\lambda_{im}-\lambda_{ml})U_{im}U_{ml}U_{lj}
\end{aligned}
$$

$$-\sum_{l\neq j}\frac{1}{a_l-a_j}\sum_m(\lambda_{lm}-\lambda_{mj})U_{il}U_{lm}U_{mj}$$

$$U_{ij}\sum_{l\neq j}\frac{\lambda_{jl}}{a_j-a_l}U_{jl}U_{lj}-\frac{\lambda_{il}}{a_i-a_l}U_{il}U_{li}U_{ij} \tag{3.24}$$

3.3 Reduction to the KdV hierarchy

In the following, two examples of reductions are discussed, which lead to the KdV equation. The first example is the Bakas-Depireux (BD) reduction [13], and the second is the Mason-Sparling (MS) reduction [25, 34]. In both cases of reductions, we assume that the Yang-Mills potentials do not depend on the coordinates $\bar{z}$ and $y-\bar{y}$.

3.3.1 Bakas-Depireux (BD) reduction

We choose the gauge potentials as follows

$$A_y=\begin{pmatrix}0&1\\u&0\end{pmatrix},\qquad A_{\bar{z}}=\begin{pmatrix}0&0\\1&0\end{pmatrix}, \tag{3.25}$$

with a function u, depending differentiably on the coordinates y, z_k, $k=2,3,\ldots$. Then (2.35), (2.36) read

$$A_{y,z_k}=(\partial_y-\mathrm{ad}A_y)L_{k-1},\qquad k=2,3,\ldots \tag{3.26}$$

and

$$\mathrm{ad}A_{\bar{z}}(L_i)=(\partial_y-adA_y)L_{i-1}+A_{y,y}\delta_{i,1},\qquad i=0,1,\ldots \tag{3.27}$$

(where $L_{-1}=0$).

1. Calculation of the first hierarchy equation

For $i=0$, the last equation reads

$$[A_{\bar{z}},L_0]=0, \tag{3.28}$$

which leads to

$$L_0=\begin{pmatrix}f&0\\h&f\end{pmatrix}, \tag{3.29}$$

with functions f,h. (3.27) for $i=1$ reads

$$[A_{\bar{z}},L_1]=(\partial_y-\mathrm{ad}A_y)\,L_0+A_{y,y}, \tag{3.30}$$

and insertion of our expression for L_0 in particular leads to $f_y = 0$. Imposing the condition $f = 0$. Then L_1 takes the form

$$L_1 = \begin{pmatrix} a & h \\ c & a - (u+h)_y \end{pmatrix}, \tag{3.31}$$

with new functions a, c. Now we insert this result in the first evolution equation (3.26) for $k = 2$. Then we obtain

$$h = -\frac{1}{2}\, u\,, \quad c = -\frac{1}{2}\, u^2 + \frac{1}{4}\, u_{yy}\,, \quad a = \frac{1}{4}\, u_y, \tag{3.32}$$

and the (noncommutative) KdV equation

$$u_z = \frac{1}{4}\, u_{yyy} - \frac{3}{4}\, (u^2)_y\ . \tag{3.33}$$

2. Calculation of the second hierarchy equation

Equation (3.27) for $i = 2$ reads

$$\mathrm{ad}A_{\bar{z}}(L_2) = (\partial_y - \mathrm{ad}A_y)L_1, \tag{3.34}$$

which leads to

$$L_2 = \begin{pmatrix} p & 0 \\ r & p - (\frac{1}{4}u_{yy} - \frac{3}{4}u^2)_y \end{pmatrix}, \tag{3.35}$$

with functions p, r. (3.26) for $k = 3$ reads

$$A_{y,z_3} = (\partial_y - \mathrm{ad}A_y)L_2. \tag{3.36}$$

The upper off-diagonal part of (3.36) reads [6]

$$(u_{yy} - 3u^2)_y = 0, \tag{3.37}$$

Furthermore from the diagonal part of (3.36) we obtain $p_y = r = 0$. This leads to $L_2 = 0$ imposing $p = 0$. As a consequence, the lower off-diagonal part of (3.36) yields

$$u_{z_3} = 0, \tag{3.38}$$

with the constraint (3.37).
Calculating the higher hierarchy equations in this way, we find

$$u_{z_k} = 0, \qquad L_k = 0, \qquad k = 3, 4, \ldots \tag{3.39}$$

[6] If u is a function, the solution of (3.37) has the form $u = \wp(y + C)$,where $\wp$ is the Weierstrass $\wp$-function and C a constant.

Hence, in case of the BD reduction, the pre-sdYM hierarchy *does not* give rise to a recursion formula to obtain a hierarchy associated with the KdV equation. Apparently, the authors of [13] overlooked the shortcoming of this reduction.

Remark. In this example, L_0, L_1 is *not* determined simply by the apparent recursion relation (2.36), but only in combination with the (apparent) evolution equation (3.27). Note that, since A_y has no diagonal part, (3.26) splits into an (apparent) evolution equation

$$(A_{y,z_k})_{\mathrm{off-diag}} = \partial_y (L_{k-1})_{\mathrm{off-diag}} - [A_y, L_{k-1}]_{\mathrm{off-diag}} \qquad k = 2, 3, \dots \tag{3.40}$$

and the constraint

$$\partial_y (L_{k-1})_{\mathrm{diag}} = [A_y, L_{k-1}]_{\mathrm{diag}} \qquad k = 2, 3, \dots \ . \tag{3.41}$$

For $k = 2$ the latter takes the form

$$\begin{pmatrix} a_y & 0 \\ 0 & a_y - (u+h)_{yy} \end{pmatrix} = \begin{pmatrix} c - hu & 0 \\ 0 & -c + uh \end{pmatrix} \tag{3.42}$$

which is equivalent to $c = (u+h)_{yy} + uh + hu$ and $a_y = uh + (u+h)_{yy}$, and we see that the resulting relations are also not sufficient to fix L_0 and L_1. The missing equation which fixes h arises from the upper right entry of (3.26) which is also a constraint, as a consequence of the "degenerate" ansatz for A_y.

This example in particular shows that the apparently obvious distinction between (3.26) and (3.27) into evolution and recursion equations, respectively, does *not* hold in general.

3.3.2 Mason-Sparling (MS) reduction

In this case, we choose the gauge potentials,

$$A_y = \begin{pmatrix} u & 1 \\ u_y - u^2 & -u \end{pmatrix}, \qquad A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.43}$$

(2.35) and (2.36) have the same form as in the BD reduction, so that

$$A_{y,z_k} = (\partial_y - \mathrm{ad} A_y) L_{k-1}, \qquad k = 2, 3, \dots \tag{3.44}$$

$$\mathrm{ad} A_{\bar{z}}(L_i) = (\partial_y - adA_y) L_{i-1} + A_{y,y} \delta_{i,1}, \qquad i = 0, 1, \dots \ , \tag{3.45}$$

and the calculation is similar.

1. Calculation of the first hierarchy equation

(3.45) for $i = 0$ reads

$$[A_{\bar{z}}, L_0] = 0, \tag{3.46}$$

which leads to

$$L_0 = \begin{pmatrix} f & 0 \\ h & f \end{pmatrix}. \tag{3.47}$$

with functions f, h. (3.45) for $i = 1$ reads

$$[A_{\bar{z}}, L_1] = (\partial_y - \mathrm{ad}A_y)\, L_0 + A_{y,y} \tag{3.48}$$

The diagonal part of (3.48) leads to $f_y = 0$. Imposing the condition $f = 0$, we have

$$L_0 = \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} a & h - u_y \\ c & a - D \end{pmatrix}, \tag{3.49}$$

with new functions a, c, and

$$D := h_y + \{u, h\} + u_{yy} - (u^2)_y, \tag{3.50}$$

with the anti-commutator $\{A, B\} = AB + BA$. (3.44) for $k = 2$ reads

$$A_{y,z_2} = (\partial_y - \mathrm{ad}A_y)L_1, \tag{3.51}$$

which leads to

$$\begin{aligned} h &= 0, \qquad a = \frac{1}{2}u_{yy} - u_y u, \\ c &= \frac{1}{4}u_{yyy} - \frac{1}{2}\{u, u_{yy}\} - \frac{1}{2}(u_y)^2 + u u_y u, \end{aligned} \tag{3.52}$$

and the (noncommutative) potential KdV equation [35] :

$$u_{z_2} = \frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2. \tag{3.53}$$

In terms of

$$v := 2u_y, \tag{3.54}$$

this gives the (noncommutative) KdV equation :

$$v_{z_2} = \frac{1}{4}v_{yyy} - \frac{3}{4}(v^2)_y. \tag{3.55}$$

2. Calculation of the second hierarchy equation

Let us calculate the next hierarchy equation. (3.45) for $i = 2$ reads

$$\mathrm{ad}A_{\bar{z}} = (\partial_y - \mathrm{ad}A_y)L_1, \tag{3.56}$$

which leads to

$$L_2 = \begin{pmatrix} p & -\frac{1}{4}u_{yyy} + \frac{3}{2}(u_y)^2 \\ r & p - Q \end{pmatrix}, \tag{3.57}$$

where p, r are functions and

$$Q : \;=\; \frac{1}{4}u_{yyyy} - \frac{3}{2}(u_y^2)_y - \frac{1}{4}\{u, u_{yyy}\} + \frac{3}{2}\{u, u_y^2\}. \tag{3.58}$$

(3.44) for $k = 3$ is

$$A_{y,z_3} = (\partial_y - \mathrm{ad}A_y)L_2, \tag{3.59}$$

Inserting (3.57) in (3.59), we obtain the following expressions for unknown functions p, r,

$$\begin{aligned} p &= \frac{1}{8}u_{yyyy} - u_y u_{yy} - \frac{1}{2}u_{yy}u_y + \frac{3}{2}(u_y)^2 u - \frac{1}{4}u_{yyy}u, \\ r &= \frac{1}{16}u_{yyyyy} - \frac{1}{8}(u_{yy})^2 + \frac{1}{2}(u_y)^2 \\ &\quad -\frac{1}{4}uu_{yyy}u - \frac{3}{2}u(u_y)^2 u - \frac{3}{8}\{u_y, u_{yyy}\} \\ &\quad -\frac{1}{8}\{u, u_{yyyy}\} + \frac{3}{4}\{u, (u_y^2)_y\} + \frac{1}{4}\{u^2, u_{yyy}\}, \end{aligned} \tag{3.60}$$

and the evolution equation

$$u_{z_3} = \frac{1}{16}\Big(u_{yyyyy} - 10(u_{yyy}u_y + u_y u_{yyy} + u_{yy}^2) + 40u_y^3\Big). \tag{3.61}$$

In terms of (3.54), this becomes the second evolution equation of the (noncommutative) KdV hierarchy [24, 36],

$$v_{z_3} = \frac{1}{16}\Big(v_{yyyy} - 5(v_{yy}v + vv_{yy} + v_y^2) + 10v^3\Big)_y \tag{3.62}$$

The commutativity of the first equations of the hierarchy can be checked by using FORM (see Sec.C.4.3).

In contrast to the BD reduction, in case of the MS reduction the pre-sdYM hierarchy indeed leads to the KdV hierarchy. At least we verified this by computing the second member of the hierarchy.

3.3.3 Recursion formula of the KdV hierarchy

Now we present a derivation of the complete (noncommutative) KdV hierarchy following the method in Sec.3.2. Let

$$L_p = \begin{pmatrix} l^{(p)} & m^{(p)} \\ n^{(p)} & h^{(p)} \end{pmatrix} \qquad p = 0, 1, \ldots . \tag{3.63}$$

Proposition 3.2 *The pre-sdYM hierarchy is equivalent, in the reduction under consideration, to*

$$u_{z_k} = -m^{(k)}, \qquad k = 2, 3, \ldots \tag{3.64}$$

with the recursion formula

$$\begin{aligned} m^{(k)} &= \frac{1}{4} m^{(k-1)}_{yy} - \frac{1}{2}\{m^{(k-1)}, u_y\} - \frac{1}{2}\int \{m^{(k-1)}_y, u_y\}\, dy \\ &\quad + \int u_y u m^{(k-1)}\, dy - \int u m^{(k-1)} u_y\, dy \\ &\quad - \int [u_y, \int u m^{(k-1)}_y\, dy + \int m^{(k-1)} u_y\, dy]\, dy, \qquad k = 2, 3, \ldots \end{aligned} \tag{3.65}$$

where $m^{(1)} = -u_y$.

Proof. Inserting (3.45) in (3.44) yields

$$A_{y,z_k} = \mathrm{ad} A_{\bar{z}}(L_k), \qquad k = 2, 3, \ldots \tag{3.66}$$

which leads to

$$u_{z_k} = -m^{(k)}, \qquad (u_y - u^2)_{z_k} = l^{(k)} - h^{(k)}, \qquad k = 2, 3, \ldots \tag{3.67}$$

Now let us calculate the recursion formula for $m^{(k)}$. The equation (3.44) leads to the following four equations

$$u_{z_k} = l^{(k-1)}_y - [u, l^{(k-1)}] - n^{(k-1)} + m^{(k-1)}(u_y - u^2), \tag{3.68}$$

$$l^{(k-1)} - h^{(k-1)} = \{u, m^{(k-1)}\} - m^{(k-1)}_y, \tag{3.69}$$

$$u_{yz_k} - \{u_{z_k}, u\} = n^{(k-1)}_y - (u_y - u^2) l^{(k-1)} + \{n^{(k-1)}, u\} + h^{(k-1)}(u_y - u^2), \tag{3.70}$$

$$-u_{z_k} = h^{(k-1)}_y + [u, h^{(k-1)}] + n^{(k-1)} - (u_y - u^2) m^{(k-1)}. \tag{3.71}$$

Adding (3.68) and (3.71), using (3.69), and integrating with respect to y, yields

$$l^{(k-1)} + h^{(k-1)} = \int [u_y, m^{(k-1)}] dy + \int [m^{(k-1)}_y, u] dy. \tag{3.72}$$

Together with (3.69), we find

$$\begin{aligned} l^{(k-1)} &= -\frac{1}{2} m^{(k-1)}_y + \int m^{(k-1)}_y u dy + \int u_y m^{(k-1)} dy, \\ h^{(k-1)} &= \frac{1}{2} m^{(k-1)}_y - \int u m^{(k-1)}_y dy - \int m^{(k-1)} u_y dy. \end{aligned} \tag{3.73}$$

Using (3.68) on the left hand side of (3.70), we obtain

$$\begin{aligned} u_{yz_k} - \{u_{z_k}, u\} &= l^{(k-1)}_{yy} - [u_y, l^{(k-1)}] - [u, l^{(k-1)}_y] - n^{(k-1)}_y \\ &\quad + m^{(k-1)}_y (u_y - u^2) + m^{(k-1)}(u_{yy} - \{u_y, u\}) - \{l^{(k-1)}_y, u\} \\ &\quad + \{u, [u, l^{(k-1)}]\} - \{u, m^{(k-1)}(u_y - u^2)\} + \{n^{(k-1)}, u\}. \end{aligned} \tag{3.74}$$

Solving (3.70) in terms of $n^{(k-1)}$ and using (3.73) we find

$$\begin{aligned} n^{(k-1)} &= -\frac{1}{4}m_{yy}^{(k-1)} - \frac{1}{2}\int\{u_y, m_y^{(k-1)}\}dy \\ &+\frac{1}{2}(\{m_y^{(k-1)}, u\} + \{u_y, m^{(k-1)}\}) \\ &-\int(um_y^{(k-1)}u + uu_y m^{(k-1)} + m^{(k-1)}u_y u)dy \end{aligned} \tag{3.75}$$

By using (3.73), (3.75) we eliminate $l^{(k-1)}$, $n^{(k-1)}$, $h^{(k-1)}$ from (3.68). Then we obtain (3.65). We get the initial condition $m^{(1)} = -u_y$ from the upper off-diagonal entry of L_1. ■

Hence, we can obtain $m^{(k)}, k = 2, 3, \ldots$, the upper off-diagonal part of L_k, recursively from (3.65) and once each $m^{(k)}$ is determined, we obtain not only the evolution equations (3.64), but also all other entries of L_k from (3.73) and (3.75). In this example, the operator $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is *not* invertible on the upper off-diagonal part of arbitrary matrices due to the reduction ansatze (3.43), however this weakened invertibility of the operator leads to the recursion relation with respect to $m^{(k)}$ and this is enough to calculate higher hierarchy equations.

For the first values of k we find

$$\begin{aligned} m^{(2)} &= -\frac{1}{4}u_{yyy} + \frac{1}{2}\{u_y, u_y\} + \frac{1}{2}\int\{u_y, u_{yy}\}dy \\ &\quad -\int u_y u u_y dy + \int u u_y^2 dy + \int[u_y, \int u u_{yy} dy + \int u_y^2 dy]dy \\ &= -\frac{1}{4}u_{yyy} + \frac{3}{2}(u_y)^2, \end{aligned} \tag{3.76}$$

$$\begin{aligned} m^{(3)} &= -\frac{1}{4}(\frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2)_{yy} \\ &\quad +\frac{1}{2}\{\frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2, u_y\} + \frac{1}{2}\int\{u_y, (\frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2)_y\}dy \\ &\quad -\int u_y u(\frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2)dy + \int u(\frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2)u_y dy \\ &\quad \int[u_y, \int u(\frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2)_y dy + \int(\frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2)u_y dy]dy \\ &= -\frac{1}{16}\Big(u_{yyyyy} - 10(u_{yy})^2 - 10\{u_{yyy}, u_y\} + 40(u_y)^3\Big), \end{aligned} \tag{3.77}$$

In terms of (3.54), these reproduce the first members of the KdV hierarchy

$$\begin{aligned} v_{z_2} &= \frac{1}{4}v_{yyy} - \frac{3}{4}(v^2)_y, \\ v_{z_3} &= \frac{1}{16}\Big(v_{yyyy} - 5(v_{yy}v + v v_{yy} + v_y^2) + 10v^3\Big)_y. \end{aligned} \tag{3.78}$$

In particular, in the commutative case (3.65) reduces to

$$\begin{aligned} m^{(k)} &= I m^{(k-1)} \\ &= I^{k-1} m^{(1)}, \qquad k = 2, 3, \dots , \end{aligned} \tag{3.79}$$

with the integro-differential operator

$$I := \partial_y^{-1} \left(\frac{1}{4} \partial_y^3 - 2 u_y \partial_y - u_{yy} \right), \tag{3.80}$$

where ∂_y^{-1} is the operator of integration with respect to the coordinate y. Inserting (3.79) in (3.64) we obtain the recursion formula for the evolution equations,

$$u_{z_k} = I^{k-1} u_y. \tag{3.81}$$

3.3.4 The relation between BD and MS reduction

In the last two subsections, we considered two reductions which both lead to the KdV equation. Whereas for the first example (BD reduction), the pre-sdYM hierarchy did not lead to the KdV hierarchy, it worked out well in the case of the second example (MS reduction). The diagram below summarizes the relations.

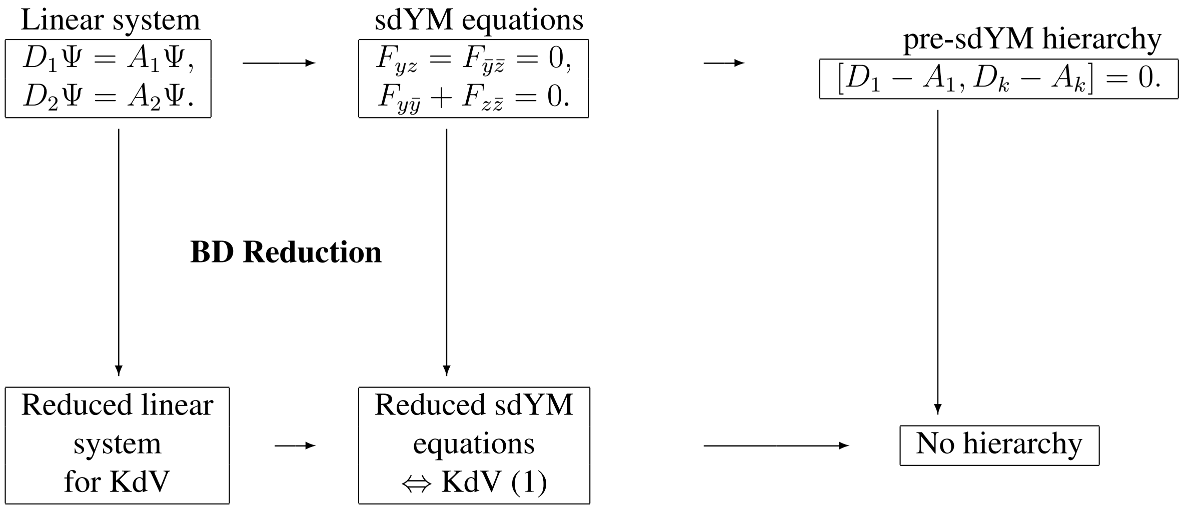

Figure 3 : BD reduction

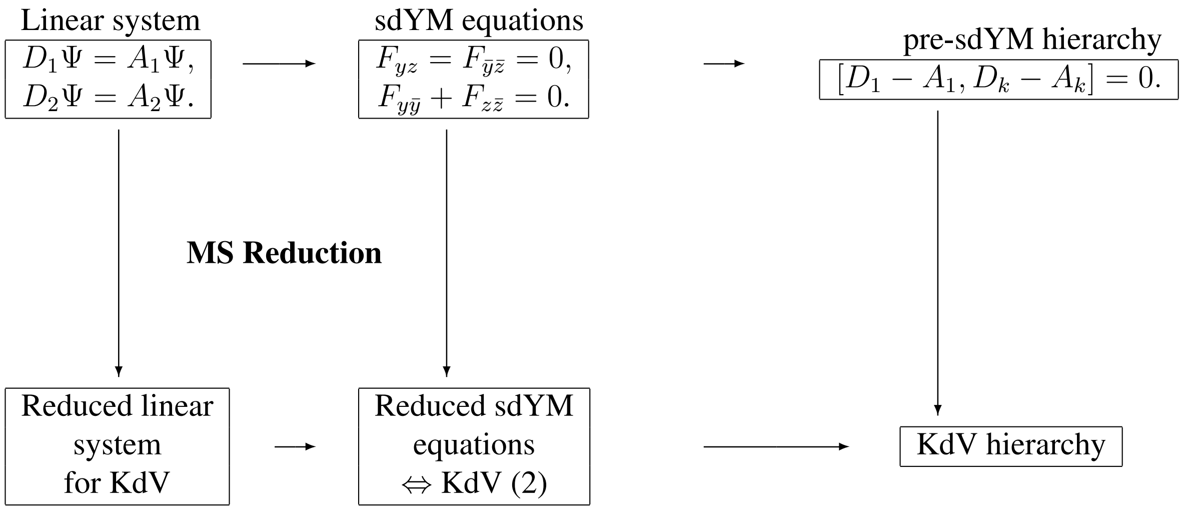

Figure 4 : MS reduction

With the identification,

$$A_{\bar{y}} := -L_0\,, \qquad A_z := L_1\,, \tag{3.82}$$

the Yang-Mills potentials corresponding to the two reductions are

$$\begin{aligned} A_z^{BD} &= \begin{pmatrix} \frac{1}{4}u_y & -\frac{1}{2}u \\ \frac{1}{4}u_{yy} - \frac{1}{2}u^2 & -\frac{1}{4}u_y \end{pmatrix}, \qquad A_{\bar{z}}^{BD} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ A_y^{BD} &= \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \qquad A_{\bar{y}}^{BD} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}u & 0 \end{pmatrix}. \end{aligned} \tag{3.83}$$

and

$$\begin{aligned} A_z^{MS} &= \begin{pmatrix} \frac{1}{2}v_{yy} - v_y v & -v_y \\ \frac{1}{4}v_{yyy} - \frac{1}{2}\{v\,,\, v_{yy}\} - \frac{1}{2}v_y^2 + v v_y v & -\frac{1}{2}v_{yy} + v v_y \end{pmatrix}, \\ A_{\bar{z}}^{MS} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_y^{MS} = \begin{pmatrix} v & 1 \\ v_y - v^2 & -u \end{pmatrix}, \qquad A_{\bar{y}}^{MS} = 0, \end{aligned} \tag{3.84}$$

where, for convenience, we renamed the function u to v in the MS case. u and v are solutions of the KdV and potential KdV equation (3.33), (3.53) respectively, i.e.,

$$\begin{aligned} u_z &= \frac{1}{4}u_{yyy} - \frac{3}{4}(u^2)_y, \\ v_z &= \frac{1}{4}v_{yyy} - \frac{3}{2}v_y^2, \end{aligned} \tag{3.85}$$

Proposition 3.3 *(See also [37]) Via the gauge transformation*

$$A_\mu \mapsto \tilde{A}_\mu = gA_\mu g^{-1} + g_\mu g^{-1}, \tag{3.86}$$

where

$$g = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, \tag{3.87}$$

the gauge potentials (3.84*) are transformed to (*3.83*), i.e., the BD and MS gauge potentials are gauge equivalent.*

Proof. By a gauge transformation (3.86), the Yang-Mills potentials (3.84) are transformed to

$$\begin{aligned}
\tilde{A}_z^{MS} &= \begin{pmatrix} \frac{1}{2}v_{yy} & -v_y \\ \frac{1}{2}v_{yyy} - 2v_y^2 & -\frac{1}{2}v_{yy} \end{pmatrix}, \qquad \tilde{A}_{\bar{z}}^{MS} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
\tilde{A}_y^{MS} &= \begin{pmatrix} 0 & 1 \\ 2v_y & 0 \end{pmatrix}, \qquad \tilde{A}_{\bar{y}}^{MS} = \begin{pmatrix} 0 & 0 \\ v_y & 0 \end{pmatrix},
\end{aligned} \tag{3.88}$$

where we made use of (3.85). Setting

$$u = 2v_y, \tag{3.89}$$

we obtain

$$\begin{aligned}
\tilde{A}_z^{MS} &= \begin{pmatrix} \frac{1}{4}u_y & -\frac{1}{2}u \\ \frac{1}{4}u_{yy} - \frac{1}{2}u^2 & -\frac{1}{4}u_y \end{pmatrix}, \\
\tilde{A}_{\bar{z}}^{MS} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{A}_y^{MS} = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \qquad \tilde{A}_{\bar{y}}^{MS} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}u & 0 \end{pmatrix}.
\end{aligned} \tag{3.90}$$

which is (3.83) (see also [37]). ■

As we have seen in Sec.3.3.1, this "new" u satisfies the first equation in (3.85). Furthermore under this gauge transformation the condition (2.34) for $k = 2$ is preserved.

