String Orbifolds on Non-Factorizable Tori

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Ph.D. thesis

Abstract

We investigate compactification of the string theory. Specifically we provide a new class of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on non-factorizable tori, whose boundary conditions are defined by Lie root lattices. Generally, point groups of these orbifolds are generated by Weyl reflections and outer automorphisms of the lattices. We evaluate the topological invariants of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds with and without discrete torsions. Then we find that some of these models have smaller Euler numbers than those of the models on the factorizable tori $T^2 \times T^2 \times T^2$. In Appendix C we give the complete classification for the abelian orbifolds on the six-dimensional Lie root lattices.

We found that the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold on the E_6 root lattice is phenomenologically interesting, because the orbifold has just three fixed tori in some twisted sectors, and there have not been known such a six-dimensional orbifold. Then we assume only two non-standard gauge embeddings and find that they lead to three-family $SU(5)$ and $SO(10)$ GUT-like models. These models also include strongly coupled sectors in the low energy and messenger states charged with both hidden and visible sectors. We observe the structures of three point interactions are changed for these models, and this is favorable to realize mass matrices with flavor mixing for quark and leptons.

Finally we apply the non-factorizable tori to the Type II orientifolds with and without its oribifolding. We explicitly calculate the Ramond-Ramond tadpole from string one-loop amplitudes, and confirm that the consistent number of orientifold planes is directly derived from the Lefschetz fixed point theorem. We furthermore classify orientifolds on non-factorizable $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, and construct new supersymmetric Type IIA orientifold models on them.

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Chapter 1

Introduction

Behind phenomena in nature there exist underlying laws, and we know that the Standard Model and general relativity provide the most elemental theory to describe the fundamental parts of nature. The Standard Model is the gauge theory with the gauge groups SU(3)×SU(2)×U(1) coupled to three generations of quarks and leptons. General relativity is the theory of gravity, whose force is extremely small in comparison with the other three forces. Both are remarkably successful theories which explain experimental data with highly precision, and related with each other. These theories describe the largest and smallest objects for us, and actually creation of matter through big bang is explained by the collaboration of them. On the other hand, we can not answer why there exist three generations of matter, existence of dark matter in the universe and how can the gravitational force is quantized as a renormalizable quantum field theory. In the understanding of renormalization, we have known that the Standard Model is an effective field theory which is obtained by integrating our high energy degree of freedom. The Einstein equation

$$(\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}) + \Lambda g_{\mu\nu} = \mathcal{T}_{\mu\nu} \tag{1.0.1}$$

suggests the quantum gravity, because we know the quantum field theory for the r.h.s of equation, but we do not know a consistent quantum field theory of gravity for l.h.s., and this also implies our luck of understanding of nature. Quantum gravity will play a crucial role in the beginning of the universe, and it is expected that our four dimensional spacetime was determined in the process of the big bang.

Superstring theory is the most promising candidate to give the explanation to them. The consistent string theory is possible only for specific dimensionalities of the spacetime. This means that space and time are the dynamical quantities by themselves. It is quite attractive feature of string theory so that it has possibilities to realize four dimensional spacetime. For the superstring theories the required dimension is ten. How are these extra dimensions compatible with our four dimensional spacetime? One possible solution is compactification in order to realize the Standard Model in the low energy effective theory of the string theory. In this scenario the six dimensions are curled due to the gravitational effect,

$$R^{10} \to R^4 \times M^6 \tag{1.0.2}$$

where M^6 is the six dimensional compact space, and geometries of the compact space determine the low energy theory as string states. It means that the geometry and inner structures are crucial to reproduce the Standard model, and investigation of them is of great importance. This is one of the main motivation of this paper, and it is a challenging issue to elucidate the origin of parameters of the Standard Model, and to understand the deep structure of nature.

There are some mysteries of the spectrum of the Standard Model. Specifically we do not know at all the meaning of the duplication of the generations of quarks and leptons, and their hierarchy of their mass structure from neutrinos to top quark,

$$m_{\nu_e} \ < \ 3 \text{ eV} \ \ll \ m_u \ \sim \ 4 \text{ MeV} \ll \ m_t \ \cong \ 174 \text{ GeV}. \tag{1.0.3}$$

The difference between the up and top quark is curious because the other quantum numbers are the same for them. The grand unified theories (GUT) based on $SU(5)$ and $SO(10)$ explain the quantum numbers of them, but say nothing about the generations. This is so fundamental problem for us that it tempts us to think that they will provide the key for the fundamental and unified theory beyond the Standard Model and it would even relate to the theory of gravity. There are some reasons why we can envisage such a expectation. As mentioned the above, it is reasonable to guess that gravitational force would play a central role for the creation of the space time, and quantum effect would be essential for it, which dominate near the Planck scale $M_{pl} = 1.2 \times 10^{19}$ GeV. Surprisingly the (supersymmetric) GUT scale $M_{GUT} \sim 10^{16}$ GeV is not so far from the Planck scale. Supersymmetry also cures the quadratic divergence of Higgs mass with respect to the renormalization. These facts suggest that the dynamics near the GUT or Planck scale generate the spectrum of the Standard Model with three generations of matter. String theory provides us attractive framework for the motivation, and it includes rich structure enough in itself as we will see the parts of them in this paper.

A dream of string theorists would be to obtain the fundamental theory of nature as the solution of string theory in non-perturbative analysis. It is generally believed that string theory include only one fundamental constant α', and the scenario looks perfect as the unified theory, but also formidable. We rather approach the realization of the Standard Model in perturbative ways. In other words we assume the compactifications and study the string spectra and interactions in the perturbative vacua. We are interested in how we can construct the string vacua with realistic spectra, and what kinds of properties are required for it.

Attempting to obtain the Standard Model from heterotic string theory [8–12] has been the earliest endeavour of such realization. Calabi-Yau compactification is a solution with $\mathcal{N} = 1$ supersymmetry [13–17]. The numbers of generations of the heterotic string models with the standard embedding are determined by half the Euler number of the Calabi-Yau three-fold $\chi/2$, and models with $\chi = 6$ were searched [18]. For non-standard gauge embeddings or inclusion of background fields, e.g. Wilson lines, we have rather rich possibilities of model constructions. Compactification on orbifold also preserve $\mathcal{N} = 1$ supersymmetry in 4D [19, 20], and blowing up the singularities of the orbifolds lead to Calabi-Yau manifolds. Despite of the singularities on the orbifolds, orbifold compactification is an exact solvable case of Calabi-Yau, and provides us an attractive playground. Three-family models compactified on orbifolds have been constructed with abelian discrete groups $\mathbb{Z}_N$ and $\mathbb{Z}_N \times \mathbb{Z}_M$ [61–69, 75–85], whose point group elements are defined by the Coxeter elements. There are also the fermionic construction of heterotic string, and three generation models are constructed [109–112], and some of these models are coincident with those of the bosonic constructions. However these models often suffer from many extra matters, and do not have realistic Yukawa matrices, and actually we have not succeeded to obtain the Standard Model or its supersymmetric extension. Then there are rather numerous models recently, and it gets more important to establish the criteria to select promising models from the landscape of string vacua [36–41, 113, 126–129].

In this thesis we generalize orbifolds through the considerations about the automorphisms on non-factorizable six-tori. In the case of $\mathbb{Z}_N \times \mathbb{Z}_M$ the Coxeter orbifolds [70, 72–74, 106, 116, 117] give the factorized compact space $T^2 \times T^2 \times T^2$. Recently, non-factorizable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds were constructed in heterotic string [44–46], and in Type IIA string [41]. Then the compact space is non-factorizable just like Calabi-Yau threefold [15, 86–88]. We recently classified $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models on non-factorizable tori [47, 48]. Non-factorizable orbifolds possess different geometries from factorizable ones because the number of fixed tori, and the Euler numbers, in six-dimensional spaces can be less than those of the factorizable ones. It means that the models based on the non-factorizable orbifolds have less number of generations, although there are often too many generations in the models on factorizable orbifolds. Moreover the structures of the interactions are different from that of the factorizable orbifolds, and this is interesting because non-trivial structure will be needed for the flavor mixing terms of the mass matrices which are generated from three-point functions. Thus, non-factorizable orbifold will provide a new tool for model constructions, and provide vacua that have not constructed. Because string models are connected to other ones by dualities and symmetries, our investigation is valued. Actually it is observed that the spectra of non-factorizable models are coincident with those of factorizable models with the generalized discrete torsion [60].

Introducing D-branes [7] provides rich possibilities for Type II string theory. The concept of branes stands for extended objects on which interactions are localized. The intersecting brane scenario contains several phenomenological appealing features. Because gravity is not forced to localize on D-branes, this means some essential difference between these two kinds of interactions. Then the gravitational interaction can be much weaker than the electroweak and strong interactions. Moreover this class of models provide the chiral spectrum and gauge group of the Standard Model [22–32] (for review see [33–35] and references therein). We can apply the non-factorizable orbifold to the intersecting brane models. Because the RR-charge of D-brane should be cancelled by O-planes, we also consider the generalization of Orientifold as it is explained in Chapter 4.

For any approaches to the Standard Model, the necessary ingredients for the model construction are as follows:

- Chiral spectrum
- Gauge group containing $SU(3) \times SU(2) \times U(1)$

- Three generations of quark and leptons
- Mass matrices with mixing (CKM and MNS)

We try to construct models for this aim in this paper. For the string models construction, it would be also require

- $\mathcal{N} = 1$ supersymmetry
- Dynamics of spontaneously supersymmetry
- Moduli stabilization

$\mathcal{N} = 1$ supersymmetry is favorable for phenomenological reasons from Higgs and cosmological constant. It is expected that supersymmetry is spontaneously broken by brane dynamics, background or gaugino condensation in the lower energy scale than the string scale. There are moduli which remains free parameters in the perturbative approach, and they should be fixed by appropriate mechanisms. Since moduli stabilization determine the size or shape of compact space, it seems that it could be related to the mechanism of compactifications. These topics will be future works.

This thesis is organized as follows. In Chapter 2 we briefly review the superstring theory in the context of orbifolds. The formulae can be used to both heterotic and Type II string theory. Chapter 3 is devoted to the non-factorizable orbifold on the Lie root lattices. We explain the detail of the automorphisms generated from the Weyl reflections and outer automorphisms. Then we can easily classify the orbifolds on the Lie root lattices. This chapter treats the geometry of orbifold, and is independent of string theory. In Chapter 4 we apply the non-factorizable orbifolds to heterotic string. Especially we present some specific $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold models, and investigate some phenomenological aspects of it. In Chapter 5 we consider orientifolds on the non-factorizable tori and orbifolds. In order to confirm the consistency conditions we mainly investigate the cancellation of the Ramond-Ramond(RR) tadpole.

We conclude in Chapter 6. Appendices also include important results. In Appendix A we collect some definitions of Lie algebra and explain that there are only sixteen distinctive six dimensional Lie root lattices. In Appendix B we sum up useful results for the calculation of the partition functions that are used in the text. In Appendix C we list all the orbifolds with their Euler and hodge numbers on the sixteen lattices, which contain new results. One can see that classification of orbifolds on Lie root lattices are easy and intuitive in this way.

The work contained in this thesis is based on the papers [46, 47] (Chapter 3,4), [48] (Chapter 4) and [26, 118] (Chapter 5). The author is grateful to my collaborators Tatsuo Kobayashi, Noriaki Kitazawa, Stefan Förste, Tetsutaro Higaki, Hiroshi Ohki, Tesuji Kimura and Mitsuhisa Ohta.

Chapter 2

A Brief Review of String Orbifolds

In this chapter we explain the basics of string orbifolds, whose formulae can be used to both the heterotic and Type II string theory for the later use. The features that depend on the specific types of the string will be given in the corresponding chapters, that is, the E_8 gauge group of the heterotic string in chapter 4 and open string of the Type II theory in chapter 5.

The quantum field theory is defined as a theory of point particles, and the divergence from the quantum effect is inevitable in the short length limit as the description of nature. String theory is a generalization that the fundamental object is spatially one dimensional [1–3] and the loop effect do not diverge owing to the modular invariance of worldsheet. The action of free string is given by

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \partial_a X^\mu \partial^a X_\mu, \tag{2.0.1}$$

in the conformal gauge. The spacetime dimension of the bosonic sting is required to be twenty six in order to cancel its conformal anomaly of two dimensional quantum field theory on the worldsheet. For superstring theory we add the action two dimensional Majorana fermion Ψ^μ, and it leads to the ten dimensional spacetime as the target space.

2.1 String orbifolds and mode expansions

As already mentioned, to construct a four-dimensional string models, we assume that six of the spacial dimensions are compactified on a torus T^6. The resulting spectrum has $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetry for heterotic and Type II string theory respectively, and it is non-chiral. It is interesting to consider orbifold [19, 20]. Then we can reduce the supersymmetry to $\mathcal{N} = 1$ or $\mathcal{N} = 2$ by the twist of orbifolds in a compact space. In Type II string theory we further implement with orientifold for the consistency of the models, and we will mainly study the models with $\mathcal{N} = 1$ supersymmetry in this thesis.

A six-torus T^6 is obtained from six-dimensional Euclidean space $\mathbb{R}^6$ divided by a lattice Λ,

$$T^6 = R^6/\Lambda. \tag{2.1.1}$$

As we will see, the structure of the lattice Λ plays an important role in the analysis of this paper. Points $\mathbf{x} \in \mathbb{R}^6$ differing by a lattice vector $\mathbf{L} \in \Lambda$ are identified as

$$\mathbf{x} \sim \mathbf{x} + 2\pi\mathbf{L}. \tag{2.1.2}$$

An orbifold is defined to be the quotient of a torus over a discrete set of isometries of the torus, called the point group P, i.e.

$$\mathcal{O} = T^6/P = R^6/S, \tag{2.1.3}$$

where S is called the space group, and is the semidirect product of the point group P and the translation group, and P must act crystallographically on the lattice Λ. A space group element (θ, l), where $\theta \in P$ is a rotation and l is a translation, acts on a coordinate $\mathbf{x}$ as

$$(\theta, l)\mathbf{x} = \theta\mathbf{x} + l. \tag{2.1.4}$$

Space group elements satisfy the following relation,

$$\begin{aligned} (\theta, l)(\phi, l') &= (\theta\phi, l + \theta l'), & (2.1.5)\\ (\theta, l)^{-1} &= (\theta^{-1}, -\theta^{-1} l). & (2.1.6) \end{aligned}$$

For a point group element θ, we can choose a basis so that the θ is diagonal, and it acts on the complex coordinate $(z_1, z_2, z_3) \in T^6$ as

$$\theta : (z_1, z_2, z_3) \to (e^{2\pi i v_1} z_1, e^{2\pi i v_2} z_2, e^{2\pi i v_3} z_3), \tag{2.1.7}$$

where (v_1, v_2, v_3) is called the twist. For a certain N, the action θ^N should be identity,

$$\theta^N = \mathbb{1}, \quad N v_i = 0 \mod 1, \quad i = 1, 2, 3, \tag{2.1.8}$$

and this defines a $\mathbb{Z}_N$ orbifold. The eigenvalues of the eight spinors of $SO(8)_{lightcone}$ are $e^{\pi i(\pm v_1 + \pm v_2 + \pm v_3)}$ under the action of θ. To preserve $\mathcal{N} = 1$ supersymmetry at least two of them should be left invariant. Without loss of generality we choose the elements of the twist to satisfy

$$v_1 + v_2 + v_3 = 0. \tag{2.1.9}$$

This condition is nothing but the $SU(3)$ holonomy for orbifolds, because the orbifolds are a singular limit of certain Calabi-Yau three-fold, where the curvature is concentrate on the fixed points. Then point group $P \subset SO(6)$ is a discrete subgroup of the $SU(3)$ holonomy group.

It is useful to define the complex string coordinates on the compact space T^6 as

$$Z^i \equiv \frac{1}{\sqrt{2}}(X^{2i+2} + iX^{2i+3}), \quad i = 1, 2, 3, \tag{2.1.10}$$

and their conjugate by $Z^{\bar{i}} \equiv \overline{Z^i}$. For the closed string, the twisted sectors appear such that the boundary of twisted strings is closed owing to the geometry of the orbifold. Such states appear around fixed point z^i_f. The periodicity of the θ-twisted sector is

$$Z^i(\sigma + 2\pi) = e^{2\pi i v_i} Z^i(\sigma) + l^i. \tag{2.1.11}$$

where l^i is the shift defined by

$$l^i = (1 - e^{2\pi i v_i}) z^i_f. \tag{2.1.12}$$

We can rewrite the condition (2.1.11) as

$$Z^i(\sigma + 2\pi) - z^i_f = e^{2\pi i v_i}(Z^i(\sigma) - z^i_f), \tag{2.1.13}$$

and it would be clear that the strings are around the fixed point z^i_f. We can label the string states localizing at z^i_f in the θ-twisted sectors as (θ, z_f), which is the corresponding space group element. Since closed strings in the twisted sectors can shrink to the fixed points, their ground energy can be zero depending on the boundary conditions. So massless states can localize at the fixed points (or fixed tori).

The mode expansions of the string in the θ-twisted sector is given by

$$\begin{aligned}
Z^i(\tau,\sigma) &= z^i_f + \frac{i}{2}\sum_{n\neq 0}\left[\frac{1}{n+v_i}\alpha^i_{n+v_i}e^{-2i(n+v_i)(\tau-\sigma)} + \frac{1}{n-v_i}\tilde{\alpha}^i_{n-v_i}e^{-2i(n-v_i)(\tau+\sigma)}\right], \\
Z^{\bar{i}}(\tau,\sigma) &= z^{\bar{i}}_f + \frac{i}{2}\sum_{n\neq 0}\left[\frac{1}{n-v_i}\alpha^{\bar{i}}_{n-v_i}e^{-2i(n-v_i)(\tau-\sigma)} + \frac{1}{n+v_i}\tilde{\alpha}^{\bar{i}}_{n+v_i}e^{-2i(n+v_i)(\tau+\sigma)}\right],
\end{aligned} \tag{2.1.14}$$

where z^i_f are the center-of-mass coordinates of the twisted states. They are split into left and right moving components as $Z^i(\tau,\sigma) = Z^i_R(\tau-\sigma) + Z^i_L(\tau+\sigma)$. For the fermionic coordinates we similarly define the complex fermions Ψ^i. The mode expansions are given by

$$\begin{aligned}
\Psi^i(\tau,\sigma) &= \sum_{r\in\mathbb{Z}+\nu}\left[\psi^i_{r+v_i}e^{-2i(r+v_i)(\tau+\sigma)} + \tilde{\psi}^i_{r-v_i}e^{-i(r-v_i)(\tau-\sigma)}\right], \\
\Psi^{\bar{i}}(\tau,\sigma) &= \sum_{r\in\mathbb{Z}+\nu}\left[\psi^{\bar{i}}_{r-v_i}e^{-2i(r-v_i)(\tau-\sigma)} + \tilde{\psi}^{\bar{i}}_{r+v_i}e^{-i(r+v_i)(\tau+\sigma)}\right],
\end{aligned} \tag{2.1.15}$$

where ν take the value 0 in the Ramond sector and 1/2 in the Neveu-Schwartz sector. They satisfy the following boundary conditions,

$$\begin{aligned} \mathrm{R}: \ \Psi^i(\sigma+\pi) &= e^{2\pi i v_i}\Psi^i(\sigma), && (2.1.16a)\\ \mathrm{NS}: \ \Psi^i(\sigma+\pi) &= -e^{2\pi i v_i}\Psi^i(\sigma), && (2.1.16b)\end{aligned}$$

and similarly for their conjugates $\Psi^{\bar{i}}$. The states are also divided to left- and right-movers,

$$\Psi^i(\tau,\sigma) = \Psi^i_R(\tau-\sigma) + \Psi^i_L(\tau+\sigma) \ . \tag{2.1.17}$$

These closed strings in a twisted sector look open on the covering space T^6, but their ends are connected by the action of the orbifold element θ. Let us show the examples in two dimensional cases. We define a $\mathbb{Z}_2$ orbifold on T^2 by the action

$$\theta: \ z \to -z. \tag{2.1.18}$$

Note that the points on this orbifold are identified up to torus lattice, e.g. $z \sim z+1 \sim z+i$, and four points $z_f = 0$, $1/2$, $i/2$ and $(1+i)/2$ are invariant under θ. Similarly a $\mathbb{Z}_3$ orbifold, whose action is $\theta: \ z \to e^{2\pi i/3}z$, have three fixed points, see Figure 2.1. Strings in the θ-twisted sectors are localizing at the fixed points. The fundamental regions are colored in the figure, and we obtain the orbifold connecting the boundaries.

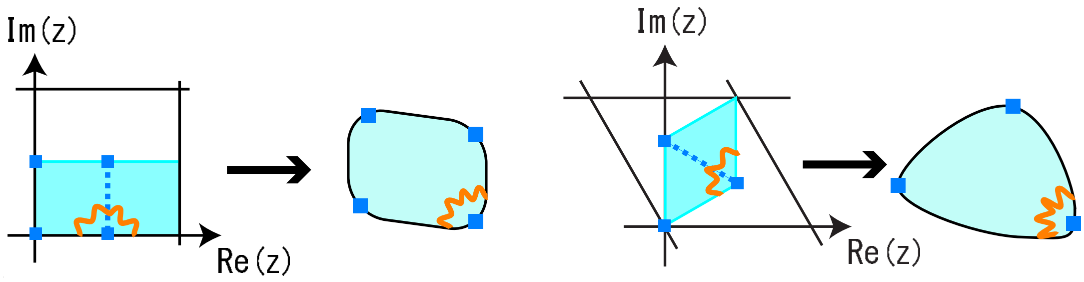

Figure 2.1: *The squares indicate fixed points. The colored area is the fundamental region of the orbifold. Its boundary is connected by folding the dashed line.*

On the other hand a closed string with the normal boundary condition ($v_1 = v_2 = v_3 = 0$) is close on T^6, and belongs to the untwisted sector. On the orbifold both the twisted and untwisted sectors appear, and their existence is actually required to satisfy the modular invariance. The untwisted sector is obtained by projecting out the string states in 10D, and contain $D = 4$ $\mathcal{N} = 1$ supergravity multiplet and vector multiplets of the corresponding gauge group.

The (anti-) commutation relations of the operators are given by

$$\begin{aligned} \left[\alpha^i_{n+v_i}, \alpha^{\bar{j}}_{m-v_j}\right] &= (n+v_i)\delta^{ij}\delta_{m+n,0}, \\ \left[\tilde{\alpha}^i_{n-v_i}, \tilde{\alpha}^{\bar{j}}_{m+v_j}\right] &= (n-v_i)\delta^{ij}\delta_{m+n,0}, \\ \{\psi^i_{r+v_i}, \psi^{\bar{j}}_{s-v_i}\} = \{\tilde{\psi}^i_{r-v_i}, \tilde{\psi}^{\bar{j}}_{s+v_i}\} &= \delta_{r+s,0}\,\delta^{ij} \ , \end{aligned} \tag{2.1.19}$$

and the others relations between left and right-moving operators vanish, i.e. $[\alpha, \tilde{\alpha}] = \{\psi, \tilde{\psi}\} = 0$. Then the mass formulae for the superstring have the general structure

$$\begin{aligned} \frac{1}{4}M_L^2 &= N_B + N_F(\nu) - a, \\ \frac{1}{4}M_R^2 &= \tilde{N}_B + \tilde{N}_F(\nu) - \tilde{a}, \end{aligned} \tag{2.1.20}$$

where a and $\tilde{a}$ are the normal ordering constants as explained later, and N's are the number operators defined by

$$\begin{aligned} N_B &= \sum_{n>0} \alpha^{\mu}_{-n}\alpha_{\mu,n} + \sum_{n+v_i>0} \alpha^{\bar{i}}_{-n-v_i}\alpha^{i}_{n+v_i} + \sum_{n-v_i>0} \alpha^{i}_{-n+v_i}\alpha^{\bar{i}}_{n-v_i}, \\ N_F &= \sum_{r>0} r\psi^{\mu}_{-r}\psi_{\mu,r} + \sum_{r+v_i>0} (r+v_i)\psi^{\bar{i}}_{-n-v_i}\psi^{i}_{n+v_i} + \sum_{r-v_i>0} (r-v_i)\psi^{i}_{-n+v_i}\psi^{\bar{i}}_{n-v_i}, \end{aligned} \tag{2.1.21}$$

where $\mu = 3, 4$ in the light-cone gauge. $\tilde{N}_B$ and $\tilde{N}_F$ for the left movers are given similarly. In heterotic string, the left movers are 26-dimensional bosonic operators whose mass formula is

$$\frac{1}{4}M_L^2 = \frac{1}{2}(p_L^I + V^I)^2 + N_B - a, \tag{2.1.22}$$

where p_L^I generates the representation **248** of E_8, and V^I is the shift on it (these are explained in Chapter 4). From the modular invariance of the string one-loop amplitude, the level matching condition is required,

$$M_L = M_R. \tag{2.1.23}$$

To calculate the zero point energy $-a$, it is useful to define $\eta_i = |v_i \pmod 1|$ so that $0 \le \eta \le 1/2$. Then the general zero-point energy for a complex degree of freedom is

$$-a(v_i) = \pm\left(\frac{1}{24} - \frac{1}{8}(2\eta_i - 1)^2\right), \tag{2.1.24}$$

where $+$ is for a complex bosonic field and $-$ is for a complex fermionic one, and in the NS sector one needs to replace v_i by $v_i - 1/2$. Note that this formula is for a complex boson of fermion, and is twice as the contribution from a single field X^{μ}. In the light-cone gauge the contributions from the operators of the non-compact directions are $-a(0) = -1/12$ (for boson), $1/12$ (for R-sector fermion) and $-1/24$ (for NS-sector fermion), and we have to add the contributions $-a(v_i)$ (i=1,2,3) from that of the compact directions.

For the stability of the theory we require to be free of tachyons in (2.1.24). Since right-movers are the same in Type II and heterotic string, we consider the condition of the mass of the ground state in the right-moving NS sector

$$\begin{aligned} \frac{1}{4}M_R &= -\frac{1}{12} + \sum_{i=1}^{3}\left(\frac{1}{24} - \frac{1}{8}(2\eta_i - 1)^2\right) - \frac{1}{24} + \sum_{i=1}^{3}\left(\frac{1}{24} - \frac{1}{8}(2\eta_i)^2\right) \\ &= \frac{1}{2}\sum_{i=1}^{3}\eta_i - \frac{1}{2}, \end{aligned} \tag{2.1.25}$$

Because the contributions from bosonic and fermionic operators just cancel in the R sector, the ground state of the R sector is massless. Then there are thirteen elements [20], which do not generate tachyonic states after implementing the GSO projection $(-1)^{\pi i \tilde{F}} = 1$ for the NS sector, as follows,

(v_1, v_2, v_3)				
$(0, \frac{1}{2}, -\frac{1}{2})$	$(0, \frac{1}{3}, -\frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$	$(0, \frac{1}{4}, -\frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, -\frac{1}{2})$
$(0, \frac{1}{6}, -\frac{1}{6})$	$(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$	$(\frac{1}{6}, \frac{1}{3}, -\frac{1}{2})$	$(\frac{1}{7}, \frac{2}{7}, -\frac{3}{7})$	$(\frac{1}{8}, \frac{1}{4}, -\frac{3}{8})$
$(\frac{1}{8}, \frac{3}{8}, -\frac{1}{2})$	$(\frac{1}{12}, \frac{1}{3}, -\frac{5}{12})$	$(\frac{1}{12}, \frac{5}{12}, -\frac{1}{2})$		

It leads to nine $\mathbb{Z}_N$ orbifolds with $\mathcal{N} = 1$ for heterotic string and $\mathcal{N} = 2$ for Type II string , see Table 2.1.

	(v_1, v_2, v_3)		(v_1, v_2, v_3)		(v_1, v_2, v_3)
$\mathbb{Z}_3$	$(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$	$\mathbb{Z}_4$	$(\frac{1}{4}, \frac{1}{4}, -\frac{1}{2})$	$\mathbb{Z}_6$-I	$(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$
$\mathbb{Z}_6$-II	$(\frac{1}{6}, \frac{1}{3}, -\frac{1}{2})$	$\mathbb{Z}_7$	$(\frac{1}{7}, \frac{2}{7}, -\frac{3}{7})$	$\mathbb{Z}_8$-I	$(\frac{1}{8}, \frac{1}{4}, -\frac{3}{8})$
$\mathbb{Z}_8$-II	$(\frac{1}{8}, \frac{3}{8}, -\frac{1}{2})$	$\mathbb{Z}_{12}$-I	$(\frac{1}{12}, \frac{1}{3}, -\frac{5}{12})$	$\mathbb{Z}_{12}$-II	$(\frac{1}{12}, \frac{5}{12}, -\frac{1}{2})$

Table 2.1: *Twists of $\mathbb{Z}_N$ orbifolds.*

In the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold, we have two generators θ and ϕ of the point group,

$$\begin{aligned} \theta: \quad (z_1, z_2, z_3) &\to (e^{2\pi i v_1} z_1, e^{2\pi i v_2} z_2, e^{2\pi i v_3} z_3), \\ \phi: \quad (z_1, z_2, z_3) &\to (e^{2\pi i w_1} z_1, e^{2\pi i w_2} z_2, e^{2\pi i w_3} z_3). \end{aligned} \tag{2.1.26}$$

The point group of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds is

$$\{\theta^k \phi^l | k = 1, \cdots, N-1, \; l = 1, \cdots, M-1\}, \tag{2.1.27}$$

with

$$\theta^N = \phi^M = \mathbb{1}, \quad N v_i = M w_i = 0 \mod 1, \quad i = 1, 2, 3,$$

and each element should be one of the element of thirteen classes so that the orbifolds preserve supersymmetry. This leads to eight $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on Table 2.2. In this construction, orbifolds with more elements, e.g. $\mathbb{Z}_N \times \mathbb{Z}_M \times \mathbb{Z}_L$, do not exist on a six-torus T^6.

	(v_1, v_2, v_3)	(w_1, w_2, w_3)		(v_1, v_2, v_3)	(w_1, w_2, w_3)
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(0, \frac{1}{2}, -\frac{1}{2})$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(0, \frac{1}{4}, -\frac{1}{4})$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(0, \frac{1}{6}, -\frac{1}{6})$	$\mathbb{Z}_2 \times \mathbb{Z}_6'$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(\frac{1}{6}, -\frac{1}{3}, \frac{1}{6})$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$(\frac{1}{3}, -\frac{1}{3}, 0)$	$(0, \frac{1}{3}, -\frac{1}{3})$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$(\frac{1}{3}, -\frac{1}{3}, 0)$	$(0, \frac{1}{6}, -\frac{1}{6})$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(\frac{1}{4}, -\frac{1}{4}, 0)$	$(0, \frac{1}{4}, -\frac{1}{4})$	$\mathbb{Z}_6 \times \mathbb{Z}_6$	$(\frac{1}{6}, -\frac{1}{6}, 0)$	$(0, \frac{1}{6}, -\frac{1}{6})$

Table 2.2: *Twists of* $\mathbb{Z}_N \times \mathbb{Z}_M$ *orbifolds.*

Chapter 3

Orbifolds on non-factorizable tori

In this chapter we study the automorphisms of the Lie root lattice, which is defined by the simple roots of the Lie algebra, and its application to non-factorizable six-tori. In this work a compactified space which cannot be represented as the direct products of two-torus T^2 is called non-factorizable. For a Lie algebra of rank l, the Lie lattice Λ is given as

$$\Lambda = \Big\{ \sum_{i=1}^{l} n_i \alpha_i \Big| n_i \in \mathbb{Z} \Big\}, \tag{3.0.1}$$

where α_i is a simple root of the Lie algebra. For example the Lie algebra A_6 defines a six-tori on the A_6 lattice. Taking a direct product of tori, we can obtain other tori, i.e. $T \equiv T_1 \times T_2$ where $\dim(T) = \dim(T_1) + \dim(T_2)$. In terms of Lie root lattices, we find that there are only twelve distinct non-factorizable six-tori and four factorizable ones [1]:

[non-factorizable tori]			
A_6	D_6	E_6	
$A_5 \times A_1$	$A_4 \times A_2$	$A_4 \times (A_1)^2$	
$D_5 \times A_1$	$D_4 \times A_2$	$D_4 \times (A_1)^2$	
$A_3 \times A_3$	$A_3 \times A_2 \times A_1$	$A_3 \times (A_1)^3$	
[factorizable tori]			
$(A_2)^3$	$(A_2)^2 \times (A_1)^2$	$A_2 \times (A_1)^4$	$(A_1)^6$

Table 3.1: *All the Lie root lattices in six dimensions.*

By the use of the Weyl reflection and the outer automorphisms, we can classify all the point groups of orbifolds and orientifold actions $\mathcal{R}$ on the tori, which crystallographically act on the Lie root lattices. In this chapter we give explicit representations of point group elements generated by the Weyl reflections and the outer automorphisms of the Lie root

[1] The definitions and details are explained in Appendix A.

lattices. Some of the point groups can be given by the Coxeter elements from the Cater diagrams or the generalized Coxeter elements as explained later [70, 72, 73, 106, 116, 117]. Beside these elements, we see that point groups $\mathbb{Z}_N \times \mathbb{Z}_M$ which are not included in the (generalized) Coxeter elements are also obtained by the classification.

We utilize these elements for the point groups of orbifolds and also for the involution $\mathcal{R}$ of orientifolds in the next chapters. Here we give the systematic way to construct orbifolds and orientifolds on the Lie root lattices.

3.1 Weyl reflection and graph automorphism

We investigate the automorphisms of the Lie root lattices. These groups can be classified in terms of the Weyl reflection and the graph automorphism acting on the simple roots of the Lie root lattice. The Weyl group $\mathcal{W}$ is generated by the following Weyl reflections r_{α_k} which associate the simple root α_k:

$$r_{\alpha_k}: \ \lambda \ \rightarrow \ \lambda - 2\frac{\alpha_k \cdot \lambda}{|\alpha_k|^2}\alpha_k. \tag{3.1.1}$$

This is simply a reflection which reverse the axis along α_k. In the case of the D_N Lie root lattice, for instance, the simple roots are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_6 = \mathbf{e}_5 + \mathbf{e}_6, \quad i = 1, \ldots, 5, \tag{3.1.2}$$

where $\mathbf{e}_i$'s are basis of Cartesian coordinates whose normalization is given as $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. The Weyl reflection r_{α_k} for $k = 2, \ldots, N-2$ is

$$r_{\alpha_k}: \begin{cases} \alpha_{k-1} & \rightarrow \quad \alpha_{k-1} + \alpha_k \\ \quad \alpha_k & \rightarrow \quad -\alpha_k \\ \alpha_{k+1} & \rightarrow \quad \alpha_{k+1} + \alpha_k \end{cases} \tag{3.1.3a}$$

and α_m (for $m \neq k-1, k, k+1$) are unchanged. For $k = 1, N-1$ and N they are

$$\begin{aligned} r_{\alpha_1}: &\ \alpha_1 \ \rightarrow -\alpha_1, & \alpha_2 \ &\rightarrow \alpha_1 + \alpha_2, \\ r_{\alpha_{N-1}}: &\ \alpha_{N-2} \rightarrow \alpha_{N-2} + \alpha_{N-1}, & \alpha_{N-1} &\rightarrow -\alpha_{N-1}, \\ r_{\alpha_N}: &\ \alpha_{N-2} \rightarrow \alpha_{N-2} + \alpha_N, & \alpha_N \ &\rightarrow -\alpha_N, \end{aligned} \tag{3.1.3b}$$

and the other α_m's are unchanged. For the classification of the automorphisms, it would be convenient to rewrite them in the basis of orthogonal unit vectors $\mathbf{e}_i$ as

$$r_{\alpha_k}: \ \mathbf{e}_k \ \leftrightarrow \ \mathbf{e}_{k+1} \qquad k = 1, \ldots, N-1, \tag{3.1.4a}$$

$$r_{\alpha_N}: \ \mathbf{e}_N \ \leftrightarrow \ -\mathbf{e}_{N-1}. \tag{3.1.4b}$$

The outer automorphism g can be read from the Cartan diagram D_N in figure 3.1, and it is represented as

$$g: \ \alpha_{N-1} \leftrightarrow \alpha_N, \tag{3.1.5}$$

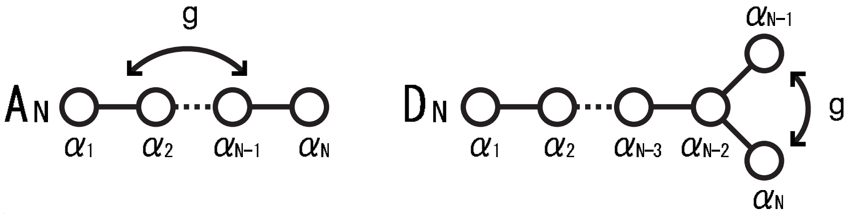

Figure 3.1: *The Dynkin diagrams A_N and D_N. g is the outer automorphism of the diagram.*

and the other simple roots are left unchanged. In the unit vector basis, it is expressed as

$$g: \ \mathbf{e}_N \to -\mathbf{e}_N. \tag{3.1.6}$$

In terms of the $\mathbf{e}_i$ basis, we can easily construct any element generated from r_{α_k} and g. For example a product of two Weyl reflections which do not commute with each other makes up $\mathbb{Z}_3$ element as

$$r_{\alpha_k} r_{\alpha_{k+1}}: \ \mathbf{e}_k \to \mathbf{e}_{k+1} \to \mathbf{e}_{k+2} \to \mathbf{e}_k, \qquad (k < N-1). \tag{3.1.7}$$

This is the permutation group S_3. Similarly the Weyl reflections r_{α_k} for $k = 1, \ldots, N-1$ generate a permutation group S_N. Adding the other elements r_{α_N} and g to S_N, the representation of the group is given by permutations with signs

$$\mathbf{e_i} \to \pm\mathbf{e_j} \to \pm\mathbf{e_k} \to \cdots \to \pm\mathbf{e_i}. \tag{3.1.8}$$

Then the order of the Weyl group $\mathcal{W}$ and $\{\mathcal{W}, g\}$ are summarized in Table 3.2:

	$\mathcal{W}$	$\{\mathcal{W}, g\}$
D_N	$2^{N-1}N!$	$2^N N!$
A_{N-1}	$N!$	$2N!$

Table 3.2: *The order of the Weyl group and the graph automorphism.*

In the case of the A_{N-1} Lie root lattice, the Weyl reflections generate permutation group S_N. Its outer automorphism of the Dynkin diagram is given as

$$g: \ \alpha_k \leftrightarrow \alpha_{N-k}, \quad k = 1, \cdots, [N/2], \tag{3.1.9}$$

and in the $\mathbf{e}_i$ basis it is

$$g: \ \mathbf{e}_i \leftrightarrow -\mathbf{e}_{N+1-i}, \quad i = 1, \cdots, N, \tag{3.1.10}$$

We can always permute g to g' by the elements of $\mathcal{W}$ such that g' change the sign of all $\mathbf{e}_i$'s, This element is expressed as an identity matrix with negative sign $-\mathbb{1}_N$, which means $\{\mathcal{W}, g\} = \{\mathcal{W}, -\mathbb{1}_N\}$. Therefore the order of $\{\mathcal{W}, g\}$ is twice as large as that of $\mathcal{W}$, as in Table 3.2.

Then it is straightforward to obtain all $\mathbb{Z}_2$ elements of the D_N lattice, and they are given by the following sub-elements

$$\begin{aligned} \mathbf{e}_i &\leftrightarrow \mathbf{e}_j, \\ \mathbf{e}_k &\leftrightarrow -\mathbf{e}_l, \\ \mathbf{e}_m &\rightarrow -\mathbf{e}_m, \\ \mathbf{e}_n &\rightarrow \mathbf{e}_n, \end{aligned} \tag{3.1.11a}$$

or in a matrix representation it corresponds to

$$\theta_{\mathbb{Z}_2} = \begin{array}{c} \begin{array}{cccccc} i & j & k & l & m & n \end{array} \\ \left(\begin{array}{cccccccc} \ddots & & & & & & & \\ & 0 & 1 & 0 & 0 & 0 & 0 & \\ & 1 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & -1 & 0 & 0 & \\ & 0 & 0 & -1 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & -1 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & 1 & \\ & & & & & & & \ddots \end{array}\right) \end{array} \begin{array}{c} \\ i \\ j \\ k \\ l \\ m \\ n \\ \\ \end{array} . \tag{3.1.11b}$$

The $\mathbb{Z}_N$ elements are constructed similarly. For example $\mathbb{Z}_3$ elements are constructed by the following sub-elements,

$$\mathbf{e}_i \rightarrow \pm\mathbf{e}_j \rightarrow \pm\mathbf{e}_k \rightarrow \pm\mathbf{e}_i, \qquad (\#\text{ of terms with } - \text{ sign is even}), \tag{3.1.12a}$$

or

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \tag{3.1.12b}$$

and their permutations. $\mathbb{Z}_4$ elements include the following sub-elements,

$$\begin{gathered} \mathbf{e}_i \rightarrow -\mathbf{e}_j \rightarrow -\mathbf{e}_i, \qquad i \neq j, \\ \mathbf{e}_i \rightarrow \pm\mathbf{e}_j \rightarrow \pm\mathbf{e}_k \rightarrow \pm\mathbf{e}_l \rightarrow \pm\mathbf{e}_i, \qquad (\#\text{ of terms with } - \text{ sign is even}), \end{gathered} \tag{3.1.12ca}$$

or

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.1.12d}$$

and their permutations. We can similarly deal with the A_N lattices.

Here we introduce an abbreviation for 6×6 element matrices as

$$(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3) \equiv \begin{pmatrix} \mathbf{m}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{m}_3 \end{pmatrix} \quad \text{with } \mathbf{m}_i \in \{\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{1}\} \tag{3.1.4}$$

and

$$\mathbf{a} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{b} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

that are very useful for the classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds and the orientifold involution $\mathcal{R}$.

Note that the roots of the A_N and D_N can be given by

$$A_N : \quad \mathbf{e}_i - \mathbf{e}_j, \tag{3.1.5a}$$
$$D_N : \quad \pm\mathbf{e}_i \pm \mathbf{e}_j, \quad i, j = 1, \cdots, N. \tag{3.1.5b}$$

They are symmetric under the permutations of i and j. Now it is apparent that on the D_6 lattice, point group elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds can be given by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{1}$ in (3.1.5). The elements with the twist $(1/2, 1/2, 0)$ are given by

$$(\underline{-\mathbf{1}, -\mathbf{1}, \mathbf{1}}), \ (\underline{-\mathbf{1}, \pm\mathbf{a}, \pm\mathbf{a}}), \ (\underline{-\mathbf{1}, \pm\mathbf{b}, \pm\mathbf{a}}), \ (\underline{-\mathbf{1}, \pm\mathbf{b}, \pm\mathbf{b}}), \tag{3.1.6}$$

where underlined entries can be permutated. The twist of each point group element of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold should be $(1/2, 1/2, 0)$, and this requirement constrains the pairs of θ and ϕ. All the results are available in Appendix C.

In this way, we can construct all the point group elements for six dimensional Lie root lattices on the Table 3.1 (except for the E_6 lattice). The allowed $\mathbb{Z}_N$ orbifolds are listed in Table 3.3 and $\mathbb{Z}_N \times \mathbb{Z}_M$ are in Table 3.4. However we have to pay special attentions to a few cases owing to additional outer automorphisms as explained in the following.

An exception occurs in the D_4 lattice, which has another outer automorphism g',

$$g' : \ \alpha_1 \to \alpha_3 \to \alpha_4 \to \alpha_1. \tag{3.1.7}$$

For example it generate an element

$$r_{\alpha_1} r_{\alpha_2} g' = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}. \tag{3.1.8}$$

This action corresponds to a rotation of $(e^{1\pi i/6}, e^{5\pi i/6})$, and actually this is the generalized Coxeter element $C^{[3]}$. For this element the classification in the $\mathbf{e}_i$ basis is inconvenient (since for example it acts as $g' : \ \mathbf{e}_1 \to (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)/2$). We comment that among

Lie root lattice	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_6$-I	$\mathbb{Z}_6$-II	$\mathbb{Z}_7$	$\mathbb{Z}_8$-I	$\mathbb{Z}_8$-II	$\mathbb{Z}_{12}$-I	$\mathbb{Z}_{12}$-II
A_6	–	–	–	–	✓	–	–	–	–
D_6	–	✓	–	–	–	✓	✓	–	–
E_6	✓	✓	✓	–	–	–	–	✓	–
$A_5 \times A_1$	–	–	–	✓	–	–	–	–	–
$D_5 \times A_1$	–	✓	–	–	–	–	✓	–	–
$A_4 \times A_2$	–	–	–	✓	–	–	–	–	–
$A_4 \times (A_1)^2$	–	–	–	–	–	–	–	–	–
$D_4 \times A_2$	–	✓	–	✓	–	–	✓	✓	✓
$D_4 \times (A_1)^2$	–	✓	–	–	–	✓	✓	–	✓
$A_3 \times A_3$	–	✓	–	–	–	✓	–	–	–
$A_3 \times A_2 \times A_1$	–	–	–	✓	–	–	–	–	–
$A_3 \times (A_1)^3$	–	✓	–	–	–	–	–	–	–
$(A_2)^3$	✓	✓	✓	✓	–	–	–	–	–
$(A_2)^2 \times (A_1)^2$	–	✓	–	✓	–	–	–	–	–
$A_2 \times (A_1)^4$	–	✓	–	–	–	–	✓	–	–
$(A_1)^6$	–	✓	–	–	–	✓	✓	–	–

Table 3.3: *Table of six-dimensional (non-)factorizable tori and possible $\mathbb{Z}_N$ orbifold models on them.*

$\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds this element generates a new orbifold only for $\mathbb{Z}_3 \times \mathbb{Z}_3$, e.g. $(C^{[3]})^4$ is rotation of $(e^{2\pi i/3}, e^{2\pi i/3})$ and that of $r_{\alpha_3} g'$ is $(1, e^{2\pi i/3})$. Then a torus on the $D_4 \times A_2$ lattice allows a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold.

The other exception occurs when the independent radii of a torus which is expressed as direct products of the same kind of tori, are equal to each other. The $A_3 \times A_3$ lattice is the case, and there is an additional outer automorphism g_{33},

$$g_{33} : \alpha_i \leftrightarrow \pm \alpha_{i+3}, \tag{3.1.9}$$

where α_i is a simple root of the first (second) A_3 for $i = 1, 2, 3$ ($i = 4, 5, 6$). From the observation of its eigenvalues, these elements do not generate another $\mathbb{Z}_N \times \mathbb{Z}_M$ elements. However the orientifold action $\mathcal{R}$ can be generated from g_{33}, which will be explained in chapter 5. Such outer automorphisms also arise in factorizable tori including sublattices

Lie root lattice	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_4 \times \mathbb{Z}_4$
A_6	–	–	–	–
D_6	✓	✓	–	✓
E_6	✓	✓	✓	–
$A_5 \times A_1$	✓	–	–	–
$D_5 \times A_1$	✓	✓	–	–
$A_4 \times A_2$	✓	–	–	–
$A_4 \times (A_1)^2$	✓	–	–	–
$D_4 \times A_2$	✓	✓	✓	–
$D_4 \times (A_1)^2$	✓	✓	–	✓
$A_3 \times A_3$	✓	✓	–	–
$A_3 \times A_2 \times A_1$	✓	–	–	–
$A_3 \times (A_1)^3$	✓	✓	–	–
$(A_2)^3$	✓	✓	✓	–
$(A_2)^2 \times (A_1)^2$	✓	✓	–	–
$A_2 \times (A_1)^4$	✓	✓	–	–
$(A_1)^6$	✓	✓	–	✓

Table 3.4: *Table of six-dimensional (non-)factorizable tori and possible $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models on them.*

$(A_2)^n$ and $(A_1)^m$. For example, $(A_2)^2$ has an outer automorphism as

$$g_{22} : \alpha_i \to -\alpha'_i \to -\alpha_i, \quad i = 1, 2, \tag{3.1.10}$$

where α_i is a simple root of the first A_2 and α'_i is one of the second A_2. The eigenvalues of this element are $(e^{\pi i/2}, e^{\pi i/2})$, and generate $\mathbb{Z}_4$ elements. In this case the factorizable tori are actually non-factorizable as orbifolds. In the tables 3.3 and 3.4 these elements are included.

Finally we comment on the automorphisms of the E_6 lattice. We take account of the outer automorphisms of E_6. [2] Due to the symmetry of the Dynkin diagram, it has a $\mathbb{Z}_2$ symmetry

$$g_2 : \ \alpha_1 \leftrightarrow \alpha_5, \ \alpha_2 \leftrightarrow \alpha_4, \ \alpha_i \to \alpha_i, \ i = 3, 6, 0. \tag{3.1.11}$$

[2]The Weyl groups of simple Lie algebras and their conjugacy classes are classified in Ref.137.

Note that this action is not included in the Weyl group of E_6. One can show that the group which generated from the Weyl group and g_2, i.e. $\{\mathcal{W}, g_2\}$, is equivalent to $\{\mathcal{W}, -\mathbb{1}\}$ [136], where $-\mathbb{1}$ act as

$$-\mathbb{1}: \ \alpha_i \to -\alpha_i, \ i = 0, 1, \cdots, 6. \tag{3.1.12}$$

There are the other outer automorphisms in the extended E_6 diagram. We define one of the element by

$$g_2': \ \alpha_5 \leftrightarrow \alpha_0, \ \alpha_4 \leftrightarrow \alpha_6, \ \alpha_i \to \alpha_i, \ i = 1, 2, 3. \tag{3.1.13}$$

Since the product of the action $g_2 g_2'$ is the $\mathbb{Z}_3$ element of the automorphism of the extended E_6 diagram, we see that g_2 and g_2' generate the whole outer automorphisms of E_6. The Weyl group of E_6 is generated from all the simple roots of the algebra, and it is given by

$$\mathcal{W} = \{r_i \ |i = 1, \cdots, 6\} = \{r_i \ |i = 2, 3, 4, 5, 6, 0\}. \tag{3.1.14}$$

It means that $\{\mathcal{W}, g_2\} = \{\mathcal{W}, g_2'\} = \{\mathcal{W}, -\mathbb{1}\}$, and we conclude that

$$\{\mathcal{W}, g_2, g_2'\} = \{\mathcal{W}, -\mathbb{1}\} \tag{3.1.15}$$

From the conjugacy classes of the Weyl groups [137] and the element $-\mathbb{1}$, we can obtain all the $\mathcal{N} = 1$ orbifolds on the E_6 torus from the automorphism $\{\mathcal{W}, -\mathbb{1}\}$.

3.2 Two dimensional orbifolds

At first we consider the orbifolds on two dimensional tori. For the action of a rotation θ, there are only two kinds of lattices in two dimensions. For the action of a reflection r, there are two kinds of orbifolds in two dimensions, and orbifolds on different lattices are continuously deformed to each other. Referring to the original papers of string orbifolds, we also explain our notations.

The boundary conditions of T^2 are defined by a lattice

$$\Lambda = \left\{ n_1 \alpha_1 + n_2 \alpha_2 \middle| \, n_1, n_2 \in \mathbb{Z} \right\}, \tag{3.2.1}$$

where α_j is a base vector of the lattice. We define the coordinate (x_1, x_2) on a two-tori T^2. The $\mathbb{Z}_2$ elements are given by a rotation

$$r: \ x_{1,2} \to -x_{1,2}, \tag{3.2.2}$$

or by a reflection

$$r: \ x_1 \to -x_1. \tag{3.2.3}$$

In any cases the elements must act on the lattice as the automorphisms.

The rotation does not fix the complex structure U,

$$U \equiv (G_{12} + i\sqrt{\det G})/G_{11} \tag{3.2.4}$$

where the metric$G$ is defined by $G_{ij} = \alpha_i \cdot \alpha_i$. This orbifold is characterized by its four fixed points as figure 2.1. However $\mathbb{Z}_N$ orbifolds ($N > 2$) fix the complex structures. Let us define the Lie root lattices by

$$\begin{array}{rll} (A_1)^2: & \alpha_1 = \sqrt{2}\mathbf{e}_1, & \alpha_2 = \sqrt{2}\mathbf{e}_2, \\ A_2: & \alpha_1 = \sqrt{2}\mathbf{e}_1, & \alpha_2 = -\frac{1}{\sqrt{2}}\mathbf{e}_1 + \sqrt{\frac{3}{2}}\mathbf{e}_2, \\ B_2: & \alpha_1 = \sqrt{2}(\mathbf{e}_1 - \mathbf{e}_2), & \alpha_2 = \sqrt{2}\mathbf{e}_2, \\ C_2: & \alpha_1 = \sqrt{2}\mathbf{e}_1, & \alpha_2 = -\sqrt{2}\mathbf{e}_1 + \sqrt{2}\mathbf{e}_2, \\ D_2: & \alpha_1 = \sqrt{2}\mathbf{e}_1, & \alpha_2 = \sqrt{2}\mathbf{e}_2, \\ G_2: & \alpha_1 = \sqrt{2}\mathbf{e}_1, & \alpha_2 = -\frac{3}{\sqrt{2}}\mathbf{e}_1 + \sqrt{\frac{3}{2}}\mathbf{e}_2, \end{array} \tag{3.2.5}$$

where for convenience we rescaled B_2 from the standard notations. From the definition of the lattices (3.2.1), the following equivalencies between these lattices are apparent,

$$\Lambda_{(A_1)^2} = \Lambda_{B_2} = \Lambda_{C_2} = \Lambda_{D_2} = \{n_1\sqrt{2}\mathbf{e}_1 + n_2\sqrt{2}\mathbf{e}_2 |\, n_1, n_2 \in \mathbb{Z}\}, \tag{3.2.6a}$$

$$\Lambda_{A_2} = \Lambda_{G_2} = \{n_1\tfrac{1}{\sqrt{2}}\mathbf{e}_1 + n_2\sqrt{\tfrac{3}{2}}\mathbf{e}_2\}. \tag{3.2.6b}$$

Some of the lattices are depicted in figure 3.2. Two dimensional orbifolds can be given by the (generalized) Coxeter elements [3]. These elements act on the lattices as rotations, and in the complex coordinate $z \equiv x_1 + x_2$ they are expressed as

$$\begin{array}{rlll} (A_1)^2 & \text{or} \; (SU(2))^2 & \equiv r_{\alpha_1} r_{\alpha_2} & : \; z \to e^{\pi i} z \\ A_2 & \text{or} \; SU(3) & \equiv r_{\alpha_1} r_{\alpha_2} & : \; z \to e^{2\pi i/3} z \\ A_2^{[2]} & \text{or} \; SU(3)^{[2]} & \equiv r_{\alpha_1} g & : \; z \to e^{\pi i/3} z \\ B_2 & \text{or} \; SO(5) & \equiv r_{\alpha_1} r_{\alpha_2} & : \; z \to e^{\pi i/2} z \\ & \quad G_2 & \equiv r_{\alpha_1} r_{\alpha_2} & : \; z \to e^{\pi i/3} z \end{array} \tag{3.2.7}$$

where α_i is the simple root of the corresponding Lie algebra, and g is the outer automorphism of A_2. The Coxeter element of D_2 is the same as $(A_1)^2$, and that of C_2 is the same as B_2. These actions simply mean that there are two distinct Lie root lattices $(A_1)^2$ and A_2, and their automorphism are given by rotations $z \to e^{2\pi i v} z$,

$$(A_1)^2: \; v = \frac{1}{2}, \frac{1}{4}, \tag{3.2.8a}$$

$$A_2: \; v = \frac{2}{3}, \frac{1}{3}. \tag{3.2.8b}$$

As mentioned, the lattice structure is not restricted to $(A_1)^2$ for the $\mathbb{Z}_2$ orbifold of $v = \frac{1}{2}$. In two dimensions these four elements generate all the possible orbifolds with rotations [97]. In a similar manner we use the six dimensional lattices on table 3.1, and classify the automorphisms on them.

[3]The definitions are given in Appendix C.