Again we show schematically this result.

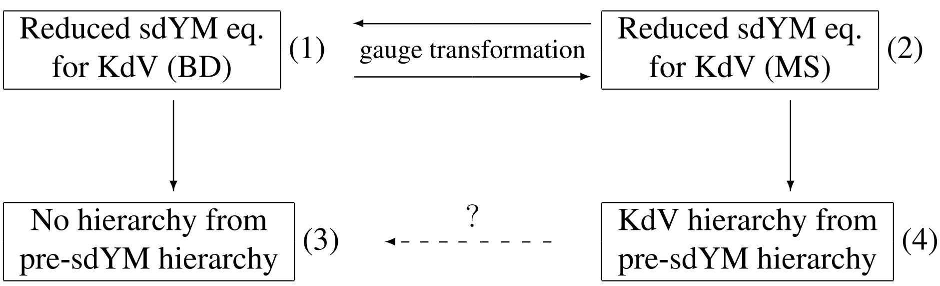

Figure 5 : Gauge equivalence between BD and MS gauge potentials

There is no clear relationship on the level between (3) and (4). The result manifests the fact that the pre-sdYM hierarchy equations (2.35), (2.36) are *not* gauge invariant.

3.4 Reduction to NLS and Zakharov system

In this subsection we consider two reductions with Yang-Mills potentials

$$A_{\bar{z}} = \begin{pmatrix} a\mathbf{1}_N & 0 \\ 0 & -a\mathbf{1}_M \end{pmatrix}, \qquad A_y = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{3.91}$$

where $a \in \mathbb{C}$ and $\mathbf{1}_N$ is the $N \times N$ unit matrix, q, r are $N \times M$ and $M \times N$ complex matrices respectively. It is known that NLS equation and Zakharov system, one of 2+1-dimensional extensions of NLS equation are obtained as a reduction of the (anti-) sdYM equations with the reduction ansätze above (see [37], for example).

3.4.1 Reduction to NLS

Let us assume that A_μ does not depend on the variables $\bar{z}$ and $y - \bar{y}$. The pre-sdYM hierarchy equations take the following form,

$$A_{y,z_k} = (\partial_y - \mathrm{ad}A_y)L_{k-1} \quad k = 2, 3, \ldots, \tag{3.92}$$

and

$$\mathrm{ad}A_{\bar{z}}(L_i) = (\partial_y - \mathrm{ad}A_y)L_{i-1} + A_{y,y}\delta_{i,1} \qquad i = 0, 1, \ldots . \tag{3.93}$$

1. Calculation of the first hierarchy equation

(3.93) for $i = 0$ reads

$$\mathrm{ad}A_{\bar{z}}(L_0) = 0, \tag{3.94}$$

which leads to

$$L_0 = \begin{pmatrix} f & 0 \\ 0 & k \end{pmatrix}, \tag{3.95}$$

with functions f, k. (3.93) for $i = 1$ reads

$$\mathrm{ad}A_{\bar{z}}(L_1) = (\partial_y - \mathrm{ad}A_y)L_0 + A_{y,y}. \tag{3.96}$$

Inserting

$$L_1 := \begin{pmatrix} \phi & \psi \\ \theta & \tau \end{pmatrix},$$

with new functions ϕ, ψ, θ, τ, we obtain from the diagonal part of (3.96),

$$f_y = k_y = 0.$$

In the following, we impose the conditions $f = k = 0$. Together with this we have

$$L_1 = \begin{pmatrix} \phi & \frac{1}{2a}q_y \\ \frac{1}{2a}r_y & \tau \end{pmatrix}, \tag{3.97}$$

(3.92) for $k = 2$ reads

$$A_{y,z_2} = (\partial_y - \mathrm{ad}A_y)L_1, \tag{3.98}$$

which leads to

$$\phi = \frac{1}{2a}qr, \qquad \tau = -\frac{1}{2a}rq, \tag{3.99}$$

$$q_{z_2} = \frac{1}{2a}(q_{yy} + 2qrq), \qquad r_{z_2} = -\frac{1}{2a}(r_{yy} + 2rqr), \tag{3.100}$$

and

$$L_1 = \frac{1}{2a}\begin{pmatrix} qr & q_y \\ r_y & -rq \end{pmatrix}. \tag{3.101}$$

Choosing

$$a = \frac{1}{2}i, \qquad r = \pm q^*, \tag{3.102}$$

where q^* denotes the adjoint, i.e., conjugate transpose of q. (3.100) yields the noncommutative NLS equation

$$iq_{z_2} = q_{yy} \pm 2qq^*q. \tag{3.103}$$

(The second of equations of (3.100) yields the adjoint of (3.103)).

2. Calculation of the second hierarchy equation

(3.93) for $i = 2$ reads

$$\mathrm{ad}A_{\bar{z}}(L_2) = (\partial_y - \mathrm{ad}A_y)L_1, \tag{3.104}$$

which leads to

$$L_2 = \begin{pmatrix} s & \frac{1}{4a^2}(q_{yy} + 2qrq) \\ -\frac{1}{4a^2}(r_{yy} + 2rqr) & v \end{pmatrix}, \tag{3.105}$$

with functions s, v. (3.92) for $k = 3$ reads

$$A_{y,z_3} = (\partial_y - \mathrm{ad}A_y)L_2, \tag{3.106}$$

which leads to

$$s = \frac{1}{4a^2}(q_y r - qr_y), \qquad v = -\frac{1}{4a^2}(rq_y - r_y q), \tag{3.107}$$

$$q_{z_3}=\frac{1}{4a^2}\Big(q_{yyy}+3(q_yrq+qrq_y)\Big),\qquad r_{z_3}=\frac{1}{4a^2}\Big(r_{yyy}+3(r_yqr+rqr_y)\Big). \tag{3.108}$$

Setting (3.102) in (3.108), the first equation of (3.108) yields the noncommutative complex modified KdV (mKdV) equation (see also [24]).

$$q_{z_3}=-q_{yyy}\mp 3(q_yq^*q+qq^*q_y). \tag{3.109}$$

3. Calculation of the third hierarchy equation

(3.93) for $i=3$ reads

$$\mathrm{ad}A_{\bar z}(L_3)=(\partial_y-\mathrm{ad}A_y)L_2, \tag{3.110}$$

which leads to

$$L_3=\begin{pmatrix} b & q_{yyy}+3(q_yrq+qrq_y)\\ r_{yyy}+3(r_yqr+rqr_y) & c\end{pmatrix}, \tag{3.111}$$

with functions b, c. (3.92) for $k=4$ reads

$$A_{y,z_4}=(\partial_y-\mathrm{ad}A_y)L_3, \tag{3.112}$$

which leads to

$$\begin{aligned} b &= q_{yy}+qr_{yy}-q_yr_y+3qrqr,\\ c &= -(r_{yy}q+rq_{yy})+r_yq_y-3rqrq, \end{aligned} \tag{3.113}$$

$$\begin{aligned} q_{z_4} &= \frac{1}{8a^3}\Big(q_{yyyy}+4(q_{yy}rq+qrq_{yy})\\ &\quad +2(q_yr_yq+qr_yq_y+qr_{yy}q)+6(q_yrq_y+qrqrq)\Big)\\ r_{z_4} &= -\frac{1}{8a^3}\Big(r_{yyyy}+4(r_{yy}qr+rqr_{yy})\\ &\quad +2(r_yq_yr+rq_yr_y+rq_{yy}r)+6(r_yqr_y+rqrqr)\Big). \end{aligned} \tag{3.114}$$

Setting (3.102) again, we obtain from the first of equations (3.114)

$$\begin{aligned} -iq_{z_4} &= q_{yyyy}\pm 2(q_yq^*_yq+qq^*_yq_y+qq^*q)\pm 4(q_{yy}q^*q+qq^*q_{yy})\\ &\quad \pm 6(q_yq^*q_y+qq^*qq^*q). \end{aligned} \tag{3.115}$$

This is the second NLS hierarchy equation [36]. Using FORM we checked the commutativity of the first/second and second/third equations (see Sec.C.4.4).

3.4.2 Reduction to the Zakharov system

Now we assume that the Yang-Mills potentials are only independent of $\bar z$. In this case, the sdYM hierarchy equations take the following form

$$\begin{aligned} A_{y,z_k} &= (\partial_y-\mathrm{ad}A_y)L_{k-1}, \qquad k=2,3,\dots && (3.116)\\ \mathrm{ad}A_{\bar z}(L_i) &= (\partial_y-\mathrm{ad}A_y)L_{i-1}+A_{y,\bar y}\delta_{i,1}, \qquad i=0,1,\dots && (3.117) \end{aligned}$$

1. Calculation of the first hierarchy equation

The equation (3.117) for $i = 0$, $\mathrm{ad}(L_0) = 0$ leads to the following form of L_0

$$L_0 = \begin{pmatrix} f & 0 \\ 0 & k \end{pmatrix} \tag{3.118}$$

with functions f, k. Subsequently, (3.117) for $i = 1$ reads

$$\mathrm{ad}A_{\bar{z}} = (\partial_y - \mathrm{ad}A_y)L_0 + A_{y,\bar{y}}. \tag{3.119}$$

The diagonal part of (3.119) leads to $f_y = k_y = 0$. Setting $f = k = 0$ and then, we can see that L_1 has a form

$$L_1 = \begin{pmatrix} \phi & \frac{1}{2a}q_{\bar{y}} \\ \frac{1}{2a}r_{\bar{y}} & \nu \end{pmatrix}. \tag{3.120}$$

with new functions ϕ and ν. The equation (3.116) for $k = 2$ reads

$$A_{y,z_2} = (\partial_y - \mathrm{ad}A_y)L_1. \tag{3.121}$$

Inserting (3.120) in this equation, the diagonal part determine the unknowns ϕ and ν such that

$$\begin{aligned} \phi &= \frac{1}{2a}\int (qr)_{\bar{y}}\, dy, \\ \nu &= \frac{1}{2a}\int (rq)_{\bar{y}}\, dy \end{aligned} \tag{3.122}$$

Inserting these in the off-diagonal part of (3.121) we obtain the following equations

$$\begin{aligned} q_{z_2} &= \frac{1}{2a}\Big(q_{\bar{y}y} + q\int (rq)_{\bar{y}}\, dy + \int (qr)_{\bar{y}}\, dy\, q\Big), \\ r_{z_2} &= -\frac{1}{2a}\Big(r_{\bar{y}y} - r\int (qr)_{\bar{y}}\, dy - \int (rq)_{\bar{y}}\, dy\, r\Big). \end{aligned} \tag{3.123}$$

Proceeding as in Sec.3.4.1, setting

$$a = \frac{i}{2}, \qquad r = \pm q^*, \tag{3.124}$$

we obtain

$$iq_{z_2} = q_{\bar{y}y} \pm \Big(q\int (q^*q)_{\bar{y}}\, dy + \int (qq^*)_{\bar{y}}\, dy\, q\Big), \tag{3.125}$$

and the adjoint of (3.125). This is the Zakharov system [16, 37]. If we set $y = \bar{y}$, the equation (3.125) reduces to the NLS equation (see Sec.3.4.1).

2. Calculation of the second hierarchy equation
The equation (3.117) for $i = 2$ reads

$$\mathrm{ad}A_{\bar{z}}(L_2) = (\partial_y - \mathrm{ad}A_y)L_1, \tag{3.126}$$

which leads to the following form of L_2

$$L_2 = \frac{1}{4a^2}\begin{pmatrix} 4a^2P & q_{\bar{y}y} + q\int(rq)_{\bar{y}}\,dy + \int(qr)_{\bar{y}}\,dy\,q \\ r_{\bar{y}y} + r\int(qr)_{\bar{y}}\,dy + \int(rq)_{\bar{y}}\,dy\,r & 4a^2S \end{pmatrix}, \tag{3.127}$$

with functions P and S. (3.116) for $k = 2$ is

$$A_{y,z_3} = (\partial_y - \mathrm{ad}A_y)L_2 \tag{3.128}$$

which determines the functions P, S,

$$P = \frac{1}{4a^2}\int\Big(q_{\bar{y}y}r - qr_{\bar{y}y} + [\int(qr)_{\bar{y}}\,dy, qr]\Big)\,dy,$$
$$S = \frac{1}{4a^2}\int\Big(r_{\bar{y}y}q - rq_{\bar{y}y} + [\int(rq)_{\bar{y}}\,dy, rq]\Big)\,dy,$$

and the second hierarchy equations

$$\begin{aligned} q_{z_3} &= \frac{1}{4a^2}\Big(q_{\bar{y}yy} + q(rq)_{\bar{y}} + (qr)_{\bar{y}}q + q_y\int(rq)_{\bar{y}}\,dy + \int(qr)_{\bar{y}}\,dy\,q_y \\ &\quad + \int(q_{\bar{y}y}r - qr_{\bar{y}y})\,dy\,q - q\int(r_{\bar{y}y}q - rq_{\bar{y}y})\,dy \\ &\quad - q\int[\int(rq)_{\bar{y}}\,dy, rq]\,dy + \int[\int(qr)_{\bar{y}}\,dy, qr]\,dy\,q\Big), \\ r_{z_3} &= \frac{1}{4a^2}\Big(r_{\bar{y}yy} + r(qr)_{\bar{y}} + (rq)_{\bar{y}}r + r_y\int(qr)_{\bar{y}}\,dy + \int(rq)_{\bar{y}}\,dy\,r_y \\ &\quad - r\int(q_{\bar{y}y}r + qr_{\bar{y}y})\,dy - \int(rq_{\bar{y}y} - r_{\bar{y}y}q)\,dy\,r \\ &\quad - r\int[\int(qr)_{\bar{y}}\,dy, qr]\,dy + \int[\int(rq)_{\bar{y}}\,dy, rq]\,dy\,r\Big) \end{aligned} \tag{3.129}$$

With the setting (3.124) we obtain

$$\begin{aligned} -q_{z_3} &= q_{\bar{y}yy} \pm \Big(q(q^*q)_{\bar{y}} + (qq^*)_{\bar{y}}q + q_y\int(q^*q)_{\bar{y}}\,dy + \int(qq^*)_{\bar{y}}\,dy\,q_y \\ &\quad + \int(q_{\bar{y}y}q^* - qq^*_{\bar{y}y})\,dy\,q - q\int(q^*_{\bar{y}y}q - q^*q_{\bar{y}y})\,dy \\ &\quad - q\int[\int(q^*q)_{\bar{y}}\,dy, q^*q]\,dy + \int[\int(qq^*)_{\bar{y}}\,dy, qq^*]\,dy\,q\Big) \end{aligned} \tag{3.130}$$

and its adjoint. If we impose $y = \bar{y}$, this equation reduces to the complex mKdV equation obtained in Sec.3.4.1.

3.4.3 Recursion formula of the NLS and Zakharov system hierarchy

Now we turn to a more complete treatment of the NLS and Zakharov system reduction, following the procedure of Sec.3.2.1 and 3.3.3. The hierarchy equations for both reductions have a similar form. As we have seen, the "evolution equation" has the same form

$$A_{y,z_k} = (\partial_y - \mathrm{ad}A_y)L_{k-1}, \qquad k = 2, 3, \ldots \tag{3.131}$$

Recalling the "recursion relation" for both reductions,

NLS

$$\mathrm{ad}A_{\bar{z}}(L_i) = (\partial_y - \mathrm{ad}A_y)L_{i-1} + A_{y,y}\delta_{i,1}, \qquad i = 0, 1, \ldots \tag{3.132}$$

Zakharov system

$$\mathrm{ad}A_{\bar{z}}(L_i) = (\partial_y - \mathrm{ad}A_y)L_{i-1} + A_{y,\bar{y}}\delta_{i,1}, \qquad i = 0, 1, \ldots \tag{3.133}$$

Obviously, the only difference between (3.132) and (3.133) is the last term on the right hand side ($A_{y,y}$ or $A_{y,\bar{y}}$). Thus, inserting (3.132) and (3.133) in (3.131), this results in the same equation,

$$A_{y,z_k} = \mathrm{ad}A_{\bar{z}}(L_k), \qquad k = 2, 3, \ldots . \tag{3.134}$$

Inserting

$$L_p = \begin{pmatrix} l^{(p)} & m^{(p)} \\ n^{(p)} & h^{(p)} \end{pmatrix} \qquad p = 0, 1, \ldots \tag{3.135}$$

in (3.134) leads to

$$q_{z_k} = 2am^{(k)}, \qquad r_{z_k} = 2an^{(k)}, \qquad k = 2, 3, \ldots . \tag{3.136}$$

The equation (3.92) leads to the following equations

$$l_y^{(k-1)} = qn^{(k-1)} - m^{(k-1)}r, \tag{3.137}$$
$$q_{z_k} = m_y^{(k-1)} - qh^{(k-1)} + l^{(k-1)}q, \tag{3.138}$$
$$-r_{z_k} = -n_y^{(k-1)} + h^{(k-1)}r - rl^{(k-1)}, \tag{3.139}$$
$$h_y^{(k-1)} = -(rm^{(k-1)} + n^{(k-1)}q). \tag{3.140}$$

Integrating (3.137) and (3.140) with respect to y we easily find

$$\begin{aligned} l^{(k-1)} &= \int qn^{(k-1)}\, dy + \int m^{(k-1)}r\, dy, \\ h^{(k-1)} &= -\Big(\int rm^{(k-1)}\, dy + \int n^{(k-1)}q\, dy\Big) \end{aligned} \tag{3.141}$$

Inserting these in (3.138) and (3.139), we obtain the recursion formula for $m^{(k)}$, $n^{(k)}$,

$$
\begin{aligned}
m^{(k)} &= \frac{1}{2a}\Big(m_y^{(k-1)} + q\int rm^{(k-1)}\,dy + \int qn^{(k-1)}\,dy\,q \\
&\quad + q\int n^{(k-1)}q\,dy + \int m^{(k-1)}r\,dy\,q\Big), \\
n^{(k)} &= -\frac{1}{2a}\Big(n_y^{(k-1)} + r\int qn^{(k-1)}\,dy + \int n^{(k-1)}q\,dy\,r \\
&\quad + r\int m^{(k-1)}r\,dy + \int rm^{(k-1)}\,dy\,r\Big), \qquad k = 2, 3, \dots
\end{aligned}
\tag{3.142}
$$

Hence, whereas both reductions lead to different systems, they yield the same recursion formula. Furthermore, due to the form of $A_{\bar{z}}$, the operator $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is *not* invertible on the diagonal part of arbitrary matrices which have non-zero entries.

Considering the equation (3.132) and (3.133), the noninvertible diagonal part enables that $l^{(k)}$, $h^{(k)}$, the diagonal entries of $L_k, k = 2, 3, \dots$ are written with respect to $m^{(k)}$, $n^{(k)}$. It helps to obtain the recursion relation of $m^{(k)}$, $n^{(k)}$. Therefore, the diagonal part of the equation (3.132) and (3.133) is an additional condition to determine the missing diagonal entries $l^{(k)}$ and $h^{(k)}$.

Using (3.136) and (3.142) we can calculate the higher hierarchy equations selecting the initial values

NLS hierarchy

$$
m^{(1)} = \frac{1}{2a}q_y, \qquad n^{(1)} = \frac{1}{2a}r_y, \tag{3.143}
$$

Zakharov system hierarchy

$$
m^{(1)} = \frac{1}{2a}q_{\bar{y}}, \qquad n^{(1)} = \frac{1}{2a}r_{\bar{y}}, \tag{3.144}
$$

NLS hierarchy For even k we find

$$
\begin{aligned}
m^{(2)} &= \frac{1}{4a^2}(q_{yy} + 2qrq), \qquad n^{(2)} = -\frac{1}{4a^2}(r_{yy} + 2rqr), \\
m^{(4)} &= \frac{1}{16a^4}\Big(q_{yyyy} + 2(q_y r_y q + q r_{yy} q + q r_y q_y) \\
&\quad + 4(q_{yy} rq + qr q_{yy}) + 6(q_y r q_y + qrqrq)\Big), \\
n^{(4)} &= -\frac{1}{16a^4}\Big(r_{yyyy} + 2(r_y q_y r + r q_{yy} r + r q_y r_y) \\
&\quad + 4(r_{yy} qr + rq r_{yy}) + 6(r_y q r_y + rqrqr)\Big),
\end{aligned}
\tag{3.145}
$$

In terms of (3.102), these reproduce the noncommutative NLS hierarchy [36]

$$
\begin{aligned}
iq_{z_2} &= q_{yy} \pm 2qq^*q, \\
-iq_{z_4} &= q_{yyyy} \pm 2(q_y q_y^* q + qq_y^* q_y + qq^* q) \pm 4(q_{yy}q^*q + qq^*q_{yy}) \\
&\quad \pm 6(q_y q^* q_y + qq^*qq^*q))
\end{aligned} \tag{3.146}
$$

For odd k we find

$$
\begin{aligned}
m^{(3)} &= \frac{1}{8a^3}\Big(q_{yyy} + 3(q_y rq + qrq_y)\Big), \qquad n^{(3)} = \frac{1}{8a^3}\Big(r_{yyy} + 3(r_y qr + rqr_y)\Big), \\
m^{(5)} &= \frac{1}{32a^5}\Big(q_{yyyyy} + 5(q_{yyy}rq + qrq_{yyy} + q_{yy}r_y q + qr_y q_{yy} + q_y r_{yy} q + qr_{yy} q_y) \\
&\quad +10(q_{yy} r q_y + q_y r q_{yy} + q_y r_y q_y) + 10(q_y rqrq + qrq_y rq + qrqrq_y)\Big), \\
n^{(5)} &= \frac{1}{32a^5}\Big(r_{yyyyy} + 5(r_{yyy}qr + rqr_{yyy} + r_{yy}q_y r + rq_y r_{yy} + r_y q_{yy} r + rq_{yy} r) \\
&\quad +10(r_{yy} q r_y + r_y q r_{yy} + r_y q_y r_y) + 10(r_y qrqr + rqr_y qr + rqrqr_y)\Big),
\end{aligned} \tag{3.147}
$$

and we obtain the noncommutative complex mKdV hierarchy (see also [36]).

$$
\begin{aligned}
q_{z_3} &= -q_{yyy} \mp 3(q_y q^* q + qq^* q_y), \\
q_{z_5} &= q_{yyyyy} \pm 5(q_{yyy}q^*q + qq^*q_{yyy} + q_{yy}q_y^* q + qq_y^* q_{yy} + q_y q_{yy}^* q + qq_{yy}^* q_y) \\
&\quad \pm 10(q_{yy}q^*q_y + q_y q^* q_{yy} + q_y q^* q_y) \\
&\quad +10(q_y q^* qq^* q + qq^* q_y q^* q + qq^* qq^* q_y),
\end{aligned} \tag{3.148}
$$

In the following we sketch the results.