Next we consider the orbifold with reflections. For the reflection r which acts crystallographically on the lattice Λ_i, without loss of generality we can set one of the basis, say α_1, along the x_1-direction. The reflection acts on the basis vectors as

$$r : \begin{cases} \alpha_1 & \to & -\alpha_1 \\ \alpha_2 & \to & \alpha_2 + n\alpha_1 \end{cases} \tag{3.2.9}$$

where $n \in \mathbb{Z}$. Then there are only two distinct solutions for the complex structure (3.2.4),

$$U = ia \quad \text{or} \quad \frac{1}{2} + ia\ , \qquad a \in \mathbb{R}, \tag{3.2.10}$$

which leave the complex structure U invariant under the action of r. The D_2 lattice in (3.2.5) corresponds to the case with $n = 0$, i.e. $\mathrm{Re}\,(U) = 0$, and the A_2 lattice to the $n = 1$ case of $\mathrm{Re}\,(U) = \frac{1}{2}$. The reflection r can also be represented by the Weyl reflections $r = r_{\alpha_1}$ for both cases. These two orbifolds have different topology, because the orbifold on the D_2 (A_2) lattice has two (one) fixed lines. In figure 3.2 the fixed line on the A_2 lattice is connected owing to the tilted structure of the lattice.

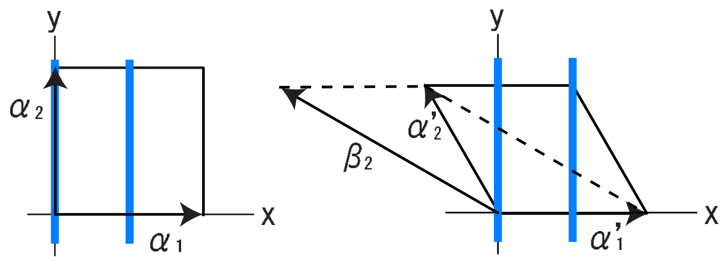

Figure 3.2: $\mathbb{Z}_2$ orbifolds on two two-tori. The left figure is the orbifold on the $(A_1)^2$ ($\sim D_2$) lattice, and the right one is that on the A_2 lattice. Blue colored lines are fixed lines. The dashed line indicate the G_2 lattice, which is equivalent to the A_2 lattice.

Let us consider the deformation of the orbifolds on the lattices. The reflection r of (3.2.9), that acts on the metric G, is given as

$$r = \begin{pmatrix} -1 & 0 \\ n & 1 \end{pmatrix}. \tag{3.2.11}$$

The metric should be invariant under the actions of orbifold elements r,

$$G = rGr^t. \tag{3.2.12}$$

For the basis of D_2 in (3.2.5), one obtains

$$\alpha_i \cdot \alpha_j = \begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix}. \tag{3.2.13}$$

This means D_2 is decomposed to two A_1 as

$$D_2 \to A_1 \times A_1 = (A_1)^2. \tag{3.2.14}$$

For the basis of A_2 in (3.2.5), one obtains

$$\alpha_i \cdot \alpha_j = \begin{pmatrix} G_{11} & -\frac{1}{2}G_{11} \\ -\frac{1}{2}G_{11} & G_{22} \end{pmatrix}. \tag{3.2.15}$$

For the Euclidean metric one obtains the Cartan matrix of A_2, whereas one can also find a metric of B_2,

$$\alpha_i \cdot \alpha_j = \begin{cases} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \text{for} \quad G_{11} = G_{22} = 2 \\ \\ \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} & \text{for} \quad \begin{matrix} G_{11} = 2, \\ G_{22} = 1. \end{matrix} \end{cases} \tag{3.2.16}$$

Then the deformation is

$$A_2 \to B_2 \sim D_2. \tag{3.2.17}$$

Note that the B_2 lattice is equivalent to the D_2 ($\sim (A_1)^2$) lattice, and the A_2 lattice can be continuously deformed to the $(A_1)^2$ lattice. However since the reflection r mixes the roots α_1 and α_2 each other, the orbifold can not be decomposed to $(A_1)^2$ and has different topology.

In the two dimensional orbifold with the reflection r, we have seen that the shape of the lattice is not important, rather the definition of r is crucial, and it leads to two different topology of orbifolds. It is also important that continuous deformation connects orbifolds on different lattices. We will see it in six dimensional cases.

3.3 Topology of orbifolds

There are some important topological invariants which relate to the string spectra on the orbifolds [13, 19, 20, 44, 50, 79–81, 102–108]. One is the Euler number,

$$\chi = \frac{1}{N} \sum_{[g,h]=0} \chi_{g,h}, \tag{3.3.1}$$

where $\chi_{g,h}$ is the Euler characteristic of the subspace left simultaneously fixed by the action of g and h, and N is the order of point group.

Hodge number is defined as the number of independent harmonic forms $h_g^{p,q} = \dim H_{g\text{-sector}}^{p,q}$, and it can expressed by the sum of the separate contributions of the untwisted sector $h_{untwisted}^{p,q}$ and g-twisted sectors $h_g^{p,q}$,

$$h^{p,q} = h_{untwisted}^{p,q} + \sum_g h_g^{p,q}. \tag{3.3.2}$$

For a Kähler manifold there is a relation between these quantities,

$$\chi = \sum_{p,q}^{N} (-1)^{p+q}\, h^{p,q}, \tag{3.3.3}$$

and for Calabi-Yau three-folds, including orbifolds, it is

$$\chi = 2(h^{1,1} - h^{2,1}). \tag{3.3.4}$$

Hodge numbers are related to the string spectra as table 3.5. Moreover in the case of the

	Gravity multiplet	Moduli	
		Kähler	Complex structure
$\mathcal{N}=2$ Type IIA	1	$h^{1,1}$	$h^{2,1}+1$
$\mathcal{N}=2$ Type IIB	1	$h^{1,1}+1$	$h^{2,1}$
$\mathcal{N}=1$ Heterotic	1	$h^{1,1}$	$h^{2,1}$

Table 3.5: *The relations between hodge numbers and the number of string states on Calabi-Yau three-folds.*

$E_8 \times E_8$ heterotic string with the standard embedding, $h^{1,1}$ corresponds to the number of chiral superfields with the representation $\overline{\mathbf{27}}$ of E_6, and $h^{2,1}$ to that of $\mathbf{27}$ of E_6. Then $\chi/2$ gives the number of generations [19].

The hodge numbers of the untwisted sectors are obtained by projecting out those of six-torus T^6. The basis for the cohomology class $H^{i,j}(T^6)$ are given by

$$\begin{array}{c} z_i, \quad \bar{z}_i \\ z_i \wedge z_j, \quad z_i \wedge \bar{z}_j, \quad \bar{z}_i \wedge \bar{z}_j, \\ z_i \wedge z_j \wedge z_k, \quad z_i \wedge z_j \wedge \bar{z}_k, \quad z_i \wedge \bar{z}_j \wedge \bar{z}_k, \quad \bar{z}_i \wedge \bar{z}_j \wedge \bar{z}_k, \\ \vdots \end{array} \tag{3.3.5}$$

It follows that $h^{1,1}(T^6) = h^{2,1}(T^6) = 9$.

For g-twisted sectors which have only fixed points, they are given by

$$h_g^{1,1} = \frac{1}{N} \sum_h \epsilon(g,h)\, \chi_{g,h}, \tag{3.3.6a}$$

$$h_g^{1,2} = h_g^{2,2} = h_g^{2,1} = 0, \tag{3.3.6b}$$

where $\tilde{\chi}_{g,h}$ is the number of g-fixed tori (points) which are invariant under the action h [79–81, 105], and $\epsilon(g,h)$ is the discrete torsion associated with B-field backgrounds of string. Without B-field we have $\epsilon(g,h) = 1$. On the other hand when the g-twisted sectors have fixed tori, the hodge numbers are given by

$$h_g^{1,1} = h_g^{2,2} = \frac{1}{N}\sum_h \epsilon(g,h)\,\tilde{\chi}_{g,h}, \tag{3.3.7a}$$

$$h_g^{1,2} = \frac{1}{N}\sum_h \epsilon(g,h)\,\tilde{\chi}_{g,h}\,e^{2\pi v_h(g)}, \tag{3.3.7b}$$

$$h_g^{2,1} = \frac{1}{N}\sum_h \epsilon(g,h)\,\tilde{\chi}_{g,h}\,e^{-2\pi v_h(g)}, \tag{3.3.7c}$$

where $v_h(g)$ is the twist of h on the plane which g acts trivially. The generalization of the Euler characteristic with discrete torsions are defined by

$$\chi_\epsilon = \frac{1}{N}\sum_{[g,h]=0} \epsilon(g,h)\,\chi_{g,h}, \tag{3.3.8}$$

where N is the order of the orbifold.

The difference between $\chi_{g,h}$ and $\tilde{\chi}_{g,h}$ will be clear in the following example. For the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orbifold on $T^2 \times T^2 \times T^2$, we give the twists of the point group as

$$\theta : (v_1, v_2, v_3) = (\frac{1}{4}, -\frac{1}{4}, 0), \quad \phi : (v_1, v_2, v_3) = (0, \frac{1}{2}, -\frac{1}{2}). \tag{3.3.9}$$

There are two fixed points on the first and second tori respectively under the θ action, and four on the second and third tori under ϕ, see figure 3.3. Since θ and ϕ leave $2 \times 2 \times 4$ points invariant, we obtain $\chi_{\theta,\phi} = 16$. The four fixed tori under θ are also invariant under ϕ, and it follows that $\tilde{\chi}_{\theta,\phi} = 4$. The action of θ does not have fixed points, $\chi_{\theta,1} = 0$, but four fixed tori $\tilde{\chi}_{\theta,1} = 4$. Whereas ϕ have 4×4 fixed tori, 2×4 of them are invariant under θ, $\tilde{\chi}_{\phi,\theta} = 8$. For the identity element 1, it is $\tilde{\chi}_{1,h} = 1$ for $h \in \{\theta, \phi\}$.

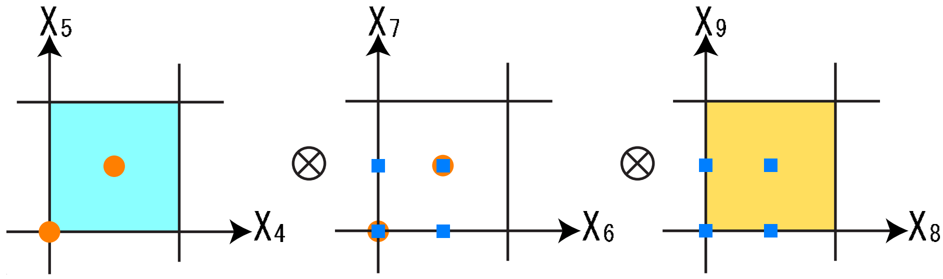

Figure 3.3: *The circles and third torus indicate fixed tori under the action of ϕ. The squares and first torus indicate fixed tori under the action of ϕ.*

3.4 $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on six-tori

Now we are ready to investigate toroidal orbifolds on non-factorizable six-tori. The hodge numbers depend on the numbers of fixed tori, and fixed tori appear when one of the point group elements leaves some directions of the compact space invariant. Therefore a non-factorizable orbifold can have different topology from the factorizable orbifolds with the same twist if they have fixed tori. This is the case of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds and $\mathbb{Z}_N$ orbifolds with non-prime N. Since $\mathbb{Z}_N$ orbifolds have classified by means of the (generalized) Coxeter elements [70,72,73,106,116,117], the new results are given in the case of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds.

We give some examples of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds in detail. We also consider Z_6-II orbifolds, where the point group is expressed as $\mathbb{Z}_2 \times \mathbb{Z}_3$.

The result of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models are examined in ref [45, 46]. The Euler number is simplified as

$$\chi = \frac{3}{2}\chi_{\theta,\phi}. \tag{3.4.1}$$

We give concrete examples of a few orbifolds on the D_6 lattice,

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_6 = \mathbf{e}_5 + \mathbf{e}_6, \quad i = 1, \ldots, 5. \tag{3.4.2}$$

Let us consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold action with the point group elements,

$$\begin{cases} \theta\text{: } (-\mathbf{1}, -\mathbf{1}, \mathbf{1}) \\ \phi\text{: } (\mathbf{1}, -\mathbf{1}, -\mathbf{1}) \end{cases}. \tag{3.4.3}$$

The common fixed points under the action of θ and ϕ are

$$\begin{aligned} &0, \quad e_1 \\ &\frac{1}{2}(e_{2i-1} \pm e_{2i}), \quad i = 1, 2, 3 \\ &\frac{1}{2}(e_1 + e_2 + e_3 \pm e_4), \ \frac{1}{2}(e_3 + e_4 + e_5 \pm e_6), \ \frac{1}{2}(e_1 + e_2 + e_5 \pm e_6), \\ &\frac{1}{2}(e_i + e_j \pm e_k), \quad i = 1, 2, \quad j = 3, 4, \quad k = 5, 6, \\ &\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 \pm e_6), \end{aligned} \tag{3.4.4}$$

and we have $\chi_{\theta,\phi} = 32$. From the relation (3.4.1) the Euler number of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is

$$\chi = \frac{1}{2}\chi_{\theta,\phi} = 48. \tag{3.4.5}$$

There are eight fixed tori under the action of θ,

$$\begin{aligned} &xe_5 + ye_6, \\ &\frac{1}{2}(e_i + e_j) + xe_5 + ye_6, \quad i, j = 1, \cdots, 4, \ (i \neq j), \\ &\frac{1}{2}(e_1 + e_2 + e_3 + e_4) + xe_5 + ye_6, \end{aligned} \tag{3.4.6}$$

where $x, y \in \mathbb{R}$ are coordinates of the fixed tori. From (3.3.7c), we obtain the hodge numbers of the orbifold as

$$h^{1,1} = 24, \quad h^{2,1} = 0, \tag{3.4.7}$$

We can construct other orbifolds on the D_6 lattice, for example we consider the point group elements,

$$\begin{cases} \theta\colon (-\mathbf{1}, -\mathbf{1}, \mathbf{1}) \\ \phi\colon (\mathbf{b}, \mathbf{b}, -\mathbf{1}) \end{cases} . \tag{3.4.8}$$

The Euler number of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is

$$\chi = \frac{1}{2}\chi_{\theta,\phi} = 24, \tag{3.4.9}$$

and the hodge numbers are

$$h^{1,1} = 14, \quad h^{2,1} = 2. \tag{3.4.10}$$

Although we have many choices for the point group elements and the lattices for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, there are only eight topologically distinct orbifolds. Actually we can deform the complex moduli of the orbifolds in the following. The metric invariat under the point group element (3.4.3) is given by

$$\alpha_i \cdot \alpha_j = \begin{pmatrix} G_{1,1} & G_{1,2} & 0 & 0 & 0 & 0 \\ G_{1,2} & G_{2,2} & -G_{3,3} - G_{3,4} & a & 0 & 0 \\ 0 & -G_{3,3} - G_{3,4} & G_{3,3} & G_{3,4} & 0 & 0 \\ 0 & a & G_{3,4} & G_{4,4} & -\frac{1}{2}G_{5,5} - \frac{1}{2}G_{5,6} & -\frac{1}{2}G_{6,6} - \frac{1}{2}G_{5,6} \\ 0 & 0 & 0 & -\frac{1}{2}G_{5,5} - \frac{1}{2}G_{5,6} & G_{5,5} & G_{5,6} \\ 0 & 0 & 0 & -\frac{1}{2}G_{6,6} - \frac{1}{2}G_{5,6} & G_{5,6} & G_{6,6} \end{pmatrix} . \tag{3.4.11}$$

where $a \equiv -G_{3,4} - G_{4,4} + \frac{1}{4}(G_{5,5} + 2G_{5,6} + G_{6,6})$. One can continuously deform $G_{1,2} \to 0$, and the torus is factorized as $D_6 \to D_5 \times A_1$, where A_1 is generated from α_1. In addition we set

$$G_{5,5} = 2, \quad G_{5,6} = -1, \quad G_{6,6} = 1. \tag{3.4.12}$$

In this case the basis can be given as

$$\begin{aligned} \alpha_1 &= \mathbf{e}_1, \\ \alpha_i &= \sum_{j=2,3,4,5} a_{i,j}\mathbf{e}_j, \quad i = 2, \ldots, 4, \quad a_{i,j} \in \mathbb{R} \\ \alpha_5 &= \mathbf{e}_5 - \mathbf{e}_6, \\ \alpha_6 &= \mathbf{e}_6. \end{aligned} \tag{3.4.13}$$

So we can redefine the some of the basis as

$$\begin{aligned} \alpha'_5 &= \mathbf{e}_5, \\ \alpha_6 &= \mathbf{e}_6. \end{aligned} \tag{3.4.14}$$

Since the orbifold elements (3.4.3) act on these basis as

$$\begin{aligned} \alpha_5' &\to \pm \mathbf{e}_5, \\ \alpha_6 &\to \pm \mathbf{e}_6, \end{aligned} \tag{3.4.15}$$

the torus is factorizable to $D_6 \to D_4 \times (A_1)^2$. Moreover the deformation with

$$G_{3,3} = 1, \quad G_{3,4} = -1, \quad G_{4,4} = 2, \tag{3.4.16}$$

factorizes the direction of α_3. To sum up the torus of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of (3.4.3) can be factorized to

$$D_6 \to A_3 \times (A_1)^3, \tag{3.4.17}$$

where we identify D_3 by A_3.

It is observed that all the six-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on Lie root lattices are obtained from the orbifolds of the A_3, A_2 and A_1 types that are defined on table 3.6, and orbifolds with the same hodge numbers would be continuously deformed to one another.

orbifold	α_i	θ	ϕ	$\theta\phi$	$T_1(\theta)$	$T_2(\phi)$	$T_3(\theta\phi)$
A_3	α_1	$-\alpha_3$	$-\alpha_1$	α_3			
	α_2	$-\alpha_2$	$\sum_i^3 \alpha_i$	$-\sum_i^3 \alpha_i$	2	2	2
	α_3	$-\alpha_1$	$-\alpha_3$	α_1			
A_2	α_1	$-\alpha_1$	$-\alpha_1$	α_1	1	4	1
	α_2	$\sum_i^2 \alpha_i$	$-\alpha_2$	$-\sum_i^2 \alpha_i$			
A_1	α_1	$-\alpha_1$	$-\alpha_1$	α_1	2	2	0

Table 3.6: Building blocks for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. $T_i(g)$ indicates the number of the fixed points or lines under the action of g.

The allowed values of discrete torsion are

$$\epsilon = \pm 1. \tag{3.4.18}$$

However these discrete torsions do not make difference of generation numbers, except its sign. For all models, the numbers of zero modes of untwisted sector are

$$h^{1,1}_{untwisted} = h^{2,1}_{untwisted} = 3. \tag{3.4.19}$$

The Hodge number of twisted sectors and the generation numbers of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models are listed in table 3.7. The factorizable model is expressed as $T^2 \times T^2 \times T^2$, because the complex structure of each torus is not fixed by orbifold action.

We also calculate the generation numbers $\chi/2$ by

$$\frac{\chi}{2} = h^{1,1}_{untwisted} - h^{2,1}_{untwisted} + h^{1,1}_{twisted} - h^{2,1}_{twisted}, \tag{3.4.20}$$

Lattice	ϵ	χ	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$T^2 \times T^2 \times T^2$	1	96	48	0
	-1	-96	0	48
$A_2 \times (A_1)^4$	1	48	28	4
	-1	-48	4	28
$A_3 \times (A_1)^3$	1	48	24	0
	-1	-48	0	24
$(A_2)^2 \times (A_1)^2 - I$	1	24	18	6
	-1	-24	6	18
$(A_2)^2 \times (A_1)^2 - II$	1	24	16	4
	-1	-24	4	16
$A_3 \times A_2 \times A_1$	1	24	14	2
	-1	-24	2	14
$(A_3)^2$	1	24	12	0
	-1	-24	0	12
$(A_2)^3$	1	12	9	3
	-1	-12	3	9

Table 3.7: $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models for standard embedding: The second column denotes values of the discrete torsion ϵ. The generation numbers are given by $\chi/2$.

where $h^{1,1}_{twisted}$ and $h^{2,1}_{twisted}$ are the numbers of chiral matter fields from twisted sectors with representation in 27 and $\overline{27}$ respectively. Similarly $h^{1,1}_{untwisted}$ and $h^{2,1}_{untwisted}$ are those of untwisted sectors, and they are independent of Lattice structure.

For the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifolds, the Euler number and the number of generations with the discrete torsion are given as

$$\begin{aligned} \chi &= \frac{1}{8}\left(24\chi_{\theta.\phi} + 12\chi_{1.\theta\phi} + 6\chi_{\theta^2.\phi}\right), \\ \chi_{-1} &= \frac{1}{8}\left(-24\chi_{\theta.\phi} + 12\chi_{1.\theta\phi} + 6\chi_{\theta^2.\phi}\right). \end{aligned} \tag{3.4.21}$$

The allowed values of discrete torsion are

$$\epsilon = \pm 1. \tag{3.4.22}$$

For all models, the numbers of zero modes of untwisted sector are

$$h^{1,1}_{untwisted} = 3, \; h^{2,1}_{untwisted} = 1. \tag{3.4.23}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold models are listed in table 3.8. The factorizable model is expressed as $D_2 \times D_2 \times T^2$.

Lattice	ϵ	χ	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$D_2 \times D_2 \times (A_1)^2$	1	120	58	0
$((A_1)^6)$	-1	24	18	8
$A_3 \times D_2 \times A_1$	1	72	36	2
$(A_3 \times (A_1)^3)$	-1	24	16	6
$(A_3)^2$	1	48	24	2
	-1	24	14	4
$D_4 \times T^2$	1	96	48	2
$(D_4 \times (A_1)^2)$	-1	0	8	10
$D_4 \times D_2$	1	72	36	2
$(D_4 \times (A_1)^2)$	-1	24	16	6
$D_5 \times A_1$	1	72	34	0
	-1	24	14	4
D_6	1	72	34	0
	-1	24	14	4

Table 3.8: $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold models for standard embedding.

For the $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifolds, the Euler number and the numbers of generations with the discrete torsions are

$$\begin{aligned}
\chi &= \frac{1}{16}\{96\chi_{\theta.\phi} + 24(\chi_{\theta^2.\phi} + \chi_{\theta.\phi^2} + \chi_{\theta\phi,\phi^2}) \\
&\quad +12(\chi_{1.\theta\phi^2} + \chi_{1.\theta\phi^3} + \chi_{1.\theta^2\phi}) + 6\chi_{\theta^2.\phi^2}\}, \\
\chi_{-1} &= \frac{1}{16}\{-96\chi_{\theta.\phi} + 24(\chi_{\theta^2.\phi} + \chi_{\theta.\phi^2} + \chi_{\theta\phi,\phi^2}) \\
&\quad +12(\chi_{1.\theta\phi^2} + \chi_{1.\theta\phi^3} + \chi_{1.\theta^2\phi}) + 6\chi_{\theta^2.\phi^2}\}, \\
\chi_{\pm i} &= \frac{1}{16}\{-24(\chi_{\theta^2.\phi} + \chi_{\theta.\phi^2} + \chi_{\theta\phi,\phi^2}) \\
&\quad +12(\chi_{1.\theta\phi^2} + \chi_{1.\theta\phi^3} + \chi_{1.\theta^2\phi}) + 6\chi_{\theta^2.\phi^2}\}.
\end{aligned} \tag{3.4.24}$$

The allowed values of discrete torsion are

$$\epsilon = \pm 1, \ \pm i. \tag{3.4.25}$$

For all models, the numbers of zero modes of untwisted sector are

$$h^{1,1}_{untwisted} = 3, \ h^{2,1}_{untwisted} = 0. \tag{3.4.26}$$

The $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifold models are listed in table 3.9. The factorizable model is expressed as $D_2 \times D_2 \times D_2$.

The $\mathbb{Z}_2 \times \mathbb{Z}_3$ elements are equivalent to the $\mathbb{Z}_6$-II elements. The Euler number is simplified as

$$\chi = 4\chi_{1.\omega}. \tag{3.4.27}$$

Lattice	ϵ	χ	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$D_2 \times D_2 \times D_2$	1	180	87	0
	-1	84	39	0
	$\pm i$	-12	3	12
$D_4 \times D_2$	1	120	58	1
	-1	72	34	1
	$\pm i$	0	6	9
D_6	1	108	51	0
	-1	60	27	0
	$\pm i$	12	9	6

Table 3.9: $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifold models for standard embedding.

The discrete torsion is trivial in this case, i.e.

$$\epsilon = 1. \tag{3.4.28}$$

For all models, the numbers of zero modes of untwisted sector are

$$h^{1,1}_{untwisted} = 3,\ h^{2,1}_{untwisted} = 1. \tag{3.4.29}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifold models are listed in table 3.10. As we mentioned before, we do not distinguish the A_2 lattice from the G_2 lattice, then the factorizable model can be expressed as $(A_2)^3$.

Lattice	ϵ	χ	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$(A_2)^3$	1	48	32	10
$A_3 \times A_2 \times A_1$	1	48	26	4
$D_4 \times A_2$	1	48	26	4
$A_4 \times A_2$	1	48	26	4

Table 3.10: $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifold models for standard embedding.

The numbers of $h^{1,1}_{twisted}$ and $h^{2,1}_{twisted}$ are the same in non-factorizable models. This implies they are in equivalent class of orbifolds, and connected by continuous deformation of geometric moduli.

Chapter 4

Heterotic orbifolds

Heterotic string is a theory of closed string, and right-moving sectors of the superstring are adjoined to the twenty-six left moving sectors of the bosonic string. The sixteen left-movers generate an internal gauge symmetry with the gauge group $E_8 \times E_8$ or $SO(32)$, whose rank is sixteen. Owing to the Green-Schwartz mechanism only these two gauge groups are consistent with anomaly cancellation and $\mathcal{N} = 1$ supersymmetry of the spacetime [4]. Thus the heterotic string naturally includes gauge groups in it, and it provides interesting possibilities for particle phenomenology. In this paper we investigate $E_8 \times E_8$ heterotic string theory. One advantage of this group is that since E_8 is exceptional Lie group the spectra have variety of its representations. For example the representation $248 \in E_8$ can be broken to its subgroup $SO(10)$, and it includes a spinor representation $16 \in SO(10)$. On the other hand breaking of the adjoint representation in $SO(32)$ leads to only two rank tensor of $SU(N)$ or $SO(N)$. Other advantage is that we can interpret the other E_8' as the hidden sector which has possibilities of spontaneous supersymmetric breaking. The hidden sector is entitled to be dark matter because if our gauge interaction is derived from the first E_8 the hidden sector gauge interaction is invisible in the massless level of string except gravitational interaction.

We give several models on the non-factorizable orbifolds. In particular a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold on the E_6 lattice have interesting feature that the orbifold includes three fixed tori, and leads to GUT-like models with three generations of matter [48].

Non-factorizable orbifolds are characterized by some specific point in the moduli space of the Narain compactification [124,125]. $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the non-factorizable orbifolds are constructed in [44–46]. We gave the generalization to $\mathbb{Z}_N \times \mathbb{Z}_M$ and a systematic classification in [47]. We find that $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ non-factorizable orbifolds are realized by the automorphisms of the Lie root lattices. We also consider the effects of discrete torsions [71], since $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models with $N, M \geq 2$ allow the addition of discrete torsions. There are also interesting coincidence between the non-factorizable models and the factorizable models with generalized discrete torsion [60].

4.1 Heterotic constructions

We briefly introduce the $E_8 \times E_8$ heterotic string. Especially in this section we concentrate on the $E_8 \times E_8$ gauge group and explain the gauge embeddings and Wilson lines, which are associating to the gauge group breaking.

4.1.1 Definitions and basics

The heterotic string can be described by the action

$$S = -\frac{1}{4\pi} \int d\tau d\sigma \left(\partial X^\mu \bar{\partial} X_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu + \lambda^A \bar{\partial} \lambda^A \right), \tag{4.1.1}$$

with the fields

$$\begin{array}{ll} X^\mu(\tau,\sigma),\ \tilde{\psi}^\mu(\tau-\sigma), & \mu = 0,\cdots,9, \\ \lambda^A(\tau+\sigma), & A = 1,\cdots,32. \end{array} \tag{4.1.2}$$

Since the mode expansions of X^μ and $\tilde{\psi}^\mu$ on orbifolds are given in chapter 2, here we explain the part λ^A.

For the $E_8 \times E_8$ heterotic string, the boundary conditions of λ^A are divided to two independent sectors as

$$\begin{aligned} \lambda^A(\sigma + 2\pi) &= \eta \lambda^A(\sigma), \quad A = 1,\cdots,16 \\ &= \eta' \lambda^A(\sigma), \quad A = 17,\cdots,32, \end{aligned} \tag{4.1.3}$$

where $\eta = \pm$ and $\eta' = \pm$ that define NS and R sectors for each state. We impose the GSO projections $\exp(\pi i F_1)$ and $\exp(\pi i F_1')$ with the conditions

$$\exp(\pi i F_1) = \exp(\pi i F_1') = 1. \tag{4.1.4}$$

The states with $\eta = \eta' = -1$ belong to the NSNS$'$ sector. The four complex bosons from X^μ and sixteen complex fermions from λ^A contribute to the normal ordering constant $\tilde{a}$ in the NSNS$'$ sector, and from (2.1.24) we have

$$\tilde{a} = 4 \times \frac{1}{12} + 16 \times \frac{1}{24} = -1. \tag{4.1.5}$$

Then the massless NSNS$'$ states consist of

$$\lambda^A_{-1/2} \lambda^B_{-1/2} |0\rangle_{NS,NS'}, \quad 1 \le A, B \le 16,\ or, 17 \le A, B \le 32. \tag{4.1.6}$$

The states with $\eta = 1$ and $\eta' = -1$ are the RNS$'$ sector. The normal ordering constant of the RNS$'$ states is

$$\tilde{a} = 4 \times \frac{1}{12} - 8 \times \frac{1}{12} + 8 \times \frac{1}{24} = 0. \tag{4.1.7}$$

Therefore fermionic operators λ^A for $A = 1, \cdots, 16$ generate fermion in the Euclidean space of the index A. The eigenstate of the spinors can be expressed as

$$|(\pm\tfrac{1}{2})^8\rangle_{R,NS'} \equiv |\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle_{R,NS'}, \tag{4.1.8}$$

where the number of the + sign is even due to the GSO projection. Similarly for $\eta = -1$ and $\eta' = 1$ we obtain the NSR$'$ states

$$|(\pm\tfrac{1}{2})^8\rangle_{NS,R'}, \tag{4.1.9}$$

where the number of the + sign is even. The boundary condition $\eta = \eta' = 1$ does not include massless state. The states (4.1.6) with $1 \le A, B \le 16$ transform as 120 in SO(16), and (4.1.8) transform as 128. These states actually constitute the adjoin representation 248 in E_8.

We can rewrite these states in the bosonic description. We define the complex fermions by

$$\lambda^{I\pm} \equiv \frac{1}{\sqrt{2}}(\lambda^{2I-1} \pm i\lambda^{2I}), \quad I = 1, \cdots, 16. \tag{4.1.10}$$

There is the relationship between the bosonic coordinates X^I_L and the complex fermions:

$$\lambda^{I\pm} \cong \exp(\pm 2iX^I_L). \tag{4.1.11}$$

We assume the normal ordering in the operator product expansions in the following. The operator product of bosonic coordinates are

$$X^I(z)X^J(0) \sim -\delta^{IJ} \ln z, \quad I, J = 1 \sim 16, \tag{4.1.12}$$

and we obtain

$$e^{X^I(z)}e^{X^J(0)} \sim \delta^{IJ}\frac{1}{z}, \tag{4.1.13}$$

using the Campbell-Baker-Hausdorff formula $e^X e^Y = e^{X+Y+[X,Y]/2+\cdots}$. The operator product of complex fermions are given by

$$\lambda^I(z)\lambda^J(0) \sim \delta^{IJ}\frac{1}{z}, \tag{4.1.14}$$

and has the same formula of (4.1.13). Therefore we confirm the equivalency (4.1.11).

The mode expansion of the bosonic coordinate X^I_L is

$$X^I_L = x^I_L + p^I_L(\tau + \sigma) + \frac{i}{2}\sum \frac{1}{n}\alpha^I_n e^{-2i(\tau+\sigma)}. \tag{4.1.15}$$

The states in fermionic formulations are mapped to the bosonic states $|p_L\rangle$ as

$$\begin{aligned} &\lambda^{I\pm}_{-1/2}\lambda^{J\pm}_{-1/2}|0\rangle_{NS,NS'}, \to \begin{cases} |p_L\rangle, & p_L = (\underline{\pm 1, \pm 1, 0, 0, 0, 0, 0, 0}) \\ \alpha^I|p_L\rangle, & p_L = (0, 0, 0, 0, 0, 0, 0, 0) \end{cases} \\ &|(\pm\tfrac{1}{2})\rangle_{R,NS'} \to |p_L\rangle, \ p_L = (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2})_{even+}, \end{aligned} \tag{4.1.16}$$

where p_L^I is a zero mode of X_L^I. This means that the bosonic coordinates X_L^I for $I = 1, \cdots, 8$ are compactified on the E_8 lattice, whose simple roots are given by

$$\begin{aligned}
\alpha_1 &= (1,-1,0,0,0,0,0,0), \\
\alpha_2 &= (0,1,-1,0,0,0,0,0), \\
&\vdots \\
\alpha_6 &= (0,0,0,0,0,1,-1,0), \\
\alpha_7 &= (0,0,0,0,0,1,1,0), \\
\alpha_8 &= (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}).
\end{aligned} \tag{4.1.17}$$

The momentum on the torus on a lattice Λ is generated from its dual lattice Λ^* ,

$$p_L = \sum_i n_i \alpha_i^*, \quad n_i \in \mathbb{Z}, \quad \alpha_i^* \in \Lambda^*, \tag{4.1.18}$$

see also (5.3.8). Since E_8 is a self dual lattice $\Lambda^* = \Lambda$, the momenta p_L^I in (4.1.16) are generated from the roots of E_8.

4.1.2 Modular invariance and discrete torsion

Heterotic orbifold models must satisfy some consistency conditions required by the modular invariance. The modular invariance guarantees the anomaly cancellation in the low energy theory [71]. For Z_N orbifolds with prime N, the level matching conditions are necessary and sufficient for modular invariance to all loops of string amplitude.

Heterotic string is left-right asymmetric as the definition, and this means that for the action of a symmetric orbifold

$$\begin{aligned}
X^i_{L,R} &\rightarrow \theta\, X^i_{L,R}, \\
\psi^i_R &\rightarrow \theta\, \psi^i_R.
\end{aligned} \tag{4.1.19}$$

The phase factors associated with the modular transformation are cancelled between world sheet bosons, however the right-moving fermions ψ_R^i do not have the counterparts, i.e. ψ_L^i. A simple way to balance the phase factors from ψ_R^i is to twist the fermionic operators in the gauge sector so that the phase factors are cancelled each other. The bosonization (4.1.11) means that a rotation θ which acts on the indices i of the fermionic coordinates corresponds to a shift for the bosonic coordinates

$$\theta : \begin{cases} \lambda^{I\pm} & \rightarrow e^{\pm 2\pi i V_I} \lambda^{I\pm}, \\ X_L^I(\sigma+\pi) & \rightarrow X_L^I(\sigma) + \pi V^I, \end{cases} \tag{4.1.20}$$

and the mode expansions are modified as

$$X_L^I = x_L^I + (p_L^I + V^I)(\tau+\sigma) + \frac{i}{2}\sum \frac{1}{n}\alpha_n^I e^{-2i(\tau+\sigma)}. \tag{4.1.21}$$

The mass formula of the left mover is given by

$$\frac{1}{4}M_L^2 = \frac{1}{2}\sum_{I=1}^{16}(p_L^I + V^I)^2 + N_B - a. \tag{4.1.22}$$

Next we derive the modular invariance conditions for these twisted sectors. Let us consider the partition function from the θ-twisted sector, [1]

$$\mathrm{Tr}_{NS}\left(q^{L_0}\bar{q}^{\bar{L}_0}\right) \to \prod_{i=1}^{3}\left(\frac{\bar{\vartheta}\left[{v_i \atop 0}\right]}{\bar{\eta}}\right)\prod_{I=1}^{16}\left(\frac{\vartheta\left[{V_I \atop 0}\right]}{\eta}\right), \tag{4.1.23}$$

where v_i is the twist under the action of θ, and V_I is the corresponding shift in the gauge sector. Because the phase factors from bosonic left- and right-movers in ten-dimensional spacetime are cancelled each other, we extract only the contributions from the gauge sectors and right-mover fermions in the above expression. This function should be invariant by a modular transformation $\tau \to \tau + N$.

$$\prod_{i=1}^{3}\left(\frac{\bar{\vartheta}\left[{v_i \atop 0}\right]}{\bar{\eta}}\right)\prod_{I=1}^{16}\left(\frac{\vartheta\left[{V_I \atop 0}\right]}{\eta}\right) \tag{4.1.24}$$

$$\to \quad \exp N\left[\sum_{i=1}^{3}v_i(1-v_i) - \sum_{I=1}^{16}V_I(1-V_I)\right]\prod_{i=1}^{3}\left(\frac{\bar{\vartheta}\left[{v_i \atop 0}\right]}{\bar{\eta}}\right)\prod_{I=1}^{16}\left(\frac{\vartheta\left[{V_I \atop 0}\right]}{\eta}\right). \tag{4.1.25}$$

Then the prefactor is required to be identity. The contribution from the R sector is obtained by replacing $v_i \to v_i + 1/2$, and leads to the same condition. We can set $N\sum_{i=1}^{3} v_i = 0$ and $N\sum_{i=1}^{16} V_I = 0 \pmod 2$, and the conditions are simplified as

$$N\left(\sum_{i=1}^{3}(v_i)^2 - \sum_{I=1}^{16}(V_I)^2\right) \equiv N\left((v_i)^2 - (V_I)^2\right) = 0 \bmod 2. \tag{4.1.26}$$

This is the modular invariance condition for orbifolds. For $v_i = V_I$ with $i = I = 1, 2, 3$, the condition is trivially satisfied, and this corresponds to the standard embedding.

For orbifolds with non-prime N, we also need to generalize the GSO projection [80]. The number of $\theta^k\phi^l$-twisted states is given by

$$D(\theta^k\phi^l) = \frac{1}{MN}\sum_{t=0}^{N-1}\sum_{s=0}^{M-1}\epsilon^{(ks-lt)}\tilde{\chi}_{\theta^k\phi^l,\theta^t\omega^s}\Delta(k,l;t,s), \tag{4.1.27}$$

where $\tilde{\chi}$ is the number of points left simultaneously fixed by $\theta^k\phi^l$ and $\theta^t\phi^s$. If $\theta^k\phi^l$ leaves unrotated any of the coordinates, $\tilde{\chi}$ must be calculated using only the sub-lattice which is rotated. $\Delta(k,l;t,s)$ is a state dependent phase,

$$\begin{aligned}\Delta(k,l;t,s) \;&= P^{(k,l)}\exp\{2\pi i[(p+kV+lW)(tV+sW) - (q+kv+lw)(tv+sw)] \\ &\qquad -\tfrac{1}{2}((kV+lW)(tV+sW) - (kv+lw)(tv+sw))\},\end{aligned} \tag{4.1.28}$$

[1]The definitions and calculations are given in Appendix B. The evaluation including all the twisted sectors are seen in [130].

where $P^{(k,l)}$ indicates a contribution of the oscillators. p is momentum of the $E_8 \times E_8'$ gauge sectors, and q is the H-momentum of the twisted states.

There are degrees of freedom to add phase factors in the partition functions. This is described as turning on background antisymmetric field $B_{\mu\nu}$ on the torus, and it introduces phases to string states [71, 80]. The general form of one-loop partition function is

$$\mathcal{Z} = \frac{1}{MN} \sum_{\theta,\phi} \epsilon(\theta,\phi) \mathcal{Z}(\theta,\phi). \tag{4.1.29}$$

The phase $\epsilon(\theta,\phi)$ is called discrete torsion. In a $\mathbb{Z}_N$ orbifold these phases are fixed by one-loop modular invariance. On the other hand in $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, where M is generally divisible by N, the phase is restricted to N-th root of unity,

$$\epsilon(\theta,\phi) \equiv \epsilon, \;\; \epsilon^N = 1. \tag{4.1.30}$$

Then the phases for general twisted sectors are given by

$$\epsilon(\theta^k \phi^l, \theta^t \omega^s) = \epsilon^{(ks-lt)}. \tag{4.1.31}$$

4.1.3 Discrete Wilson lines

If we implement Wilson lines as the background in the compact space, we can break the gauge group of the models further and have smaller number of generations of matter **??**. Discrete Wilson lines are defined by the embedding of the shifts in a six dimensional torus into the gauge degrees of freedom as

$$\sum_{i=1}^{6} r_i \alpha_i \to \sum_{i=1}^{6} r_i a_i^I, \quad I = 1, \cdots, 16, \tag{4.1.32}$$

where $r_i \in \mathbb{R}$ depends on the location of the fixed points, and $a_i^I \in \Lambda_{E_8 \times E_8'}$ is a Wilson line. In the untwisted sector, the states that are invariant under the action of Wilson lines survive, and it leads to breaking of the gauge group. In the twisted sectors, the numbers of the degenerate states are reduced because Wilson lines distinguish the fixed points or tori. The mode expansion is modified as

$$X_L^I = x_L^I + (p_L^I + V^I + r_\rho a_\rho^I)(\tau+\sigma) + \frac{1}{2}\sum \frac{1}{n} \tilde{\alpha}_n^I e^{-2\pi i(\tau+\sigma)}. \tag{4.1.33}$$

This means that the mass formula and the conditions for the modular invariance are obtained by the replacement of the momentum,

$$p_L^I \to p_L^I + V^I + r_\rho a_\rho^I. \tag{4.1.34}$$

from the case with no Wilson lines. Then the generalization of the conditions to the θ^n-twisted sector with the Wilson lines are

$$N\left(n^2 \sum_{i=1}^{3} (v^i)^2 - \sum_{I=1}^{16} (nV^I + r_\rho a_\rho^I)^2\right) = 0 \bmod 2. \tag{4.1.35}$$

Since there are independent conditions for n and r_ρ, these conditions are written as follows,

$$N\left(\sum_{I=1}^{16}(V^I)^2 - \sum_{i=1}^{3}(v^i)^2\right) \equiv N(V^2 - v^2) = 0 \bmod 2, \tag{4.1.36a}$$

and

$$N\sum_{I=1}^{16}(a_\rho^I)^2 \equiv Na_\rho^2 = 0 \bmod 2, \tag{4.1.36b}$$

$$N\sum_{I=1}^{16} a_\rho^I a_\sigma^I \equiv Na_\rho \cdot a_\sigma = 0 \bmod 1, \quad \rho \neq \sigma, \tag{4.1.36c}$$

$$N\sum_{I=1}^{16} V^I a_\rho^I \equiv NV \cdot a_\rho = 0 \bmod 1. \tag{4.1.36d}$$

4.2 $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models on non-factorizable tori

For the $\theta^k\phi^l$-twisted sector, the level matching condition is

$$\begin{gathered} N'[(kV+lW)^2 - (kv+lw)^2] = 0 \bmod 2, \\ k = 0, \cdots, N-1, \quad l = 0, \cdots, M-1, \end{gathered} \tag{4.2.1}$$

where N' is the order of the twist $\theta^k\phi^l$, and V and W are the shift vectors in the gauge sector associated to θ and ϕ respectively.

In this section we consider $E_8 \times E_8'$ heterotic string models with the standard embeddings, where the shifts on the gauge sector are given as

$$\begin{aligned} v = (n, -n, 0) \quad &\to \quad V = (n, -n, 0, 0, 0, 0, 0, 0), \\ w = (0, m, -m) \quad &\to \quad W = (0, m, -m, 0, 0, 0, 0, 0). \end{aligned} \tag{4.2.2}$$

Thus the level matching condition is trivially satisfied in the standard embeddings. This corresponds to embedding the spin connection in the gauge connection.

4.2.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models

$\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is phenomenologically interesting model, because some three generation models are presented with the aid of Wilson lines, and three generations may be associated to three complex dimension of compact space [63].

In heterotic orbifold models there are two classes of string states. One is untwisted sector in bulk and the other is twisted sector which localizes at fixed torus. The untwisted sector in the **27** of E_6 reads from

$$h^{1,1}_{untwist} = h^{2,1}_{untwist} = 3. \tag{4.2.3}$$

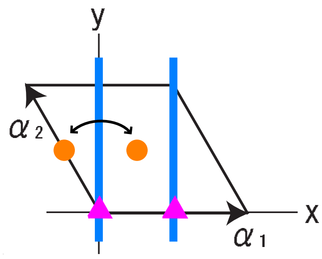

Figure 4.1: $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The blue colored lines are fixed tori by action of θ. The circles and triangles are fixed points by action of ϕ. The twisted states on the circles are mapped to the other by θ, and the linear combinations of states are eigenstates of orbifold.

The chirality of untwisted sector of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is left-right symmetric. so it does not contribute to the number of generations.

The number of zero modes of twisted sector is related to the number of fixed tori. In a factorizable model with the action θ and ϕ, whose shift vectors are $v = (\frac{1}{2}, -\frac{1}{2}, 0)$ and $w = (0, \frac{1}{2}, -\frac{1}{2})$ respectively, the number of tori of θ-twisted sector is 4^2, because there are four fixed points in the first and second tori, and the third torus is free from the action of θ. Therefore the total number of zero modes of three twisted sectors is 48, and this corresponds to the generation numbers of $\mathcal{N} = 1$ chiral matter which have gauge charge $27 \in \mathrm{E}_6$ in the standard embedding.

We can confirm this result by calculating the Euler number, because the number of generations is equal to a half of Euler number [19], For $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, this equation is simplified to

$$\chi = \frac{3}{2}\chi_{\theta,\phi}. \tag{4.2.4}$$

Here, $\chi_{\theta,\phi}$ is the number of points left simultaneously fixed by θ and ϕ, and is equal to 4^3. Then we have $\chi = 96$, and this agrees with the former result.

In the case of non-factorizable model it is easier to use the Lefschetz fixed point theorem. The number of fixed tori (#FT) of θ-twisted sector is

$$\#FT = \frac{vol((1-\theta)\Lambda)}{vol(N)}, \tag{4.2.5}$$

where N is the lattice normal to the sub-lattice invariant by the action.

As an example we consider the orbifold model on the $A_2 \times D_4$ lattice, whose basis is given by simple roots,

$$\begin{aligned}
\alpha_1 &= \sqrt{2}(1,0,0,0,0,0),\\
\alpha_2 &= \sqrt{2}\left(-\frac{1}{2},\frac{\sqrt{3}}{2},0,0,0,0\right),\\
\alpha_3 &= (0,0,1,-1,0,0),\\
\alpha_4 &= (0,0,0,1,-1,0),\\
\alpha_5 &= (0,0,0,0,1,-1),\\
\alpha_6 &= (0,0,0,0,1,1),
\end{aligned} \tag{4.2.6}$$

and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold actions, θ, ϕ, are given by

$$\theta = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \phi = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.2.7}$$

The common fixed points by the actions of θ and ϕ are,

$$\begin{gathered}
(0,0,0,0,0,0),\ (0,0,1,0,0,0),\ (0,0,0,0,\tfrac{1}{2},\pm\tfrac{1}{2}),\\
(\tfrac{\sqrt{2}}{2},0,0,0,0,0),\ (\tfrac{\sqrt{2}}{2},0,1,0,0,0),\ (\tfrac{\sqrt{2}}{2},0,0,0,\tfrac{1}{2},\pm\tfrac{1}{2}),\\
(0,0,\tfrac{1}{2},\underline{\tfrac{1}{2},\pm\tfrac{1}{2},0}),\ (\tfrac{\sqrt{2}}{2},0,\tfrac{1}{2},\underline{\tfrac{1}{2},\pm\tfrac{1}{2},0}),
\end{gathered} \tag{4.2.8}$$

where the underlined entries can be permuted. This leads to $\chi_{\theta,\phi} = 16$ and $\chi = 24$. Then the generation number is twelve. We can reconfirm this result by counting the number of fixed tori as follows. There are four independent fixed tori of the θ-twisted sector,

$$\begin{gathered}
(0,x,y,0,0,0),\\
(0,x,y,\underline{\tfrac{1}{2},\tfrac{1}{2},0}).
\end{gathered} \tag{4.2.9}$$

Note that these tori are identified by the sub lattice $(1-\theta)\Lambda$. The $\theta\phi$-twisted sector also has four fixed tori. In the ϕ-twisted sector there are eight fixed tori,

$$\begin{gathered}
(0,0,0,0,x,y),\ (0,0,\tfrac{1}{2},\tfrac{1}{2},x,y), (\tfrac{\sqrt{2}}{2},0,0,0,x,y),\ (\tfrac{\sqrt{2}}{2},0,\tfrac{1}{2},\tfrac{1}{2},x,y),\\
(\pm\tfrac{\sqrt{2}}{4},\tfrac{1}{2}\sqrt{\tfrac{3}{2}},0,0,x,y),\ (\pm\tfrac{\sqrt{2}}{4},\tfrac{1}{2}\sqrt{\tfrac{3}{2}},\tfrac{1}{2},\tfrac{1}{2},x,y).
\end{gathered} \tag{4.2.10}$$

Then the total number of tori is 16, and does not match the generation number.

This is because two of fixed tori of the ϕ-twisted sector are not invariant by the action of θ. On the A_2 torus in figure 4.1, there are four fixed points, and can be labeled by shift vectors, $v = n\alpha_1 + m\alpha_2,\ n,m = 0,1$. The θ-invariant states are

$$|0\rangle,\quad |\alpha_1\rangle. \tag{4.2.11}$$

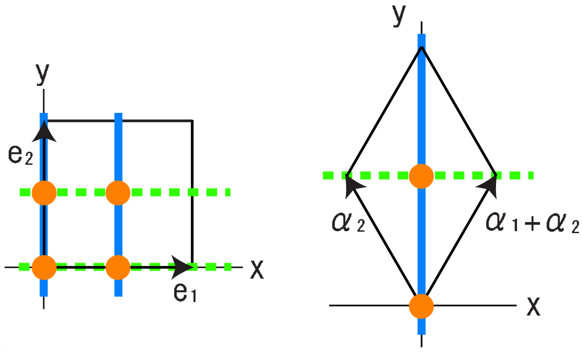

Figure 4.2: $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. Blue colored lines are fixed tori by θ action, and green colored dotted lines are by $\theta\phi$ action. The circles are common fixed points.

These are charged matter of representation 27. We take linear combinations of remaining two states as eigenstates of action of θ,

$$\begin{aligned} +1: &\quad |\alpha_2\rangle + |\alpha_1 + \alpha_2\rangle, \\ -1: &\quad |\alpha_2\rangle - |\alpha_1 + \alpha_2\rangle, \end{aligned} \tag{4.2.12}$$

where ± 1 denote the eigenvalue of these states under the action of θ. The phase of physical states should be cancelled with Δ-phase (4.1.28). These are the same chirality states with the charge of representation in 27 and $\overline{27}$ respectively, and they do not contribute to the number of generations [45]. The generation number from the ϕ-sector is four, and we have twelve generations from three twisted sectors, which is equal to a half of χ. This is significantly small compared to the generation number of factorizable model.

The diminution of fixed tori and fixed points in the A_2 torus can be seen in figure 4.2. In this figure the basis of the A_2 root lattice is changed to $\alpha_1+\alpha_2$ and α_1, and that makes it easier to draw the orbifold action. In this figure we can see that the decrease of θ fixed tori of the non-factorizable model is related to direction which is left invariant by the action of θ. Therefore the diminution does not occur in non-degenerate orbifold such as Coxeter orbifold which rotates the whole space of T^6.

4.2.2 $\mathbb{Z}_2 \times \mathbb{Z}_4$ models

As an example we consider a model on the D_6 root lattice (3.1.2). The only consistent point group action on this lattice, except its conjugate representation, is

$$\begin{aligned} \theta &= \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ \phi &= diag(1,1,-1,-1,-1,-1). \end{aligned} \tag{4.2.13}$$

We count the number of states with representations 27 and $\overline{27}$ by the use of the coordinates of fixed points and fixed tori [2]. The $\theta^i\phi^j$-twisted sector T_{ij} localizes at fixed points or tori as follows,

$$\begin{aligned} T_{01}: &\ (x,y,0,0,0,0),\ (x,y,\tfrac{1}{2},\tfrac{1}{2},0,0),\ (x,y,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}), \\ T_{10}: &\ (0,0,x,y,0,0),\ (\tfrac{1}{2},\tfrac{1}{2},x,y,\tfrac{1}{2},\tfrac{1}{2}), \\ T_{11}: &\ (0,0,0,0,0,0),\ (1,0,0,0,0,0),\ (\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\pm\tfrac{1}{2}), \\ &\ (\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2},0,0,0),\ (0,0,\tfrac{1}{2},0,\tfrac{1}{2},\pm\tfrac{1}{2}), \\ &\ (0,0,\tfrac{1}{2},\pm\tfrac{1}{2},0,0),\ (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2}), \\ T_{20}: &\ (0,0,x,y,0,0),\ (\tfrac{1}{2},\tfrac{1}{2},x,y,0,0),\ (0,0,x,y,\tfrac{1}{2},\tfrac{1}{2}) \\ &\ (\tfrac{1}{2},0,x,y,\tfrac{1}{2},0),\ (\tfrac{1}{2},\tfrac{1}{2},x,y,\tfrac{1}{2},\tfrac{1}{2}), \\ T_{21}: &\ (0,0,0,0,x,y),\ (\tfrac{1}{2},\tfrac{1}{2},0,0,x,y),\ (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},x,y). \end{aligned} \tag{4.2.14}$$

In the orbifolds of $\mathbb{Z}_N$ with non-prime N, physical states of θ^k-sector are generally linear combinations of states at fixed points by the action θ^k [70, 72]. If f_k is a fixed point of θ^k such that l is the smallest number giving $\theta^l f_k = f_k + u,\ u \in \Lambda$, then the eigenstates of θ are

$$|p\rangle = \sum_{r=0}^{n-1} e^{i\gamma r} |\theta^r f_k\rangle, \tag{4.2.15}$$

with $\gamma = 2\pi/l$, $p = 0, 1, \cdots, N-1$. Then the physical states of T_{01} sector by orbifold are linear combinations of them,

$$\begin{aligned} 1: &\ |(x,y,0,0,0,0)\rangle,\ |(x,y,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})\rangle, \\ &\ |(x,y,\tfrac{1}{2},\tfrac{1}{2},0,0)\rangle,\ |(x,y,0,0,\tfrac{1}{2},\tfrac{1}{2})\rangle, \\ &\ |(x,y,\tfrac{1}{2},0,\tfrac{1}{2},0)\rangle + |(x,y,0,\tfrac{1}{2},\tfrac{1}{2},0)\rangle, \\ &\ |(x,y,\tfrac{1}{2},0,0,\tfrac{1}{2})\rangle + |(x,y,0,\tfrac{1}{2},0,\tfrac{1}{2})\rangle, \end{aligned} \tag{4.2.16}$$

$$\begin{aligned} -1: &\ |(x,y,\tfrac{1}{2},0,\tfrac{1}{2},0)\rangle - |(x,y,0,\tfrac{1}{2},\tfrac{1}{2},0)\rangle, \\ &\ |(x,y,\tfrac{1}{2},0,0,\tfrac{1}{2})\rangle - |(x,y,0,\tfrac{1}{2},0,\tfrac{1}{2})\rangle, \end{aligned} \tag{4.2.17}$$

[2] In this approach we can observe the twisted states explicitly. However we can systematically count these numbers by (4.1.27).

where ± 1 denote eigenvalues under the action of θ. Then there are six states of $27 \in E_6$, but the negative eigenvalue state does not make $\overline{27}$ state because it does not cancel the Δ-phase (4.1.28). In this way we confirm the number of 27 states is 34, and that of $\overline{27}$ states is 0. The untwisted sector is the same as the factorizable model, that is $h^{1,1}_{untwisted} = 3$ and $h^{2,1}_{untwisted} = 1$. This is because the untwisted sector is determined by local action of orbifold and not affected by global structure of Lie lattice.