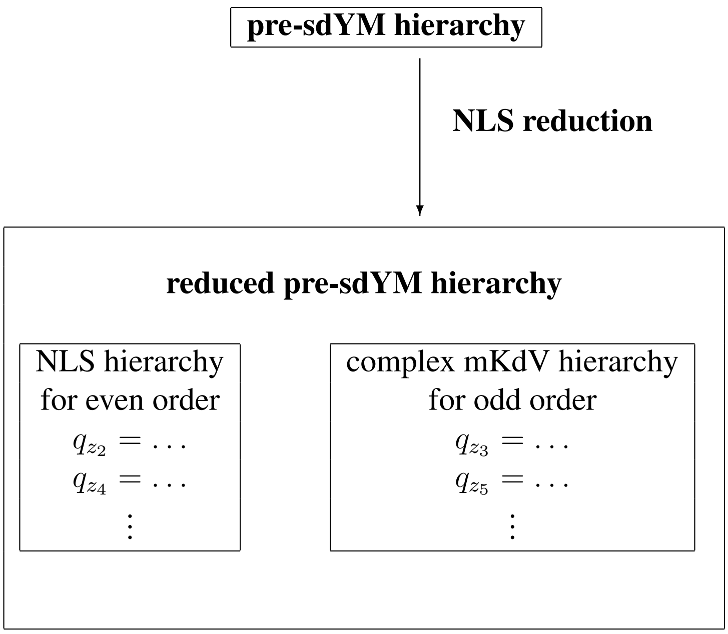

Figure 6 : NLS Reduction

Hence, from the pre-sdYM hierarchy we obtain a hierarchy which consists of the two hierarchies associated with NLS and complex mKdV equations. [7].

Zakharov system hierarchy From (3.142) we obtain

$$
\begin{aligned}
m^{(2)} &= \frac{1}{4a^2}\Big(q_{\bar{y}y} + q\int (rq)_{\bar{y}}\,dy + \int (qr)_{\bar{y}}\,dy\,q\Big),\\
n^{(2)} &= -\frac{1}{4a^2}\Big(r_{\bar{y}y} + r\int (qr)_{\bar{y}}\,dy + \int (rq)_{\bar{y}}\,dy\,r\Big),\\
m^{(3)} &= \frac{1}{8a^3}\Big(q_{\bar{y}yy} + q(rq)_{\bar{y}} + (qr)_{\bar{y}}q + q_y\int (rq)_{\bar{y}}\,dy + \int (qr)_{\bar{y}}\,dy\,q_y\\
&\quad + \int (q_{\bar{y}y}r - qr_{\bar{y}y})\,dy\,q + q\int (rq_{\bar{y}y} - r_{\bar{y}y}q)\,dy\\
&\quad + q\int [rq, \int (rq)_{\bar{y}}\,dy]\,dy + \int [\int (qr)_{\bar{y}}\,dy, qr]\,dy\,q\Big),\\
n^{(3)} &= \frac{1}{8a^3}\Big(r_{\bar{y}yy} + r(qr)_{\bar{y}} + (rq)_{\bar{y}}r + r_y\int (qr)_{\bar{y}}\,dy + \int (rq)_{\bar{y}}\,dy\,r_y\\
&\quad - r\int (q_{\bar{y}y}r - qr_{\bar{y}y})\,dy - \int (rq_{\bar{y}y} - r_{\bar{y}y}q)\,dy\,r\\
&\quad - r\int [\int (qr)_{\bar{y}}\,dy, qr]\,dy - \int [rq, \int (rq)_{\bar{y}}]\,dy\,r\Big),
\end{aligned}
\tag{3.149}
$$

which reproduces the Zakharov system hierarchy

$$
\begin{aligned}
q_{z_2} &= 2am^{(2)}\\
&= \frac{1}{2a}\Big(q_{\bar{y}y} + q\int (rq)_{\bar{y}}\,dy + \int (qr)_{\bar{y}}q\Big),\\
r_{z_2} &= 2an^{(2)}\\
&= -\frac{1}{2a}\Big(r_{\bar{y}y} + r\int (qr)_{\bar{y}}\,dy + \int (rq)_{\bar{y}}\,dy\,r\Big),\\
q_{z_3} &= 2am^{(3)}\\
&\quad \frac{1}{4a^2}\Big(q_{\bar{y}yy} + q_y\int (rq)_{\bar{y}} + \int (qr)_{\bar{y}}\,dy\,q_y + q(rq)_{\bar{y}} + (qr)_{\bar{y}}q\\
&\quad + q\int (rq_{\bar{y}y} - r_{\bar{y}y}q)\,dy + \int (q_{\bar{y}y}r - qr_{\bar{y}y})\,dy\,q\\
&\quad q\int [rq, \int (rq)_{\bar{y}} + \int [\int (qr)_{\bar{y}}\,dy, qr]\,dy\,q\Big),\\
r_{z_3} &= 2an^{(3)}\\
&\quad \frac{1}{4a^2}\Big(r_{\bar{y}yy}r_y\int (qr)_{\bar{y}}\,dy + \int (rq)_{\bar{y}}\,dy\,r_y + r(qr)_{\bar{y}} + (rq)_{\bar{y}}r\\
&\quad - r\int (q_{\bar{y}y} - qr_{\bar{y}y})\,dy - \int (rq_{\bar{y}y} - r_{\bar{y}y}q)\,dy\,r
\end{aligned}
$$

[7] As a reduction of the AKNS-D hierarchy, a generalization of the AKNS hierarchy, it is known that the NLS and mKdV equation are obtained as its first and second flow. See [24], Chapter 9.

$$-r\int[\int(qr)_{\bar{y}},qr]\,dy-\int[rq,\int(rq)_{\bar{y}}\,dy\,r\Big), \tag{3.150}$$

In particular, considering the commutative case, (3.142) can be written in vector form

$$\begin{pmatrix} m^{(k)} \\ n^{(k)} \end{pmatrix} = I\begin{pmatrix} m^{(k-1)} \\ n^{(k-1)} \end{pmatrix} = I^{k-1}\begin{pmatrix} m^{(1)} \\ n^{(1)} \end{pmatrix}, \qquad k=2,3,\dots, \tag{3.151}$$

where I is defined by

$$I := \frac{1}{2a}\begin{pmatrix} \partial_y + 2q\partial_y^{-1}r & 2q\partial_y^{-1}q \\ -2r\partial_y^{-1}r & -\partial_y - 2r\partial_y^{-1}q \end{pmatrix} \tag{3.152}$$

and ∂_y^{-1} is the operator of integration with respect to the variable y. Inserting (3.136) in (3.151), we obtain the following recursion formula for the hierarchy equations,

$$\begin{pmatrix} q_{z_k} \\ r_{z_k} \end{pmatrix} = 2aI^{k-1}\begin{pmatrix} m^{(1)} \\ n^{(1)} \end{pmatrix}, \qquad k=2,3,\dots \tag{3.153}$$

In [38], the NLS hierarchy which is in the form (3.153) is studied as a reduction of the AKNS hierarchy. Selecting the initial values $m^{(1)}$, $n^{(1)}$ with (3.143) or (3.144), the recursion formula (3.153) reproduces the commutative NLS or Zakharov system hierarchy.

3.5 Reduction to Sine-Gordon

In [25], a reduction of the (anti-) sdYM equations which leads to the sine-Gordon equation is presented. Now we turn to apply this reduction ansatz to the pre-sdYM hierarchy. We choose the gauge potentials

$$A_y = -\frac{i}{2}\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \qquad A_{\bar{z}} = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.154}$$

where c is a function which does not depend on z, $\bar{z}$. The pre-sdYM hierarchy equations take the following form

$$A_{y,z_k} = (\partial_y - \mathrm{ad}A_y)L_{k-1}, \qquad k=3,4,\dots \tag{3.155}$$

with

$$(\partial_y - \mathrm{ad}A_y)L_1 = 0, \tag{3.156}$$

and

$$\mathrm{ad}A_{\bar{z}}(L_i) = (\partial_y - \mathrm{ad}A_y)L_{i-1} + A_{y,\bar{y}}\delta_{i,1} + A_{\bar{z},\bar{y}}\delta_{i,0}, \qquad i=0,1,\dots \tag{3.157}$$

1. Calculation of the first hierarchy equation

The equation (3.157) for $i = 0$ reads

$$\mathrm{ad}A_{\bar{z}}(L_0) = 0, \tag{3.158}$$

which leads to

$$L_0 = \begin{pmatrix} f & g \\ g & f \end{pmatrix}, \tag{3.159}$$

with functions f, g. (3.157) for $i = 1$ reads

$$\mathrm{ad}A_{\bar{z}}(L_1) = (\partial_y - \mathrm{ad}A_y)L_0 + A_{y,\bar{y}}. \tag{3.160}$$

Inserting

$$L_1 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \tag{3.161}$$

equation (3.160) yields these following equations

$$\frac{i}{2}(q - r) \;=\; f_y - \frac{i}{2}[f, c] - \frac{i}{2}c_{\bar{y}} = -f_y + \frac{i}{2}[c, f] - \frac{i}{2}c_{\bar{y}}, \tag{3.162}$$

$$\frac{i}{2}(p - s) \;=\; g_y + \frac{i}{2}\{g, c\} = -g_y + \frac{i}{2}\{g, c\}, \tag{3.163}$$

From (3.162) and (3.163) we find $f_y = 0$, $g_y = 0$. We set $f = g = 0$. Then we have

$$q - r = -c_{\bar{y}}, \qquad p - s = 0. \tag{3.164}$$

We find that L_1 takes the form

$$L_1 = \begin{pmatrix} p & q \\ q - c_{\bar{y}} & p \end{pmatrix}. \tag{3.165}$$

Inserting this in (3.156), we obtain

$$p_y - \frac{i}{2}[p, c] \;=\; p_y - \frac{i}{2}[c, p] = 0, \tag{3.166}$$

$$q_y + \frac{i}{2}\{c, q\} \;=\; q_y - c_{\bar{y}y} - \frac{i}{2}\{c, q - c_{\bar{y}}\} = 0. \tag{3.167}$$

The equation (3.166) implies $p_y = 0$. We set $p = 0$. Adding first and second equations in (3.167) and integrating with respect to y, we obtain

$$q = \frac{1}{2}c_{\bar{y}} - \frac{i}{4}\int (c^2)_{\bar{y}}\, dy. \tag{3.168}$$

Introducing

$$a := \frac{1}{2}\int \{(c^2)_{\bar{y}}\}\, dy, \qquad b := -c_{\bar{y}}, \tag{3.169}$$

we have $q = -\frac{1}{2}(b + ia)$. Inserting this in the first equation (3.167), we obtain

$$a_y = -\frac{1}{2}\{b, c\}, \qquad b_y = \frac{1}{2}\{a, c\}. \tag{3.170}$$

In the commutative case, we can see that $a_y = -bc, b_y = ac$. This implies that the quantity $a^2 + b^2$ does not depend on y. Choosing $a^2 + b^2 = r^2$, $r \in \mathbb{R}$, $r \neq 0$ and parametrizing

$$a = r \cos\, u, \qquad b = r \sin\, u, \tag{3.171}$$

where u is a function of y, $\bar{y}$. Inserting (3.171) in (3.170) this gives $c = u_y$. As a consequence the second equation (3.169) yields the sine-Gordon equation

$$u_{\bar{y}y} = -r \sin\, u. \tag{3.172}$$

2. Calculation of the second hierarchy equation

Now let us calculate the next hierarchy equation. (3.157) for $i = 2$ reads

$$\mathrm{ad}A_{\bar{z}}(L_2) = (\partial_y - \mathrm{ad}A_y)L_1. \tag{3.173}$$

Inserting

$$L_2 = \begin{pmatrix} s & t \\ v & w \end{pmatrix} \tag{3.174}$$

with functions s, t, v, w. The off-diagonal part of the equation (3.173) leads to

$$s - w = ib_y + a_y + \frac{1}{2}\{c, b\} - \frac{i}{2}\{c, a\} = ib_y - a_y + \frac{1}{2}[c, b] - \frac{i}{2}\{c, a\}. \tag{3.175}$$

This gives $cc_{\bar{y}} = 0$. This condition leads $a = b = 0, L_1 = 0$. Hence, in the case of the sine-Gordon reduction, the pre-sdYM hierarchy does not produce a recursion formula which leads to a hierarchy associated with the sine-Gordon equation.

4 Towards hierarchies in 2+1 dimensions

In this section we consider how to obtain 2+1-dimensional (1 time variable and 2 space variables) integrable systems such as 2+1-dimensional N-wave system, KP equation as a reduction of the sdYM hierarchy. As an example how to obtain a 2+1-dimensional integrable system, we start from the following linear system

$$\begin{aligned} \partial_y \Psi &= X\Psi, \\ \partial_z \Psi &= Y\Psi, \end{aligned} \tag{4.1}$$

where X, Y are elemtns in the Lie algebra of objects which are polinomials in an indeterminate λ and formal power series in its inverse λ^{-1} with coefficients in an associative algebra. Choosing

$$X = \begin{pmatrix} 0 & 1 \\ u+\lambda & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} \frac{1}{4}u_y & -\frac{1}{2}u+\lambda \\ \frac{1}{4}u_{yy} - \frac{1}{2}u^2 + \frac{1}{2}u\lambda + \lambda^2 & -\frac{1}{4}u_y \end{pmatrix}. \tag{4.2}$$

The compatibility condition of (4.1) is equivalent to the (noncommutative) KdV equation

$$u_z = \frac{1}{4}u_{yyy} - \frac{3}{4}(u^2)_y. \tag{4.3}$$

Next we assume that X, Y are elements in the Lie algebra of matrix differential operators with respect to a new variable s :

$$\mathcal{G} = \left\{ \sum_{j=0}^{\ll\infty} a_j\, \partial_s^j \right\} \tag{4.4}$$

where a_j are $N\times N$ matrices with entries from an associative algebra $\mathcal{A}$ over $\mathbb{C}$ (typically an algebra of matrices of functions). Furthermore, $\hat{\mathcal{G}}$ is the Lie algebra of objects which are polynomials in the differential operator ∂_s and formal power series in its inverse ∂_s^{-1} with coefficients in $\mathcal{G}$. By $(\ )_\pm$ we denote the projections to the regular (+), respective singular (-) part of such a series. We choose

$$X = \begin{pmatrix} 0 & 1 \\ u+\partial_s & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} \frac{1}{4}u_y & -\frac{1}{2}u+\partial_s \\ \frac{1}{4}u_{yy} - \frac{1}{2}u^2 + \frac{1}{2}u\partial_s + \partial_s^2 & -\frac{1}{4}u_y \end{pmatrix}. \tag{4.5}$$

which is indeed equivalent to (4.2) in terms of replacing λ by ∂_s. In this case, the compatibility condition of (4.1) is equivalent to the noncommutative KP equation

$$u_{z_3} = \frac{1}{4}u_{yyy} - \frac{3}{4}(u^2)_y + \frac{3}{4}[u, \int u_s\, dy] + \frac{3}{4}\int u_{ss}\, dy. \tag{4.6}$$

In the following, we consider a sdYM hierarchy and its reductions assuming that all gauge potentials are in $\mathcal{G}$. In this case, we can not apply the pre-sdYM hierarchy simply replacing λ by ∂_s. The compatibility condition (2.13) in Sec.2.1 with the replacement, $\lambda \to \partial_s$ reads

$$[\,\partial_y + \partial_s\partial_{\tilde{z}} - A_y - A_{\tilde{z}}\partial_s\,,\, \partial_{t_n} + \partial_s^n\partial_{\bar{y}} - \sum_{i=1}^{n-1} B_{n-i}\partial_s^i - A_{\bar{y}}\partial_s^n\,] = 0 \qquad n = 1, 2, \ldots \tag{4.7}$$

which does not give rise to a recursion relation since second order of differential operators appear, such as $\partial_s\partial_{\tilde{z}}$.

4.1 A sdYM hierarchy with gauge potentials in a Lie algebra of differential operators

In this section we consider a sdYM hierarchy with gauge potentials in $\mathcal{G}$ introduced in [13] which can be a candidate of an extension of the sdYM hierarchy which reduces to hierarchies associated with 2+1-dimensional integrable systems and its reductions.

4.1.1 Definition of the hierarchy

In this subsection, we consider a sdYM hierarchy for $2+1$-dimensions introduced in [13]. Introducing the following differential operators

$$\begin{aligned} D_1 &:= \partial_y + \partial_{\bar{z}}, \qquad (4.8)\\ D_k &:= \partial_{z_k} - \partial_{\bar{y}}, \qquad k = 2, 3, \ldots \qquad (4.9) \end{aligned}$$

and considering the system defined by

$$[D_1 - A_1, D_k - A_k] = 0, \qquad k = 2, 3, \ldots \tag{4.10}$$

where A_1, $A_k \in \mathcal{G}$. Now we choose them as follows

$$\begin{aligned} A_1 &= U + A\partial_s, \qquad (4.11)\\ A_k &= \sum_{j=0}^{k-1} L_j^{(k)} \partial_s^{k-j-1}, \qquad k = 2, 3, \ldots \qquad (4.12) \end{aligned}$$

where U, A and $L_j^{(k)}$, $k = 2, 3, \ldots$ are in $\mathcal{A}$ and A is a constant matrix. Moreover, the index above L_j specifies the number of "flow". Inserting these A_1 and A_k, the system (4.10) takes the following form

$$\begin{aligned} &U_{z_k} - U_{\bar{y}} + \sum_{j=0}^{k-1}(UL_j^{(k)} + AL_{j,s}^{(k)} - L_{j,y}^{(k)} - L_{j,\bar{z}}^{(k)})\partial_{s^{k-j-1}} \\ &-\sum_{j=0}^{k-1} L_j^{(k)} \sum_{l=0}^{k-j-1} \begin{pmatrix} k-j-1 \\ m \end{pmatrix} U_{s^{k-j-m-1}} \partial_s^m + \sum_{j=0}^{k-1}[A, L_j]\partial_s^{k-j} = 0 \qquad (4.13) \end{aligned}$$

where $U_{s^j} = U_{\underbrace{\text{s}\ldots\text{s}}_{\text{j-times}}}$, i.e., j-times differentiation of U with respect to the variable s. Now we equate (4.13) with respect to ∂_s. For each order of ∂_s we obtain

$$\begin{aligned} U_{z_k} &= U_{\bar{y}} + (\partial_y + \partial_{\bar{z}})L_{k-1}^{(k)} - UL_{k-1}^{(k)} - AL_{k-1,s}^{(k)} \\ &\quad + \sum_{j=0}^{k-1} L_j^{(k)} U_{s^{k-j-1}}, \qquad k = 2, 3, \ldots \qquad (4.14)\\ \mathrm{ad}A(L_i^{(k)}) &= (\partial_y + \partial_{\bar{z}})L_{i-1}^{(k)} - UL_{i-1}^{(k)} - AL_{i-1,s}^{(k)} \\ &\quad + \sum_{j=0}^{k-1} \begin{pmatrix} k-j-1 \\ k-i \end{pmatrix} L_j^{(k)} U_{s^{i-j-1}}, \qquad i = 1, 2, \ldots, k \qquad (4.15)\\ \mathrm{ad}A(L_0^{(k)}) &= 0, \qquad k = 2, 3, \ldots \qquad (4.16) \end{aligned}$$

This is the sdYM hierarchy for 2+1-dimensions introduced in [13] . One can see that

$$L^{(k)} = \sum_{j=0}^{\infty} L_j^{(k)} \partial_s^{k-j-1} \tag{4.17}$$

satisfies the following differentials equation

$$D_1(L^{(k)}) = [\, U + A\partial_s \, , \, L^{(k)} \,]. \tag{4.18}$$

Remark. The "recursion relation" (4.15) can be solved exactly if the operator $\mathrm{ad}A$ is invertible, there are no examples in reductions, however. In examples of reductions, we show that we obtain a recursion relation without the condition that the operator $\mathrm{ad}A$ is invertible (see Sec.4.2.2, Sec.4.2.4).

To see the relationship between the system (4.10) and the sdYM equations, firstly we consider the sdYM equations with the following gauge choices $A_{\bar{y}} = A_{\bar{z}} = 0$

$$\partial_y A_z - \partial_z A_y - [A_y, A_z] = 0, \qquad \partial_{\bar{y}} A_y + \partial_{\bar{z}} A_z = 0. \tag{4.19}$$

Furthermore we assume

$$A_y = U + A\partial_s, \qquad A_z = V + B\partial_s \tag{4.20}$$

V and B are also in $\mathcal{A}$ and we assume that B is a constant matrix. Then the sdYM equations (4.19) yield

$$\begin{aligned} V_y - U_z - [U, V] + BU_s - AV_s &= 0, && (4.21)\\ [U, B] + [A, V] &= 0, && (4.22)\\ [A, B] &= 0, && (4.23)\\ U_{\bar{y}} + V_{\bar{z}} &= 0. && (4.24) \end{aligned}$$

From (4.24) we can see that A and B are commuting matrices. The "first members" of the system (4.10), i.e., for $k = 2$ we have

$$[D_1 - A_1, D_2 - A_2] = 0 \tag{4.25}$$

which is equivalent to the compatibility condition of the following linear system

$$D_1\Psi = A_1\Psi, \qquad D_2\Psi = A_2\Psi \tag{4.26}$$

where

$$A_2 = \sum_{j=0}^{1} L_j^{(2)} \partial_s^{k-j-1} = L_0^{(2)}\partial_s + L_1^{(2)}. \tag{4.27}$$

The system (4.25) yields the following equations

$$\begin{aligned} L_{1,y}^{(2)} - U_z - U_{\bar{y}} + L_0^{(2)} U_s - AL_{1,s}^{(2)} - [U, L_1^{(2)}] + L_{1,\bar{z}}^{(2)} - U_{\bar{y}} &= 0, && (4.28)\\ L_{0,y}^{(2)} + L_{0,\bar{z}}^{(2)} - AL_{0,s}^{(2)} + [U, L_0^{(2)}] + [A, L_1^{(2)}] &= 0, && (4.29)\\ [A, L_0^{(2)}] &= 0. && (4.30) \end{aligned}$$

With the Setting

$$L_0^{(2)} := B, \qquad L_1^{(2)} := V, \tag{4.31}$$

the equations (4.30) and (4.30) become (4.23) and (4.24), respectively. However, the equation (4.29) becomes a linear combination of (4.22) and (4.24). Imposing the condition

$$U_{\bar{y}} + V_{\bar{z}} = U_{\bar{y}} + L^{(2)}_{1,\bar{z}} = 0, \tag{4.32}$$

Then the system (4.25) is equivalent to sdYM equations with the special gauge choice $A_{\bar{y}} = A_{\bar{z}} = 0$, $A_y = U + A\partial_s$, $A_z = V + B\partial_s$. In general, the above condition (4.32) is too strong and rarely satisfied in the process of reductions. In Sec.4.2, we consider examples of reductions in which the condition (4.32) is indeed satisfied.

4.1.2 Some general results

In this subsection we show that the sdYM hierarchy for 2+1 dimensions (4.10) implies a hierarchy under the special condition that the operator $\mathrm{ad}A$ is invertible, though we have no examples for reductions in which the condition is satisfied.

Lemma 4.1 *If $\mathrm{ad}A$ is invertible, there is a $W \in \hat{\mathcal{G}}_-$ in the form*

$$W = \mathbf{1} + \sum_{k=1}^{\infty} W_k \partial_s^{-k} \tag{4.33}$$

which satisfies

$$D_1 - A_1 = \hat{W}(D_1 - A\partial_s)\hat{W^{-1}} \tag{4.34}$$

where $\hat{W}$ is the multiplication operator associated with W.

Proof. We find that (4.34) yields

$$A_1 W = D_1(W) + WA\partial_s. \tag{4.35}$$

Inserting (4.33) in (4.35), we find the recursion relation with respect to W_k

$$\mathrm{ad}A(W_i) = (\partial_y + \partial_{\bar{z}})W_{i-1} - UW_{i-1} - AW_{i-1,s}, \qquad i = 2, 3, \ldots \tag{4.36}$$

with the initial condition

$$\mathrm{ad}A(W_1) = -U. \tag{4.37}$$

Hence, W_k is determined recursively if $\mathrm{ad}A$ is invertible. ■

Now we define

$$L^{(k)} := W(\partial_{\bar{y}} + B\partial_s^{k-1})W^{-1} \tag{4.38}$$

where B is a constant matrix and commutes with A.

Proposition 4.1 *The sdYM hierarchy for $(2+1)$-dimension (4.10) implies the following equation,*

$$W_{z_k} = -(L^{(k)})_- W \tag{4.39}$$

if $\mathrm{ad}A$ *is invertible.*

Proof. First we rewrite (4.38) as follows.

$$L^{(k)} = -W_{\bar{y}} W^{-1} + W(B\partial_s^{k-1})W^{-1} \tag{4.40}$$

Inserting the equation (4.34) in (4.10), we have

$$[\hat{W^{-1}}(D_1 - A\partial_s)\hat{W}, D_k - A_k] = 0. \tag{4.41}$$

Multiplying $\hat{W^{-1}}, \hat{W}$ from the right and left side in (4.41), we obtain

$$[D_1 - A\partial_s, \hat{W^{-1}}(D_k - A_k)\hat{W}] = 0 \tag{4.42}$$

where

$$\begin{aligned}(D_k - A_k)\hat{W} &= W_{z_k} - \underbrace{W_{\bar{y}}}_{-L^{(k)}W + WB\partial_s^{k-1}} - A_k W + W(\partial_{z_k} - \partial_{\bar{y}}) \\ &= W_{z_k} + (L^{(k)})_- W + WB\partial_s^{k-1} + W(\partial_{z_k} - \partial_{\bar{y}})\end{aligned} \tag{4.43}$$

Then the equation (4.42) yields

$$\begin{aligned}0 &= [D_1 - A\partial_s, W^{-1}D_k(W) - W^{-1}A_k W] \\ &= (\partial_y + \partial_{\bar{z}})W^{-1}(W_{z_k} + (L^{(k)})_- W) - \\ &\quad -A\partial_s\Big(W^{-1}(W_{z_k} + (L^{(k)})_- W)\Big) - \mathrm{ad}A\Big(W^{-1}(W_{z_k} + (L^{(k)})_- W)\Big)\partial_s.\end{aligned} \tag{4.44}$$

Then (4.44) implies (4.39) if $\mathrm{ad}A$ is invertible. In the following we assume the invertibility of this operator.

The equation (4.39) together with (4.38) can be rewritten as

$$D_k - A_k = \hat{W}(D_k - B\partial_s^{k-1})\hat{W^{-1}}. \tag{4.45}$$

We rewrite (4.38) in the form

$$\partial_{\bar{y}} + \hat{L}^{(k)} = \hat{W}(\partial_{\bar{y}} + B\partial_s^{k-1})\hat{W^{-1}} \tag{4.46}$$

and considering the following commutator and applying (4.34) as well as (4.45), we find

$$\begin{aligned}[\, D_1 - A_1 \,,\, \partial_{\bar{y}} + \hat{L}^{(k)} \,] &= \hat{W}[\, D_1 - A\partial_s \,,\, \partial_{\bar{y}} + B\partial_s^{k-1} \,]\hat{W^{-1}} \\ &= 0,\end{aligned} \tag{4.47}$$

$$\begin{aligned}[\, D_k - A_k \,,\, \partial_{\bar{y}} + \hat{L}^{(k)} \,] &= \hat{W}[\, D_k - B\partial_s^{k-1} \,,\, \partial_{\bar{y}} + B\partial_s^{k-1} \,]\hat{W^{-1}} \\ &= 0.\end{aligned} \tag{4.48}$$

which lead to

$$\begin{aligned} D_1(L^{(k)}) &= [\,U + A\partial_s\,,\, L^{(k)}\,] - U_{\bar y}, && (4.49)\\ D_k(L^{(k)}) &= [\,A_k\,,\, L^{(k)}\,] - A_{k,\bar y}. && (4.50)\end{aligned}$$

Remark. In [13], $L^{(k)}$ is defined by

$$L^{(k)} = W(B\partial_s^{k-1})W^{-1}. \tag{4.51}$$

If we assume (4.51), (4.43) has the following form

$$(D_k - A_k)\hat W = D_k(W) - A_k W + W(\partial_{z_k} - \partial_{\bar y}). \tag{4.52}$$

Then (4.44) reads

$$\begin{aligned} 0 \;=\; & (\partial_y + \partial_{\bar z})W^{-1}(D_k(W) - A_k W) - A\partial_s\Big(W^{-1}(D_k(W) - A_k W)\Big)\\ & -\mathrm{ad}A\Big(W^{-1}(D_k(W) - A_k W)\Big)\partial_s \end{aligned} \tag{4.53}$$

If $\mathrm{ad}A$ is invertible, we have

$$\begin{aligned} D_k(W) &= A_k W\\ &= (L^{(k)})_+ W. \end{aligned} \tag{4.54}$$

Then

$$\begin{aligned} \hat W(D_k - B\partial_s^{k-1})\hat W^{-1} &= D_k - D_k(W)W^{-1} - W(B\partial_s^{k-1})W^{-1}\\ &= D_k - A_k - L^{(k)}\\ &\neq D_k - A_k. \end{aligned} \tag{4.55}$$

Hence, the necessary condition to obtain (4.45) is that $L^{(k)}$ must be defined by (4.38). In the following, we take (4.38) as a definition of $L^{(k)}$.
Moreover, the equation (4.49) *does not* reproduce the recursion relation (4.15) and (4.16) due to the additional term $-U_{\bar y}$. If the gauge potentials do not depend on $\bar y$, (4.49) leads to (4.16). In examples of reductions which we consider in Sec.4.2, this condition is always satisfied. In this subsection, we assume that all gauge potentials depend on $\bar y$ and one can show that the system (4.10) has the hierarchy property, i.e., arbitrary flows of (4.10) commute (under the condition that the operator $\mathrm{ad}A$ is invertible).

Theorem 4.1 *The flows given by (4.39) commute.*

Proof.

$$W_{z_k z_l} - W_{z_l z_k} = \Big((L^{(l)})_{-,z_l} - (L^{(k)})_{-,z_l} + [\,(L^{(k)})_-\,,\,(L^{(l)})_-\,]\Big)W \tag{4.56}$$

Considering following commutators

$$\begin{aligned} [\,D_k - A_k\,,\, \partial_{\bar y} + \hat L^{(l)}\,] &= \hat W[\,D_k - B\partial_s^{k-1}\,,\, \partial_{\bar y} + B\partial_s^{l-1}\,]\hat W^{-1}\\ &= 0 \end{aligned} \tag{4.57}$$

and

$$
\begin{aligned}
[\, D_l - A_l\,,\, \partial_{\bar{y}} + \hat{L}^{(k)}\,] &= \hat{W}[\, D_l - B\partial_s^{l-1}\,,\, \partial_{\bar{y}} + B\partial_s^{k-1}\,]\hat{W}^{-1} \\
&= 0,
\end{aligned} \tag{4.58}
$$

since B is constant. This yields

$$
\begin{aligned}
D_k(L^{(l)}) &= [\,(L^{(k)})_+\,,\, L^{(l)}\,] - \partial_{\bar{y}}(L^{(k)})_+, \\
D_l(L^{(k)}) &= [\,(L^{(l)})_+\,,\, L^{(k)}\,] - \partial_{\bar{y}}(L^{(l)})_+.
\end{aligned} \tag{4.59}
$$

Projecting on $\hat{G}_-$, we find

$$
\begin{aligned}
(L^{(l)})_{-,z_k} - (L^{(k)})_{-,z_l} &= [\,(L^{(k)})_+\,,\,(L^{(l)})_-\,]_- - [\,(L^{(l)})_+\,,\,(L^{(k)})_-\,]_- \\
&\quad + \partial_{\bar{y}}(L^{(l)})_- - \partial_{\bar{y}}(L^{(k)})_-.
\end{aligned} \tag{4.60}
$$

To eliminate the commutators on the right side, we look at the following commutator

$$
\begin{aligned}
[\,\partial_{\bar{y}} + \hat{L}^{(k)}\,,\, \partial_{\bar{y}} + \hat{L}^{(l)}\,] &= \hat{W}[\partial_{\bar{y}} + B\partial_s^{k-1}\,,\, \partial_{\bar{y}} + B\partial_s^{l-1}\,]\hat{W}^{-1} \\
&= 0.
\end{aligned} \tag{4.61}
$$

Hence,

$$
\partial_{\bar{y}}(L^{(l)}) - \partial_{\bar{y}}(L^{(k)}) + [\,(L^{(k)})\,,\,(L^{(l)})\,] = 0 \tag{4.62}
$$

Projecting (4.62) on $\hat{G}_-$, we obtain

$$
\begin{aligned}
[\,(L^{(k)})_+\,,\,(L^{(l)})_-\,]_- - [\,(L^{(l)})_+\,,\,(L^{(k)})_-\,]_- &= [\,(L^{(l)})_-\,,\,(L^{(k)})_-\,] \\
&\quad + \partial_{\bar{y}}(L^{(k)})_- - \partial_{\bar{y}}(L^{(l)})_-.
\end{aligned} \tag{4.63}
$$

Then (4.60) yields

$$
(L^{(l)})_{-,z_k} - (L^{(k)})_{-,z_l} = -[\,(L^{(k)})_-\,,\,(L^{(l)})_-\,] \tag{4.64}
$$

Inserting this in (4.56), we obtain

$$
W_{z_k z_l} - W_{z_l z_k} = 0 \tag{4.65}
$$

■

Theorem 4.2 *The arbitrary flows of the sdYM hierarchy for 2+1- dimensions commute such that*

$$
[D_k - A_k, D_l - A_l] = 0, \qquad k, l = 2, 3, \ldots \tag{4.66}
$$

Proof:

$$
\begin{aligned}
[\, D_k - A_k\,,\, D_l - A_l\,] &= D_l(A_k) - D_k(A_l) + [\, A_k\,,\,,\, A_l\,] \\
&= D_l(L^{(k)})_+ - D_k(L^{(l)})_+ + [\,(L^{(k)})_+\,,\,(L^{(l)})_+\,]
\end{aligned} \tag{4.67}
$$

Considering the projection of (4.62) on $\hat{g}_+$, we find

$$\begin{aligned}\partial_{\bar{y}}(L^{(k)})_+ - \partial_{\bar{y}}(L^{(l)})_+ &= [\,(L^{(k)})_+\,,\,(L^{(l)})_+\,] + [\,(L^{(k)})_+\,,\,(L^{(l)})_-\,] \\ &\quad +[(L^{(k)})_-\,,\,(L^{(l)})_+\,]. \end{aligned} \tag{4.68}$$

Then

$$\begin{aligned} D_l(L^{(k)})_+ - D_k(L^{(l)})_+ &= 2[\,(L^{(l)})_+\,,\,(L^{(k)})_+\,] + [\,(L^{(l)})_+\,,\,(L^{(k)})_-\,]_+ \\ &\quad -[\,(L^{(k)})_+\,,\,(L^{(l)})_-\,]_+ + \underbrace{\partial_{\bar{y}}(L^{(k)})_+ - \partial_{\bar{y}}(L^{(l)})_+}_{(4.68)} \\ &= [\,(L^{(l)})_+\,,\,(L^{(k)})_+\,] \end{aligned} \tag{4.69}$$

Inserting this in (4.67), we have (4.66). ■

4.2 Reductions

In this section we consider the reductions of the sdYM hierarchy defined in Sec.4.1.1 . We consider two examples of reductions which lead to the 2+1-dimensional N-wave system hierarchy and the KP hierarchy.
In both examples, we assume that the gauge potentials do not depend on $\bar{y}$ and $\bar{z}$.

4.2.1 Reduction to 2+1-dimensional N-wave system

Choosing the gauge potentials

$$A = \mathrm{diag}(a_1, \ldots, a_n) \tag{4.1}$$

with all eigenvalues different from each other. The hierarchy equations read

$$\begin{aligned} U_{z_k} &= L^{(k)}_{k-1,y} - UL^{(k)}_{k-1} - AL^{(k)}_{k-1,s} + \sum_{j=0}^{k-1} L^{(k)}_j U_{s^{k-j-1}}, \qquad k = 2, 3, \ldots & (4.2) \\ \mathrm{ad}A(L^{(k)}_i) &= L^{(k)}_{i-1,y} - UL^{(k)}_{i-1} - AL^{(k)}_{i-1,s} \\ &\quad + \sum_{j=0}^{k-1} \binom{k-j-1}{k-i} L^{(k)}_j U_{s^{i-j-1}}, \qquad i = 1, 2, \ldots, k, \qquad k = 2, 3, \ldots & (4.3) \\ \mathrm{ad}A(L^{(k)}_0) &= 0, \qquad k = 2, 3, \ldots & (4.4) \end{aligned}$$

1. Calculation of the first hierarchy equation From (4.4) we can see that $L^{(k)}_0$ is a diagonal matrix. Assuming

$$L^{(k)}_0 = \mathrm{diag}(b_1, b_2, \ldots, b_n), \qquad k = 2, 3, \ldots \tag{4.5}$$

with different eigenvalues b_i, $i = 1, 2, \ldots n$ [8]. (4.3) for $i = 1$, $k = 2$ reads

$$\mathrm{ad}A(L^{(2)}_1) = L^{(2)}_{0,y} - \mathrm{ad}U(L^{(2)}_0) - AL^{(2)}_{0,s} \tag{4.6}$$

which leads to the off-diagonal parts of L_1,

$$(L^{(2)}_1)_{ij} = \lambda_{ij} U_{ij}, \qquad i \neq j \tag{4.7}$$

where

$$\lambda_{ij} := \frac{b_i - b_j}{a_i - a_j} = \lambda_{ji}, \tag{4.8}$$

The equation (4.2) for $k = 2$ reads

$$U_{z_2} = L^{(2)}_{1,y} + L^{(2)}_0 U_s - [U, L^{(2)}_1] - AL^{(2)}_{1,s}. \tag{4.9}$$

[8] In general, all $L^{(k)}_0$, $k = 2, 3, \ldots$ can differ from each other. However, by a suitable gauge transformation it is possible to transform $L^{(k)}$ in the form (4.5).

This yields the $2+1$ dimensional N-wave system [34]

$$U_{ij,z_2} = \lambda_{ij} U_{ij} - \mu_{ij} U_{ij} + \sum_{l=1}^{n} (\lambda_{in} - \lambda_{nj}) U_{ik} U_{kj} \tag{4.10}$$

where

$$\mu := a_i \lambda_{ij} - b_i. \tag{4.11}$$

Ignoring the dependency of the additional variable s, the equation (4.10) reduces to the N-wave interaction equation which we obtained in Sec.3.2.

2. Calculation of the second hierarchy equation (4.3) for $i=1$, $k=3$ reads

$$\mathrm{ad}A(L_1^{(3)}) = L_{0,y}^{(3)} - [U, L_0^{(3)}] - AL_{0,s}^{(3)}. \tag{4.12}$$

This equation hat the same form as (4.6) except for the difference of index above L. This gives

$$(L_1^{(3)})_{ij} = \lambda_{ij} U_{ij} \tag{4.13}$$

The equation (4.3) for $i=2$, $k=3$ reads

$$\mathrm{ad}A(L_2^{(3)}) = L_{1,y}^{(3)} - [U, L_1^{(3)}] - AL_{1,s}^{(3)} + 2L_0^{(3)} U_s, \qquad i \neq j \tag{4.14}$$

which determines the off-diagonal parts of L_2

$$L_{2,ij}^{(3)} = \frac{1}{a_i - a_j} [\lambda_{ij} U_{ij,y} + (2b_i - a_i \lambda_{ij}) U_{ij,s} + \sum_{m=1}^{n} (\lambda_{im} - \lambda_{mj}) U_{im} U_{mj}]. \tag{4.15}$$

The equation (4.2) for $k=3$ reads

$$U_{z_3} = L_{2,y}^{(3)} - [U, L_2^{(3)}] - AL_{2,s}^{(3)} + L_0^{(3)} U_{ss} + L_1^{(3)} U_s. \tag{4.16}$$

Splitting this into diagonal and off-diagonal parts, the diagonal part read

$$\begin{aligned}(\partial_y - a_i \partial_s)(L_2^{(3)})_{ii} &= [U, L_2^{(3)}]_{ii} - (L_1^{(3)} U_s)_{ii} \\ &= \sum_{k=1}^{n} \Big(\frac{\lambda_{ik}}{a_i - a_k} (U_{ik} U_{ki})_y + \frac{2b_i - a_i \lambda_{ik}}{a_i - a_k} (U_{ik} U_{ki})_s \\ &\quad -2\lambda_{ik} U_{ik} U_{ki,s} \Big),\end{aligned} \tag{4.17}$$

and the off-diagonal part yields the second hierarchy equation

$$\begin{aligned}U_{ij,z_3} &= \frac{1}{a_i - a_j} \Big[\lambda_{ij} U_{ij,yy} + 2(b_i - a_i) \lambda_{ij} U_{ij,ys} + (a_i^2 \lambda_{ij} - b_i (a_i + a_j)) U_{ij,ss} \Big] \\ &\quad + \sum_{k=1}^{n} \Big(\frac{\lambda_{ik}}{a_i - a_k} U_{ik,y} U_{kj} - \frac{\lambda_{kj}}{a_k - a_j} U_{ik} U_{kj} \Big)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\Big(\frac{2b_i-a_i\lambda_{ik}}{a_i-a_k}U_{ik,s}U_{kj}-\frac{2b_k-a_k\lambda_{kj}}{a_k-a_j}U_{ik}U_{kj,s}\Big)\\
&+\sum_{k=1}^{n}\frac{1}{a_i-a_k}\sum_{l=1}^{n}(\lambda_{il}-\lambda_{lk})U_{il}U_{lk}U_{kj}-\sum_{k=1}^{n}\frac{1}{a_k-a_j}U_{ik}\sum_{l=1}^{n}(\lambda_{kl}-\lambda_{lj})U_{kl}U_{lj}\\
&+\sum_{k=1}^{n}\lambda_{ik}U_{ik}U_{kj,s}+\frac{1}{a_i-a_j}\sum_{k=1}^{n}(\lambda_{ik}-\lambda_{kj})\Big((U_{ik}U_{kj})_y-a_i(U_{ik}U_{kj})_s\Big)\\
&+(L_2^{(3)})_{ii}U_{ij}-U_{ij}(L^{(3)})_{jj}. \qquad (4.18)
\end{aligned}
$$

Due to the dependency of gauge potential on s, the diagonal part $(L_2^{(3)})_{ii}$ contains integrals with respect to y and s (cf. Sec.3.2). If we assume that the gauge potentials do not depend on s, the equation (4.18) yields the second hierarchy equation of the N-wave system in Sec.3.2

4.2.2 Recursion formula of the 2+1-dimensional N-wave system

In the following, we treat the 2+1-dimensional N-wave system more exactly by deriving a recursion formula of the reduced sdYM hierarchy.

Proposition 4.2 *The reduced sdYM hierarchy is equivalent to*

$$U_{ij,z_k}=(a_i-a_j)(L_k^{(k)})_{ij}, \qquad k=2,3,\dots \qquad (4.19)$$

where $L_p^{(k)}$, $p=1,2,\dots,k$ is determined by the following recursion formula

$$
\begin{aligned}
(L_p^{(k)})_{ij} &= \frac{1}{a_i-a_j}\Big((L_{p-1}^{(k)})_{ij,y}-\sum_{m=1}^{n}U_{im}(L_{p-1}^{(k)})_{mi}-a_i(L_{p-1}^{(k)})_{ij,s}\\
&\qquad+\sum_{j=0}^{k-1}\begin{pmatrix}k-j-1\\k-p\end{pmatrix}\sum_{m=1}^{n}(L_j^{(k)})_{im}U_{mj,s^{p-j-1}}\Big),\\
&\qquad p=1,2,\dots,k \qquad (4.20)\\
(\partial_y-a_i\partial_s)(L_{p-1}^{(k)})_{ii} &= \sum_{m=1}^{n}U_{im}(L_{p-1}^{(k)})_{mi}-\sum_{j=0}^{k-1}\begin{pmatrix}k-j-1\\k-p\end{pmatrix}\sum_{m=1}^{n}(L_j^{(k)})_{im}U_{mi,s^{p-j-1}},\\
&\qquad i=1,2,\dots,k \qquad (4.21)
\end{aligned}
$$

with the initial condition

$$\mathrm{ad}A(L_0^{(k)})=0 \qquad (4.22)$$

Proof. The right-hand side of (4.2) and (4.3) for $p=k$ are equivalent, therefore we find

$$U_{z_k}=\mathrm{ad}A(L_k^{(k)}) \qquad (4.23)$$

which yields (4.19). Moreover, the diagonal and off-diagonal part of (4.3) yield (4.21) and (4.20) respectively. ■

For the first value of k we find

k = 2

$$
\begin{aligned}
L_0^{(2)} &= \text{diag}(b_1, b_2, \ldots, b_n), \\
(L_1^{(2)})_{ij} &= \frac{1}{a_i - a_j}\Big((L_0^{(2)})_{ij} - \sum_{m=1}^{n} U_{im}(L_0^{(2)})_{mj} - a_i(L_0^{(2)})_{ij,s} \\
&\quad + \sum_{j=0}^{1}\begin{pmatrix} 1-j \\ 1 \end{pmatrix}\sum_{m1}^{n}(L_j^{(2)})_{im}U_{mj,s^{-j}}\Big) \\
&= \lambda_{ij}U_{ij}, \\
(\partial_y - a_i\partial_s)(L_1^{(2)})_{ii} &= 0 \qquad \Rightarrow \qquad (L_1^{(2)})_{ii} = 0 \\
U_{ij,z_2} &= (L_2^{(2)})_{ij} \\
&= \frac{1}{a_i - a_j}\Big((L_1^{(2)})_{ij,y} - \sum_{m=1}^{n} U_{im}(L_1^{(2)})_{mj} - a_i(L_1^{(2)})_{ij,s} \\
&\quad + \sum_{l=0}^{1}\begin{pmatrix} 1-l \\ 0 \end{pmatrix}\sum_{m=1}^{n}(L_l^{(2)})_{im}U_{mj,s^{1-l}}\Big).
\end{aligned}
$$

k=3

$$
\begin{aligned}
L_0^{(3)} &= \text{diag}(b_1, b_2, \ldots, b_n), \\
(L_1^{(3)})_{ij} &= \frac{1}{a_i - a_j}\Big((L_1^{(3)})_{ij,y} - a_i(L^{(3)})_{ij,s} - \sum_{m=1}^{n} U_{im}(L_{mj}^{(3)} \\
&\quad + \sum_{j=0}^{2}\begin{pmatrix} 2-j \\ 2 \end{pmatrix}\sum_{m=1}^{n}(L_j^{(3)})_{im}U_{mj,s^{-j}}\Big), \\
&= \lambda_{ij}U_{ij}, \\
(L_1^{(3)})_{ii} &= (\partial_y - a_i\partial_s)^{-1}\Big(\sum_{m=1}^{n} U_{im}(L_1^{(3)})_{mi} - \sum_{j=0}^{2}\begin{pmatrix} 2-j \\ 1 \end{pmatrix}\sum_{m=1}^{n}(L_j^{(3)})_{im}U_{mi,s^{1-j}}\Big) \\
&= 0, \\
(L_2^{(3)})_{ij} &= \frac{1}{a_i - a_j}\Big((L_0^{(3)})_{ij,y} - a_i(L_0^{(3)})_{ij,s} - \sum_{m=1}^{n} U_{im}(L_0^{(3)})_{mj} \\
&\quad + \sum_{j=0}^{2}\begin{pmatrix} 2-j \\ 2 \end{pmatrix}\sum_{m=1}^{n}(L_j^{(3)})_{im}U_{mj,s^{-j}}\Big) \\
&= \frac{1}{a_i - a_j}[\lambda_{ij}U_{ij,y} + (2b_i - a_i\lambda_{ij})U_{ij,s} + \sum_{m=1}^{n}(\lambda_{im} - \lambda_{mj})U_{im}U_{mj}], \\
(L_2^{(3)})_{ii} &= (\partial_y - a_i\partial_s)^{-1}\Big(\sum_{m=1}^{n} U_{im}(L_2^{(3)})_{mi} - \sum_{j=0}^{2}\sum_{m=1}^{n}(L^{(3)})_{im}U_{mi,s^{2-j}}\Big) \\
&= (\partial_y - a_i\partial_s)^{-1}\Big(\sum_{m=1}^{n}\frac{\lambda_{im}}{a_m - a_i}(U_{im}U_{mi})_y
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{2b_i - a_i\lambda_{ik}}{a_i - a_k}(U_{ik}U_{ki})_s - 2\lambda_{ik}U_{ik}U_{ki,s}\Big), \\
U_{ij,z_3} &= (L_3^{(3)})_{ij} \\
&= \frac{1}{a_i - a_j}\Big((L_2^{(3)})_{ij,y} - a_i(L_2^{(3)})_{ij,s} - \sum_{m=1}^{n} U_{im}(L_2^{(3)})_{mj} \\
&\quad + \sum_{j=0}^{2}\sum_{m=1}^{n}(L_j^{(3)})_{im}U_{mj,s^{2-j}}\Big) \\
&= \frac{1}{a_i - a_j}\Big[\lambda_{ij}U_{ij,yy} + 2(b_i - a_i)\lambda_{ij}U_{ij,ys} + (a_i^2\lambda_{ij} - b_i(a_i + a_j))U_{ij,ss}\Big] \\
&\quad + \sum_{k=1}^{n}\Big(\frac{\lambda_{ik}}{a_i - a_k}U_{ik,y}U_{kj} - \frac{\lambda_{kj}}{a_k - a_j}U_{ik}U_{kj}\Big) \\
&\quad + \sum_{k=1}^{n}\Big(\frac{2b_i - a_i\lambda_{ik}}{a_i - a_k}U_{ik,s}U_{kj} - \frac{2b_k - a_k\lambda_{kj}}{a_k - a_j}U_{ik}U_{kj,s}\Big) \\
&\quad + \sum_{k=1}^{n}\frac{1}{a_i - a_k}\sum_{l=1}^{n}(\lambda_{il} - \lambda_{lk})U_{il}U_{lk}U_{kj} - \sum_{k=1}^{n}\frac{1}{a_k - a_j}U_{ik}\sum_{l=1}^{n}(\lambda_{kl} - \lambda_{lj})U_{kl}U_{lj} \\
&\quad + \sum_{k=1}^{n}\lambda_{ik}U_{ik}U_{kj,s} + \frac{1}{a_i - a_j}\sum_{k=1}^{n}(\lambda_{ik} - \lambda_{kj})\Big((U_{ik}U_{kj})_y - a_i(U_{ik}U_{kj})_s\Big) \\
&\quad + (L_2^{(3)})_{ii}U_{ij} - U_{ij}(L^{(3)})_{jj}.
\end{aligned}
$$

4.2.3 Reduction to the KP hierarchy

In this case, we choose the gauge potentials

$$
A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}. \tag{4.24}
$$

The hierarchy equations have same form as in the 2+1-dimensional N-wave case, so that

$$
U_{z_k} = L_{k-1,y}^{(k)} - UL_{k-1}^{(k)} - AL_{k-1,s} + \sum_{l=0}^{k-1} L_j^{(k)}U_{s^{k-l-1}}, \qquad k = 2, 3, \ldots \tag{4.25}
$$

$$
\begin{aligned}
\mathrm{ad}A(L_i^{(k)}) &= L_{i-1,y}^{(k)} - UL_{i-1}^{(k)} - AL_{i-1,s}^{k} \\
&\quad + \sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix} L_l^{(k)}U_{s^{i-l-1}}, \qquad i = 1, 2, \ldots, k, \qquad k = 2, 3, \ldots
\end{aligned} \tag{4.26}
$$

$$
\mathrm{ad}A(L_0^{(k)}) = 0 \qquad k = 2, 3, \ldots \tag{4.27}
$$

(4.27) means that $L_0^{(k)}$ has a following form

$$
L_0^{(k)} = \begin{pmatrix} \phi & 0 \\ \psi & \phi \end{pmatrix}, \tag{4.28}
$$

with functions ϕ, ψ.