This result is confirmed by the Euler number (3.3.1),

$$\begin{aligned} \chi &= \frac{1}{8}\left(24\chi_{\theta.\phi} + \chi_{1.\theta\phi} + \chi_{\theta^2.\phi}\right), \\ \chi_{-1} &= \frac{1}{8}\left(-24\chi_{\theta.\phi} + \chi_{1.\theta\phi} + \chi_{\theta^2.\phi}\right). \end{aligned} \tag{4.2.18}$$

Here, $\chi_{-1}/2$ is the generation number of the model with discrete torsion $\epsilon = -1$. The fixed points by the action of θ and ϕ are

$$\begin{gathered} (0,0,0,0,0,0),\ (1,0,0,0,0,0),\ (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2}), \\ (0,0,\tfrac{1}{2},\pm\tfrac{1}{2},0,0),\ (\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\pm\tfrac{1}{2}). \end{gathered} \tag{4.2.19}$$

The fixed points of action of $\theta\phi$ are

$$\begin{gathered} (0,0,0,0,0,0),\ (1,0,0,0,0,0),\ (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2}), \\ (0,0,\tfrac{1}{2},\pm\tfrac{1}{2},0,0),\ (\tfrac{1}{2},\tfrac{1}{2},0,0,\tfrac{1}{2},\pm\tfrac{1}{2}), \\ (\tfrac{1}{2},\tfrac{1}{2},\underline{\pm\tfrac{1}{2},0},0,0),\ (0,0,\underline{\tfrac{1}{2},0},\tfrac{1}{2},\pm\tfrac{1}{2}). \end{gathered} \tag{4.2.20}$$

The fixed points of action of θ^2 and ϕ are

$$\begin{gathered} (0,0,0,0,0,0),\ (1,0,0,0,0,0),\ (\underline{\tfrac{1}{2},0},\underline{\tfrac{1}{2},0},\underline{\pm\tfrac{1}{2},0}), \\ (\tfrac{1}{2},\pm\tfrac{1}{2},0,0,0,0),\ (0,0,\tfrac{1}{2},\pm\tfrac{1}{2},0,0),\ (0,0,0,0,\tfrac{1}{2},\pm\tfrac{1}{2}), \\ (\tfrac{1}{2},\tfrac{1}{2},\underline{\pm\tfrac{1}{2},0},0,0),\ (0,0,\underline{\tfrac{1}{2},0},\tfrac{1}{2},\pm\tfrac{1}{2}),\ (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\pm\tfrac{1}{2}). \end{gathered} \tag{4.2.21}$$

As a result we have $\chi_{\theta.\phi} = 8$, $\chi_{1.\theta\phi} = 16$ and $\chi_{\theta^2.\phi} = 32$. The generation numbers in these models are

$$\begin{aligned} \chi/2 &= 36, \\ \chi_{-1}/2 &= 12. \end{aligned} \tag{4.2.22}$$

This result is different from that of factorizable tori $D_2 \times D_2 \times T^2$, i.e. $\chi/2 = 60$ and $\chi_{-1}/2 = 12$. All other models on non-factorizable tori are listed in Table 3.8 of Appendix.A.

4.2.3 $\mathbb{Z}_6$-II models

The point group elements θ, ϕ of $\mathbb{Z}_6$-II ($\mathbb{Z}_2 \times \mathbb{Z}_3$) orbifold can be expressed by one element $\theta\phi \in \mathbb{Z}_6$ which is non-degenerate. Let ω be defined by $\omega \equiv \theta\phi$. Then we have $\theta = \omega^4$ and

$\phi = \omega^3$. This implies $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifold is essentially non-degenerate and does not provide new models which have different Euler number compared to the factorizable model. The Euler characteristic of $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifold is evaluated by (3.3.1), and it is simplified to

$$\chi = 4\chi_{1.\omega}. \tag{4.2.23}$$

We see that only non-degenerate element $\omega \in \mathbb{Z}_6$ contributes to the generation numbers of these models. However the Hodge numbers are dependent on lattices as we see below.

For example the basis of six dimensional tori $A_3 \times A_2 \times A_1$ is given by

$$\begin{aligned} \alpha_1 &= (1,-1,0,0,0,0,0), \\ \alpha_2 &= (0,1,-1,0,0,0,0), \\ \alpha_3 &= (0,1,1,0,0,0,0), \\ \alpha_4 &= (0,0,0,1,-1,0,0), \\ \alpha_5 &= (0,0,0,0,1,-1,0), \\ \alpha_6 &= (0,0,0,0,0,0,1). \end{aligned} \tag{4.2.24}$$

In the $A_3 \times A_2 \times A_1$ lattice the only consistent point group action, except its conjugate, is

$$\begin{aligned} \theta &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ \phi &= diag(-1,-1,-1,1,1,1,-1). \end{aligned} \tag{4.2.25}$$

In the D_3 lattice subspace there is only one fixed tori, which is depicted as figure 4.3.

The Hodge numbers are calculated as $h^{1,1}_{twisted} = 26$ and $h^{2,1}_{twisted} = 4$ from the ω^i-twisted sector T_i localizing at fixed points or tori. The untwisted sector is the same as factorizable model, that is $h^{1,1}_{untwisted} = 3$ and $h^{2,1}_{untwisted} = 1$. As we have mentioned the number of generations is 24, which is the same as the factorizable model. However this model has different Hodge numbers from that of the factorizable model on $(A_2)^3$ root lattice.

Similarly the other two $\mathbb{Z}_2 \times \mathbb{Z}_3$ models are examined, and the results are listed in Appendix A. The non-factorizable models are not the same orbifolds as factorizable one and the structure of Yukawa coupling can be different from the factorizable model.

4.3 $\mathbb{Z}_3 \times \mathbb{Z}_3$ Orbifold models on the E_6 root lattice

4.3.1 An Orbifold with three fixed tori

Some non-factorizable orbifolds are highly symmetric in six dimensions, especially for E_6. Interestingly it contains twisted sectors localized at the three fixed tori respectively. Then

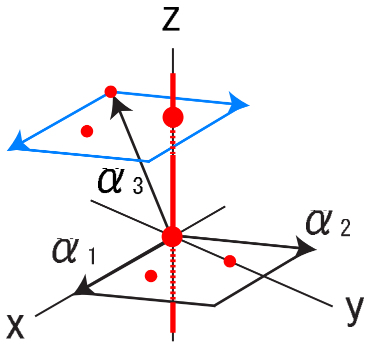

Figure 4.3: $\mathbb{Z}_2 \times \mathbb{Z}_3$ non-factorizable orbifold. Circles are points on the θ fixed tori which are parallel to z-axis. Red colored line on z-axis is a fixed tori by θ action, and we can see two circles are on the line. Due to lattice structure all circles are on the same fixed torus. So there is only one fixed torus in θ-twisted sector.

the models based on it can naturally lead to three generations of matter. We investigate the general structure of interaction of this orbifold, and find that some interactions allow flavor mixing terms. These features seem favorable for phenomenological motivation. In this section we explain the detail of a orbifold on the E_6 root lattice, and give examples of three-family GUT-like models and general consideration of the three point interactions.

We consider the case that Λ is the E_6 root lattice whose basis vectors are given by the simple roots[3]

$$\begin{aligned}
\alpha_1 &= (1,0,0,0,0,0)\,, \\
\alpha_2 &= \left(-\frac{1}{2},\frac{\sqrt{3}}{2},0,0,0,0\right), \\
\alpha_3 &= \left(0,-\frac{1}{\sqrt{3}},0,-\frac{1}{\sqrt{3}},0,-\frac{1}{\sqrt{3}}\right), \\
\alpha_4 &= \left(0,0,-\frac{1}{2},\frac{\sqrt{3}}{2},0,0\right), \\
\alpha_5 &= (0,0,1,0,0,0)\,, \\
\alpha_6 &= \left(0,0,0,0,-\frac{1}{2},\frac{\sqrt{3}}{2}\right).
\end{aligned} \tag{4.3.1}$$

Then we define the E_6 *torus*, $T^6_{E_6} = \mathbb{R}^6/\Lambda_{E_6}$. It may look a little complicated object, however from the viewpoint of six dimensions it is highly symmetric space as we will see.

[3]Note that despite it is usual that the lengths of the simple roots are $\sqrt{2}$, here they have length 1 for simplicity.

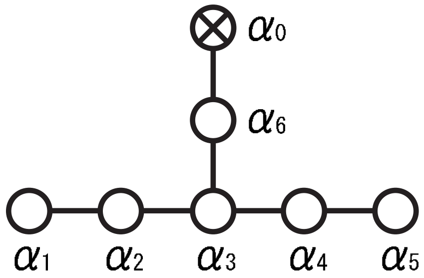

Figure 4.4: Extended E_6 diagram

We define the lowest root,

$$\begin{aligned} \alpha_0 &\equiv -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - 2\alpha_6 \\ &= (0,0,0,0,1,0)\,, \end{aligned} \tag{4.3.2}$$

and they form the E_6 extended Dynkin diagram, see figure 4.4. We give the point group elements of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold by the elements of the Weyl group as follows,

$$\begin{aligned} \theta &\equiv r_{\alpha_1} r_{\alpha_2} r_{\alpha_4} r_{\alpha_5}\,, \\ \phi &\equiv r_{\alpha_5} r_{\alpha_4} r_{\alpha_6} r_{\alpha_0}\,. \end{aligned} \tag{4.3.3}$$

They act on the simple roots as

$$\begin{array}{llll} \theta: & \alpha_1 \to \alpha_2, & \alpha_2 \to -\alpha_1 - \alpha_2, & \alpha_3 \to \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\ & \alpha_4 \to \alpha_5, & \alpha_5 \to -\alpha_4 - \alpha_5, & \alpha_6 \to \alpha_6, \quad \alpha_0 \to \alpha_0, \\ \phi: & \alpha_1 \to \alpha_1, & \alpha_2 \to \alpha_2, & \alpha_3 \to \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\ & \alpha_4 \to -\alpha_4 - \alpha_5, & \alpha_5 \to \alpha_4, & \alpha_6 \to \alpha_0, \quad \alpha_0 \to -\alpha_6 - \alpha_0. \end{array} \tag{4.3.4}$$

In complex coordinates $z_i = x_{2i-1} + ix_{2i}$, $i = 1, 2, 3$, we can rewrite the point group action as

$$\begin{aligned} \theta: &\quad z_i \to e^{2\pi i v_i} z_i, \quad v = \left(\tfrac{1}{3}, -\tfrac{1}{3}, 0\right), \\ \phi: &\quad z_i \to e^{2\pi i w_i} z_i, \quad w = \left(0, \tfrac{1}{3}, -\tfrac{1}{3}\right), \end{aligned} \tag{4.3.5}$$

where v and w are the shift vectors of the point group. These shifts project out six components of the spinor, and leave two chiral spinors $|\pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\rangle$ invariant. After the GSO projection, $\mathcal{N} = 1$ supersymmetry is unbroken in four dimension. We can also construct $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the E_6 torus, which satisfies $\mathcal{N} = 1$ supersymmetric condition.

The orbifold action leaves sets of points invariant, i.e. these points differ from their orbifold image by a shift of torus lattice Λ. For the action of θ, ϕ or $\theta\phi$ these sets appear as two dimensional fixed tori. All the fixed tori by the action of θ are given as

$$\begin{aligned} &(0,0,0,0,x,y),\\ &(0,\tfrac{1}{\sqrt{3}},0,0,x,y),\\ &(0,0,0,\tfrac{1}{\sqrt{3}},x,y), \end{aligned} \tag{4.3.6}$$

where $x, y \in \mathbb{R}$. We can confirm the number of the fixed tori by the use of Lefschetz fixed point theorem. The number of fixed tori (#FT) by the action of θ is

$$\#FT = \frac{vol((1-\theta)\Lambda)}{vol(N)}, \tag{4.3.7}$$

where N is the lattice normal to the sub-lattice invariant by the action. Then we have $\#FT = 3$ for the θ-twisted sector. Note that by the shift of the E_6 root lattice the following tori are identified;

$$(0,0,0,0,x,y) \simeq (0,0,0,0,x,y) - \alpha_3 \simeq \left(0,\tfrac{1}{\sqrt{3}},0,\tfrac{1}{\sqrt{3}},x,y\right). \tag{4.3.8}$$

These nontrivial structure leads to diminution of the number of the fixed tori compared to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold on factorizable torus $T^2 \times T^2 \times T^2$.

For the action of $\theta\phi^2$, 27 fixed points are left invariant. Generally the numbers of fixed *points* are the same between non-factorizable and factorizable orbifolds. However in the orbifold on the E_6 torus some of the $\theta\phi^2$ fixed points are not invariant for the action of θ and ϕ, and the states of their linear combinations are left by the projections. By explicit calculation we have 15 states, which are invariant by θ and ϕ, and 6 states with a phase of $e^{2\pi i/3}$ and $e^{-2\pi i/3}$ by the actions respectively, see table 4.5.

By embedding two twists into the gauge sector [4], we can break one E_8 to GUT gauge group [91], however not to $SU(3) \times SU(2) \times U(1)$.

Shifts V^I	Gauge Group
$(\frac{2}{3},0^7)$, $(\frac{1}{3}^4,0^4)$, $(\frac{1}{3}^6,\frac{2}{3},0)$	$SO(14) \times U(1)$
$(\frac{1}{3}^2,0^6)$, $(\frac{1}{3}^8)$, $(\frac{1}{6}^8)$	$E_7 \times U(1)$
$(\frac{1}{3}^2,\frac{2}{3},0^5)$, $(\frac{1}{3}^6,0^2)$	$E_6 \times SU(3)$
$(\frac{1}{3}^4,\frac{2}{3},0^3)$, $(\frac{1}{6}^7,\frac{5}{6})$	$SU(9)$

Table 4.1: Shifts and the correspondent gauge groups from E_8.

For the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold, the shifts of the torus lattice are identified as $\theta\alpha_1 = \alpha_2$, $\theta\alpha_4 = \alpha_5$ and $\phi\alpha_6 = \alpha_0$. It means for the Wilson lines we have

$$a_1^I = a_2^I, \quad a_4^I = a_5^I, \quad a_6^I = a_0^I. \tag{4.3.9}$$

[4]Some relations about shifts and Dynkin diagrams are investigated in Ref.89, 90

Moreover in the θ-twisted sector on the E_6 torus, the fixed tori which have different shifts are identical,

$$\left(0,0,0,-\tfrac{1}{\sqrt{3}},x,y\right) \simeq \left(0,\tfrac{1}{\sqrt{3}},0,0,x,y\right). \tag{4.3.10}$$

We can read the shifts by the action of θ from

$$\begin{aligned}(1-\theta)\left(0,0,0,-\tfrac{1}{\sqrt{3}},x,y\right) &= \left(0,0,0,-\tfrac{1}{\sqrt{3}},x,y\right)+\alpha_4+\alpha_5,\\ (1-\theta)\left(0,\tfrac{1}{\sqrt{3}},0,0,x,y\right) &= \left(0,\tfrac{1}{\sqrt{3}},0,0,x,y\right)+\alpha_1+\alpha_2.\end{aligned} \tag{4.3.11}$$

It follows that the relation for the Wilson lines,

$$a_4^I + a_5^I = a_1^I + a_2^I. \tag{4.3.12}$$

For a shift α_3 we have a Wilson line a_3 from the definition of α_0. From (4.3.9) and (4.3.12) we have only one independent Wilson line, $\alpha^I \equiv \alpha_i^I,\ i=0,1,\cdots,6$, for the E_6 torus. It implies the orbifold on the E_6 torus is highly symmetric space.

When we include a Wilson line, the degeneracy of the fixed tori is reduced from 3 to 1, and that of the fixed points of the $\theta\phi^2$-sector is from 27 to 9. It seems a Wilson line breaks the three-family structure of the orbifold, and we do not include Wilson lines in this paper. At first sight the states on the 27 fixed points are too many as a low energy spectrum. We will see later that for some non-standard gauge embedding the $\theta\phi^2$-twisted sector contains only hidden sector states and singlets. Then we realize three family structure states from fixed tori.

4.4 $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold models on E_6 torus

4.4.1 A model with standard embedding

At first we consider an $E_8 \times E_8'$ heterotic orbifold model from $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold on the E_6 torus with the standard embedding (4.2.3),

$$\begin{aligned}V &= \left(\frac{1}{3},-\frac{1}{3},0,0,0,0,0,0\right)(0,0,0,0,0,0,0,0),\\ W &= \left(0,\frac{1}{3},-\frac{1}{3},0,0,0,0,0\right)(0,0,0,0,0,0,0,0).\end{aligned} \tag{4.4.1}$$

By counting the fixed points and tori, we easily find the spectrum of this model. The gauge group is broken to

$$E_6 \times U(1)^2 \times E_8'. \tag{4.4.2}$$

There are twisted sectors

$$\theta,\ \theta^2,\ \phi,\ \phi^2,\ \theta\phi,\ \theta^2\phi^2,\ \theta\phi^2, \tag{4.4.3}$$

or A, $\bar{A}$, B, $\bar{B}$, C, $\bar{C}$, D, respectively. These states are distinguished by their H-momenta (listed on the table 4 and 5). There are also three untwisted sector states, U_1, U_2 and U_3, which H-momenta of their bosonic states are $q_1 = (0,1,0,0)$,$q_2 = (0,0,1,0)$ and $q_3 = (0,0,0,1)$ respectively. Then the matter content of the model is

$$\begin{array}{rlrlrlrl} U_1, U_2, U_3\!: & 3\times\mathbf{27}, & A\!: & 3\times\mathbf{27}, & B\!: & 3\times\mathbf{27}, & C\!: & 3\times\mathbf{27}, \\ D\!: & 15\times\mathbf{27}, & \bar{A}\!: & 3\times\mathbf{27}, & \bar{B}\!: & 3\times\mathbf{27}, & \bar{C}\!: & 3\times\mathbf{27}, \end{array}$$

and singlets. Then we have 36 generations of matter. This number is coincident with half of the Euler number χ of this orbifold, where $\chi = \sum_{[\theta,\phi]=0}\chi_{\theta,\phi}$ [19, 20]. Since the generation number of the $\mathbb{Z}_3\times\mathbb{Z}_3$ orbifold model on the factorizable torus is 84, we observe the generation number is decreased in the non-factorizable model.

4.4.2 An $SO(10)$ GUT-like model

We have seen that the $\mathbb{Z}_3\times\mathbb{Z}_3$ orbifold on the E_6 torus has phenomenologically interesting feature that it has three fixed tori in the θ, ϕ and $\theta\phi$-twisted sectors respectively. We present an example of the models with $SO(10)$ GUT group. Our assumption is very simple, that is the $E_8\times E_8'$ heterotic string from the $\mathbb{Z}_3\times\mathbb{Z}_3$ orbifold on the E_6 torus and the non-standard gauge embeddings,

$$\begin{aligned} V &= \left(\frac{1}{3},\frac{1}{3},\frac{2}{3},0,0,0,0,0\right)\left(\frac{1}{3},\frac{1}{3},0,0,0,0,0,0\right), \\ W &= \left(-\frac{2}{3},0,0,0,0,0,0,0\right)\left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,0\right). \end{aligned} \tag{4.4.4}$$

The surviving gauge group in d=4 is

$$SO(10)\times SU(2)\times U(1)^2\times[SU(7)\times U(1)^2]'. \tag{4.4.5}$$

The $U(1)$ directions [81] are

$$\begin{aligned} Q_1 &= 6(1,0,0,0,0,0,0,0)(0^8), \\ Q_2 &= 6(0,1,-1,0,0,0,0,0)(0^8), \\ Q_3 &= 6(0^8)(1,1,0,0,0,0,0,0), \\ Q_4 &= 6(0^8)(1,0,1,1,1,0,0,0), \end{aligned} \tag{4.4.6}$$

and the linear combination $Q_A = 3Q_1 - 2Q_2 - Q_3 + 3Q_4$ is anomalous. The matter content is

$$3\times(\mathbf{16},\mathbf{1}) + \text{vector-like}. \tag{4.4.7}$$

Then we have the three-family matter of the $SO(10)$ GUT model. We also have Higgs states, $(\mathbf{10},\mathbf{1})$, however no adjoint Higgs, see table 4.2.

	untwisted sector		twisted sector			
		representation		representation		representation
visible sector	U_3	$(\mathbf{16},\mathbf{1})_{3,6,0,0}$	A	$3(\mathbf{16},\mathbf{1})_{-1,-2,4,2}$	$\bar{B}$	$3(\mathbf{10},\mathbf{1})_{-2,0,-2,-8}$
	U_2	$(\overline{\mathbf{16}},\mathbf{1})_{-3,6,0,0}$	$\bar{A}$	$3(\mathbf{10},\mathbf{1})_{-2,-4,-4,-2}$	B	$3(\mathbf{1},\mathbf{2})_{2,-6,2,8}$
	U_1	$(\mathbf{10},\mathbf{2})_{0,6,0,0}$	$\bar{A}$	$3(\mathbf{1},\mathbf{2})_{-2,2,-4,-2}$	$\bar{C}$	$3(\mathbf{1},\mathbf{2})_{2,2,6,-4}$
	U_2	$(\mathbf{1},\mathbf{2})_{6,6,0,0}$				
	U_3	$(\mathbf{1},\mathbf{2})_{-6,6,0,0}$				
messenger sector	U_1	$\mathbf{1}_{0,-12,0,0}$	C	$3(\mathbf{1},\mathbf{2})(\mathbf{7})'_{-2,-2,0,-2}$	D	$27\times\mathbf{1}_{0,4,8,18}$
	U_2	$\mathbf{1}_{0,0,12,6}$	C	$3\times\mathbf{1}_{4,4,6,10}$	D	$15\times\mathbf{1}_{0,-8,2,-6}$
			A	$3\times\mathbf{1}_{-4,4,4,2}$	B	$3\times\mathbf{1}_{-4,0,2,8}$
			A	$3\times\mathbf{1}_{-4,4,-8,-4}$		
hidden sector	U_3	$(\mathbf{35})'_{0,0,0,-6}$	B	$3(\bar{\mathbf{7}})'_{-4,0,2,-4}$	D	$6(\mathbf{7})'_{0,4,-4,0}$
	U_1	$(\overline{\mathbf{21}})'_{0,0,6,0}$	$\bar{C}$	$3(\bar{\mathbf{7}})'_{-4,-4,0,2}$	D	$6(\bar{\mathbf{7}})'_{0,4,2,6}$
	U_2	$(\mathbf{7})'_{0,0,-6,-12}$				
	U_3	$(\mathbf{7})'_{0,0,0,12}$				
	U_2	$(\bar{\mathbf{7}})'_{0,0,-6,6}$				

Table 4.2: All of the massless chiral states of the $SO(10)\times SU(2)\times U(1)^2\times[SU(7)\times U(1)^2]'$ model. The entries of the representations are given as No.$\times$(Repts.)$_{Q_1,Q_2,Q_3,Q_4}$.

We observe in the $\theta\phi$-twisted sector three states charged with both the visible and hidden sectors appear. We call these states as messenger sector, because these messenger states have the potential to mediate the SUSY breaking effect by the strong dynamics of the hidden sector. The running coupling of $SU(7)'$ at scale μ is

$$\frac{1}{\alpha'_{GUT}} = \frac{1}{\alpha'(\mu)} - \frac{b}{2\pi}\ln\left|\frac{M_{GUT}}{\mu}\right|. \tag{4.4.8}$$

If all the $(\mathbf{7})$ and $(\bar{\mathbf{7}})$ generate large mass terms, we have $-b = 8$ for $SU(7)'$. The confining scale is defined by the scale μ where $\alpha'(\mu) = 1$. If we have $M_{GUT} = 2\times 10^{16}$ and $\alpha'_{GUT} = 1/25$, the hidden sector scale is estimated as

$$\Lambda_{hidden} \sim 1.3\times 10^{8}\text{GeV}. \tag{4.4.9}$$

This leads to confinement and gaugino condensation. The hidden sector of this model may cause the gauge mediated SUSY breaking [68, 93–96].

4.4.3 An $SU(5)$ GUT-like model

We also construct an $SU(5)$ model. We take the gauge embeddings[5],

$$\begin{aligned} V &= \left(0,0,0,0,0,\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)\left(0,0,0,0,0,0,\frac{1}{3},\frac{1}{3}\right), \\ W &= \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6},\frac{5}{6},-\frac{5}{6},\frac{5}{6}\right)\left(\frac{2}{3},\frac{1}{3},\frac{1}{3},0,0,0,0,0\right). \end{aligned} \tag{4.4.10}$$

The gauge group of this model is

$$SU(5) \times SU(2)_L \times SU(2)_R \times U(1)^2 \times [SU(6) \times SU(3) \times U(1)]'. \tag{4.4.11}$$

The $U(1)$ directions are

$$\begin{aligned} Q_1 &= 3(0,0,0,-1,1,2,0,0)(0^8), \\ Q_2 &= 3(0,0,0,0,0,2,-1,1)(0^8), \\ Q_3 &= 3(0^8)(0,0,0,0,0,0,1,1), \end{aligned} \tag{4.4.12}$$

and the linear combination $Q_A = 8Q_1 - 3Q_2 + 2Q_3$ is anomalous.

We have totally three chiral **10**s and $\bar{\mathbf{5}}$s of $SO(10)$, which correspond to the spectrum of the Standard Model. Then the states, $(\mathbf{5},\mathbf{1},\mathbf{1})$ and $(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$, have just the quantum numbers of higgs, see table 4.3.

By counting the hidden sector states charged with $SU(6)'$, and if pairs of **6** and $\bar{\mathbf{6}}$ generate large mass terms we have $-b = 10$ for $SU(6)'$. The confining scale is estimated as

$$\Lambda_{hidden} \sim 5.6 \times 10^9 \text{GeV}. \tag{4.4.13}$$

This value is compatible with the hidden sector scale of SUSY breaking mediation scenario.

4.4.4 Yukawa coupling

We discuss three point functions of twisted state, which give rise to Yukawa couplings in superpotential [97–101]. A trilinear coupling of a boson and two fermions, $\phi\psi\psi$, corresponds to a three point correlation function of twisted states

$$\langle V^B V^F V^F \rangle, \tag{4.4.14}$$

where V^B and V^F are the vertex operators of a boson and a fermion, respectively.

The ground states of the α-twisted state localize at f_α are defined by twisted state $|\sigma_{\alpha,f_\alpha}\rangle$. The twisted ground state $|\sigma_{\alpha,f_\alpha}\rangle = \sigma(0)_{\alpha,f_\alpha}|0\rangle$ is annihilated by all the positive frequency mode operators

$$\alpha_{m-k/N}|\sigma_{\alpha,f_\alpha}\rangle = 0, \quad \bar{\alpha}_{m+k/N}|\sigma_{\alpha,f_\alpha}\rangle = 0, \quad m > 0. \tag{4.4.15}$$

[5]It may seem that this shift can be reduced to more simple form by adding lattice vectors. However it generally leads to a different model. This freedom of shift vectors is related to the discrete torsions [60].

	untwisted sector		twisted sector			
		representation		representation		representation
visible sector	U_2	$(\mathbf{5},\mathbf{1},\mathbf{1})_{0,-6,0}$	$\bar{A}$	$3(\mathbf{10},\mathbf{1},\mathbf{1})_{-2,0,-2}$	A	$3(\mathbf{1},\mathbf{1},\mathbf{2})_{2,3,-4}$
	U_3	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})_{6,0,0}$	A	$3(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})_{-4,0,2}$	A	$3(\mathbf{1},\mathbf{1},\mathbf{2})_{2,3,2}$
	U_2	$(\mathbf{10},\mathbf{1},\mathbf{2})_{0,3,0}$	A	$3(\mathbf{1},\mathbf{2},\mathbf{1})_{5,0,2}$	$\bar{A}$	$3(\mathbf{1},\mathbf{1},\mathbf{2})_{-2,-3,-2}$
	U_3	$(\overline{\mathbf{10}},\mathbf{2},\mathbf{1})_{-3,0,0}$				
	U_1	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{2})_{-6,-3,0}$				
	U_1	$(\mathbf{5},\mathbf{2},\mathbf{1})_{3,6,0}$				
	U_1	$(\mathbf{1},\mathbf{2},\mathbf{2})_{3,-3,0}$				
	U_2	$(\mathbf{1},\mathbf{2},\mathbf{1})_{-9,-6,0}$				
	U_3	$(\mathbf{1},\mathbf{1},\mathbf{2})_{6,9,0}$				
messenger sector	U_1	$\mathbf{1}_{0,0,-6}$	B	$3(\mathbf{1},\mathbf{2},\mathbf{1})(\mathbf{1},\mathbf{3})'_{2,4,0}$	C	$3(\mathbf{1},\mathbf{2},\mathbf{1})(\mathbf{6},\mathbf{1})'_{-3,-2,-1}$
			$\bar{B}$	$3(\mathbf{1},\mathbf{1},\mathbf{2})(\mathbf{1},\bar{\mathbf{3}})'_{-4,-1,0}$	A	$3\times\mathbf{1}_{-4,-6,2}$
hidden sector	U_1	$(\mathbf{20},\mathbf{1})'_{0,0,3}$	D	$15(\bar{\mathbf{6}},\mathbf{1})'_{-2,2,-1}$	$\bar{C}$	$3(\mathbf{1},\bar{\mathbf{3}})'_{-6,-4,-2}$
	U_3	$(\mathbf{15},\mathbf{3})'_{0,0,0}$	D	$6(\mathbf{1},\bar{\mathbf{3}})'_{-2,2,2}$		
	U_2	$(\mathbf{6},\bar{\mathbf{3}})'_{0,0,-3}$				

Table 4.3: All of the massless chiral states of the $SU(5)\times SU(2)_L\times SU(2)_R\times U(1)^2\times [SU(6)\times SU(3)\times U(1)]'$ model. The entries of the representations are given as No.$\times$(Repts.)$_{Q_1,Q_2,Q_3}$.

Then we have vertex operators of a twisted state by use of twisted state $\sigma_{\alpha,f_\alpha}(z,\bar{z})$ and untwisted vertex operators V_{untw}:

$$V(z,\bar{z}) = V_{untw}(z,\bar{z})\ \sigma_{\alpha,f_\alpha}(z,\bar{z}). \tag{4.4.16}$$

In (4.4.14) calculation of the untwisted part is straightforward. For twisted sector ground state we have to estimate the expectation value

$$\mathcal{Z} \equiv \langle \sigma_{\alpha,f_\alpha}\sigma_{\beta,f_\beta}\sigma_{\gamma,f_\gamma}\rangle. \tag{4.4.17}$$

The string coordinates can be divided to the classical part

$$Z_{cl}(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \theta Z_{cl}(z,\bar{z}) + v, \tag{4.4.18}$$

and the quantum part

$$Z_{qu}(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = \theta Z_{qu}(z,\bar{z}). \tag{4.4.19}$$

The action is given by

$$S = \frac{1}{\pi}\int dz^2 \left(\partial Z^i \bar{\partial} Z^{\bar{i}} + \bar{\partial} Z^i \partial Z^{\bar{i}}\right), \tag{4.4.20}$$

where we take the sum for the complex coordinates $i = 1,2,3$ of the compact space. The quantum contribution is independent of the classical part, and the correlation function factorizes as

$$\mathcal{Z} = \mathcal{Z}_{qu} \sum_{\langle X_{cl} \rangle} e^{-S_{cl}}, \tag{4.4.21}$$

where we take all the contribution of X_{cl} which satisfies all the constraint. Since the equations of motion

$$\frac{\partial^2 Z^i}{\partial z \partial \bar{z}} = 0, \tag{4.4.22}$$

are functions of z or $\bar{z}$ alone, Z^i split into a holomorphic and an antiholomorphic pieces. The holomorphic part is

$$Z_R = z_R + \frac{i}{2} \sum_{n=1}^{\infty} \frac{\beta_{n-k/N}}{n-k/N} z^{n-k/N} - \frac{i}{2} \sum_{n=0}^{\infty} \frac{\gamma^\dagger_{n+k/N}}{n+k/N} z^{n+k/N}, \tag{4.4.23}$$

where z_R is a fixed point. For $z \to 0$

$$\partial Z |\sigma_{k/N, z_R}\rangle \to -\frac{i}{2} z^{k/N-1} \gamma^\dagger_{n+k/N} |\sigma_{k/N, z_R}\rangle, \tag{4.4.24}$$

then we can read the behavior of these operators near the twist operators. Similarly we have the OPEs as follows,

$$\begin{aligned}
\partial Z \sigma_{k/N}(w, \bar{w}) &\sim (z-w)^{-(1-k/N)} \tau_{k/N}(w, \bar{w}), \\
\partial \bar{Z} \sigma_{k/N}(w, \bar{w}) &\sim (z-w)^{-k/N} \tau'_{k/N}(w, \bar{w}), \\
\bar{\partial} Z \sigma_{k/N}(w, \bar{w}) &\sim (\bar{z}-\bar{w})^{-k/N} \tilde{\tau}_{k/N}(w, \bar{w}), \\
\bar{\partial} \bar{Z} \sigma_{k/N}(w, \bar{w}) &\sim (\bar{z}-\bar{w})^{-(1-k/N)} \tilde{\tau}'_{k/N}(w, \bar{w}).
\end{aligned} \tag{4.4.25}$$

The classical solutions with the correct behavior at the twist insertion points $z \to z_1, z_2, z_3$ are of the form

$$\begin{aligned}
\partial Z^i &= a^i (z-z_1)^{-(1-k/N)} (z-z_2)^{-(1-l/N)} (z-z_3)^{-(k/N+l/N)}, \\
\bar{\partial} \bar{Z}^i &= \bar{a}^i (\bar{z}-\bar{z}_1)^{-(1-k/N)} (\bar{z}-\bar{z}_2)^{-(1-l/N)} (\bar{z}-\bar{z}_3)^{-(k/N+l/N)}, \\
\bar{\partial} Z^i &= b^i (\bar{z}-\bar{z}_1)^{-k/N} (\bar{z}-\bar{z}_2)^{-l/N} (\bar{z}-\bar{z}_3)^{-1+(k/N+l/N)}, \\
\partial \bar{Z}^i &= \bar{b}^i (z-z_1)^{-k/N} (z-z_2)^{-l/N} (z-z_3)^{-1+(k/N+l/N)},
\end{aligned} \tag{4.4.26}$$

where a^i and b^i, $i = 1, 2, 3$, are constants. Since the action with non-zero b_i diverges $S_{cl} \to \infty$, these solution do not contribute to the Yukawa coupling. So we set b_i to be zero. The remaining solutions have the form of the Schwartz-Christoffel map from z to Z:

$$\partial Z = a(z-z_1)^{-\frac{\alpha_1}{\pi}} (z-z_2)^{-\frac{\alpha_2}{\pi}} \cdots (z-z_{n-1})^{-\frac{\alpha_{n-1}}{\pi}}, \tag{4.4.27}$$

with $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = 2\pi$. The function $Z(z)$ maps the upper half plane of z onto the polygon which the turning angle of α_i at the vertexes.

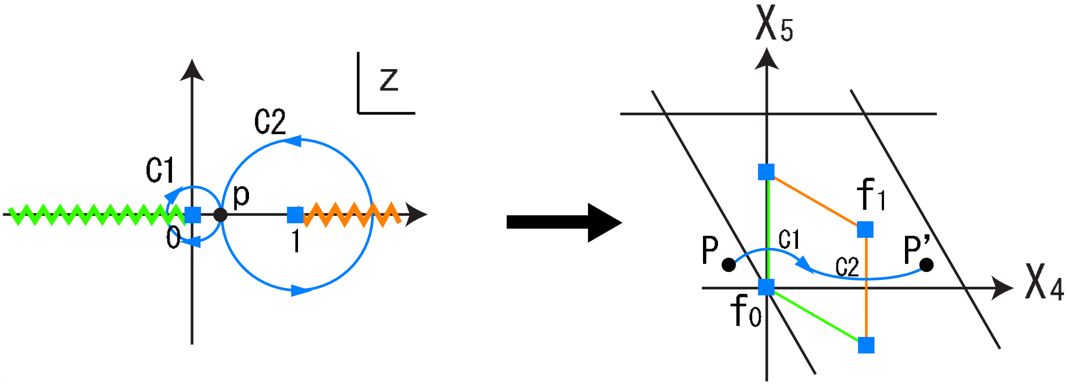

Figure 4.5: *The Schwartz-Christoffel map from the complex plane z to $Z = X_4 + iX_5$. The wavy lines are cut of the Riemann surface, which corresponds to the colored lines on the Z plane.*

Using $SL(2,\mathbb{C})$ symmetry on the world sheet, in (4.4.26) we take

$$z_1 = 0, \quad z_2 = 1, \quad z_3 = \infty, \tag{4.4.28}$$

and normalize $(z - z_3)^{-(k/N+l/N)}$ in the definition of a^i. The map is

$$\partial Z^i = a^i z^{-2/3}(z-1)^{-2/3}. \tag{4.4.29}$$

The upper half plane of z is mapped to a triangle in each Z^i plane, and a leaf of the Riemann surface is mapped to the fundamental region of the orbifold, as described in figure 4.5. In this figure we can intuitively see the monodoromy conditions,

$$\int_C dz(\partial Z^i) = v. \tag{4.4.30}$$

We take the path C as it closes on the Rieman surface so that X is shifted by v but not rotated. For example we define $C1$ to go round z_1 clockwise and $C2$ to go round z_2 counterclockwise, and $C = C1 + C2$, see left side of figure 4.5. We calculate the shift v from the products of the space group. From (2.1.4) a space group element associating with a fixed point f is given by (θ, l), with $l = (1-\theta)f$. The coordinates of twisted states at z_1 and z_2 are mapped to the fixed points f_1^i and f_2^i on the target space Z^i, respectively. We can set $(1-\theta)f_1 = 0$ and $\alpha \equiv (1-\theta)f_2$. Then the integration on C corresponds to the space group elements

$$(\theta, \alpha)(\theta^{-1}, 0) = (1, \alpha) \tag{4.4.31}$$

The action of space group element gives a rotation around the fixed point associated to the elements because

$$\begin{aligned} Z' = (\theta, (1-\theta)f)Z &= \theta Z + (1-\theta)f, \\ \to Z' - f &= \theta(Z - f). \end{aligned} \tag{4.4.32}$$

We take an arbitrary point, say p on z and its map P. A movement along $C1$ corresponds to a clockwise rotation by angle of $2\pi/3$ round $Z(z_1)$ on the image plane, and similarly $C2$ to a counterclockwise rotation by angle of $2\pi/3$ round $Z(z_2)$. Then we obtain

$$\begin{aligned}\int_C dz(\partial Z_{cl}) &= \int_C az^{-(1-k/N)}(z-1)^{-(1-l/N)}(-z_\infty)^{-(k/N+l/N)} \\ &= \int -2i(-z_\infty)^{-(k/N+l/N)}\sin(kl\pi/N)\frac{\Gamma(k/N)\Gamma(l/N)}{\Gamma((k+l)/N))} \\ &= v. \end{aligned} \tag{4.4.33}$$

The coefficient a is determined as

$$a = \frac{iv(-z_\infty)^{(k/N+l/N)}\Gamma(k/N)\Gamma(l/N)}{2\sin(kl\pi/N)\Gamma((k+l)/N))}. \tag{4.4.34}$$

Inserting these to the action, the classical action is given by

$$S_{cl} = \frac{|v|^2|\sin(k\pi/N)||\sin(l\pi/N)|}{4\pi\sin^2(kl\pi/N)|\sin((k+l)\pi/N)|}. \tag{4.4.35}$$

This equation means the instanton effect generates exponentially suppressed interaction.

We also see that the classical solution gives the map from world sheet to orbifold space, and by monodoromy condition we tune the map to fit to the geometry of orbifold on target space. The classical string action is equivalent to the Nambu-Goto action

$$\begin{aligned} S_{cl} = S_{NG} &= -T\int_C d\tau d\sigma\sqrt{-\det\left(\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X_\mu}{\partial\sigma^b}\right)}, \\ &= \int_C\sqrt{-\frac{1}{2}dS^{\mu\nu}dS_{\mu\nu}} = \int_C dS, \end{aligned} \tag{4.4.36}$$

where

$$dS^{\mu\nu} \equiv d\tau d\sigma\left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\tau} - \frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\sigma}\right) = dX^\mu\wedge dX^\nu \tag{4.4.37}$$

is the area element. Then the action gives precisely the area $Z(C)$ formed by fixed points.

4.4.5 Three point functions

Finally we consider the Yukawa couplings of the $\mathbb{Z}_3\times\mathbb{Z}_3$ orbifold on the E_6 torus. To see the allowed interactions, we should take account of the H-momentum conservation and the space group selection rule. In order to see the general structure of the interactions of the orbifold, we ignore its gauge groups in the following. The constraint obtained here is common to models for any gauge embedding of the $\mathbb{Z}_3\times\mathbb{Z}_3$ orbifold on the E_6 torus.

		location	shift			location	shift
θ-sector	A_0	$(0,0,0,0,x,y)$	0	θ^2-sector	$\bar{A}_0$	$(0,0,0,0,x,y)$	0
$\frac{1}{3}(0,1,2,0)$	A_1	$(0,0,0,\frac{1}{\sqrt{3}},x,y)$	α	$\frac{1}{3}(0,2,1,0)$	$\bar{A}_1$	$(0,\frac{1}{\sqrt{3}},0,0,x,y)$	α
	A_2	$(0,\frac{1}{\sqrt{3}},0,0,x,y)$	$-\alpha$		$\bar{A}_2$	$(0,0,0,\frac{1}{\sqrt{3}},x,y)$	$-\alpha$
ϕ-sector	B_0	$(x,y,0,0,0,0)$	0	ϕ^2-sector	$\bar{B}_0$	$(x,y,0,0,0,0)$	0
$\frac{1}{3}(0,0,1,2)$	B_1	$(x,y,0,0,0,\frac{1}{\sqrt{3}})$	α	$\frac{1}{3}(0,0,2,1)$	$\bar{B}_1$	$(x,y,0,\frac{1}{\sqrt{3}},0,0)$	α
	B_2	$(x,y,0,\frac{1}{\sqrt{3}},0,0)$	$-\alpha$		$\bar{B}_2$	$(x,y,0,0,0,\frac{1}{\sqrt{3}})$	$-\alpha$
$\theta\phi$-sector	C_0	$(0,0,x,y,0,0)$	0	$\theta^2\phi^2$-sector	$\bar{C}_0$	$(0,0,x,y,0,0)$	0
$\frac{1}{3}(0,1,0,2)$	C_1	$(0,0,x,y,0,\frac{1}{\sqrt{3}})$	α	$\frac{1}{3}(0,2,0,1)$	$\bar{C}_1$	$(0,\frac{1}{\sqrt{3}},x,y,0,0)$	α
	C_2	$(0,\frac{1}{\sqrt{3}},x,y,0,0)$	$-\alpha$		$\bar{C}_2$	$(0,0,x,y,0,\frac{1}{\sqrt{3}})$	$-\alpha$

Table 4.4: The H-momenta of the bosonic states, the coordinates and the shifts of the θ, ϕ and $\theta\phi$-sectors.

We distinguish the individual states in each twisted sector, A, $\bar{A}$, B, $\bar{B}$, C, $\bar{C}$ and D, by its space group. For example in the θ-twisted sector (labeled as A), we can assign a state which localize at f_θ a space group element (θ, l). l is the shift of the state defined by

$$l = (\mathbb{1} - \theta) f_\theta \quad (\mathrm{mod}\ (\mathbb{1} - \theta)\Lambda), \tag{4.4.38}$$

where we have $l \in \Lambda$ by the definition. The space group corresponds to the boundary condition of the closed string as $X(2\pi) = (\theta, l)X(0) = \theta X(0) + l$. Therefore the boundary condition of the string interaction of three twisted states are

$$\begin{aligned} X(2\pi) &= (\theta_1, l_1)(\theta_2, l_2)(\theta_3, l_3)X(0), \\ &= \theta_1\theta_2\theta_3 X(0) + l_1 + \theta_1 l_2 + \theta_1\theta_2 l_3. \end{aligned} \tag{4.4.39}$$

Then we have the space group selection rules,

$$\begin{aligned} &\theta_1\theta_2\theta_3 = I, \\ &l_1 + \theta_1 l_2 + \theta_1\theta_2 l_3 = 0. \end{aligned} \tag{4.4.40}$$

We call the former equation the point group selection rule. Since the shifts are defined up to the sublattice $(\mathbb{1} - \theta)\Lambda$, the latter condition is simplified to

$$l_1 + l_2 + l_3 = 0 \quad (\mathrm{mod}\ \sum_{i=1}^{3}(\mathbb{1} - \theta_i)\Lambda). \tag{4.4.41}$$

Actually these are the conditions required for non-zero correlation functions. The three point interactions consistent with the point group selection rule and the H-momentum conservation [70, 72, 76] are

$$\begin{aligned} &U_1U_2U_3,\ \ U_3A\bar{A},\ \ U_1B\bar{B},\ \ U_2C\bar{C},\ \ DDD, \\ &\bar{A}BD,\ \ ACD,\ \ \bar{B}\bar{C}D,\ \ AB\bar{C},\ \ \bar{A}\bar{B}C, \end{aligned} \tag{4.4.42}$$

	location	shift		location	shift		location	shift
D_{00}	$(0,0,0)$	0	D_{01}	$(-,+,0)$	α_2 -α_4	D_{02}	$(+,-,0)$	-α_2 -α_4
D_{10}	$(-,0,0)$	α_2	D_{11}	$(0,-,0)$	α_4	D_{12}	$(0,0,-)$	-α_2 -α_4
D_{20}	$(+,0,0)$	-α_2	D_{21}	$(0,+,0)$	-α_4	D_{22}	$(0,0,+)$	α_2+α_4
$D^{'}_{00}$	(a,a,a)	-α_3	$D^{'}_{01}$	(a,b,c)	-α_3+α_2 -α_4	$D^{'}_{02}$	(a,c,b)	-α_3 -α_2+α_4
$D^{'}_{10}$	(b,a,a)	-α_3+α_2	$D^{'}_{11}$	(a,b,a)	-α_3+α_4	$D^{'}_{12}$	(a,a,b)	-α_3 -α_2 -α_4
$D^{'}_{20}$	(c,a,a)	-α_3 -α_2	$D^{'}_{21}$	(a,c,a)	-α_3 -α_4	$D^{'}_{22}$	(a,a,c)	-α_3+α_2+α_4
$D^{''}_{00}$	$(\bar{a},\bar{a},\bar{a})$	α_3	$D^{''}_{01}$	$(\bar{a},\bar{b},\bar{c})$	α_3+α_2 -α_4	$D^{''}_{02}$	$(\bar{a},\bar{c},\bar{b})$	α_3 -α_2+α_4
$D^{''}_{10}$	$(\bar{b},\bar{a},\bar{a})$	α_3+α_2	$D^{''}_{11}$	$(\bar{a},\bar{b},\bar{a})$	α_3+α_4	$D^{''}_{12}$	$(\bar{a},\bar{a},\bar{b})$	α_3 -α_2 -α_4
$D^{''}_{20}$	$(\bar{c},\bar{a},\bar{a})$	α_3 -α_2	$D^{''}_{21}$	$(\bar{a},\bar{c},\bar{a})$	α_3 -α_4	$D^{''}_{22}$	$(\bar{a},\bar{a},\bar{c})$	α_3+α_2+α_4

Table 4.5: The coordinates and the shifts of 27 states in the $\theta\phi^2$-sector. The H-momentum is $\frac{1}{3}(0,1,1,2)$ for all states. On this table we use abbreviations in the coordinates as follows, $+ = (\frac{1}{\sqrt{3}},0)$, $- = (-\frac{1}{\sqrt{3}},0)$, $a = (\frac{1}{3},0)$, $b = (-\frac{1}{6},\frac{1}{2\sqrt{3}})$ and $c = (-\frac{1}{6},-\frac{1}{2\sqrt{3}})$. And the bars on the characters mean negative entries of them, i.e., $\bar{a} \equiv -a$. For example we have a coordinate $(a,b,c) = (\frac{1}{3},0,-\frac{1}{6},\frac{1}{2\sqrt{3}},-\frac{1}{6},-\frac{1}{2\sqrt{3}})$. Note that under the identification of $\theta\phi^2$ we observe $(a,a,a) = (b,b,b) = (c,c,c)$, etc.

where U_1, U_2 and U_3 are untwisted sector states. We should also take account of the constraint from (4.4.41).

At first we consider the coupling of $\bar{A}BD$. The sum of the sublattices generated by the action of θ, ϕ and $\theta\phi^2$ is

$$\begin{aligned}&(\mathbb{1}-\theta)\Lambda + (\mathbb{1}-\phi)\Lambda + (\mathbb{1}-\theta\phi^2)\Lambda \\ &= \{\alpha_1-\alpha_2,\ \alpha_2-\alpha_4,\ \alpha_4-\alpha_5,\ \alpha_4-\alpha_6,\ \alpha_6-\alpha_0,\ 3\alpha_2\}.\end{aligned} \tag{4.4.43}$$

Therefore in the space group selection rule of $\bar{A}BD$, we can identify

$$\alpha \equiv \alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_0, \quad 3\alpha = 3\alpha_3 \simeq 0, \tag{4.4.44}$$

up to the sublattices. The last identity is obtained from the definition of α_0, however we have $\alpha \neq \alpha_3$. The space group elements of the three states in the θ-twisted sector are given as $(\theta,0)$, (θ,α), $(\theta,-\alpha)$ respectively, and this is similar for the ϕ and $\theta\phi$-sectors. The shifts and the coordinates of these states are listed on table 4.4. The states in the $\theta\phi^2$-sectors are on the table 4.5. According to the space group selection rule $\bar{A}BD$ couplings are allowed for D states with no α_3 shift, that is D_{ij}, $i,j = 0,1,2$. The couplings with $D^{'}_{ij}$ and $D^{''}_{ij}$ are forbidden. Then in the interactions of $AB\bar{C}$, the coupling of

$$\bar{A}_i B_j D_{kl}, \quad i+j+k = 0 \pmod 3,\ l = 0,1,2, \tag{4.4.45}$$

can be non-zero values. The interaction is allowed for any l, and such interactions do not appear in factorizable orbifold models. In this meaning $\bar{A}BD$ couplings include nontrivial structures[6]. This is the same for the couplings of ACD and $\bar{B}\bar{C}D$. This interaction is generated because three fixed points of D are on the same fixed tori of $\bar{A}$ or B. Some of these are not contact interactions, and generated by the effect of the world sheet instanton [70]. Then the couplings of the three point interactions are suppressed by instanton action ε. The suppression factor relates to the distance of the three associated fixed points, and the dominant contribution is from the instanton effects on the minimum distance of them. Taking it consideration we can estimate the coupling of three point functions. Suppose B_j and D_{2l} are interpreted as the three-family matter states and $\bar{A}_i$ is Higgs H_i $(i = 0, 1, 2)$, the following interactions are allowed,

$$\bar{A}_0 B_1 D_{2l}, \quad \bar{A}_1 B_0 D_{2l}, \quad \bar{A}_2 B_2 D_{2l}, \quad l = 0, 1, 2. \tag{4.4.46}$$

Then the Yukawa matrix is

$$Y = \begin{pmatrix} H_1 & \varepsilon H_1 & \varepsilon H_1 \\ \varepsilon H_0 & \varepsilon H_0 & H_0 \\ \varepsilon H_2 & H_2 & \varepsilon H_2 \end{pmatrix}, \tag{4.4.47}$$

with order one factor from quantum correction. So in this case we can realize the Yukawa matrix with mixing! It is also notable that three generation structure naturally arises in the $Z_3 \times Z_3$ orbifold on the E_6 torus.

Next we consider the couplings of $AB\bar{C}$. The sum of the sublattices

$$\begin{aligned} &(\mathbb{1} - \theta)\Lambda + (\mathbb{1} - \phi)\Lambda + (\mathbb{1} - \theta\phi)\Lambda \\ &= \{\alpha_1 - \alpha_2,\ \alpha_2 - \alpha_4,\ \alpha_4 - \alpha_5,\ \alpha_4 - \alpha_6,\ \alpha_6 - \alpha_0,\ 3\alpha_2\} \end{aligned} \tag{4.4.48}$$

is the same as (4.4.43), so we can use the relations (4.4.44) in this case. Then the coupling of

$$A_i B_j \bar{C}_k, \quad i + j + k = 0 \pmod 3, \tag{4.4.49}$$

can be non-zero values. Assuming that A_i and B_j are the three-family matter states and $\bar{C}_k$ is Higgs H_k $(k = 1, 2, 3)$, the Yukawa matrix is given by

$$Y = \begin{pmatrix} H_0 & H_2 & H_1 \\ H_2 & H_1 & H_0 \\ H_1 & H_0 & H_2 \end{pmatrix}, \tag{4.4.50}$$

with order one factor from the quantum correction. These are contact interactions and not suppressed by ε. We have the same structures for $\bar{A}\bar{B}C$.

Finally we consider the interactions of DDD type. One complication is that the fixed points in table 4.5 are not invariant by the actions of θ and ϕ. Then we should take the linear combinations of these states in order to have the eigenstates of the orbifold actions as follows,

$$\begin{aligned} D_j^{'k} &\equiv D_{j0}^{'} + \omega^k D_{j1}^{'} + \omega^{2k} D_{j2}^{'}, \\ D_j^{''k} &\equiv D_{j0}^{''} + \omega^k D_{j1}^{''} + \omega^{2k} D_{j2}^{''}, \end{aligned} \tag{4.4.51}$$

[6] We observed similar phenomena in $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on non-factorizable lattices [46].

where $\omega = e^{2\pi/3i}$ and j,k=0,1,2. The allowed interactions by the selection rule (4.4.41) in terms of α_3 are DDD, $D^{'}D^{'}D^{'}$, $D^{''}D^{''}D^{''}$ and $DD^{'}D^{''}$ types. It is straightforward to see the following self-interactions can exist

$$D_{ij}D_{ij}D_{ij},\ D^{'}_{ij}D^{'}_{ij}D^{'}_{ij},\ D^{''}_{ij}D^{''}_{ij}D^{''}_{ij}, \tag{4.4.52}$$

here $i, j = 0, 1, 2$ and we do not sum up their indices. These are contact interactions, so the couplings are of order one. There are other couplings for

$$D_{0i}D_{1j}D_{2k},\quad D^{'i}_{0}D^{'j}_{1}D^{'k}_{2},\quad D^{''i}_{0}D^{''j}_{1}D^{''k}_{2},\quad i+j+k=0 \pmod 3. \tag{4.4.53}$$

These are not contact interactions, and the minimum distance associated to the instanton effect are the same for the interactions above. If the state D_{0i} for $i = 0, 1, 2$ is higgs, we have a Yukawa matrix (4.4.50) with suppression factor ε. We have also non-zero couplings for

$$D_{il}D^{'1}_{j}D^{''2}_{k},\quad D_{il}D^{'2}_{j}D^{''1}_{k},\quad i+j+k=0 \pmod 3),\ l=0,1,2. \tag{4.4.54}$$

Here we again observe the degenerate interactions for l. These couplings are generated by instantons, and suppressed by the factor of ε. In this case the minimum distances for the instanton effect are the same for all.

We considered the allowed coupling from the space group selection rules here. We have the interactions (4.4.45) and (4.4.54), and these are new structures in the non-factorizable orbifold models. Surprisingly the coupling of $\bar{A}BD$ can have mixing terms between flavors. The 27 fixed states in the $\theta\phi^2$-sector are divided to the interactions between three flavors. For further model construction, there is the possibility to realize three-family states from this sector with the flavor mixing interactions.

Chapter 5

Type IIA Orientifolds

Introducing D-branes [7] provide rich possibilities for Type II string theory. This class of models provide the chiral spectrum and gauge group of the Standard Model [22–32] (for review see [33–35] and references therein). Type IIA models compactified on six-dimensional orbifolds have been constructed with the point group of $\mathbb{Z}_N$ [27, 28], $\mathbb{Z}_2 \times \mathbb{Z}_2$ [21, 29, 30] and $\mathbb{Z}_4 \times \mathbb{Z}_2$ [31, 32], We generalize $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifolds to non-factorizable tori in this chpter.