1. Calculation of the first hierarchy equation

(4.26) for $i = 1$, $k = 2$ reads

$$\mathrm{ad}A(L_1^{(2)}) = L_{0,y}^{(2)} - [U, L_0^{(2)}] - AL_{0,s}^{(2)}. \tag{4.29}$$

Inserting

$$L_0^{(2)} := \begin{pmatrix} f & 0 \\ h & f \end{pmatrix}, \qquad L_1^{(2)} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.30}$$

with unknowns a, b, c, d, f, h, the equation (4.29) leads to

$$b = h, \qquad a - d = 0. \tag{4.31}$$

The equation (4.25) for $k = 2$ reads

$$U_{z_2} = L_{1,y}^{(2)} - [U, L_1^{(2)}] - AL_{1,s}^{(2)} + L_0^{(2)} U_s, \tag{4.32}$$

which leads to the following four equations

$$a_y - c + hu = 0 \tag{4.33}$$
$$2h_y = 0 \tag{4.34}$$
$$u_{z_2} = c_y - [u, a] \tag{4.35}$$
$$a_y - uh + c \tag{4.36}$$

Adding (4.33) and (4.36) we see that $a_y = 0$. Now we impose $a = 0$. Inserting this in (4.33) and (4.35) we obtain

$$c = hu, \qquad u_{z_2} = hu_y. \tag{4.37}$$

Setting

$$L_0^{(k)} = A, \qquad k = 2, 3, \dots, \tag{4.38}$$

we obtain $c = u$ and the linear equation

$$u_{z_2} = u_y. \tag{4.39}$$

2. Calculation of the second hierarchy equation

The equation (4.26) for $i = 1$, $k = 3$ reads

$$\mathrm{ad}A(L_1^{(3)}) = L_{0,y}^{(3)} - [U, L_0^{(3)}] - AL_{0,s}^{(3)}. \tag{4.40}$$

Inserting

$$L_0^{(3)} := \begin{pmatrix} n & 0 \\ m & n \end{pmatrix}, \qquad L^{(3)} := \begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{4.41}$$

The diagonal parts of (4.40) lead to $n_y = 0$, $q = h$. As a consequence, $L_1^{(3)}$ has the form

$$L_1^{(3)} = \begin{pmatrix} p & n \\ r & p \end{pmatrix} \tag{4.42}$$

(4.26) for $i = 2$, $k = 3$ is

$$\mathrm{ad}A(L_2^{(3)}) = L_{1,y}^{(3)} - [U, L_1^{(3)}] - AL_{1,s}^{(3)} + 2L_0^{(3)}U_s. \tag{4.43}$$

Inserting

$$L_2^{(3)} := \begin{pmatrix} \alpha & \beta \\ \delta & \epsilon \end{pmatrix} \tag{4.44}$$

the upper off-diagonal part of this equation implies $m_y = 0$. Requiring (4.38), the remaining equations in (4.43) lead to $p_y = 0$, $\beta = r - u$.
Then we have

$$L_0^{(3)} = A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad L_1^{(3)} = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}, \qquad L_2^{(3)} = \begin{pmatrix} \alpha & r - u \\ \delta & \alpha - r_y \end{pmatrix}. \tag{4.45}$$

Inserting (4.45) in (4.25) for $k = 3$ yields

$$U_{z_3} = L_2^{(3)} - [U, L^{(3)}] - AL_{2,s}^{(3)} + L_{0,ss}^{(3)} + L_1^{(3)}U_s, \tag{4.46}$$

and this leads to the following four equations

$$\alpha_y + ru + u_s - \delta - u^2 = 0, \tag{4.47}$$
$$r = \frac{1}{2}u, \tag{4.48}$$
$$u_{z_3} = \delta_y - \alpha_s - r_y u - [u, \alpha] \tag{4.49}$$
$$\alpha_y + \delta - r_s - ur - r_{yy} + u_s + u^2 = 0 \tag{4.50}$$
$$\tag{4.51}$$

Adding (4.47) and (4.50) arrive at

$$\alpha = \frac{1}{4}u_y - \frac{3}{4}\int u_s\, dy, \qquad \delta = \frac{1}{4}u_{yy} + \frac{1}{4}u_s - \frac{1}{2}u^2. \tag{4.52}$$

Inserting these in (4.49) we obtain the *noncommutative* KP equation [39]

$$u_{z_3} = \frac{1}{4}u_{yyy} - \frac{3}{4}(u^2)_y + \frac{3}{4}[u, \int u_s\, dy] + \frac{3}{4}\int u_{ss}\, dy. \tag{4.53}$$

We can see that the first and second flows of this hierarchy commute

$$u_{z_2 z_3} = u_{z_3 z_2} \tag{4.54}$$

4.2.4 Recursion formula of the KP hierarchy

We turn to a more complete treatment of the KP hierarchy to derive the recursion formula of the reduced sdYM hierarchy. First we show that the upper off-diagonal entry of $L_i^{(k)}$ determines all other entries recursively.

Proposition 4.3 *Writing*

$$L_i^{(k)} = \begin{pmatrix} l_i^{(k)} & m_i^{(k)} \\ n_i^{(k)} & h_i^{(k)} \end{pmatrix}, \qquad k = 2, 3, \ldots, \qquad i = 0, 1, \ldots, k, \tag{4.55}$$

the upper off-diagonal entry $m_i^{(k)}$ is determined by the recursion formula

$$\begin{aligned} m_i^{(k)} &= \frac{1}{4}\Big[m_{i-1,yy}^{(k)} - um_{i-1}^{(k)} - 2m_{i-1,s}^{(k)} - 2\sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix} m_l^{(k)} u_{s^{i-l-1}} \\ &\quad + \int (u\int um_{i-1}^{(k)}\,dy)\,dy + \int (u\int m_{i-1,s}^{(k)}\,dy)\,dy - \int um_{i-1,y}^{(k)}\,dy \\ &\quad - \sum_{l-0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix} \int (u\int m_l^{(k)} u_{s^{i-l-1}}\,dy)\,dy \\ &\quad + \int \Big(\int (um_{i-1}^{(k)})_s\,dy + \int m_{i-1,s}^{(k)}\,dy - \sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix}\int m_l^{(k)} u_{s^{i-l-1}}\,dy\Big)\,dy \\ &\quad - \sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix}\Big(\int m_{l,y}^{(k)} u_{s^{i-l-1}}\,dy + \int (\int um_l^{(k)}\,dy) u_{s^{i-l-1}}\,dy \\ &\quad - \sum_{p=0}^{k-1}\begin{pmatrix} k-p-1 \\ k-l-1 \end{pmatrix}\int (\int m_p^{(k)} u_{s^{l-p}}\,dy) u_{s^{i-l-1}}\,dy\Big)\Big] \\ &\quad \textit{with} \qquad m_0^{(k)} = 0, \qquad m_1^{(k)} = 1. \end{aligned} \tag{4.56}$$

Proof. Inserting (4.55) in (4.26), we find that (4.26) yields the following equations

$$-m_i^{(k)} = l_{i-1,y}^{(k)} - n_{i-1}^{(k)} + \sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix} m_l^{(k)} u_{s^{i-l-1}}, \tag{4.57}$$

$$l_{i-1}^{(k)} - h_{i-1}^{(k)} = m_{i-1,y}^{(k)}, \tag{4.58}$$

$$l_i^{(k)} - h_i^{(k)} = n_{i-1,y}^{(k)} - ul_{i-1}^{(k)} - l_{i-1,s}^{(k)} + \sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix} h_l^{(k)} u_{s^{i-l-1}}, \tag{4.59}$$

$$m_i^{(k)} = h_{i-1,y}^{(k)} - um_{i-1}^{(k)} - m_{i-1,s}^{(k)} + n_{i-1}^{(k)}. \tag{4.60}$$

Adding (4.57) and (4.60), then integrating with respect to y, we can see that

$$l_{i-1}^{(k)} + h_{i-1}^{(k)} = \int um_{i-1}^{(k)}\,dy + \int m_{i-1,s}^{(k)}\,dy - \sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix}\int m_l^{(k)} u_{s^{i-l-1}}\,dy. \tag{4.61}$$

Together with (4.58), we obtain

$$\begin{aligned} l^{(k)}_{i-1} &= \frac{1}{2}\Big(-m^{(k)}_{i-1,y} + \int u m^{(k)}_{i-1}\,dy + \int m^{(k)}_{i-1,s}\,dy \\ &\quad - \sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix}\int m^{(k)}_l u_{s^{i-l-1}}\,dy\Big), && (4.62) \\ h^{(k)}_{i-1} &= \frac{1}{2}\Big(m^{(k)}_{i-1,y} + \int u m^{(k)}_{i-1}\,dy + \int m^{(k)}_{i-1,s}\,dy \\ &\quad - \sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix}\int m^{(k)}_l u_{s^{i-l-1}}\,dy\Big). && (4.63) \end{aligned}$$

Inserting (4.62) and (4.63) in (4.57), we find

$$n^{(k)}_{i-1} = m^{(k)}_i + \frac{1}{2}\Big(u m^{(k)}_{i-1} + m^{(k)}_{i-1,s} - m^{(k)}_{i-1,yy}\Big)\sum_{l=0}^{k-1}\begin{pmatrix} k-l-1 \\ k-i \end{pmatrix} m^{(k)}_l u_{s^{i-l-1}}. \tag{4.64}$$

Inserting $l^{(k)}_{i-1}$, $h^{(k)}_{i-1}$ and $n^{(k)}_{i-1}$ in (4.59), we have (4.56). Due to the requirement (4.38), i.e.

$$L^{(k)}_0 = A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.65}$$

we must set

$$m^{(k)}_0 = 0, \qquad n^{(k)}_0 = 1$$

Inserting this in (4.64) for $i = 1$, we have $m^{(k)}_1 = 1$. ■

Next we show that the reduced sdYM hierarchy is determined recursively, once the upper off-diagonal entry of $L^{(k)}_i$ is calculated by (4.56).

Proposition 4.4 *The reduced sdYM hierarchy is equivalent to*

$$\begin{aligned} u_{z_k} &= \frac{1}{2}\Big[-m^{(k)}_{k-1,yyy} + 2m^{(k)}_{k-1,sy} + u m^{(k)}_{k-1,y} + \{u_y\,,\, m^{(k)}_{k-1}\} + \{u\,,\, m^{(k)}_{k-1,y}\} \\ &\quad + \sum_{l=0}^{k-2}(m^{(k)}_l u_s)_y - u\int[\,u\,,\, m^{(k)}_{k-1}\,]\,dy - u\int m^{(k)}_{k-1,s}\,dy - \int m^{(k)}_{k-1,ss}\,dy \\ &\quad + \sum_{l=0}^{k-2} u\int m^{(k)}_l u_{s^{k-l-1}}\,dy - \int[\,u_s\,,\, m^{(k)}_{k-1}\,]\,dy - \int[\,u\,,\, m^{(k)}_{k-1,s}\,]\,dy \\ &\quad + \sum_{l=0}^{k-2}\int(m^{(k)}_l u_{s^{k-l-1}})_s\,dy - \sum_{l=0}^{k-1}\sum_{p=0}^{k-2}\begin{pmatrix} k-p-1 \\ k-l-1 \end{pmatrix}\int m^{(k)}_p u_{s^{l-p}}\,dy u_{s^{k-l-1}} \\ &\quad + \sum_{l=0}^{k-1}\Big(m^{(k)}_{l,y} + \int[\,u\,,\, m^{(k)}_l\,]\,dy + \int m^{(k)}_{l,s}\,dy\Big)u_{s^{k-l-1}}\Big]. \end{aligned} \tag{4.66}$$

Proof, By inserting (4.55), we find that the lower off-diagonal part of (4.25) yields

$$u_{z_k} = n^{(k)}_{k-1,y} - ul^{(k)}_{k-1} - l^{(k)}_{k-1,s} - \sum_{l=0}^{k-1} h^{(k)}_l u_{s^{k-l-1}}. \tag{4.67}$$

Applying (4.62), (4.64) for $i = k$ and (4.63) for $i - 1 = l$ to (4.67), we obtain (4.66). ■

Hence, the upper off-diagonal entry $m^{(k)}_i$ determines the evolution equations as well as the other entries of $L^{(k)}_i$ recursively. For the first k we find

k = 2

$$\begin{aligned} m^{(2)}_0 &= 0, \qquad m^{(2)}_1 = 1, \\ u_{z_2} &= u_y. \end{aligned} \tag{4.68}$$

k = 3

$$\begin{aligned}
m^{(3)}_0 &= 0, \qquad m^{(3)}_1 = 1, \\
m^{(3)}_2 &= \frac{1}{4}\Big[- um^{(3)}_1 - 2\sum_{l=0}^{1} \binom{2-l}{1} m^{(3)}_l u_{s^{1-l}} \\
&\quad + \int (u \int um^{(3)}_1 \, dy) \, dy - \int um^{(3)}_{1,y} \, dy - \sum_{l=0}^{1} \binom{2-l}{1} \int (u \int m^{(3)}_l u_{s^{1-l}} \, dy) \, dy \\
&\quad + \int \Big(\int (um^{(3)}_1)_s \, dy - \sum_{l=0}^{1} \binom{2-l}{1} \int m^{(3)}_l u_{s^{1-l}} \, dy \Big) \, dy \\
&\quad - \sum_{l=0}^{1} \binom{2-l}{1} \Big(\int (\int um^{(3)}_l \, dy) u_{s^{1-l}} \, dy \\
&\quad - \sum_{p=0}^{1} \binom{2-p}{2-l} \int (\int m^{(3)}_p u_{s^{l-p}} \, dy) u_{s^{1-l}} \, dy \Big) \Big] \\
&= -\frac{1}{2} u, \\
u_{z_3} &= \frac{1}{2}\Big[- m^{(3)}_{2,yyy} + 2m^{(3)}_{2,sy} + um^{(3)}_{2,y} + \{u_y \, , \, m^{(3)}_2\} + \{u \, , \, m^{(3)}_{2,y}\} \\
&\quad + \sum_{l=0}^{1} (m^{(3)}_l u_s)_y - u \int [\, u \, , \, m^{(3)}_2 \,] \, dy - u \int m^{(3)}_{2,s} \, dy - \int m^{(3)}_{2,ss} \, dy \\
&\quad + \sum_{l=0}^{1} u \int m^{(3)}_l u_{s^{2-l}} \, dy - \int [\, u_s \, , \, m^{(3)}_2 \,] \, dy - \int [\, u \, , \, m^{(3)}_{2,s} \,] \, dy \\
&\quad + \sum_{l=0}^{1} \int (m^{(3)}_l u_{s^{2-l}})_s \, dy - \sum_{l=0}^{2} \sum_{p=0}^{1} \binom{2-p}{2-l} \int m^{(3)}_p u_{s^{l-p}} \, dy u_{s^{2-l}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=0}^{2}\Big(m_{l,y}^{(3)}+\int[\,u\,,\,m_{l}^{(3)}\,]\,dy+\int m_{l,s}^{(3)}\,dy\Big)u_{s^{k-l-1}}\Big]\\
=\;&\frac{1}{4}u_{yyy}-\frac{3}{4}(u^2)_y+\frac{3}{4}[u,\int u_s\,dy]+\frac{3}{4}\int u_{ss}\,dy. \qquad (4.69)
\end{aligned}
$$

Conclusion

In this thesis we explored a definition of a sdYM hierarchy, starting from the infinite linear system (2.1), (2.2), and showed that it includes (or implies) some other versions of sdYM hierarchies. Furthermore, we explored a subsystem of the sdYM hierarchy introduced in Sec.2.1 (pre-sdYM hierarchy) and showed that the pre-sdYM hierarchy reduces to "possible" hierarchies (it is necessary to check that all equation of the hierarchy commute with each other) associated with the following well-known (noncommutative) integrable systems by a suitable choice of the gauge potentials, and we obtained recursion formulae for the reduced pre-sdYM hierarchies. We summarize our results in the following table

Considered PDE	Reduction ansatz	PDE from sdYM?	Recursion formula from pre-sdYM hierarchy?
(1) N-wave system	$\partial_{\bar{y}} = \partial_{\bar{z}} = 0$ $A_{\bar{z}} = \mathrm{diag}(a_1, a_2, \ldots)$	Yes	Yes
KdV equation (2) (BD reduction)	$\partial_{\bar{z}} = 0, y = \bar{y}$ $A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A_y = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}$	Yes	No
(3) (MS reduction)	$A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A_y = \begin{pmatrix} u & 1 \\ u_y - u^2 & -u \end{pmatrix}$	Yes	Yes
(4) NLS equation	$\partial_{\bar{z}} = 0, y = \bar{y}$ $A_{\bar{z}} = \begin{pmatrix} a\mathbf{1}_N & 0 \\ 0 & -a\mathbf{1}_M \end{pmatrix}$ $A_y = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}$	Yes	Yes
(5) Zakharov system	$\partial_{\bar{z}} = 0$ $A_{\bar{z}} = \begin{pmatrix} a\mathbf{1}_N & 0 \\ 0 & -a\mathbf{1}_M \end{pmatrix}$ $A_y = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}$	Yes	Yes
(6) Sine-Gordon equation	$\partial_z = \partial_{\bar{z}} = 0$ $A_{\bar{z}} = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $A_y = -\frac{i}{2}\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$	Yes (in the commutative case)	No

Table 1 : Reductions of the pre-sdYM hierarchy

Considered PDE	Reduction ansatz	PDE from sdYM?	Recursion formula from pre-sdYM hierarchy?
(7) 2+1-dimensional N-wave system	$\partial_{\bar{y}} = \partial_{\bar{z}} = 0$ $A = \mathrm{diag}(a_1, a_2, \ldots)$	Yes	Yes
(8) KP equation	$\partial_{\bar{y}} = \partial_{\bar{z}} = 0$ $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $U = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}$	Yes	Yes

Table 2 : Reductions of the sdYM hierarchy with gauge potentials in a Lie algebra of differential operators

In the tables above, the reductions of the pre-sdYM hierarchy (1), (2), (4), (7), (8) can be found in [13] (in the commutative cases). The reductions (3), (5), (6) are performed applying the reduction ansätze which lead from the sdYM equation to the KdV equation, Zakharov system and Sine-Gordon equation, respectively (see [25, 37]). The noncommutative calculations as well as derivation of the (noncommutative) recursion formulae for the reduced pre-sdYM hierarchies appear to be new. Moreover, we showed that via the reduction (3), the pre-sdYM hierarchy reduces to a hierarchy consisting of the two hierarchies associated with the NLS equation and complexified mKdV equation.

All formalisms in [13] were derived under the condition that the operator $\partial_{\bar{z}} - \mathrm{ad}A_{\bar{z}}$ is invertible (see Sec.2.2 and Sec.2.3), though there are no examples of reductions in [13] for which this turns out to be the case that the authors of [13] do not formulate this condition so strictly and leave alternatives open. But this condition is strongly used in the development of the theory and no concrete alternative is mentioned. Moreover, we showed that recursion relations for reduced pre-sdYM hierarchies could be derived in cases where this operator is *not* invertible. Hence, we expect that the condition is too strong and there should be other conditions with similar implication. We need further research to find out under which conditions we indeed obtain a recursion relation to calculate higher equations of reduced pre-sdYM hierarchies. For this purpose, our results should be helpful.

In the 2+1-dimensional case, we showed that the sdYM hierarchy introduced in [13] ((4.10) in Sec.4.1.1) yields hierarchies of 2+1-dimensional integrable systems. However, there are still problems left open. Firstly, the authors of [13] developed the formalisms under the condition that the operator $\mathrm{ad}A$ is invertible (see Sec.4.1.2) which, however, is not satisfied in examples of reductions. This is the same problem as in the 1+1-dimensional case.

There are also other points left for future research. Firstly, the pre-sdYM hierarchy is only an example of the subsystem of the sdYM hierarchy defined in Sec.2.1. We can find another version of "smaller" sdYM hierarchies which reduce to another integrable systems (see [27], for example). Secondly, as we mentioned in Sec.2.1, the form of the pre-sdYM hierarchy is *not* gauge invariant. Alternatively, we can consider another subsystem of the sdYM hierarchy, such as the compatibility condition of the following linear system (all notations follow in Sec.2.1)

$$(\lambda\partial_{x_0} - \partial_{x_1})\Psi = (A_0\lambda - D_1)\Psi,$$

$$(\lambda^n \partial_{t_0} - \partial_{t_n})\Psi \;=\; (\sum_{i=1}^{n} C_{n-i}\lambda^i - \sum_{i=0}^{n-1} B_{n-i}\lambda^i)\Psi, \qquad n = 1, 2, \ldots \tag{4.70}$$

which is covariant under the gauge transformation (2.3) :

$$\begin{aligned} \Psi \mapsto \tilde{\Psi} &= g^{-1}\Psi, \\ A_\mu \mapsto \tilde{A}_\mu &= g^{-1}A_\mu g - g^{-1}g_\mu, \end{aligned} \tag{4.71}$$

where g, A_μ are in a (Lie) algebra of $N \times N$ matrices. We expect that we can obtain a gauge invariant pre-sdYM hierarchy from the compatibility condition of this linear system and its recursion relation by a suitable reduction.

Acknowledgement

At the end of this thesis, the author would like to express gratitudes to all those who supported him during this thesis. The author especially wants to thank apl. Prof. Dr. Folkert Müller-Hoissen, for many helpful advices, discussions, and the suggestion of the theme of this thesis, apl. Prof. Dr. Karl-Henning Rehren for taking the burden of refereeing this thesis, Prof. Aristophanes Dimakis, Dr. Kouich Toda and Dr. Masashi Hamanaka for stimulating discussions and advice.

This work has been carried out at the Max-Planck-Institute for Dynamcs and Self-Organization in Göttingen.

Thanks to the MPI for the very good research environment. Finally, many thanks to the author's families and friends for kindness and patience.