For the consistency of theory the tadpole cancellation is required (see [24, 50–54], and for review, [55–57]). We explicitly calculate string one-loop amplitudes on the Klein bottle, the annulus and the Möbius strip on non-factorizable tori and orbifolds, and confirm that the consistent number of orientifold planes (O-planes) is directly derived from the Lefschetz fixed point theorem via the cancellations of Ramond-Ramond (RR) tadpole. We give a systematic way to construct various models on non-factorizable orbifolds. Interestingly, we further find new feature of non-factorizable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, in which the numbers of O-planes depend on three-cycles.

5.1 D-brane and Open string

The Hamiltonian is given for strings that stretch between two D-branes that intersect at an angle of θ [114, 115]:

$$\partial_\sigma X^\mu|_{\sigma=0} = 0, \quad \partial_\sigma X^\mu|_{\sigma=\pi} = 0, \quad \mu = 0, 1, 2, 3$$

for non-compact dimensions, and

$$\begin{aligned} Re(e^{-i\theta_i^a}\partial_\sigma Z^i)|_{\sigma=0} = 0, \quad Im(e^{-i\theta_i^a}\partial_\tau Z^i)|_{\sigma=0} = 0, \\ Re(e^{-i\theta_i^b}\partial_\sigma Z^i)|_{\sigma=\pi} = 0, \quad Im(e^{-i\theta_i^b}\partial_\tau Z^i)|_{\sigma=\pi} = 0, \end{aligned} \tag{5.1.1}$$

for compact dimensions. The mode expansions are

$$\begin{aligned} X^\mu &= p^\mu \tau + \sum_{n\in\mathbb{Z}} \frac{1}{n}\alpha^\mu_n e^{-in\tau}\cos n\sigma, \quad \mu = 0,1,2,3 \\ Z^i &= \sum \frac{1}{n+v_i}\alpha^i_{n+v_i} e^{-i(n+v_i)(\tau-\sigma)} + \sum \frac{1}{n-v_i}\alpha^i_{n-v_i} e^{-i(n-v_i)(\tau+\sigma)}, \end{aligned} \tag{5.1.2}$$

where $v_i \equiv (\theta^b_i - \theta^a_i)/\pi$. In the case of parallel D-branes, say $\theta^a_i = \theta^b_i = 0$, we set the boundary condition as $X^{2i}|_{\sigma=0} = 0$ and $X^{2i}|_{\sigma=\pi} = Y^{2i}$. The mode expansions are given by

$$\begin{aligned} X^{2i} &= Y^{2i}\frac{\sigma}{\pi} + \sum_{n\in\mathbb{Z}} \frac{1}{n}\alpha^\mu_n e^{-in\tau}\sin n\sigma, \\ X^{2i+1} &= p^{2i+1}\tau + \sum_{n\in\mathbb{Z}} \frac{1}{n}\alpha^\mu_n e^{-in\tau}\cos n\sigma. \end{aligned} \tag{5.1.3}$$

For the fermionic states, mode expansions are given as

$$\Psi^i = \sum_{r\in\mathbb{Z}+\nu} \left[\psi^i_{n+v_i} e^{-i(n+v_i)(\tau+\sigma)} + \psi^i_{n-v_i} e^{-i(n-v_i)(\tau-\sigma)}\right]. \tag{5.1.4}$$

The GSO projection is implemented by imposing $\sum_i r = \text{odd}$.

The number operators are

$$\begin{aligned} N(\nu) &= \sum_{n>0} \alpha^\mu_{-n}\alpha_{\mu,n} + \sum_{n+v_i>0} \alpha^i_{-n-v_i}\alpha^i_{n+v_i} + \sum_{n-v_i>0} \alpha^i_{-n+v_i}\alpha^i_{n-v_i} \\ &\quad + \sum_{r>0} r\psi^\mu_{-r}\psi_{\mu,r} + \sum_{r+v_i>0} (r+v_i)\psi^i_{-r-v_i}\psi^i_{r+v_i} + \sum_{r-v_i>0} (r-v_i)\psi^i_{-r+v_i}\psi^i_{r-v_i} \end{aligned} \tag{5.1.5}$$

We calculate these quantities in the light-cone gauge. The general mass operator for the string stretching between two D-branes in the distance of Y and angle of θ_i are given by

$$\alpha' M^2_{ab} = \frac{Y^2}{4\pi^2\alpha'} + N_\nu + \nu\left(\sum_i \eta_i - 1\right), \tag{5.1.6}$$

where $0 \le \eta < 1/2$.

5.2 orientifold

We begin from the Type IIB orientifold with O9-planes compactified on a six tori $T^6 = \mathbb{R}^6/\Lambda$. By T-duality we can obtain Type IIA theory. Under the T duality $\mathcal{T}_i$ in the x_i direction, the bosonic coordinates are transformed as

$$\mathcal{T}_i: \ X^i_L \to -X^i_L, \quad X^i_R \to X^i_R, \tag{5.2.1}$$

where Ω is the worldsheet parity operator. For the involution of orientifold it generates inversion of the coordinate,

$$\mathcal{T}_i \Omega \mathcal{T}_i^{-1} = \Omega \mathcal{R}_i \tag{5.2.2}$$

where

$$\Omega \mathcal{R}_i : \ X^i_{L,R} \to -X^i_{L,R}. \tag{5.2.3}$$

Then $\mathcal{R}_i$ is the orientifold involution which indicates the reflection in the x_i direction. We mainly consider the Type IIA models compactified on a six-torus in this paper. Type IIA theory with O6-planes is obtained by

$$\text{Type IIB on } \frac{T^6}{\{1+\Omega\}} \xrightarrow{\mathcal{T}_5\mathcal{T}_7\mathcal{T}_9} \text{Type IIA on } \frac{T^6}{\{1+\Omega\mathcal{R}\}}, \tag{5.2.4}$$

where $\mathcal{R} \equiv \mathcal{R}_5\mathcal{R}_7\mathcal{R}_9$. Usually the action $\mathcal{R}$ can be given as

$$\mathcal{R} : \ z_i \to \bar{z}_i, \quad i = 1,2,3. \tag{5.2.5}$$

In order to construct consistent effective theories in four-dimensional spacetime, we study the tadpole cancellation condition in the presence of orientifolds. The tadpole amplitude is derived from the string one-loop graphs whose topologies are the Klein bottle, the annulus, and the Möbius strip. These amplitudes are represented as $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$,

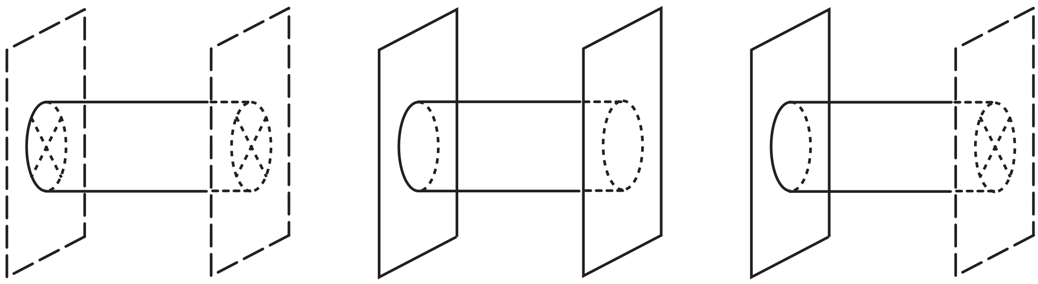

Figure 5.1: *The one-loop graphs of the Klein bottle, Annulus and Möbius strip. The cross indicates the boundary condition given by the cross cap.*

respectively. Here let us explicitly describe their amplitudes in terms of a modulus t in the loop channel as follows:

$$\mathcal{K} = 4c \int_0^\infty \frac{dt}{t^3} \mathrm{Tr}_{\mathrm{closed}} \left(\frac{\Omega\mathcal{R}}{2} \mathbf{P} \left(\frac{1+(-1)^F}{2} \right) (-1)^{\mathbf{s}} e^{-2\pi t \left(L_0 + \bar{L}_0\right)} \right), \tag{5.2.6a}$$

$$\mathcal{A} = c \int_0^\infty \frac{dt}{t^3} \mathrm{Tr}_{\mathrm{open}} \left(\frac{1}{2} \mathbf{P} \left(\frac{1+(-1)^F}{2} \right) (-1)^{\mathbf{s}} e^{-2\pi t L_0} \right), \tag{5.2.6b}$$

$$\mathcal{M} = c \int_0^\infty \frac{dt}{t^3} \mathrm{Tr}_{\mathrm{open}} \left(\frac{\Omega\mathcal{R}}{2} \mathbf{P} \left(\frac{1+(-1)^F}{2} \right) (-1)^{\mathbf{s}} e^{-2\pi t L_0} \right), \tag{5.2.6c}$$

where F and $\mathbf{s}$ denote the fermion numbers in the worldsheet and in the spacetime, respectively; the overall coefficient c is given by $c \equiv V_4/(8\pi^2\alpha')^2$, where V_4 is from the integration over momenta in non-compact directions, and $\mathbf{P}$ is the insertion of orbifold actions. Since the divergence from the RR-tadpole should be evaluated in the tree channel, which is described by the l-modulus, we should rewrite them via the modular transformation, even though the computations of the amplitudes are easier in the loop channel given by t-modulus. The RR-sectors in the tree channel which we should evaluate in order to see the tadpole cancellation in the presence of orientifold planes and D-branes, correspond to the states with the following insertions in the loop channel [49]:

$$\begin{aligned} \text{Klein bottle} &: \text{closed string, NS-NS sector, } (-1)^F \\ \text{annulus} &: \text{open string, R sector} \\ \text{Möbius strip} &: \text{open string, NS sector, } (-1)^F. \end{aligned} \tag{5.2.7}$$

In the next section 5.3 we describe the tadpole cancellation condition on generic non-factorizable tori. In this analysis the Lefschetz fixed point theorem makes the cancellation condition simplified and provides an intuitive picture. We apply this formula to orientifold models which have been already well-investigated. In section 5.4 we explicitly construct Type IIA orientifolds on $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the D_6 Lie root lattice. Because the contributions of untwisted sector in orbifolds are given by the same forms of those in tori, the formula, which is derived from the Lefschetz fixed point theorem, provides a necessary condition on non-factorizable orbifolds. We describe general features of orientifold constructions on non-factorizable orbifolds. In appendix A we summarize a set of useful conventions to describe non-factorizable tori in terms of the lattice space and its dual. In appendix B we briefly review the string one-loop amplitudes which are given by the Klein bottle, the annulus and the Möbius strip as the worldsheet topologies.

5.3 Orientifold on non-factorizable torus

In this section we will evaluate the RR-tadpole cancellation conditions of torus compactification in Type IIA string theory in the presence of D6-branes and orientifold planes (O6-planes). We will show a method to analyze the orientifold models on non-factorizable tori, which can be applied to any kind of torus compactifications. We introduce a set of general formula for the tadpole amplitudes in the RR-sector on non-factorizable tori, which are defined by the Lie root lattices. Utilizing the Lefschetz fixed point theorem, we can check the tadpole cancellation condition not only on the usual factorizable tori but also on the non-factorizable ones in a quite simple way. We will further apply this method to orbifold models in section 5.4.

5.3.1 RR-tadpole and the Lefschetz fixed point theorem

In this section we calculate these amplitudes for the cases that D-branes are parallel on O-planes. Then the amplitudes in (5.2.6) can be written in the form as follows:

$$\mathcal{K} = c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty \frac{dt}{t^3} \frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^4}{\eta^{12}} \mathcal{L}_{\mathcal{K}}, \tag{5.3.1a}$$

$$\mathcal{A} = \frac{c}{4}(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}})\{(\mathrm{tr}(\gamma_1))^2\} \int_0^\infty \frac{dt}{t^3} \frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^4}{\eta^{12}} \mathcal{L}_{\mathcal{A}}, \tag{5.3.1b}$$

$$\mathcal{M} = -\frac{c}{4}(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}})\{\mathrm{tr}(\gamma_{\Omega\mathcal{R}}^{-1}\gamma_{\Omega\mathcal{R}}^T)\} \int_0^\infty \frac{dt}{t^3} \frac{\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4}{\eta^{12}} \mathcal{L}_{\mathcal{M}}, \tag{5.3.1c}$$

where the string oscillation modes are represented with respect to the ϑ-function and the Dedekind η-function, while the zero modes are given by $\mathcal{L}_{\mathcal{K}}$, $\mathcal{L}_{\mathcal{A}}$ and $\mathcal{L}_{\mathcal{M}}$. The γ matrices are orientifold actions on the Chan-Paton factors in the notation of [54]. Due to the spacetime supersymmetry, the total amplitudes from the RR- and NSNS-sectors should be cancelled to each other, as seen the factor $(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}})$ on each amplitude in (5.3.1). The mapping between the two different moduli t and l in these channels is also given as

$$\text{Klein bottle}: \ t = \frac{l}{4}, \qquad \text{annulus}: \ t = \frac{l}{2}, \qquad \text{Möbius strip}: \ t = \frac{l}{8}. \tag{5.3.2}$$

To evaluate the RR-tadpole generated by the orientifold, we extract only the contributions from the RR-sector in the tree channel. In the IR limit $l \to \infty$ the divergence from the RR-tadpole should be cancelled,

$$\tilde{\mathcal{K}}_{\mathrm{RR}} + \tilde{\mathcal{A}}_{\mathrm{RR}} + \tilde{\mathcal{M}}_{\mathrm{RR}} \to 0, \tag{5.3.3}$$

where $\tilde{\mathcal{K}}_{\mathrm{RR}}$, $\tilde{\mathcal{A}}_{\mathrm{RR}}$ and $\tilde{\mathcal{M}}_{\mathrm{RR}}$ are RR-tadpole contributions in the tree channel mapped from $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$ in the loop channel under the modular transformation, respectively.

Now let us evaluate the zero mode contributions $\mathcal{L}_{\mathcal{K},\mathcal{A},\mathcal{M}}$ in (5.3.1) given by the momentum modes and the winding modes. General momenta $\mathbf{p}$ and winding modes $\mathbf{w}$ can be written in terms of a set of certain basis vectors $\{\mathbf{p}_i\}$ and $\{\mathbf{w}_i\}$, respectively:

$$\mathbf{p} = \sum_i n_i \mathbf{p}_i, \qquad \mathbf{w} = \sum_i m_i \mathbf{w}_i, \qquad m_i, n_i \in \mathbb{Z}. \tag{5.3.4}$$

The zero mode contribution to the loop channel amplitudes is

$$\mathcal{L} \equiv \sum_{n_i} \exp\Big(- \delta\pi t n_i M_{ij} n_j \Big) \cdot \sum_{m_i} \exp\Big(- \delta\pi t m_i W_{ij} m_j \Big), \tag{5.3.5}$$

where n_i, $m_i \in \mathbb{Z}$, $M_{ij} = \mathbf{p}_i \cdot \mathbf{p}_j$, $W_{ij} = \mathbf{w}_i \cdot \mathbf{w}_j$ and $\delta = 1$ for Klein bottle, $\delta = 2$ for annulus and Möbius strip. Using the generalized Poisson resummation formula, we can rewrite

$$\sum_{n_i} \exp\Big(- \pi t n_i A_{ij} n_j \Big) = \frac{1}{t^{\frac{\dim(A)}{2}} (\det A)^{\frac{1}{2}}} \sum_{n_i} \exp\Big(- \frac{\pi}{t} n_i A_{ij}^{-1} n_j \Big). \tag{5.3.6}$$

When we move to the tree channel by using (5.3.2), the zero mode contribution $\mathcal{L}$ is

$$\mathcal{L} = \sum_{n_i} \frac{\left(\frac{\alpha l}{\delta}\right)^3}{\sqrt{\det M \det W}} \exp\Big(- \pi \frac{\alpha l}{\delta} t n_i M^{-1}_{ij} n_j \Big) \cdot \sum_{m_i} \exp\Big(- \pi \frac{\alpha l}{\delta} m_i W^{-1}_{ij} m_j \Big), \quad (5.3.7)$$

which goes to $\frac{(\frac{\alpha l}{\delta})^3}{\sqrt{\det M \det W}}$ in the IR limit $l \to \infty$.

We consider a six-torus T^6 on a lattice Λ. Then two different points in T^6 are identified in terms of the lattice shift vector $r\alpha_i \in \Lambda$ as

$$T_{\alpha_i}: \ \mathbf{x} \to \mathbf{x} + r\alpha_i, \quad (5.3.8)$$

where r is a radius of T^6. For simplicity we set $r = 1$ in the following in this paper . Translation operator acting on the momentum states $|\mathbf{p}\rangle$ is given by

$$T_{\alpha_i}|\mathbf{p}\rangle = \exp(2\pi i \mathbf{p} \cdot \alpha_i)|\mathbf{p}\rangle. \quad (5.3.9)$$

Then the momentum modes are expressed by dual vector $\alpha_i^* \in \Lambda^*$,

$$\alpha_i \cdot \alpha_j^* = \delta_{ij}. \quad (5.3.10)$$

In the Klein bottle amplitude, the momentum modes should be invariant under the action of $\Omega\mathcal{R}$. Thus the vector α_i^* consists of the $\mathcal{R}$ invariant sublattice in the dual lattice Λ^*, and we have [41][1]

$$\sqrt{\det M^{\mathcal{K}}} = \mathrm{Vol}(\Lambda^*_{\mathcal{R},\mathrm{inv}}). \quad (5.3.11)$$

In the same way, the winding modes $\mathbf{w}_i$ are given by the lattice vector α_i invariant under the action $-\mathcal{R}$ on the lattice space Λ (with the constant $\alpha' = 1$). Then we obtain

$$\sqrt{\det W^{\mathcal{K}}} = \mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}}). \quad (5.3.12)$$

One of the simplest way to cancel the RR-tadpole of the O6-plane is to add D6-branes parallel to the O6-planes. Since the O6-planes lie on the $\mathcal{R}$ fixed locus, the basis vectors which describe three-cycles of the O6-plane are generated from $\mathcal{R}$-invariant sublattice $\Lambda_{\mathcal{R},\mathrm{inv}}$. Then, in the case of annulus amplitude, the momentum modes are described by the vector in the dual lattice $(\Lambda_{\mathcal{R},\mathrm{inv}})^*$. The winding modes are related to the distances between these D6-branes, and they are the sublattice projected by $-\mathcal{R}$, i.e., $\Lambda_{-\mathcal{R},\perp} \equiv \frac{1-\mathcal{R}}{2}\Lambda$. In the Möbius strip amplitude the momentum modes are the same as ones of the annulus amplitude. On the other hand, the winding modes should be invariant sublattice under $-\Omega\mathcal{R}$, and it is given by $\Lambda_{-\mathcal{R},\mathrm{inv}}$. Summarizing the above, we obtain the following descriptions:

$$\sqrt{\det M^{\mathcal{K}}} = \mathrm{Vol}(\Lambda^*_{\mathcal{R},\mathrm{inv}}), \quad (5.3.13a)$$
$$\sqrt{\det M^{\mathcal{A}}} = \sqrt{\det M^{\mathcal{M}}} = \mathrm{Vol}(\Lambda^*_{\mathcal{R},\perp}), \quad (5.3.13b)$$
$$\sqrt{\det W^{\mathcal{K}}} = \sqrt{\det W^{\mathcal{M}}} = \mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}}), \quad (5.3.13c)$$
$$\sqrt{\det W^{\mathcal{A}}} = \mathrm{Vol}(\Lambda_{-\mathcal{R},\perp}), \quad (5.3.13d)$$

[1]See appendix A for the definition of $\Lambda_{\mathcal{R},\mathrm{inv}}$ and $\Lambda_{\mathcal{R},\perp}$.

where we used the following relations:

$$\Lambda^*_{\mathcal{R},\perp} = (\Lambda_{\mathcal{R},\mathrm{inv}})^*, \qquad \mathrm{Vol}(\Lambda) = \mathrm{Vol}(\Lambda_{\mathcal{R},\mathrm{inv}}) \cdot \mathrm{Vol}(\Lambda_{-\mathcal{R},\perp}). \tag{5.3.14}$$

For the contributions to Chan-Paton factors, we have $\gamma_1 = \mathbb{1}$ so that $\mathrm{tr}(\gamma_1) = N$ is the number of D6-branes. Furthermore we require $\gamma^{-1}_{\Omega\mathcal{R}}\gamma^T_{\Omega\mathcal{R}} = \mathbb{1}$ in order to cancel the RR-tadpole.

Now we are ready to obtain the RR-tadpole cancellation condition. The sum of RR-tadpole contributions for large l is asymptotically

$$\begin{aligned}
&\tilde{\mathcal{K}}_{\mathrm{RR}} + \tilde{\mathcal{A}}_{\mathrm{RR}} + \tilde{\mathcal{M}}_{\mathrm{RR}} \\
&\quad = c\int^\infty dl\,\left(\frac{64}{\sqrt{\det M^{\mathcal{K}}\det W^{\mathcal{K}}}} + \frac{N^2}{16\sqrt{\det M^{\mathcal{A}}\det W^{\mathcal{A}}}} - \frac{4N}{\sqrt{\det M^{\mathcal{M}}\det W^{\mathcal{M}}}}\right) \\
&\quad = c\int^\infty dl\,\frac{1}{16\mathrm{Vol}(\Lambda^*_{\mathcal{R},\perp})\mathrm{Vol}(\Lambda_{-\mathcal{R},\perp})}\left(N - 4N_{\mathrm{O6}}\right)^2,
\end{aligned} \tag{5.3.15}$$

where N_{O6} is the number of the O6-planes according to the Lefschetz fixed point theorem [133]:

$$N_{\mathrm{O6}} \equiv \frac{\mathrm{Vol}((1-\mathcal{R})\Lambda)}{\mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}})} = 2^3 \cdot \frac{\mathrm{Vol}(\Lambda_{-\mathcal{R},\perp})}{\mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}})}. \tag{5.3.16}$$

The equation (5.3.15) indicates that the RR-tadpole is cancelled by D6-branes whose number is four times as many as that of O6-planes. Therefore we find that it is enough to count the number of O6-planes in (5.3.16) instead of calculating individual amplitudes. For factorizable models, we have $\mathrm{Vol}(\Lambda_{-\mathcal{R},\perp})/\mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}}) = 1$. The condition (5.3.15) is also expressed as

$$N\Pi - 4\Pi_{\mathrm{O6}} = 0, \tag{5.3.17}$$

where Π and Π_{O6} denote three-cycles in D6-branes and O6-planes, respectively.

This is the case for O6-planes in Type IIA theory. We can generalize this tadpole cancellation condition to an Oq-plane in type IIA/IIB theory in such a way as

$$\left(N - 2^{q-4}N_{\mathrm{O}q}\right)^2 = 0, \tag{5.3.18}$$

where the number of Oq-planes is given by

$$N_{\mathrm{O}q} \equiv \frac{\mathrm{Vol}((1-\mathcal{R})\Lambda)}{\mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}})} = 2^{9-q} \cdot \frac{\mathrm{Vol}(\Lambda_{-\mathcal{R},\perp})}{\mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}})}. \tag{5.3.19}$$

In the case of an O9-plane,the orientifold action is given by Ω, i.e., $\mathcal{R} = \mathbb{1}$, and the above equation is ill-defined, however we can calculate it in the same way. Then it is appropriate to set $\mathrm{Vol}(\Lambda_{-\mathcal{R},\perp})/\mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}}) = 1$ for O9-plane.

5.3.2 Orientifold models on the Lie root lattices

Here let us first review the Type IIA orientifold on a factorizable torus $T^2 \times T^2 \times T^2$ to fix our notation. There are two ways to implement $\Omega\mathcal{R}$ of (5.2.5) in each T^2. The lattice Λ_i which defines the boundary condition of the i-th T^2 is given by

$$\Lambda_i = \left\{ n_{2i-1}\alpha_{2i-1} + n_{2i}\alpha_{2i} \middle| n_{2i-1}, n_{2i} \in \mathbb{Z} \right\}, \quad i = 1, 2, 3, \tag{5.3.20}$$

where, for simplicity, we set $r = 1$ in (5.3.8); α_j is a simple root of the lattice. Without loss of generality we can define α_{2i} along the x^{2i}-direction for the orientifold action $\Omega\mathcal{R}$ in (5.2.5), which acts crystallographically on the lattice Λ_i. Therefore the complex structure U_i on the i-th torus T^2 should satisfy $\mathcal{R}U_i = U_i$ modulo the shift given by Λ_i. Then there are only two solutions

$$U_i = ia \ \text{ or } \ \frac{1}{2} + ia \ , \qquad a \in \mathbb{R}, \tag{5.3.21}$$

which indicates that there are two distinct lattices for the $\mathcal{R}$ action[2]. The one is called **A**-type lattice [58], whose lattice vector is given by

$$\alpha_1^{\mathbf{A}} = \sqrt{2}\mathbf{e}_1, \quad \alpha_2^{\mathbf{A}} = \sqrt{2}\mathbf{e}_2. \tag{5.3.22}$$

Notice that in this case the complex structure of the torus is given by $U = ia$. The other is called **B**-type lattice, which is given by

$$\alpha_1^{\mathbf{B}} = \mathbf{e}_1 - \mathbf{e}_2, \quad \alpha_2^{\mathbf{B}} = \mathbf{e}_1 + \mathbf{e}_2. \tag{5.3.23}$$

This corresponds to the case $U = \frac{1}{2} + ia$. We can see it by the re-definition of the vector $\alpha_2^{\mathbf{B}} \to -\alpha_1^{\mathbf{B}} + \alpha_2^{\mathbf{B}}$. Then we have two distinct theories which depend on the choice of **A**-type or **B**-type lattices in Figure 5.2. For example, the number of fixed loci given by the action of $\mathcal{R}$ is two (for the **A**-type) and one (for the **B**-type), which associate the total O6-plane charges. Instead of using the **B**-type lattice, we define an equivalent orientifold

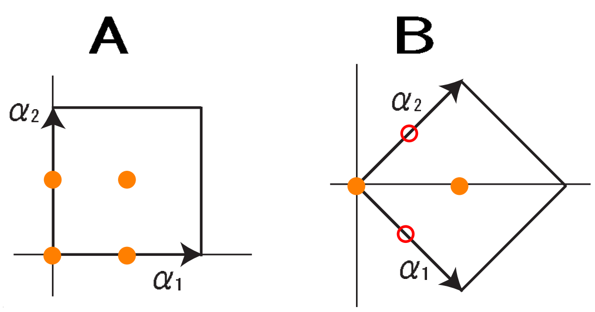

Figure 5.2: **A**-*type lattice and* **B**-*type lattice in a factorizable torus.*

[2]By T-dualizing this torus this corresponds to B-field which is frozen NS-NS closed moduli [34, 53].

by an alternative definition for $\mathcal{R}$ on the lattice (5.3.22),

$$\mathcal{R}: \; z_j \to i\bar{z}_j. \tag{5.3.24}$$

In order to distinguish the actions on non-factorizable tori from the ones on factorizable torus, let us attach a label to the action (5.3.24) as **D**, and to the one (5.2.5) in the previous subsection as **C** [41]. For example we call the models by following $\mathcal{R}$ action **CCD** model,

$$\mathcal{R}: \; z_1 \to \bar{z}_1, \quad z_2 \to \bar{z}_2, \quad z_3 \to i\bar{z}_3. \tag{5.3.25}$$

In this chapter we will explain that these actions provide convenient tools for the classifications of orientifold orbifolds on the Lie root lattices.