Appendix A: Another approach to 2+1-dimensional hierarchies

A.1 Another 2+1-dimensional system

In the following, we show another "possibility" to obtain hierarchies of 2+1-dimensional systems. As mentioned in Sec.2.1, the linear system (2.11)

$$\begin{array}{rcl} (\lambda^m \partial_{x_0} - \partial_{x_m}) \ \Psi &=& (\sum_{i=1}^{m} A_{m-i}\lambda^i - \sum_{i=0}^{m-1} D_{m-i}\lambda^i) \ \ \Psi, \\ (\lambda^n \partial_{t_0} - \partial_{t_n}) \ \Psi &=& (\sum_{i=1}^{n} C_{n-i}\lambda^i - \sum_{i=0}^{n-1} B_{n-i}\lambda^i) \ \ \Psi \end{array} \tag{A.1}$$

is a large source of hierarchies of integrable systems. Now we consider $(A.1)$ in a different algebra. The process introduced in this subsection still needs further reserches. We show the current results. Considering $(A.1)$ for $m = 1$ in Sec.2.1 :

$$\begin{array}{rcl} (\partial_y + \lambda \partial_{\bar{z}})\Psi &=& (A_y + A_{\bar{z}}\lambda)\Psi, \\ (\partial_{z_{n+1}} - \lambda^n \partial_{\bar{y}})\Psi &=& (\displaystyle\sum_{i=1}^{n} C_{n-i}\lambda^i - \sum_{i=0}^{n-1} B_{n-i}\lambda^i)\Psi. \qquad n = 1, 2, \ldots \end{array} \tag{A.2}$$

whose compatibility condition is equivalent to the sdYM hierarchy introduced in [13]. Note that we use the identification

$$\begin{array}{llll} x_0 = \bar{z}, & x_1 = -y, & t_0 = -\bar{y}, & t_n = -z_{n+1} \ , \\ A_0 = A_{\bar{z}}, & B_1 = -A_z, & C_0 = -A_{\bar{y}}, & D_1 = -A_y \ , \end{array} \tag{A.3}$$

for easy comparison with the notations in Sec.2.2.

By imposing the following conditions on $(A.2)$, we obtain the following reduced systems :

(1) $\partial_{\bar{z}} = 0$**,** $\partial_{\bar{y}} = 0$ [9]

$$\begin{array}{rcl} \partial_y \Psi &=& (A_y + A_{\bar{z}}\lambda)\Psi, \\ \partial_{z_{n+1}}\Psi &=& (\displaystyle\sum_{i=1}^{n} C_{n-i}\lambda^i - \sum_{i=0}^{n-1} B_{n-i}\lambda^i)\Psi, \qquad n = 1, 2, \ldots \end{array} \tag{A.4}$$

(2) $\partial_{\bar{z}} = 0$**,** $y = \bar{y}$

$$\begin{array}{rcl} (\partial_y + \partial_s \partial_{\bar{z}})\Psi &=& (A_y + A_{\bar{z}}\partial_s)\Psi, \\ \partial_{z_{n+1}}\Psi &=& (\displaystyle\sum_{i=1}^{n} C_{n-i}\partial_s^i - \sum_{i=0}^{n-1} B_{n-i}\partial_s^i + A_y \partial_s^n + A_{\bar{z}}\partial_s^{n+1})\Psi \\ && \qquad\qquad n = 1, 2, \ldots \end{array} \tag{A.5}$$

[9] The notation $\partial_{\bar{z}} = 0$ means that all gauge potentials do not depend on the variable $\bar{z}$

For these reduced systems, we consider another "version" whose gauge potentials are in (4.4), i.e., the Lie algebra of matrix differential operators with respect to the variable s such that

$$\mathcal{G} = \left\{ \sum_{j=0}^{\ll\infty} a_j\, \partial_s^j \right\} \tag{A.6}$$

where a_j are $N \times N$ matrices with entries from an associative algebra $\mathcal{A}$ over $\mathbb{C}$ (typically an algebra of matrices of functions). By replacing $\lambda \to \partial_s$, the systems (1) and (2) yield

(1)'

$$\begin{aligned} \partial_y \Psi &= (A_y + A_{\bar{z}}\partial_s)\Psi, \\ \partial_{z_{n+1}} \Psi &= (\sum_{i=1}^{n} C_{n-i}\partial_s^i - \sum_{i=0}^{n-1} B_{n-i}\partial_s^i)\Psi, \qquad n = 1, 2, \ldots \end{aligned} \tag{A.7}$$

(2)'

$$\begin{aligned} \partial_y \Psi &= (A_y + A_{\bar{z}}\partial_s)\Psi, \\ \partial_{z_{n+1}} \Psi &= (\sum_{i=1}^{n} C_{n-i}\partial_s^i - \sum_{i=0}^{n-1} B_{n-i}\partial_s^i + A_y\partial_s^n + A_{\bar{z}}\partial_s^{n+1})\Psi, \qquad n = 1, 2, \ldots \end{aligned} \tag{A.8}$$

which lead to (2+1)-dimensional N-wave system and KP equation respectively, with suitable choice of A_y and $A_{\bar{z}}$.

In contrast to the case where the gauge potentials are in the λ-dependent Lie algebra, we have not succeeded to obtain a recursion relation. We only succeeded to obtain the first hierarchy equation.

Remark. After replacing $\lambda \to \partial_s$, the systems $(1)'$ and $(2)'$ have no longer a clear relationship with the sdYM equations. For example, the compatibility condition of $(1)'$ reads

$$A_{y,z_2} = (\partial_y - \mathrm{ad}A_y)A_z + A_{\bar{y}}A_{y,s} + A_{\bar{z}}A_{z,s}, \tag{A.9}$$

$$[\, A_{\bar{z}}\, ,\, A_z\,] = -(\partial_y + \mathrm{ad}A_y)A_{\bar{y}} - A_{\bar{y}}A_{\bar{z},s} + A_{\bar{z}}A_{\bar{y},s}, \tag{A.10}$$

$$[\, A_{\bar{z}}\; A_{\bar{y}}\,] = 0, \tag{A.11}$$

where we use the identification $(A.3)$. These equations do not imply the sdYM equations.

A.2 Reduction to 2+1 dimensional N-wave system

Choosing the gauge potentials

$$A_{\bar{z}} = \mathrm{diag}(a_1, \ldots, a_n) \tag{A.12}$$

with all eigenvalues different from each other.

The equation $(A.7)$ for $n = 1$ reads

$$\begin{aligned} \partial_y \Psi &= (U + A_{\bar{z}} \partial_s)\Psi, \\ \partial_z \Psi &= (A_z - A_{\bar{y}} \partial_s)\Psi, \end{aligned} \tag{A.13}$$

whose compatibility (integrability) condition is

$$[\, \partial_y - U - A_{\bar{z}} \partial_s \, , \, \partial_{z_2} - A_z + A_{\bar{y}} \partial_s \,] = 0. \tag{A.14}$$

where we write U instead of A_y, following the way in Sec.3.2. $(A.13)$ leads to the following three equations

$$U_{z_2} = (\partial_y - \mathrm{ad} U) A_z + A_{\bar{y}} U_s + A_{\bar{z}} A_{z,s}, \tag{A.15}$$

$$[\, A_{\bar{z}} \, , \, A_z \,] = -(\partial_y + \mathrm{ad} U) A_{\bar{y}} - A_{\bar{y}} A_{\bar{z},s} + A_{\bar{z}} A_{\bar{y},s}, \tag{A.16}$$

$$[\, A_{\bar{z}} \; A_{\bar{y}} \,] = 0, \tag{A.17}$$

The equation $(A.17)$ leads to

$$A_{\bar{y}} = \mathrm{diag}(b_1, \ldots, b_n). \tag{A.18}$$

$(A.16)$ reads

$$[\, U \, , \, A_{\bar{y}} \,] + [\, A_{\bar{z}} \, , \, A_z \,] = 0, \tag{A.19}$$

which leads to

$$(A_z)_{ij} = \lambda_{ij} U_{ij} \tag{A.20}$$

where $\lambda_{ij} = \frac{bi - bj}{ai - aj}$. Then $(A.15)$ yields the 2+1-dimensional N-wave system [25]

$$U_{ij,z_2} = \lambda_{ij} U_{ij,y} + (b_i - a_i \lambda_{ij}) U_{ij,s} + \sum_{k=1}^{n} (\lambda_{ik} - \lambda_{kj}) U_{ik} U_{kj} \tag{A.21}$$

Remark. Considering the compatibility condition of $(A.7)$ for $n = 2$, we obtain the following equations

$$U_{z_3} = [\, U \, , \, B_2 \,] + (C_1 + A_z) U_s + A_{\bar{z}} B_{2,s} - B_{2,y}, \tag{A.22}$$

$$[\, A_{\bar{z}} \, , \, C_1] = -(\partial_y - \mathrm{ad} U) A_{\bar{y}} + A_{\bar{z}} A_{\bar{y},s} - 2 A_{\bar{y}} A_{\bar{z},s} - [\, A_{\bar{z}} \, , \, A_z \,], \tag{A.23}$$

$$\begin{aligned} [\, A_{\bar{z}} \, , \, B_2 \,] &= [\, U \, , \, C_1 + A_z \,] + 2 A_{\bar{y}} U_s - (C_{1,y} + A_{z,y}) + A_{\bar{y}} A_{\bar{z},ss} \\ &\quad + A_{\bar{z}} (C_{1,s} + A_{z,s}) - (C_1 + A_z) A_{\bar{z},s}. \end{aligned} \tag{A.24}$$

From $(A.23)$ and $(A.24)$ we find

$$\begin{aligned}
C_1 &= 0, \\
(B_2)_{ij} &= \frac{1}{a_i - a_j}\{(2b_i + a_i\lambda_{ij})U_{ij,s} - \lambda_{ij}U_{ij,y} - \sum_{k=1}^{n}(\lambda_{ik} - \lambda_{kl})U_{ik}U_{kj}\} \\
(\partial_y - a_i\partial_s)B_{2,ii} &= -\sum_{k=1}^{n}\frac{\lambda_{ik}}{a_k - a_i}(U_{ik}U_{ki})_y + \sum_{k=1}^{n}\frac{(2b_i + a_i\lambda_{ik})}{a_k - a_i}(U_{ik}U_{ki})_s \\
&\quad -4\sum_{k=1}^{n}\lambda_{ik}U_{ik}U_{ki,s},
\end{aligned} \tag{A.25}$$

and the off-diagonal part of $(A.22)$ yields the second evolution equation

$$\begin{aligned}
U_{ij,z_3} &= \sum_{k=1}^{n}\Big(\frac{\lambda_{kj}}{a_k - aj}U_{ik}U_{kj,y} - \frac{\lambda_{ik}}{a_i - a_k}U_{ik,y}U_{kj}\Big) \\
&+\sum_{k=1}^{n}\Big(\frac{b_k - a_k\lambda_{kj}}{a_k - a_j}U_{ik}U_{kj,s} - \frac{b_i - a_i\lambda_{ik}}{a_i - a_k}U_{ik,s}U_{kj}\Big) \\
&+\sum_{k=1}^{n}\frac{1}{a_k - a_j}U_{ik}\sum_{l=1}^{n}(\lambda_{kl} - \lambda_{lj})U_{kl}U_{lj} - \sum_{k=1}^{n}\frac{1}{a_i - a_k}\sum_{l=1}^{n}(\lambda_{il} - \lambda_{lk})U_{il}U_{lk}U_{kj} \\
&-\sum_{k=1}^{n}\lambda_{ik}U_{ik}U_{kj,s} \\
&+\frac{1}{a_i - a_j}\Big((2a_1\lambda_{ij} - b_i)U_{ij,ys} - \lambda_{ij}U_{ij,yy} + q_i(b_i - a_i\lambda_{ij})U_{ij,ss} \\
&+a_i\sum_{k=1}^{n}(\lambda_{ik} - \lambda_{kj})(U_{ik}U_{kj})_s - \sum_{k=1}^{n}(\lambda_{ik} - \lambda_{kj})(U_{ik}U_{kj})_y\Big) \\
&-(B_2)_{ii}U_{ij} + U_{ij}(B_2)_{jj}
\end{aligned} \tag{A.26}$$

If we ignore the dependency on the variable s, this equation reduces to the second hierarchy equation of the N-wave system (3.17) which is obtained in Sec.3.2. Comparing with the second hierarchy equation of the 2+1-dimensional N-wave system which we obtained in Sec.4.18, there are no clear relationships between $(A.26)$ and (4.18).

A.3 Reduction to KP

In this subsection we consider the system $(A.8)$ with two different choice of the gauge potentials.

(1)

$$A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_y = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}. \tag{A.27}$$

(2)

$$A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_y = \begin{pmatrix} u & 1 \\ u_y - u^2 & -u \end{pmatrix}. \tag{A.28}$$

Furthermore, we assume that the gauge potentials do not depend on the variables $\bar{z}$, $y + \bar{y}$.

In this case, $(A.8)$ for $n = 1$ reads

$$\begin{aligned} \partial_y \Psi &= (A_y + A_{\bar{z}})\Psi, \\ \partial_z \Psi &= (A_z + (A_y - A_{\bar{y}})\partial_s + A_{\bar{z}}\partial_s^2)\Psi \end{aligned} \tag{A.29}$$

which leads to

$$A_{y,z} = A_{z,y} - [\, A_y \,,\, A_z \,] + (A_y - A_{\bar{y}})A_{y,s} + A_{\bar{z}}A_{y,ss} - A_{\bar{z}}A_{z,s}, \tag{A.30}$$

$$[\, A_{\bar{z}} \,,\, A_z \,] = A_{y,y} + [\, A_y \,,\, A_{\bar{y}} \,] + A_{\bar{z}}A_{y,s} + A_{\bar{z}}A_{\bar{y},s}, \tag{A.31}$$

$$[\, A_{\bar{z}} \,,\, A_{\bar{y}} \,] = 0. \tag{A.32}$$

A.3.1 Reduction (1)

First, the equation $(A.32)$ leads to the form of $A_{\bar{z}}$,

$$A_{\bar{y}} = \begin{pmatrix} f & 0 \\ h & f \end{pmatrix} \tag{A.33}$$

with functions f, h. By inserting $(A.33)$ in $(A.31)$ we obtain $f_y = 0$. In the following we set $f = 0$. As the consequence, we have

$$A_z = \begin{pmatrix} p & -h \\ r & p - u_y + h_y \end{pmatrix} \tag{A.34}$$

with new functions p, r. Now we insert $(A.33)$ and $(A.34)$ in $(A.30)$ and find the following four equations,

$$\begin{aligned} 0 &= p_y - r - hu + u_s, && \text{(A.35)} \\ h &= \frac{1}{2}u, && \text{(A.36)} \\ u_z &= r_y - up + (p - u_y + h_y) - p_s, && \text{(A.37)} \\ 0 &= p_y - u_{yy} + h_{yy} + uh + r + h_s. && \text{(A.38)} \end{aligned}$$

Adding $(A.35)$ and $(A.38)$ we can determine p,

$$p = \frac{1}{4}u_y - \frac{3}{4}\int u_s \, dy. \tag{A.39}$$

Inserting this in $(A.35)$ we find

$$r = \frac{1}{4}u_{yy} + \frac{1}{4}u_s - \frac{1}{2}u^2. \tag{A.40}$$

By insertion of $(A.39)$ and $(A.40)$ in $(A.37)$ we obtain the noncommutative KP equation [39]

$$u_z = \frac{1}{4}u_{yyy} - \frac{3}{4}(u^2)_y + \frac{3}{4}[\, u \,,\, \int u_s \, dy \,] + \frac{3}{4}\int u_{ss} \, dy. \tag{A.41}$$

A.3.2 Reduction (2)

The Yang-Mills potential $A_{\bar{y}}$ takes a same form as $(A.33)$. Let

$$A_z := \begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{A.42}$$

with functions p, q, r, s. The equation $(A.31)$ yields

$$\begin{aligned} -q &= u_y - f_y + [\,u\,,\,f\,] + h, && \text{(A.43)} \\ p - s &= u_{yy} - \{u\,,\,u_y\} - h_y + [\,u_y - u^2\,,\,f\,] - \{u\,,\,h\} + u_s + f_s, && \text{(A.44)} \\ q &= -u_y - f_y + [\,f\,,\,u\,] - h. && \text{(A.45)} \end{aligned}$$

Adding $(A.43)$ and $(A.45)$ we find $f_y = 0$. In the following, we set $f = 0$. Then we have $q = -u_y - h$. As the result, we obtain

$$A_z = \begin{pmatrix} p & -u_y - h \\ r & p - D \end{pmatrix} \tag{A.46}$$

where

$$D := u_{yy} - \{u\,,\,u_y\} - h_y + u_s - \{u\,,\,h\}. \tag{A.47}$$

The equation $(A.30)$ leads to the following four equations

$$\begin{aligned} u_z &= p_y - [\,u\,,\,p\,] - r - (u_y + h)(u_y - u^2) + u_{ys} - u_s u_{},, && \text{(A.48)} \\ h_y &= 0, && \text{(A.49)} \\ u_{yz} - \{u_z\,,\,u\} &= r_y - (u_y - u^2)p + \{u\,,\,r\} \\ &\quad +(p - D)(u_y - u^2) + u_y u_s - u u_{ys} + u u_s u - h u_s + u_{ss} - p_s, && \text{(A.50)} \\ -u_z &= p_y - D_y + (u_y - u^2)(u_y + h) - [\,p\,,\,u\,] - [\,u\,,\,D\,] + r + u u_s \\ &\quad +u_{ys} + h_s. && \text{(A.51)} \end{aligned}$$

In $(A.49)$ we impose $h = 0$. Adding $(A.48)$ and $(A.51)$ we obtain

$$p = \frac{1}{2}u_{yy} - u_y u - \frac{1}{2}u_s \tag{A.52}$$

Inserting $(A.48)$ in $(A.50)$, we have

$$\begin{aligned} r &= \frac{1}{4}u_{yyy} - \frac{1}{2}\{u\,,\,u_{yy}\} - \frac{1}{2}(u_y)^2 + \frac{1}{2}u_{ys} + u u_y u \\ &\quad -\frac{3}{2}\int u_{ys} u\,dy - \int u_y u_s\,dy + \frac{1}{2}\int u u_{ys}\,dy - \frac{3}{4}\int u_{ss}\,dy && \text{(A.53)} \end{aligned}$$

As a result, we obtain the noncommutative potential KP equation

$$\begin{aligned} u_z &= \frac{1}{4}u_{yyy} - \frac{3}{2}(u_y)^2 + \frac{1}{2}u u_s - \frac{3}{2}u_s u \\ &\quad +\frac{3}{2}\int u_{ys} u\,dy + \int u_y u_s\,dy - \frac{1}{2}\int u u_{ys}\,dy + \frac{3}{4}\int u_{ss}\,dy. && \text{(A.54)} \end{aligned}$$

In terms of

$$v := 2u_y, \tag{A.55}$$

we find the noncommutative KP equation [39]

$$v_z = \frac{1}{4}u_{yyy} - \frac{3}{4}(v^2)_y + \frac{3}{4}[\, v\,, \int v_s\, dy\,] + \frac{3}{4}\int v_{ss}\, dy. \tag{A.56}$$

Appendix B: True/Fake "zero-curvature" conditions

In the following, we consider some examples to show that not all zero-curvature (Zakharov-Shabat) condition which is equivalent to the considered PDE are good ones. For convenience, all calculations in the following are performed "commutatively".

Firstly, we consider the following linear system with (spectral) parameter λ

$$\psi_{xx} = -(\frac{1}{6}u + \lambda)\psi, \qquad \psi_t = (-\frac{1}{6}u_x + 4i\lambda^{\frac{3}{2}})\psi + (-\frac{1}{3}u + 4\lambda)\psi_x, \tag{B.1}$$

whose compatibity condition is equivalent to the KdV equation

$$u_t = (u_{xx} + \frac{1}{2}u^2)_x. \tag{B.2}$$

Next let us assume that f, g are arbitrary (differentiable) functions. Calogero and Nucci [3] showed that the equation

$$f_t = g_x, \tag{B.3}$$

which has the form of a conservation law, is equivalent to the compatibility condition of the following linear system

$$\psi_{xx} = \lambda f^2\psi + (\mu f + \frac{f_x}{f})\psi_x, \qquad \psi_t = \nu\psi + \frac{(g+\rho)}{f}\psi_x. \tag{B.4}$$

where λ, μ, ν, ρ are spectral parameters. By direct calculation one can verify that the compatibility condition

$$\psi_{xxt} = \psi_{txx} \tag{B.5}$$

for this linear system is indeed equivalent to $(B.3)$. Obviously, not all equations of the form $(B.3)$ can be integrable. Hence, the existence of a zero-curvature formulation does not imply an aspect of integrability in some sense. One has to distinguish somehow between good and bad zero-curvature formulations and some methods have been proposed (see [40], for example), but so far there are no satisfactory methods.

As a consequence of the above result, the KdV equation

$$u_t = au_{xxx} + buu_x = (au_{xx} + bu^2)_x, \qquad a, b \in \mathbb{R}\,(\,or\,\mathbb{C}) \tag{B.6}$$

is equivalent to the linear systems given by

1)

$$f = u, \qquad g = au_{xx} + bu^2$$

$$\psi_{xx} = \lambda u^2\psi + \Big(\mu u + \frac{u_x}{u}\Big)\psi_x, \qquad \psi_t = \nu\psi + \Big(a\frac{u_{xx}}{u} + bu + \frac{\rho}{u}\Big)\psi_x \tag{B.7}$$

2)

$$f = u^2, \qquad g = 2auu_{xx} - au_{xx}^2 + \frac{2}{3}bu^3,$$

$$\psi_{xx} = \lambda u^4 \psi + \left(\mu u^2 + 2\frac{u_x}{u}\right)\psi_x, \qquad \psi_t = \nu\psi + \left(2a\frac{u_{xx}}{u^2} - a\frac{u_x^2}{u^2} + \frac{2}{3}bu + \frac{\rho}{u^2}\right)\psi_x \tag{B.8}$$

We expect that the compatibility condition of such linear systems, which are generated by the above recipe, will turn out to be "fake" in the sense that no integrability aspects can be deduced from them. In this thesis we shall not discuss further details since this is a sideaspect of our work.

Appendix C: FORM

FORM - Computer algebra program

C.1 What is FORM?

In the following, we briefly describe the FORM, a computer algebra program which we used in this thesis. FORM has been developed by J.A.M. Vermaseren at NIKHEF (the Dutch Institute for Nuclear and High-Energy Physics) by the end of the 80's and it was widely used in the field of mathematics and physics to perform lengthy algebraic calculations.
Binaries for various platforms are available for download on the web page of the developer (see http://www.nikhef.nl/ form/license.html/). We do not discuss FORM in detail and only mention some the characteristics of FORM from our point of view, based on what we have learned by using FORM (for more information on FORM, see [41]). In this thesis we used the version 3.1 of FORM.

C.2 How to work with FORM?

Now let us review the basic things which are necessary to use FORM. In order to use FORM, we have to write a program, store it in a file with extension .frm. To run the FORM program, using UNIX system, putting in a console window the following command

```
form -l filename.frm
```

where the option '-l' generates a log file "filename.log". Each statement ends with a semicolon except for special instructions such as `.sort` (sort the result and continue), `.store` (store global variables, delete local variables and continue) and `.end` (terminate the program). Furthermore, like in well-known computer languages such as C, C++, the preprocessor instructions starting with # , such as `#define` , `#do` , do not end with a semicolon either. Comment lines start with *. To understand quickly how to work with FORM, we should better at look a little example rather than tackle a thick reference manual. The following program calculates the rational sum

$$\sum_{n=1}^{100} \frac{1}{n} \tag{C.1}$$

```
*
*   Example : Calculation of summation
*   SUM = 1 + 1/2 + ... + 1/100
*

#define N "100"

Function f;
Symbol i;

Local SUM = f('N');
```

```
repeat;
id f(1) = 1;
#do i = 2, 'N'
id f('i') = 1/'i' + f('i'-1);
#enddo
endrepeat;
print SUM;
.end
```

The statement `Function` declares noncommuting objects. Alternatively, for a commutative calculation one has to use `CFunction`. The `Symbol` statement declares objects used typically as arguments and arbitrary constants. These declarations are available to be used with an arbitrary number of arguments separated by commas. The `Local` statement defines a local variable `SUM` which is finally evaluated in this program. The commands

```
repeat;

⋮

endrepeat;
```

and the preprocessor instruction

```
#do

⋮

#enddo
```

are iterative procedures. The sentences between `#do` - `#enddo` are calculated iteratively up to `N`, but no substitutions are done. The results are

$$f(2) = \frac{1}{2} + f(1), \qquad f(3) = \frac{1}{3} + f(2), \qquad \ldots \tag{C.2}$$

The `repeat` - `endrepeat` statement substitutes values in each f until values for all f are substituted. As a consequence, we obtain

$$f(2) = \frac{1}{2} + 1 = \frac{3}{2}, \qquad f(3) = \frac{1}{3} + \frac{3}{2} = \frac{11}{6}, \qquad \ldots \tag{C.3}$$

The output of this program is

```
FORM by J.Vermaseren,version 3.1(Feb  8 2005)
Run at: Thu Jun  1 17:48:24 2006
    *
    *   Example : Calculation of summation
    *   SUM = 1 + 1/2 + ... + 1/100
    *

    #define N "100"
```

```
Function f;
Symbol i;

Local SUM = f('N');

repeat;
id f(1) = 1;
#do i = 2, 'N'
id f('i') = 1/'i' + f('i'-1);
#enddo
endrepeat;
print SUM;
.end

Time =       0.00 sec    Generated terms =         10
              SUM        Terms in output =          1
                         Bytes used      =         10

   SUM =
      7381/2520;
```

FORM displays automatically informations of running times, a number of generated terms while FORM is running, a number of output terms, and used bytes. These informations can be suppressed by use of `nwrite statistics;` or `off statistics;`.

In this thesis we used frequently FORM to apply differentiation and FORM fits to achieve this purpose. [10] The following program is just a small sample how to achieve differentiation in FORM, however the basic concept of calculation which is used in the programs shown in Sec.$C.4$ is same.

```
*
*   Example : Differentiation
*
*   The function u depends on the variables x,
*   u = u(x).
*

Function u, dx ;
Symbol x;

*
* The argument stands for the differentiation
* with respect to x, i.g.,
* u(2) = uxx (second order derivative of u
* with respect of x).
*
```

[10] In the context of integrable systems, FORM has been used in [36], see also [42].

```
Local F = u(3) - 2*u(2)^2 + 5*u(1);
Local Fx = dx*F;

* Define and perform the differentiation
* with respect to x.
repeat;
        id dx*u?(x?) = u(x+1) + u(x)*dx;
endrepeat;
id dx = 0;
print +s Fx;
.end
```

The statement

```
repeat;
        id dx*u?(x?) = u(x+1) + u(x)*dx;
endrepeat;
```

defines the operation rule of the differential operator `dx` . `dx` operates on the function `u` from left to right iteratively until there are no objects on the right hand side of `dx` . After differentiations and substitutions, we set `dx=0` to eliminate the remaining differential operator. The result is

```
   Fx =
       + 5*u(2)
       - 2*u(2)*u(3)
       - 2*u(3)*u(2)
       + u(4)
      ;
```

C.3 Features of FORM

As mentioned in Sec.C.1, FORM has an extensive and powerful functionality. In this thesis we mainly applied FORM to check the commutativity of hierarchy equations (see also [36]). Such calculations are usually enormously long and laborious. FORM enables a quick evaluation of them with low overheads on computers. Additionally, FORM fits to noncommutative calculations since, as we mentioned in Sec.C.2, the objects declared by `Function` are noncommutative objects and switching in `CFunction` declaration, we can easily check our calculation in the commutative case (this is a powerful tool to debug the programs).

However, on the other side, by using FORM there are points which we should take care of. Firstly, FORM does not simplify rational expressions. Let us look at a example.

```
nwrite statistics;
Symbols l, m, n;

Local eq1 = 1/(l - m)/(m - n);
Local eq2 = 1/(m - n)/(l - m);
```

```
Local diff = eq1 - eq2;

.sort

print +s diff;
.end
```

This program calculates

$$\frac{1}{(l-m)(m-n)} - \frac{1}{(m-n)(l-m)} \tag{C.4}$$

where l, m, n are arbitrary constants. The result must be 0. However, as a matter of fact, the result of FORM calculation is

```
   diff =
       - 1/( - n + m)/( - m + l)
       + 1/( - m + l)/( - n + m)
      ;
```

For such a rational calculation we used the computer algebra system MuPAD to simplify the expression :

```
>> - 1/( - n + m)/( - m + l)
   + 1/( - m + l)/( - n + m)

                                                0
```

We faced this "rational calculation problem" in the commutativity check of the N-wave flows (see Sec.C.4.2). To deal with this problem, as mentioned above, we put the outputs of FORM into MuPAD.

C.4 FORM programs

In the following we list the FORM programs we developed for the check of commutativity of hierarchy equations which we obtained by examples of reductions.

C.4.1 Symmetry verification of the KdV equation

This program is for a verification that the KdV equation (1.11) considered in Sec.1.3

$$u_t = u_{xxx} + uu_x \tag{C.5}$$

has the symmetry (note that u is a element of a commutative algebra)

$$u_\tau = \frac{1}{6}u_{xxxxx} + \frac{5}{18}uu_{xxx} + \frac{5}{9}u_x u_{xx} + \frac{5}{36}u^2 u_x. \tag{C.6}$$

This is a simple FORM program but the process of this program is a basis of the program we used for the commutativity checks of the hierarchy equations we obtained in each example of reductions (only if there exist a reduced pre-sdYM hierarchy associated with the considered system). The processes of this program are as follows.

1. Differentiate the function u_τ (named f in the following FORM program) with respect to t.
2. Differentiate the KdV equation ($C.5$) with respect to x and substitute the result from step1.
3. Calculate the function

$$H = (u_\tau)_t - (u_\tau)_{xxx} + u_\tau u_x + u(u_\tau)_x \tag{C.7}$$

and check whether $H = 0$.

```
* Symmetry verification of KdV equation
* The arguments of u stand for the differentiation
* with respect to x, t respectively so that
* u = u(x, t).

Functions dy, dt;
Functions u;
Functions x, t;
Symbols i, j;

* Symmetry of KdV equation
* f = u differentiated with respect to tau
Local f = u(5, 0)/6 + 5*u(0, 0)*u(3, 0)/18
+ 5*u(1, 0)*u(2, 0)/9 + 5*u(0, 0)*u(0, 0)*u(1, 0)/36;

Local fx = dy*f;
Local fxx = dy*fy;
Local fxxx = dy*fyy;
Local ft = dt*f;

Local G1 = u(3, 0) + u(0, 0)*u(1, 0); * KdV equation
Local G2 = dx*G1;
Local G3 = dx*G2;
Local G4 = dx*G3;
Local G5 = dx*G4;
Local G6 = dx*G5;

Local H = ft - fxxx - f*u(1,0) - u(0,0)*fy;

repeat;
* definitions of differentiation with respect to y and t
id dx*u(i?, j?) = u(i+1, j) + u(i, j)*dx;
id dt*u(i?, j?) = u(i, j+1) + u(i, j)*dt;
endrepeat;

id dx = 0;
```

```
id dt = 0;

.sort

repeat;
 id u(0, 1) = G1;
 id u(1, 1) = G2;
 id u(2, 1) = G3;
 id u(3, 1) = G4;
 id u(4, 1) = G5;
 id u(5, 1) = G6;
endrepeat;

print +s H;
.end
```

The result is

```
> ./form -l symm.frm
FORM by J.Vermaseren,version 3.1(Feb  8 2005)
Run at: Sun Jun 11 18:43:14 2006


   H =
       - 5/36*u(0,0)*u(1,0)*u(3,0)
       + 5/12*u(0,0)*u(2,0)*u(2,0)
       + 5/36*u(1,0)*u(0,0)*u(3,0)
       + 5/6*u(1,0)*u(1,0)*u(2,0)
       - 5/12*u(1,0)*u(2,0)*u(1,0)
       - 5/12*u(2,0)*u(0,0)*u(2,0)
       - 5/12*u(2,0)*u(1,0)*u(1,0)
       - 5/6*u(2,0)*u(4,0)
       + 5/6*u(4,0)*u(2,0)
      ;
```

Putting this result into MuPAD, we can see that $H = 0$, i.e., this FORM program verifies that the function f $(= u_\tau)$ is a symmetry of the KdV equation (1.11).

Remark. Since we assumed that the function u is an element of a commutative algebra, we put the output of FORM into MuPAD directory (all calculations are performed commutatively in MuPAD). In the noncommutative calculation, we must pick up coefficients of each term and verify whether the coefficients indeed vanish.

C.4.2 Commutativity of the first two N-wave flows

This program is developed to check the commutativity of the first two flows of N-wave hierarchy which we obtained in Sec.3.2. This program consists of three parts. In `PART 1` (`2`), we differentiate the expression of the right hand side of the first (second) hierarchy equations named `eq3` (`eq2`) with respect of the variable z_3 (z_2). Then, in `PART 3` we calculate the difference. Furthermore, the coefficients

$$a_i, \qquad i = 1, 2, \ldots \qquad b_j, \qquad j = 1, 2, \ldots \qquad \lambda_{ij} = \frac{b_i - b_j}{a_i - b_j}, \qquad i \neq j \tag{C.8}$$

are defined as `CFunction` in order to be allowed to possesse arguments (objects defined as `Symbol` can not have any arguments). In this program we define them as follows

$$\texttt{a('i','j')} := \frac{1}{a_i - a_j}, \qquad \texttt{lambda('i','j')} := \lambda_{ij} \tag{C.9}$$

to avoid rational calculations. As we have seen in Sec.3.2, since the N-wave hierarchy equations (3.12) and (3.17) are written in the form of each off-diagonal entry of U, we have to compute all those entries of U with `#do` - `#enddo` instructions.

```
* N-wave system
* check of commutativity of the first two flows

* fixing the dimension of the matrices:
#define N "4"



**** PART 1 ****
nwrite statistics;
Functions U, dz3, eq2;
CFunctions lambda, a;
Symbols i, j, k, l, m, y, z2, z3;
* the symbols i, ..., m are summation indices
* y, z2, z3 stand for the numbers of derivatives w.r. to
* the variables y, z2, z3, respectively.

* computing derivative of rhs of
* first equation U_z2 = eq2 with respect to z3
#do i=1,'N'
#do j=1,'N'
   Global expr'i''j' = dz3*eq2('i','j');
#enddo
#enddo

* defining the right hand side of the first equation
#do i=1,'N'
```

```
#do j=1,'N'
id eq2('i','j') = lambda('i','j')*U('i','j',1,0,0)
+ sum_(m,1,'N',
(lambda('i',m)-lambda(m,'j'))
*U('i',m,0,0,0)*U(m,'j',0,0,0));
#enddo
#enddo

repeat;
* definition of differentiation
* with respect to z3
id dz3*U(i?,j?,y?,z2?,z3?) =
U(i,j,y,z2,z3+1) + U(i,j,y,z2,z3)*dz3;
endrepeat;

* take into account that U has
* zero diagonal components
repeat;
id U(i?,i?,y?,z2?,z3?) = 0;
endrepeat;

.sort

id dz3=0;
* The variables expr12 etc now contain
* the derivative of eq2 w.r. to z3.

* defining the right hand side of
* the second equation U_z3 = eq3
#do i=1,'N'-1
#do j='i'+1,'N'
id U('i','j',0,0,1) = a('i','j')*
( lambda('i','j')*U('i','j',2,0,0)
+ sum_(k,1,'N', (lambda('i',k) - lambda(k,'j'))*(
    U('i',k,1,0,0)*U(k,'j',0,0,0)
+U('i',k,0,0,0)*U(k,'j',1,0,0) ) ) )
+ sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,1,0,0)*U(k,'j',0,0,0) )
- sum_(k,1,'N',
lambda(k,'j')*a(k,'j')
*U('i',k,0,0,0)*U(k,'j',1,0,0) )
+ sum_(k,1,'N', sum_(l,1,'N',
  (lambda('i',k)-lambda(k,l))*a('i',l)
*U('i',k,0,0,0)*U(k,l,0,0,0)*U(l,'j',0,0,0)))
```

```
- sum_(k,1,'N', sum_(l,1,'N',
  (lambda(l,k)-lambda(k,'j'))*a(l,'j')
*U('i',l,0,0,0)*U(l,k,0,0,0)*U(k,'j',0,0,0)))
+ U('i','j',0,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,0,0,0)*U(k,'j',0,0,0) )
- sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,0,0,0)*U(k,'i',0,0,0) )*U('i','j',0,0,0);

* and its first derivative w.r. to y
id U('i','j',1,0,1) =
a('i','j')*( lambda('i','j')*U('i','j',3,0,0)
+ sum_(k,1,'N', (lambda('i',k) - lambda(k,'j'))*(
          U('i',k,2,0,0)*U(k,'j',0,0,0)
+2*U('i',k,1,0,0)*U(k,'j',1,0,0)
         +U('i',k,0,0,0)*U(k,'j',2,0,0) ) ) )
+ sum_(k,1,'N', lambda('i',k)*a('i',k)
*(U('i',k,2,0,0)*U(k,'j',0,0,0)
        +U('i',k,1,0,0)*U(k,'j',1,0,0) ) )
- sum_(k,1,'N',
lambda(k,'j')*a(k,'j')
*(U('i',k,1,0,0)*U(k,'j',1,0,0)
        +U('i',k,0,0,0)*U(k,'j',2,0,0) ) )
+ sum_(k,1,'N', sum_(l,1,'N',
(lambda('i',k)-lambda(k,l))*a('i',l)*(
        U('i',k,1,0,0)*U(k,l,0,0,0)*U(l,'j',0,0,0)
       +U('i',k,0,0,0)*U(k,l,1,0,0)*U(l,'j',0,0,0)
       +U('i',k,0,0,0)*U(k,l,0,0,0)*U(l,'j',1,0,0) )))
- sum_(k,1,'N', sum_(l,1,'N',
(lambda(l,k)-lambda(k,'j'))*a(l,'j')*(
        U('i',l,1,0,0)*U(l,k,0,0,0)*U(k,'j',0,0,0)
       +U('i',l,0,0,0)*U(l,k,1,0,0)*U(k,'j',0,0,0)
       +U('i',l,0,0,0)*U(l,k,0,0,0)*U(k,'j',1,0,0) )))
+ U('i','j',1,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,0,0,0)*U(k,'j',0,0,0) )
+ U('i','j',0,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,1,0,0)*U(k,'j',0,0,0) )
+ U('i','j',0,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,0,0,0)*U(k,'j',1,0,0) )
- sum_(k,1,'N',
lambda('i',k)*a('i',k)
```

```
*U('i',k,1,0,0)*U(k,'i',0,0,0)
)*U('i','j',0,0,0)
- sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,0,0,0)*U(k,'i',1,0,0)
)*U('i','j',0,0,0)
- sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,0,0,0)*U(k,'i',0,0,0)
)*U('i','j',1,0,0);
#enddo
#enddo


#do i=2,'N'
#do j=1,'i'-1
id U('i','j',0,0,1) =
a('i','j')*( lambda('i','j')
*U('i','j',2,0,0)
+ sum_(k,1,'N',
(lambda('i',k) - lambda(k,'j'))*(
    U('i',k,1,0,0)*U(k,'j',0,0,0)
+U('i',k,0,0,0)*U(k,'j',1,0,0) ) ) )
+ sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,1,0,0)*U(k,'j',0,0,0) )
- sum_(k,1,'N',
lambda(k,'j')*a(k,'j')
*U('i',k,0,0,0)*U(k,'j',1,0,0) )
+ sum_(k,1,'N', sum_(l,1,'N',
  (lambda('i',k)-lambda(k,l))*a('i',l)
*U('i',k,0,0,0)*U(k,l,0,0,0)*U(l,'j',0,0,0)))
- sum_(k,1,'N', sum_(l,1,'N',
  (lambda(l,k)-lambda(k,'j'))*a(l,'j')
*U('i',l,0,0,0)*U(l,k,0,0,0)*U(k,'j',0,0,0)))
+ U('i','j',0,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,0,0,0)*U(k,'j',0,0,0) )
- sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,0,0,0)*U(k,'i',0,0,0)
)*U('i','j',0,0,0);

* and its first derivative w.r.t. to y
id U('i','j',1,0,1) =
```

```
a('i','j')*( lambda('i','j')*U('i','j',3,0,0)
+ sum_(k,1,'N', (lambda('i',k) - lambda(k,'j'))*(
          U('i',k,2,0,0)*U(k,'j',0,0,0)
+2*U('i',k,1,0,0)*U(k,'j',1,0,0)
         +U('i',k,0,0,0)*U(k,'j',2,0,0) ) ) )
+ sum_(k,1,'N', lambda('i',k)*a('i',k)
*(U('i',k,2,0,0)*U(k,'j',0,0,0)
        +U('i',k,1,0,0)*U(k,'j',1,0,0) ) )
- sum_(k,1,'N',
lambda(k,'j')*a(k,'j')
*(U('i',k,1,0,0)*U(k,'j',1,0,0)
       +U('i',k,0,0,0)*U(k,'j',2,0,0) ) )
+ sum_(k,1,'N', sum_(l,1,'N',
(lambda('i',k)-lambda(k,l))*a('i',l)*(
        U('i',k,1,0,0)*U(k,l,0,0,0)*U(l,'j',0,0,0)
       +U('i',k,0,0,0)*U(k,l,1,0,0)*U(l,'j',0,0,0)
       +U('i',k,0,0,0)*U(k,l,0,0,0)*U(l,'j',1,0,0) )))
- sum_(k,1,'N', sum_(l,1,'N',
(lambda(l,k)-lambda(k,'j'))*a(l,'j')*(
        U('i',l,1,0,0)*U(l,k,0,0,0)*U(k,'j',0,0,0)
       +U('i',l,0,0,0)*U(l,k,1,0,0)*U(k,'j',0,0,0)
       +U('i',l,0,0,0)*U(l,k,0,0,0)*U(k,'j',1,0,0) )))
+ U('i','j',1,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,0,0,0)*U(k,'j',0,0,0) )
+ U('i','j',0,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,1,0,0)*U(k,'j',0,0,0) )
+ U('i','j',0,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,0,0,0)*U(k,'j',1,0,0) )
- sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,1,0,0)*U(k,'i',0,0,0)
)*U('i','j',0,0,0)
- sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,0,0,0)*U(k,'i',1,0,0)
)*U('i','j',0,0,0)
- sum_(k,1,'N',
lambda('i',k)*a('i',k)
*U('i',k,0,0,0)*U(k,'i',0,0,0)
)*U('i','j',1,0,0);
#enddo
#enddo
```

```
* Here we set a(i,j) = 1/(a(i)-a(j))

repeat;
id U(i?,i?,j?,k?,l?) = 0;
endrepeat;

* lambda(i,i) should not appear,
* we set it to zero.
repeat;
#do i=1,'N'
id lambda('i','i') = 0;
#enddo
endrepeat;

* The lambda(i,j) are symmetric.
repeat;
#do i=1,'N'-1
#do j='i'+1,'N'
id lambda('j','i') = lambda('i','j');
#enddo
#enddo
endrepeat;

* The a(i,j) are antisymmetric.
repeat;
#do i=1,'N'-1
#do j='i'+1,'N'
id a('j','i') = -a('i','j');
#enddo
#enddo
endrepeat;

* #do i=1,'N'
* #do j=1,'N'
* print +s expr'i''j';
* #enddo
* #enddo


.store

**** PART 2 ****
nwrite statistics;
Functions U, dz2, eq3;
CFunctions lambda, a;
```

```
Symbols i, j, k, l, m, n, y, z2, z3;

* computing the derivative of rhs of
* first equation U_z3 = eq3 with respect to z2
#do i=1,'N'
#do j=1,'N'
   Global exprr'i''j' = dz2*eq3('i','j');
#enddo
#enddo

* defining the right hand side of the second equation
#do i=1,'N'-1
#do j='i'+1,'N'
id eq3('i','j') =
a('i','j')*( lambda('i','j')*U('i','j',2,0,0)
+ sum_(m,1,'N', (lambda('i',m) - lambda(m,'j'))*(
          U('i',m,1,0,0)*U(m,'j',0,0,0)
+U('i',m,0,0,0)*U(m,'j',1,0,0) ) ) )
+ sum_(m,1,'N',
lambda('i',m)*a('i',m)
*U('i',m,1,0,0)*U(m,'j',0,0,0) )
- sum_(m,1,'N',
lambda(m,'j')*a(m,'j')
*U('i',m,0,0,0)*U(m,'j',1,0,0) )
+ sum_(m,1,'N', sum_(l,1,'N',
  (lambda('i',m)-lambda(m,l))*a('i',l)
*U('i',m,0,0,0)*U(m,l,0,0,0)*U(l,'j',0,0,0)))
- sum_(m,1,'N', sum_(l,1,'N',
  (lambda(l,m)-lambda(m,'j'))*a(l,'j')
*U('i',l,0,0,0)*U(l,m,0,0,0)*U(m,'j',0,0,0)))
+ U('i','j',0,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)*
U('j',k,0,0,0)*U(k,'j',0,0,0) )
- sum_(k,1,'N', lambda('i',k)*a('i',k)
*U('i',k,0,0,0)*U(k,'i',0,0,0)
)*U('i','j',0,0,0);
#enddo
#enddo

#do i=2,'N'
#do j=1,'i'-1
id eq3('i','j') =
a('i','j')*( lambda('i','j')*U('i','j',2,0,0)
+ sum_(m,1,'N', (lambda('i',m) - lambda(m,'j'))*(
    U('i',m,1,0,0)*U(m,'j',0,0,0)
```

```
+U('i',m,0,0,0)*U(m,'j',1,0,0) ) ) )
+ sum_(m,1,'N', lambda('i',m)*a('i',m)
*U('i',m,1,0,0)*U(m,'j',0,0,0) )
- sum_(m,1,'N', lambda(m,'j')*a(m,'j')
*U('i',m,0,0,0)*U(m,'j',1,0,0) )
+ sum_(m,1,'N', sum_(l,1,'N',
  (lambda('i',m)-lambda(m,l))*a('i',l)
*U('i',m,0,0,0)*U(m,l,0,0,0)*U(l,'j',0,0,0)))
- sum_(m,1,'N', sum_(l,1,'N',
  (lambda(l,m)-lambda(m,'j'))*a(l,'j')
*U('i',l,0,0,0)*U(l,m,0,0,0)*U(m,'j',0,0,0)))
+ U('i','j',0,0,0)*sum_(k,1,'N',
lambda('j',k)*a('j',k)
*U('j',k,0,0,0)*U(k,'j',0,0,0) )
- sum_(k,1,'N', lambda('i',k)*a('i',k)
*U('i',k,0,0,0)*U(k,'i',0,0,0)
)*U('i','j',0,0,0);
#enddo
#enddo

* Note that a(i,j) = 1/(a(i)-a(j))

repeat;
id U(i?,i?,y?,z2?,z3?) = 0;
endrepeat;

* definition of differentiation
* with respect to z2
repeat;
id dz2*U(i?,j?,y?,z2?,z3?) =
U(i,j,y,z2+1,z3) + U(i,j,y,z2,z3)*dz2;
endrepeat;

.sort

id dz2=0;
* The variables expr12 etc now contain
* the derivative of eq3 w.r. to z2.


* use first equation U_z2 = eq2 and
* its first two derivative w.r. to y
id U(i?,j?,0,1,0) = lambda(i,j)*U(i,j,1,0,0)
  + sum_(k,1,'N',
   (lambda(i,k)-lambda(k,j))
```

```
  *U(i,k,0,0,0)*U(k,j,0,0,0));
id U(i?,j?,1,1,0) = lambda(i,j)*U(i,j,2,0,0)
  + sum_(k,1,'N', (lambda(i,k)-lambda(k,j))*(
     U(i,k,1,0,0)*U(k,j,0,0,0)
    +U(i,k,0,0,0)*U(k,j,1,0,0) ));
id U(i?,j?,2,1,0) = lambda(i,j)*U(i,j,3,0,0)
  + sum_(k,1,'N', (lambda(i,k)-lambda(k,j))*(
     U(i,k,2,0,0)*U(k,j,0,0,0)
    +2*U(i,k,1,0,0)*U(k,j,1,0,0)
    +U(i,k,0,0,0)*U(k,j,2,0,0)  ));

repeat;
id U(i?,i?,j?,k?,l?) = 0;
endrepeat;

* lambda(i,i) should not appear,
* we set it to zero.
repeat;
#do i=1,'N'
id lambda('i','i') = 0;
#enddo
endrepeat;

* The lambda(i,j) are symmetric.
repeat;
#do i=1,'N'-1
#do j='i'+1,'N'
id lambda('j','i') = lambda('i','j');
#enddo
#enddo
endrepeat;

* The a(i,j) are antisymmetric.
repeat;
#do i=1,'N'-1
#do j='i'+1,'N'
id a('j','i') = -a('i','j');
#enddo
#enddo
endrepeat;


* #do i=1,'N'
* #do j=1,'N'
* print +s exprr'i''j';
```

```
* #enddo
* #enddo

.store


**** PART 3 ****
nwrite statistics;

Functions U;
CFunctions lambda, a, b;
Symbols i, j, tau;

Global diff12 = expr12-exprr12;
* This is only ONE of the set of
* equations to be checked

* insert the definitions of
* lambda(i,j) and a(i,j)
repeat;
id lambda(i?,j?) = (b(i)-b(j))/(a(i)-a(j));
id a(i?,j?) = 1/(a(i)-a(j));
endrepeat;

Bracket U;

print +s diff12;
*   vanishes by pasting coefficient
* into MuPAD (after simplify) !!!

.end
```

The output from FORM includes vast terms and it is hard to see whether all terms vanish. The statement `Bracket U;` enables us to pack coefficients of each term in bracket such that

```
diff12 =

     + U(1,2,0,0,0)*U(2,1,0,0,0)*U(1,3,0,0,0)*U(3,2,0,0,0) * (
        - 1/(b(1) - b(2))/(b(1) - b(2))/(b(1) - b(3))*a(1)^2
        + 2/(b(1) - b(2))/(b(1) - b(2))/(b(1) - b(3))*a(1)*a(2)
        - 1/(b(1) - b(2))/(b(1) - b(2))/(b(1) - b(3))*a(2)^2
        + 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(2))*a(1)^2
        - 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(2))*a(1)*a(2)
        - 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(2))*a(1)*a(3)
        + 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(2))*a(2)*a(3)
        + 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(3))*a(1)^2
        - 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(3))*a(1)*a(2)
        - 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(3))*a(1)*a(3)
        + 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(3))*a(2)*a(3)
        - 1/(b(1) - b(2))/(b(1) - b(3))/(b(2) - b(3))*a(1)^2
        + 1/(b(1) - b(2))/(b(1) - b(3))/(b(2) - b(3))*a(1)*a(2)
        + 1/(b(1) - b(2))/(b(1) - b(3))/(b(2) - b(3))*a(1)*a(3)
        - 1/(b(1) - b(2))/(b(1) - b(3))/(b(2) - b(3))*a(2)*a(3)
        - 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(2))*a(1)*a(2)
        + 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(2))*a(1)*a(3)
        + 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(2))*a(2)^2
        - 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(2))*a(2)*a(3)
        + 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(3))*a(1)*a(2)
        - 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(3))*a(1)*a(3)
```

```
         - 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(3))*a(2)^2
         + 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(3))*a(2)*a(3)
         + 1/(b(1) - b(3))/(b(1) - b(3))/(b(2) - b(3))*a(1)^2
         - 2/(b(1) - b(3))/(b(1) - b(3))/(b(2) - b(3))*a(1)*a(3)
         + 1/(b(1) - b(3))/(b(1) - b(3))/(b(2) - b(3))*a(3)^2
         - 1/(b(1) - b(3))/(b(2) - b(3))/(b(1) - b(3))*a(1)*a(2)
         + 1/(b(1) - b(3))/(b(2) - b(3))/(b(1) - b(3))*a(1)*a(3)
         + 1/(b(1) - b(3))/(b(2) - b(3))/(b(1) - b(3))*a(2)*a(3)
         - 1/(b(1) - b(3))/(b(2) - b(3))/(b(1) - b(3))*a(3)^2
         )
```

$$\vdots$$

Putting these coefficients into MuPAD, we find that coefficients of all terms yield 0 after a `simplify` command.

```
>> - 1/(b(1) - b(2))/(b(1) - b(2))/(b(1) - b(3))*a(1)^2
         + 2/(b(1) - b(2))/(b(1) - b(2))/(b(1) - b(3))*a(1)*a(2)
         - 1/(b(1) - b(2))/(b(1) - b(2))/(b(1) - b(3))*a(2)^2
         + 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(2))*a(1)^2
         - 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(2))*a(1)*a(2)
         - 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(2))*a(1)*a(3)
         + 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(2))*a(2)*a(3)
         + 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(3))*a(1)^2
         - 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(3))*a(1)*a(2)
         - 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(3))*a(1)*a(3)
         + 1/(b(1) - b(2))/(b(1) - b(3))/(b(1) - b(3))*a(2)*a(3)
         - 1/(b(1) - b(2))/(b(1) - b(3))/(b(2) - b(3))*a(1)^2
         + 1/(b(1) - b(2))/(b(1) - b(3))/(b(2) - b(3))*a(1)*a(2)
         + 1/(b(1) - b(2))/(b(1) - b(3))/(b(2) - b(3))*a(1)*a(3)
         - 1/(b(1) - b(2))/(b(1) - b(3))/(b(2) - b(3))*a(2)*a(3)
         - 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(2))*a(1)*a(2)
         + 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(2))*a(1)*a(3)
         + 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(2))*a(2)^2
         - 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(2))*a(2)*a(3)
         + 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(3))*a(1)*a(2)
         - 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(3))*a(1)*a(3)
         - 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(3))*a(2)^2
         + 1/(b(1) - b(2))/(b(2) - b(3))/(b(1) - b(3))*a(2)*a(3)
         + 1/(b(1) - b(3))/(b(1) - b(3))/(b(2) - b(3))*a(1)^2
         - 2/(b(1) - b(3))/(b(1) - b(3))/(b(2) - b(3))*a(1)*a(3)
         + 1/(b(1) - b(3))/(b(1) - b(3))/(b(2) - b(3))*a(3)^2
         - 1/(b(1) - b(3))/(b(2) - b(3))/(b(1) - b(3))*a(1)*a(2)
         + 1/(b(1) - b(3))/(b(2) - b(3))/(b(1) - b(3))*a(1)*a(3)
         + 1/(b(1) - b(3))/(b(2) - b(3))/(b(1) - b(3))*a(2)*a(3)
         - 1/(b(1) - b(3))/(b(2) - b(3))/(b(1) - b(3))*a(3)^2

               2                              2
           a(1)                           a(2)
--------------------------- - --------------------------- +
                          2                2
(b(1) - b(2)) (b(1) - b(3))    (b(1) - b(2))  (b(1) - b(3))

                  2                              2
              a(1)                           a(2)
   --------------------------- + --------------------------- -
                2                              2
   (b(1) - b(3))  (b(2) - b(3))   (b(1) - b(2))  (b(2) - b(3))

                        2
                    a(1)
   ----------------------------------------- -
   (b(1) - b(2)) (b(1) - b(3)) (b(2) - b(3))

                        2
                    a(2)
   ----------------------------------------- -
   (b(1) - b(2)) (b(1) - b(3)) (b(2) - b(3))

            a(1) a(2)                      a(1) a(2)
   --------------------------- + --------------------------- -
                             2                 2
   (b(1) - b(2)) (b(1) - b(3))    (b(1) - b(2))  (b(1) - b(3))

            a(1) a(2)                      a(1) a(3)
   --------------------------- - --------------------------- -
                2                                          2
   (b(1) - b(2))  (b(2) - b(3))   (b(1) - b(2)) (b(1) - b(3))

            a(1) a(3)                      a(1) a(2)
   --------------------------- - --------------------------- +
                2                              2
   (b(1) - b(2))  (b(1) - b(3))   (b(1) - b(3))  (b(2) - b(3))

            a(1) a(3)                      a(2) a(3)
   --------------------------- + --------------------------- +
                2                                          2
   (b(1) - b(2))  (b(2) - b(3))   (b(1) - b(2)) (b(1) - b(3))
```

```
         a(2) a(3)                      a(1) a(3)
   ---------------------------- - ---------------------------- -
               2                              2
   (b(1) - b(2))  (b(1) - b(3))   (b(1) - b(3))  (b(2) - b(3))

         a(2) a(3)                      a(2) a(3)
   ---------------------------- + ---------------------------- +
               2                              2
   (b(1) - b(2))  (b(2) - b(3))   (b(1) - b(3))  (b(2) - b(3))

              2 a(1) a(2)
   -----------------------------------------
   (b(1) - b(2)) (b(1) - b(3)) (b(2) - b(3))
>> simplify(%)

                                   0
>>
```

In this way, we can check that also the other terms vanish.

C.4.3 Commutativity of the first two KdV flows

Following the procedure in the case of N-wave hierarchy, we developed this FORM program to check the commutativity of the first two flows of the KdV hierarchy which we obtained from the pre-sdYM hierarchy. In contrast to the case of N-wave, we do not need to put the output from FORM into MuPAD to check whether all coefficients vanish.

```
*
*             KdV_check.frm
*
* u has two slots which contain
* the number of partial derivatives
* with respect to y and t.
* u((number of derivatives with respect of y),
* (number of derivatives with respect to t))

Functions dy, dz2, dz3;
Functions u, y, t;
Symbols i, j, k;

* first hierarchy equation of KdV hierarchy in Sec.4.4.2
Local Uz2 = u(3, 0, 0)/4 - 3*u(1, 0, 0)*u(1, 0, 0)/2;

* differentiation with respect of dz3
Global expr1 = dz3*Uz2;

Local F1 = (u(5, 0, 0) - 10*(u(3, 0, 0)*u(1, 0, 0)
+ u(1, 0, 0)*u(3, 0, 0) + u(2, 0, 0)*u(2, 0, 0))
+ 40*u(1, 0, 0)*u(1, 0, 0)*u(1, 0, 0))/16;
Local F2 = dy*F1;
Local F3 = dy*F2;
Local F4 = dy*F3;
Local F5 = dy*F4;
Local F6 = dy*F5;
Local F7 = dy*F6;
```

```
Local F8 = dy*F7;

repeat;
* definitions of differentiation with respect to y and dz3
id dz3*u(i?, j?, k?) = u(i, j, k+1) + u(i, j, k)*dz3;
id dy*u(i?, j?, k?) = u(i+1, j, k) + u(i, j, k)*dy;
endrepeat;

id dy = 0;
id dz3 = 0;

.sort


repeat;
 id u(0, 0, 1) = F1;
 id u(1, 0, 1) = F2;
 id u(2, 0, 1) = F3;
 id u(3, 0, 1) = F4;
 id u(4, 0, 1) = F5;
 id u(5, 0, 1) = F6;
 id u(6, 0, 1) = F7;
 id u(7, 0, 1) = F8;
endrepeat;

print +s expr1;

.store
Functions dy, dz2, dz3;
Functions u, y, t;
Symbols i, j, k;

* second hierarchy equation of KdV hierarchy in Sec.4.4.2
Local Uz3 = (u(5, 0, 0) - 10*(u(3, 0, 0)*u(1, 0, 0)
+ u(1, 0, 0)*u(3, 0, 0) + u(2, 0, 0)*u(2, 0, 0))
+ 40*u(1, 0, 0)*u(1, 0, 0)*u(1, 0, 0))/16;

* differentiation with respect of dz2
Global expr2 = dz2*Uz3;

Local G1 = u(3, 0, 0)/4 - 3*u(1, 0, 0)*u(1, 0, 0)/2;
Local G2 = dy*G1;
Local G3 = dy*G2;
Local G4 = dy*G3;
Local G5 = dy*G4;
```

```
Local G6 = dy*G5;
Local G7 = dy*G6;
Local G8 = dy*G7;

repeat;
* definitions of differentiation with respect to y and dz2
id dz2*u(i?, j?, k?) = u(i, j+1, k) + u(i, j, k)*dz2;
id dy*u(i?, j?, k?) = u(i+1, j, k) + u(i, j, k)*dy;
endrepeat;

id dy = 0;
id dz2 = 0;
.sort


repeat;
 id u(0, 1, 0) = G1;
 id u(1, 1, 0) = G2;
 id u(2, 1, 0) = G3;
 id u(3, 1, 0) = G4;
 id u(4, 1, 0) = G5;
 id u(5, 1, 0) = G6;
 id u(6, 1, 0) = G7;
 id u(7, 1, 0) = G8;
endrepeat;

print +s expr2;

.store

* If the flows commute, diff = 0.
Local diff = expr1 - expr2;
print +s diff;
.end
```

The output of this program yields 0,

```
> ./form -l kdv_check.frm



   diff = 0;
```

C.4.4 Commutativity of the nonlinear Schrödinger flows

In this subsection we present two FORM programs. One is developed for the commutativity check between first and second flow, the other is for the commutativity check between second and third flow of the NLS hierarchy we obtained. As mentioned in Sec.3.4.1 and 3.4.3, we obtained from the reduced pre-sdYM hierarchy the following results,

- first hierarchy equation : NLS equation,
- second hierarchy equation : complex mKdV equation,
- third hierarchy equation : second hierarchy equation of NLS hierarchy.

Apparently, it is not clear whether these equations are symmetries of each other, so that the reduced pre-sdYM hierarchy would be a hierarchy . The first FORM program checks the commutativity between NLS equation and complex mKdV equation (note that we did not rstrict a and r to $a = \frac{1}{2}, \qquad r = \pm q^*$).

Commutativity check between the first and second flow.

```
* Nonlinear Schroedinger hierarchy
* check of commutativity of the
* first and second equation

**** PART 1 ****
nwrite statistics;
Functions r, q, eq2, dz3;
Symbols a, i, j, k;
* The function q, r have 3 sockets
which mean the differentiation
* with respect to y, z2, z3 respectively.
* e.g. r(y, z2, z3).

* computing derivative of
* rhs of first equation q_z2 = eq2
* with respect to z3
Global expr1 = dz3*eq2;

* definition the right hand side of the first equation
id eq2 =
(q(2, 0, 0) + 2*q(0, 0, 0)*r(0, 0, 0)*q(0, 0, 0))/(2*a);

repeat;
* definition of differentiation with respect to z3
id dz3*q(i?, j?, k?) =
q(i, j, k + 1) + q(i, j, k)*dz3;
id dz3*r(i?, j?, k?) =
```

```
r(i, j, k + 1) + r(i, j, k)*dz3;
endrepeat;

id dz3=0;
.sort


repeat;
id q(0, 0, 1) = (q(3, 0, 0) +
3*q(1, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 3*q(0, 0, 0)*r(0, 0, 0)*q(1, 0, 0))/(4*a^2);
id q(1, 0, 1) =
(q(4, 0, 0) + 3*q(2, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 3*q(1, 0, 0)*r(1, 0, 0)*q(0, 0, 0)
+ 6*q(1, 0, 0)*r(0, 0, 0)*q(1, 0, 0)
+ 3*q(0, 0, 0)*r(1, 0, 0)*q(1, 0, 0)
+ 3*q(0, 0, 0)*r(0, 0, 0)*q(2, 0, 0))/(4*a^2);
id q(2, 0, 1) = (q(5, 0, 0)
+ 3*q(3, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 6*q(2, 0, 0)*r(1, 0, 0)*q(0, 0, 0)
+ 9*q(2, 0, 0)*r(0, 0, 0)*q(1, 0, 0)
+ 3*q(1, 0, 0)*r(2, 0, 0)*q(0, 0, 0)
+ 12*q(1, 0, 0)*r(1, 0, 0)*q(1, 0, 0)
+ 9*q(1, 0, 0)*r(0, 0, 0)*q(2, 0, 0)
+ 3*q(0, 0, 0)*r(2, 0, 0)*q(1, 0, 0)
+ 6*q(0, 0, 0)*r(1, 0, 0)*q(2, 0, 0)
+ 3*q(0, 0, 0)*r(0, 0, 0)*q(3, 0, 0))/(4*a^2);

id r(0, 0, 1) =
(r(3, 0, 0)
        + 3*r(1, 0, 0)*q(0, 0, 0)*r(0, 0, 0)
+ 3*r(0, 0, 0)*q(0, 0, 0)*r(1, 0, 0))/(4*a^2);
endrepeat;

print +s expr1;

.store

**** PART 2 ****
nwrite statistics;
Functions r, q, eq3, dz2;
Symbols a, i, j, k;

* computing the derivative of rhs
* of first equation q_z3 = eq3
```

```
* with respect to z2
Global expr2 = dz2*eq3;

* defining the right hand side of the second equation
id eq3 = (q(3, 0, 0) + 3*q(1, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 3*q(0, 0, 0)*r(0, 0, 0)*q(1, 0, 0))/(4*a^2);


* definition of differentiation with respect to z2
repeat;
id dz2*q(i?, j?, k?) = q(i, j + 1, k) + q(i, j, k)*dz2;
id dz2*r(i?, j?, k?) = r(i, j + 1, k) + r(i, j, k)*dz2;
endrepeat;

.sort

id dz2=0;
repeat;
id q(0, 1, 0) =
(q(2, 0, 0)
+ 2*q(0, 0, 0)*r(0, 0, 0)*q(0, 0, 0))/(2*a);
id q(1, 1, 0) = (q(3, 0, 0)
+ 2*q(1, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 2*q(0, 0, 0)*r(1, 0, 0)*q(0, 0, 0)
+ 2*q(0, 0, 0)*r(0, 0, 0)*q(1, 0, 0))/(2*a);
id q(2, 1, 0) =
(q(4, 0, 0)
+ 2*q(2, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 4*q(1, 0, 0)*r(1, 0, 0)*q(0, 0, 0)
+ 4*q(1, 0, 0)*r(0, 0, 0)*q(1, 0, 0)
+ 2*q(0, 0, 0)*r(2, 0, 0)*q(0, 0, 0)
+ 4*q(0, 0, 0)*r(1, 0, 0)*q(1, 0, 0)
+ 2*q(0, 0, 0)*r(0, 0, 0)*q(2, 0, 0))/(2*a);
id q(3, 1, 0) =
(q(5, 0, 0)
+ 2*q(3, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 6*q(2, 0, 0)*r(1, 0, 0)*q(0, 0, 0)
+ 6*q(2, 0, 0)*r(0, 0, 0)*q(1, 0, 0)
+ 6*q(1, 0, 0)*r(2, 0, 0)*q(0, 0, 0)
+ 12*q(1, 0, 0)*r(1, 0, 0)*q(1, 0, 0)
+ 6*q(1, 0, 0)*r(0, 0, 0)*q(2, 0, 0)
+ 2*q(0, 0, 0)*r(3, 0, 0)*q(0, 0, 0)
+ 6*q(0, 0, 0)*r(2, 0, 0)*q(1, 0, 0)
+ 6*q(0, 0, 0)*r(1, 0, 0)*q(2, 0, 0)
+ 2*q(0, 0, 0)*r(0, 0, 0)*q(3, 0, 0))/(2*a);
```

```
id r(0, 1, 0) =
(-r(2, 0, 0)
- 2*r(0, 0, 0)*q(0, 0, 0)*r(0, 0, 0))/(2*a);
endrepeat;


print +s expr2;

.store

**** PART 3 ****
nwrite statistics;

* Let's restrict to commuting objects
Functions r, q;
Symbols a;

Global diff12 = expr1 - expr2;
print +s diff12;

.sort


print +s diff12;

.end
```

Commutativity check between the second and third flow.

```
* Nonlinear Schroedinger hierarchy
* check of commutativity of the
* second and third equation

**** PART 1 ****
nwrite statistics;
Functions r, q, eq3, eq4, dz4, dy;
Symbols a, i, j, k;
* The functions q, r have 3 sockets
* which mean the differentiation
* with respect to y, z3, z4 respectively.
* e.g. r(y, z3, z4).

* computing derivative of rhs of first equation
```

```
q_z3 = eq3 with respect to z3
Global expr1 = dz4*eq3;

* The differentiations of q_z4 with respect to y
Local qz4 = eq4;
Local qyz4 = dy*eq4;
Local qyyz4 = dy*qyz4;
Local qyyyz4 = dy*qyyz4;

* defining the right hand side of the first equation
id eq3 = (q(3, 0, 0) + 3*q(1, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 3*q(0, 0, 0)*r(0, 0, 0)*q(1, 0, 0))/(4*a^2);

id eq4 =
(q(4, 0, 0)
+ 4*q(2, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 2*q(1, 0, 0)*r(1, 0, 0)*q(0, 0, 0)
+ 6*q(1, 0, 0)*r(0, 0, 0)*q(1, 0, 0)
+ 2*q(0, 0, 0)*r(1, 0, 0)*q(1, 0, 0)
+ 4*q(0, 0, 0)*r(0, 0, 0)*q(2, 0, 0)
+ 2*q(0, 0, 0)*r(2, 0, 0)*q(0, 0, 0)
+ 6*q(0, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
*r(0, 0, 0)*q(0, 0, 0))/(8*a^3);


repeat;
* definition of differentiation with respect to z4
id dz4*q(i?, j?, k?) =
q(i, j, k + 1) + q(i, j, k)*dz4;
id dz4*r(i?, j?, k?) =
r(i, j, k + 1) + r(i, j, k)*dz4;

* definition of differentiation with respect to y
id dy*q(i?, j?, k?) =
q(i + 1, j, k) + q(i, j, k)*dy;
id dy*r(i?, j?, k?) =
r(i + 1, j, k) + r(i, j, k)*dy;

endrepeat;

id dy = 0;
id dz4 = 0;
.sort
```

```
repeat;
id q(0, 0, 1) = qz4;
id q(1, 0, 1) = qyz4;
id q(2, 0, 1) = qyyz4;
id q(3, 0, 1) = qyyyz4;
id r(0, 0, 1) =
-(r(4, 0, 0)
+ 4*r(2, 0, 0)*q(0, 0, 0)*r(0, 0 ,0)
+ 2*r(1, 0, 0)*q(1, 0, 0)*r(0, 0, 0)
+ 6*r(1, 0, 0)*q(0, 0, 0)*r(1, 0, 0)
+ 2*r(0, 0, 0)*q(1, 0, 0)*r(1, 0, 0)
+ 4*r(0, 0, 0)*q(0, 0, 0)*r(2, 0, 0)
+ 2*r(0, 0, 0)*q(2, 0, 0)*r(0, 0, 0)
+ 6*r(0, 0, 0)*q(0, 0, 0)*r(0, 0, 0)
*q(0, 0, 0)*r(0, 0, 0))/(8*a^3);
endrepeat;

print +s expr1;

.store

**** PART 2 ****
nwrite statistics;
Functions dy, r, q, eq3, eq4, dz3;
Symbols a, i, j, k;

* computing the derivative of
* rhs of first equation q_z4 = eq4
* with respect to z3
Global expr2 = dz3*eq4;

* The differentiations of
* q_z3 with respect to y
Local qz3 = eq3;
Local qyz3 = dy*eq3;
Local qyyz3 = dy*qyz3;
Local qyyyz3 = dy*qyyz3;
Local qyyyyz3 = dy*qyyyz3;


* defining the right hand side of the
* second equation
id eq3 =
(q(3, 0, 0)
+ 3*q(1, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
```

```
+ 3*q(0, 0, 0)*r(0, 0, 0)*q(1, 0, 0))/(4*a^2);

id eq4 =
(q(4, 0, 0)
+ 4*q(2, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
+ 2*q(1, 0, 0)*r(1, 0, 0)*q(0, 0, 0)
+ 6*q(1, 0, 0)*r(0, 0, 0)*q(1, 0, 0)
+ 2*q(0, 0, 0)*r(1, 0, 0)*q(1, 0, 0)
+ 4*q(0, 0, 0)*r(0, 0, 0)*q(2, 0, 0)
+ 2*q(0, 0, 0)*r(2, 0, 0)*q(0, 0, 0)
+ 6*q(0, 0, 0)*r(0, 0, 0)*q(0, 0, 0)
*r(0, 0, 0)*q(0, 0, 0))/(8*a^3);



* definition of differentiation with respect to z2
repeat;
id dz3*q(i?, j?, k?) =
q(i, j + 1, k) + q(i, j, k)*dz3;
id dz3*r(i?, j?, k?) =
r(i, j + 1, k) + r(i, j, k)*dz3;

* definition of differentiation with respect to y
id dy*q(i?, j?, k?) =
q(i + 1, j, k) + q(i, j, k)*dy;
id dy*r(i?, j?, k?) =
r(i + 1, j, k) + r(i, j, k)*dy;
endrepeat;

id dy = 0;
id dz3 = 0;
.sort


repeat;
id q(0, 1, 0) = qz3;
id q(1, 1, 0) = qyz3;
id q(2, 1, 0) = qyyz3;
id q(3, 1, 0) = qyyyz3;
id q(4, 1, 0) = qyyyyz3;
id r(0, 1, 0) =
(r(3, 0, 0)
+ 3*r(1, 0, 0)*q(0, 0, 0)*r(0, 0, 0)
+ 3*r(0, 0, 0)*q(0, 0, 0)*r(1, 0, 0))/(4*a^2);
```

```
id r(1, 1, 0) =
(r(4, 0, 0)
+ 3*r(2, 0, 0)*q(0, 0, 0)*r(0, 0, 0)
+ 3*r(1, 0, 0)*q(1, 0, 0)*r(0, 0, 0)
+ 6*r(1, 0, 0)*q(0, 0, 0)*r(1, 0, 0)
+ 3*r(0, 0, 0)*q(1, 0, 0)*r(1, 0, 0)
+ 3*r(0, 0, 0)*q(0, 0, 0)*r(2, 0, 0))/(4*a^2);
id r(2, 1, 0) =
(r(5, 0, 0)
+ 3*r(3, 0, 0)*q(0, 0, 0)*r(0, 0, 0)
+ 6*r(2, 0, 0)*q(1, 0, 0)*r(0, 0, 0)
+ 9*r(2, 0, 0)*q(0, 0, 0)*r(1, 0, 0)
+ 3*r(1, 0, 0)*q(2, 0, 0)*r(0, 0, 0)
+ 12*r(1, 0, 0)*q(1, 0, 0)*r(1, 0, 0)
+ 9*r(1, 0, 0)*q(0, 0, 0)*r(2, 0, 0)
+ 3*r(0, 0, 0)*q(2, 0, 0)*r(1, 0, 0)
+ 6*r(0, 0, 0)*q(1, 0, 0)*r(2, 0, 0)
+ 3*r(0, 0, 0)*q(0, 0, 0)*r(3, 0, 0))/(4*a^2);
endrepeat;


print +s expr2;

.store

**** PART 3 ****
nwrite statistics;

* Let's restrict to commuting objects
Functions r, q;
Symbols a;

Global diff12 = expr1 - expr2;
print +s diff12;

.sort


print +s diff12;

.end
```

The results of both programs are

```
diff12 = 0;
```

Same as the commutativity check of the KdV flows, we do not need to put the output from FORM into another computer algebra system such as MuPAD to check whether all coefficients vanish.

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