First let us consider the RR-tadpole cancellation conditions in the factorizable models. Instead of the direct calculations of the zero mode contribution on each T^2 and of the oscillator modes in the Klein bottle, the annulus and the Möbius strip amplitudes, it is enough to count the number of O6-planes from (5.3.15): The numbers of O6-planes are $N_{O6} = 8$ (for **AAA**), 4 (for **AAB**), 2 (for **ABB**) and 1 (for **BBB**). The types of the actions in the $T^2 \times T^2 \times T^2$ are illustrated in Figure 5.3. Here we obtain the RR-tadpole

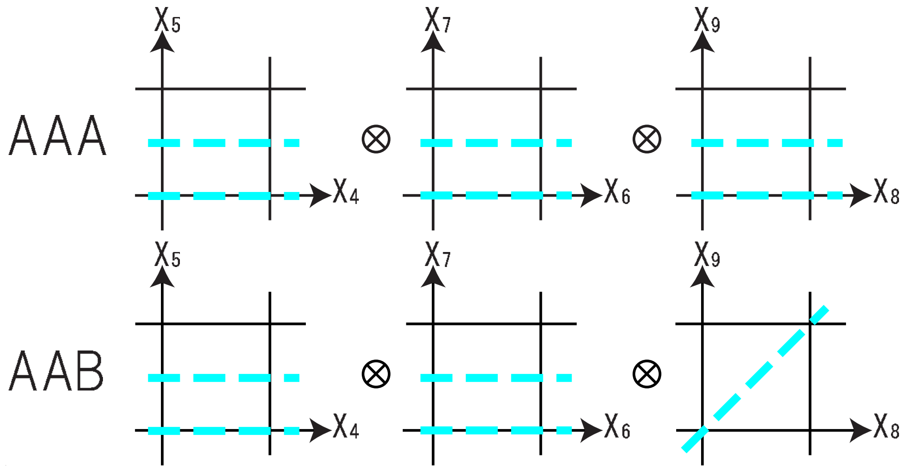

Figure 5.3: **AAA** *and* **AAB** *models on a factorizable torus. The orientifold planes lie on the dashed blue lines. In this Figure we used a label* **B** *as the* **C***-action on the* **A***-lattice.*

cancellation conditions[3]

$$\begin{aligned} \mathbf{AAA}: &\quad (N-32)^2 = 0, \\ \mathbf{AAB}: &\quad (N-16)^2 = 0, \\ \mathbf{ABB}: &\quad (N-8)^2 = 0, \\ \mathbf{BBB}: &\quad (N-4)^2 = 0. \end{aligned} \tag{5.3.26}$$

[3]Because these are the models on factorizable tori, and the **C**- and **D**-actions lead to the **A**- and **B**-models, respectively.

These are trivial results which have already been known. We emphasize that for the classification of orientifold models on non-factorizable tori and orbifolds it is convenient to fix the lattices and distinguish the models with respect to the definitions of $\mathcal{R}$.

Next we analyze some typical models on a non-factorizable torus T^6, which cannot be expressed as the direct product $T^2 \times T^2 \times T^2$. As an example we consider an orientifold model on a non-factorizable torus given by the Lie root lattice D_6. In this model the lattice D_6 can be given by the simple roots

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_6 = \mathbf{e}_5 + \mathbf{e}_6, \quad i = 1, \dots, 5, \tag{5.3.27}$$

where $\mathbf{e}_i$'s are basis of Cartesian coordinates whose normalization is given as $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. The orientifold action $\mathcal{R}$ of the **CCC**-model is

$$\mathcal{R} : \mathbf{e}_{2i-1} \to \mathbf{e}_{2i-1}, \quad \mathbf{e}_{2i} \to -\mathbf{e}_{2i}, \quad i = 1, 2, 3. \tag{5.3.28}$$

The number of O6-planes is obtained by means of (5.3.16). In order to evaluate the Lefschetz fixed point theorem, we should fix the sublattice spaces $\Lambda_{-\mathcal{R},\perp}$ and $\Lambda_{-\mathcal{R},\text{inv}}$. $\Lambda_{-\mathcal{R},\perp}$ is a lattice space projected out by $-\mathcal{R}$, and given by

$$\Lambda_{-\mathcal{R},\perp} = \Big\{ \sum_{i=1}^{3} n_{\perp,i}\alpha_{\perp,i} \Big| n_{\perp,i} \in \mathbb{Z} \Big\}, \tag{5.3.29}$$

whose basis vectors are given by

$$\alpha_{\perp,1} = \mathbf{e}_2, \quad \alpha_{\perp,2} = \mathbf{e}_4, \quad \alpha_{\perp,3} = \mathbf{e}_6. \tag{5.3.30}$$

On the other hand, the sublattice $\Lambda_{-\mathcal{R},\text{inv}}$, which is invariant under $-\mathcal{R}$, is given by

$$\begin{gathered} \Lambda_{-\mathcal{R},\text{inv}} = \Big\{ \sum_{i=1}^{3} n_{\text{inv},i}\alpha_{\text{inv},i} \Big| n_{\text{inv},i} \in \mathbb{Z} \Big\}, \\ \alpha_{\text{inv},1} = \mathbf{e}_2 - \mathbf{e}_4, \quad \alpha_{\text{inv},2} = \mathbf{e}_4 - \mathbf{e}_6, \quad \alpha_{\text{inv},3} = \mathbf{e}_4 + \mathbf{e}_6. \end{gathered} \tag{5.3.31}$$

Then we can easily evaluate the number of the O6-planes for the **CCC** model as

$$N_{\text{O6}} = 2^3 \cdot \frac{\text{Vol}(\Lambda_{-\mathcal{R},\perp})}{\text{Vol}(\Lambda_{-\mathcal{R},\text{inv}})} = 4\ . \tag{5.3.32}$$

In the same way, we consider the **CCD** model. The lattice $\Lambda_{-\mathcal{R},\perp}$ is given by

$$\begin{gathered} \Lambda_{-\mathcal{R},\perp} = \Big\{ \sum_{i=1}^{3} n_{\perp,i}\alpha_{\perp,i} \Big| n_{\perp,i} \in \mathbb{Z} \Big\}, \\ \alpha_{\perp,1} = \mathbf{e}_2, \quad \alpha_{\perp,2} = \mathbf{e}_4, \quad \alpha_{\perp,3} = \frac{1}{2}(\mathbf{e}_5 - \mathbf{e}_6), \end{gathered} \tag{5.3.33}$$

and $\Lambda_{-\mathcal{R},\text{inv}}$ is given by

$$\Lambda_{-\mathcal{R},\text{inv}} = \Big\{ \sum_{i=1}^{3} n_{\text{inv},i}\alpha_{\text{inv},i} \,\Big|\, n_{\text{inv},i} \in \mathbb{Z} \Big\}, \qquad \alpha_{\text{inv},1} = \mathbf{e}_2 - \mathbf{e}_4, \quad \alpha_{\text{inv},2} = \mathbf{e}_2 + \mathbf{e}_4, \quad \alpha_{\text{inv},3} = \mathbf{e}_5 - \mathbf{e}_6. \tag{5.3.34}$$

Then we obtain $N_{\text{O6}} = 2$. Substituting these numbers into the RR-tadpole cancellation condition (5.3.15), we easily obtain the number of D-branes. Here we summarize the data of the orientifolds on the non-factorizable D_6 lattice:

$$\begin{aligned} \textbf{CCC}: &\ (N-16)^2 = 0, \\ \textbf{CCD}: &\ (N-8)^2 = 0, \\ \textbf{CDD}: &\ (N-4)^2 = 0, \\ \textbf{DDD}: &\ (N-8)^2 = 0. \end{aligned} \tag{5.3.35}$$

These results completely agree with the ones in [41]. The gauge groups of these models are $SO(16)$, $SO(8)$, $SO(4)$ and $SO(8)$, respectively. For models on non-factorizable tori, the closed string spectra are the same as that of factorizable models.

We evaluated the number of O6-planes N_{O6} according to the Lefschetz fixed point theorem, and from (5.3.16) this gives the necessary and sufficient condition for the RR-tadpole condition. This analysis is generic and provides quite a simple rule to calculate the number of O-planes and D-branes in orientifold models on non-factorizable tori in Type II string theory.

5.4 Supersymmetric $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold models

In this section let us consider Type IIA supersymmetric orientifold models on orbifolds and describe the way to deal with orientifolds on non-factorizable lattices. Since the contributions of the RR-tadpole from untwisted states are calculated in the same way as the ones of the orientifolds on tori, we can easily count the numbers of D-branes via the Lefschetz fixed point theorem (5.3.16). We also provide detail calculations of the RR-tadpole cancellation condition on $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds.

5.4.1 Orbifolds and orientifolds

In the previous section we showed general tadpole cancellation conditions for orientifolds on non-factorizable tori (5.3.15). Here let us consider orientifold models on orbifolds given by

$$\frac{\text{Type IIA on } T^6}{\Omega\mathcal{R} \times \mathbb{Z}_N \times \mathbb{Z}_M}. \tag{5.4.1}$$

An orbifold is defined as a quotient of torus over a discrete set of isometries of the torus [20], called the point group P, i.e.,

$$\mathcal{O} = T^6/P = \mathbb{R}^6/S. \tag{5.4.2}$$

On the complex coordinates of the torus T^6, the point group elements of the orbifold act in such a way as

$$\begin{aligned} \theta : (z_1,\ z_2,\ z_3) &\to (e^{2\pi i v_1} z_1,\ e^{2\pi i v_2} z_2,\ e^{2\pi i v_3} z_3), \\ \phi : (z_1,\ z_2,\ z_3) &\to (e^{2\pi i w_1} z_1,\ e^{2\pi i w_2} z_2,\ e^{2\pi i w_3} z_3), \end{aligned} \tag{5.4.3}$$

where (v_1, v_2, v_3) and (w_1, w_2, w_3) are twists of an orbifold. We consider orientifold models with $\mathcal{N} = 1$ supersymmetry as follows: The requirement of $SU(3)$ holonomy can be phrased as invariance of the $(3, 0)$-form $\Omega = dz_1 \wedge dz_2 \wedge dz_3$, and leads to

$$v_1 + v_2 + v_3 = w_1 + w_2 + w_3 = 0. \tag{5.4.4}$$

The twists of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds which are compatible with $\mathcal{N} = 1$ supersymmetric orientifolds are listed in Table 2.2. Then only a holomorphic $(3, 0)$-form $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ and a anti-holomorphic $(0, 3)$-form $\bar{\Omega}$ are left invariant, and the other three forms on a six-tori are generally projected out. The orientifold action $\mathcal{R}$ of O6-plane, which preserves $\mathcal{N} = 1$ supersymmetry, should act as

$$\mathcal{R} : (z_1,\ z_2,\ z_3) \to (a\bar{z}_1,\ b\bar{z}_2,\ c\bar{z}_3), \tag{5.4.5}$$

where a, b and c are phase factors. Then every orientifold group element including $\mathcal{R}$ generates fixed loci of O6-planes.

For their classification we again use the abbreviations $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{1}$ in (3.1.5). For the D_6 lattice we have $\mathbb{Z}_2 \times \mathbb{Z}_2$ elements as $\theta = (-\mathbf{1}, -\mathbf{1}, \mathbf{1})$ and $\phi = (\mathbf{1}, -\mathbf{1}, -\mathbf{1})$. The orientifold actions which are compatible with this orbifold are

$$(\pm\mathbf{a}, \pm\mathbf{a}, \pm\mathbf{a}),\ (\underline{\pm\mathbf{a}, \pm\mathbf{a}, \pm\mathbf{b}}),\ (\underline{\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{b}}),\ (\pm\mathbf{b}, \pm\mathbf{b}, \pm\mathbf{b}), \tag{5.4.6}$$

where the underlined entries are permuted. For the orbifold elements $\theta = (-\mathbf{1}, \mathbf{a}, \mathbf{b})$, $\phi = (\mathbf{1}, -\mathbf{1}, -\mathbf{1})$, the compatible orientifold actions are [4]

$$\pm(\mathbf{a}, \mathbf{a}, -\mathbf{b}),\ \pm(\mathbf{a}, -\mathbf{a}, \mathbf{b}),\ \pm(-\mathbf{b}, \mathbf{a}, -\mathbf{b}),\ \pm(\mathbf{b}, -\mathbf{a}, \mathbf{b}). \tag{5.4.7}$$

In other words, the restriction is that the eigenvalues of each orientifold group element $\mathcal{R}$, $\mathcal{R}\theta$, $\mathcal{R}\phi$ and $\mathcal{R}\theta\phi$ should be $(-1, -1, -1, 1, 1, 1)$. Note that there are some equivalent actions due to the symmetry of the lattice. These considerations lead to Table 5.6 for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models on the D_6 lattice.

As explained in Appendix A there are twelve distinct classes of non-factorizable lattices, see Table 3.1. The $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ orbifolds are allowed on these non-factorizable lattices, see Table 3.4. The series of generators θ and ϕ of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold as well as the action $\Omega\mathcal{R}$ consist of the orientifold group:

$$\left\{ \theta^{k_1}\phi^{k_2},\ \Omega\mathcal{R}\theta^{k_1}\phi^{k_2} \,\middle|\, k_1 = 0, \ldots, N;\ k_2 = 0, \ldots, M \right\}. \tag{5.4.8}$$

[4] Note that for this orbifold elements the basis is different from (5.4.5).

These elements appear in the following string one-loop amplitudes in (5.2.6) with insertions [43],

$$\mathbf{P} = \left(\frac{1+\theta+\cdots+\theta^{N-1}}{N}\right)\left(\frac{1+\phi+\cdots+\phi^{M-1}}{M}\right). \tag{5.4.9}$$

After extracting the RR-tadpoles, the insertion of $\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}$ in the Klein bottle amplitude corresponds to the contribution from O-planes fixed by $\mathcal{R}\theta^{k_1}\phi^{k_2}$. Since in the $\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}$ insertion the contributions from untwisted sectors are calculated in the same way as the cases of tori in section 5.3, we obtain the necessary condition (5.3.15) for the RR-tadpole cancellation by D-branes parallel to the O-planes. From this necessary condition, we obtain all the numbers of O-planes and D-branes on the orbifold. In the next section we will demonstrate a few examples of $\mathbb{Z}_4\times\mathbb{Z}_2$ orientifold models, and evaluate the RR-tadpole cancellation condition.

The $\Omega\mathcal{R}$ invariant states consist of oscillators

$$\begin{aligned}(\Omega\mathcal{R})\left(\alpha_{n+kv_i}\tilde{\overline{\alpha}}_{n+kv_i}\right)(\Omega\mathcal{R})^{-1} &= \left(\alpha_{n+kv_i}\tilde{\overline{\alpha}}_{n+kv_i}\right),\\ (\Omega\mathcal{R})\left(\overline{\alpha}_{n-kv_i}\tilde{\alpha}_{n-kv_i}\right)(\Omega\mathcal{R})^{-1} &= \left(\overline{\alpha}_{n-kv_i}\tilde{\alpha}_{n-kv_i}\right)\end{aligned} \tag{5.4.10}$$

where α can be replaced by fermionic operators. These states are also invariant under the $\mathbb{Z}_N$ action

$$\begin{aligned}\theta\left(\alpha_{n+kv_i}\tilde{\overline{\alpha}}_{n+kv_i}\right)\theta^{-1} &= \left(\alpha_{n+kv_i}\tilde{\overline{\alpha}}_{n+kv_i}\right),\\ \theta\left(\overline{\alpha}_{n-kv_i}\tilde{\alpha}_{n-kv_i}\right)\theta^{-1} &= \left(\overline{\alpha}_{n-kv_i}\tilde{\alpha}_{n-kv_i}\right).\end{aligned} \tag{5.4.11}$$

Thus all twisted sectors contribute to the Klein bottle amplitude.

5.4.2 $\mathbb{Z}_4 \times \mathbb{Z}_2$ model

Here we discuss the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold model on the Lie root lattice D_6 (5.3.27) in detail because in this case all possible subtleties show up.

There exists only one distinct $\mathbb{Z}_4 \times \mathbb{Z}_2$ orbifold on D_6, whose point group elements θ and ϕ are given by

$$\theta : \left\{\begin{array}{lllll}\mathbf{e}_1 & \to & \mathbf{e}_2 & \to -\mathbf{e}_1 \\ \mathbf{e}_3 & \to & -\mathbf{e}_4 & \to -\mathbf{e}_3 \\ \mathbf{e}_5 & \to & \mathbf{e}_5 & \\ \mathbf{e}_6 & \to & \mathbf{e}_6 & \end{array}\right. \qquad \phi : \left\{\begin{array}{llll}\mathbf{e}_1 & \to & \mathbf{e}_1 & \\ \mathbf{e}_2 & \to & \mathbf{e}_2 & \\ \mathbf{e}_i & \to & -\mathbf{e}_i & i=3,4,5,6\end{array}\right. \tag{5.4.12a}$$

or, in matrix representation, by

$$\theta : \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \phi : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.4.12b}$$

By using the above elements we can show all the orientifold actions which preserve $\mathcal{N}=1$ supersymmetry by means of **C** and **D** actions. For example, the reflection $\mathcal{R}$ on the **DDC** model is given by

$$\mathcal{R}: \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \equiv (\mathbf{b},\mathbf{b},\mathbf{a}). \tag{5.4.13}$$

From the Lefschetz fixed point theorem (5.3.16), the number of O6-plane fixed by $\mathcal{R}$ is given as $N_{\mathrm{O6}}=1$. If we put four D-branes parallel to this $\mathcal{R}$-fixed O6-plane, the RR-tadpole of this model will be cancelled. Similarly, the element $\mathcal{R}\theta=(\mathbf{a},-\mathbf{a},\mathbf{a})$ gives $N_{\mathrm{O6}}=4$, whose tadpole is cancelled by sixteen D-branes parallel to these four $\mathcal{R}\theta$-fixed O6-planes. We similarly evaluate the cases for the other elements of the orientifold group. The relations between the orientifold group elements and the numbers of O-planes are summarized in Table 5.1.

Orientifold elements of $\mathcal{R}$	# of O6-planes
$(\pm\mathbf{a},\pm\mathbf{a},\pm\mathbf{a})$, $(\underline{\mathbf{1},-\mathbf{1},\pm\mathbf{a}})$	4
$(\underline{\pm\mathbf{a},\pm\mathbf{a},\pm\mathbf{b}})$, $(\underline{\mathbf{1},-\mathbf{1},\pm\mathbf{b}})$ $(\pm\mathbf{b},\pm\mathbf{b},\pm\mathbf{b})$	2
$(\underline{\pm\mathbf{a},\pm\mathbf{b},\pm\mathbf{b}})$	1

Table 5.1: *Orientifold group elements and the numbers of O6-planes on the D_6 lattice. The underline indicates a symmetry under the cyclic permutation.*

Since the $\mathbb{Z}_4$ action changes the directions of the O-planes by angle of $\theta^{1/2}$ in the following way:

$$\mathcal{R}\theta=\theta^{-1/2}\mathcal{R}\theta^{1/2}. \tag{5.4.14}$$

This action generates the exchange between the action **C** and **D** each other. Then we can see that **CCC** and **DDC**, **CCD** and **DDD**, **CDD** and **DCD** models are equivalent with each other, respectively. In the case of the **CCC** model, for example, two different numbers of O6-planes appear since the orientifold group elements in $\mathcal{R}$, $\mathcal{R}\theta^2$, $\mathcal{R}\phi$ and $\mathcal{R}\theta^2\phi$ are given by $(\pm\mathbf{a},\pm\mathbf{a},\pm\mathbf{a})$, whereas the elements in $\mathcal{R}\theta$, $\mathcal{R}\theta^3$, $\mathcal{R}\theta\phi$ and $\mathcal{R}\theta^3\phi$ are given by $(\pm\mathbf{b},\pm\mathbf{b},\pm\mathbf{a})$. Analyzing such actions, we obtain all the models for $\mathbb{Z}_4\times\mathbb{Z}_2$ orientifolds on the D_6 lattice, listed in Table 5.2.

We estimated the RR-tadpole cancellation by counting the O-planes from the equation (5.3.15), which is the necessary condition in the case of the orbifold model. However it is expected that the RR-tadpoles are cancelled even in the orbifold model. These countings also give correct results for well-investigated non-factorizable models on $\mathbb{Z}_N$ orbifolds in [42]

Lattice	Label	reps. of $\mathcal{R}$	# of O6-planes	
			$\mathcal{R}$, $\mathcal{R}\theta^2$, $\mathcal{R}\phi$, $\mathcal{R}\theta^2\phi$	$\mathcal{R}\theta$, $\mathcal{R}\theta^3$, $\mathcal{R}\theta\phi$, $\mathcal{R}\theta^3\phi$
D_6	**CCC**	$(\mathbf{a},\mathbf{a},\mathbf{a})$	4	1
	CCD	$(\mathbf{a},\mathbf{a},\mathbf{b})$	2	2
	CDD	$(\mathbf{a},\mathbf{b},\mathbf{b})$	2	2
	DCC	$(\mathbf{b},\mathbf{a},\mathbf{a})$	2	2

Table 5.2: *All the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold models on the D_6 lattice.*

and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds in [41]. We give the explicit results of the RR-tadpole cancellation for a few models in the following.

5.4.3 Klein bottle amplitude

First let us evaluate the Klein bottle amplitude of $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold model on the D_6 lattice (5.3.27) with the orientifold action

$$\mathcal{R} = (\mathbf{b}, \mathbf{a}, \mathbf{a}), \tag{5.4.15}$$

which gives the **DCC** model. The contributions of the oscillator modes are equal in any insertions of the orientifold group because they act as the unit operator in (5.2.6a). In the $\theta^{n_1}\phi^{n_2}$-twisted sector, the oscillator contribution is given by $\mathcal{K}^{(n_1,n_2)} \equiv \mathcal{K}^{(n_1,k_1)(n_2,k_2)}$ (see, for the notation, [43]). We also need the multiplicities $\chi_{\mathcal{K}}^{(n_1,k_1)(n_2,k_2)}$ of the $\theta^{n_1}\phi^{n_2}$-twisted fixed sectors, which are invariant under the insertion $\Omega\mathcal{R}\theta^{n_1}\phi^{n_2}$, see table 5.3.

When an action $\theta^{n_1}\phi^{n_2}$ does not fix certain directions in the compact space, the Kaluza-Klein momentum modes and the winding modes appear as the zero modes in the $\theta^{n_1}\phi^{n_2}$-fixed sector. Let us evaluate such zero modes in the θ-twisted sector. The θ invariant sublattice Λ^θ is expanded in terms of the basis

$$\big\{\mathbf{e}_5 + \mathbf{e}_6,\ \mathbf{e}_5 - \mathbf{e}_6\big\}. \tag{5.4.16}$$

We can see that the $\mathcal{R}$ invariant dual sublattice $(\Lambda^\theta)^*_{\mathcal{R},\mathrm{inv}}$, whose basis is given by $\{2\mathbf{e}_5\}$, and the $-\mathcal{R}$ invariant sublattice $(\Lambda^\theta)_{-\mathcal{R},\mathrm{inv}}$, with its basis $\{2\mathbf{e}_6\}$, yield the momentum modes and the winding modes in this sector, respectively. However there are two subtleties in this evaluation, one of which is caused by the momentum doubling, and the other from the appearance of the half winding states [42].

The former subtlety is caused by the shifts associated to the $\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}$ insertions. In the θ-twisted sector we have two fixed tori given by

$$x\mathbf{e}_5,\ \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) + x\mathbf{e}_5, \tag{5.4.17}$$

multiplicities $\chi_{\mathcal{K}}^{(n_1,k_1)(n_2,k_2)}$	**CCC**	**CCD**	**CDD**	**DCC**
$(0,k_1)(0,k_2)$	1	1	1	1
$(2n_1+1,k_1)(0,k_2)$	2	2	2	2
$(2n_1,2k_1+1)(0,k_2)$	4	4	4	4
$(2n_1,2k_1)(0,k_2)$	8	8	4	4
$(0,2k_1+1)(1,k_2)$	4	4	4	4
$(0,2k_1)(1,k_2)$	8	4	4	4
$(2n_1+1,k_1)(1,k_2)$	8	8	4	8
$(2n_1,2k_1+1)(1,k_2)$	4	4	4	4
$(2n_1,2k_1)(1,k_2)$	8	4	4	4

Table 5.3: *Multiplicities of the fixed points for the* **DCC** *and* **CCC** *models.*

where $x \in \mathbb{R}$ is a coordinate on the fixed tori. Note that the invariance of fixed points or fixed tori under $\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}$ is defined modulo the translation generated by the lattice Λ. The $\mathcal{R}$ insertion acts on the two fixed tori in such a way as

$$\mathcal{R}: \begin{cases} x\mathbf{e}_5 \;\rightarrow\; x\mathbf{e}_5, \\ \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)+x\mathbf{e}_5 \;\rightarrow\; \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_4)+x\mathbf{e}_5 \\ \qquad\qquad = \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)+(-1+x)\mathbf{e}_5. \end{cases} \tag{5.4.18}$$

In the latter case, the translation of a lattice shift $\alpha_4 = \mathbf{e}_4 - \mathbf{e}_5$ is accompanied. Because a momentum mode $\langle\mathbf{p}$ picks up a phase factor $e^{2\pi\mathbf{p}\cdot l}$ under the translation by l, we generally need phase factors in the amplitudes. In the case of (5.4.18), the phase factor is $\mathbf{p}\cdot l = 2\mathbf{e}_5\cdot\mathbf{e}_5 = 2$, and does not affect the amplitudes. If the phase factor is given as -1, the momentum modes are effectively doubled by interference between modes with and without shifts:

$$\sum_n (-1)^n \exp(-\pi t n^2 \mathbf{p}^2) + \sum_n \exp(-\pi t n^2 \mathbf{p}^2) = 2\sum_n \exp(-4\pi t n^2 \mathbf{p}^2). \tag{5.4.19}$$

The latter subtlety occurs in the winding modes. There are special points with the following property:

$$\theta: \; \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_2) \rightarrow \frac{1}{2}(-\mathbf{e}_1+\mathbf{e}_2) = \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_2)+\mathbf{e}_6, \tag{5.4.20}$$

where we used a lattice shift given by $\mathbf{e}_1+\mathbf{e}_6$. The point does not lie on the θ-fixed tori, whereas this shift *does* generate the winding modes:

$$X(\sigma,\tau) = \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_2) + \frac{\sigma}{2\pi}\mathbf{e}_6 + (\tau \text{ dependence}). \tag{5.4.21}$$

There are two points $\frac{1}{2}(\mathbf{e}_1 \pm \mathbf{e}_2)$ which are invariant under the action $\mathcal{R}$, and the multiplicity is equal to that of the θ-fixed tori which are also invariant under $\mathcal{R}$.

Therefore we conclude that the zero modes in the θ-twisted sector with $\mathcal{R}$ insertion are given by the following vectors:

$$\mathbf{p} = 2n\mathbf{e}_5, \qquad \mathbf{w} = m\mathbf{e}_6, \tag{5.4.22}$$

where $n, m \in \mathbb{Z}$. In the notation of (B.3.4), the zero mode contribution in the Klein bottle amplitude is $\mathcal{L}_{2,\frac{1}{2}}$. In a similar way we can evaluate the other twisted sectors in the orbifold model. Note that for non-factorizable orbifolds the zero mode contributions depend on the insertion $\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}$. In the ϕ-twisted sector we have $\mathcal{L}_{2,\frac{1}{2}}$ for the $\Omega\mathcal{R}\theta^{2k_1}\phi^{k_2}$ insertions, and $\mathcal{L}_{4,1}$ for the $\Omega\mathcal{R}\theta^{2k_1+1}\phi^{k_2}$ insertions.

Next we evaluate the zero mode contribution from the untwisted sector given in (5.3.13a). The basis of dual lattice $\alpha^* \in \Lambda^*$, which is defined by $\alpha_i^* \cdot \alpha_j = \delta_{ij}$, is given as

$$\begin{array}{rcl}
\alpha_1^* &=& \mathbf{e}_1, \\
\alpha_2^* &=& \mathbf{e}_1 + \mathbf{e}_2, \\
\alpha_3^* &=& \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \\
\alpha_4^* &=& \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4, \\
\alpha_5^* &=& \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 - \mathbf{e}_6), \\
\alpha_6^* &=& \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6).
\end{array} \tag{5.4.23}$$

Then the $\mathcal{R}$ invariant dual sublattice $\Lambda^*_{\mathcal{R},\mathrm{inv}}$ in (5.3.13a), which yields the momentum modes in the Kaluza-Klein states, is expanded by the basis

$$\{\mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_5\}. \tag{5.4.24}$$

In the same way, the $-\mathcal{R}$ invariant lattice $\Lambda_{-\mathcal{R},\mathrm{inv}}$ in (5.3.13c) yielding the winding states is expanded by

$$\{\mathbf{e}_1 - \mathbf{e}_2,\ 2\mathbf{e}_4,\ 2\mathbf{e}_6\}. \tag{5.4.25}$$

Substituting these elements into (5.3.5), we obtain the zero mode contribution $\mathcal{L}_{\mathcal{K}}^{(0,0)}$. Its modular transformation is given by the factors

$$\begin{aligned}
\sqrt{\det M^{\mathcal{K}}} &= \mathrm{Vol}(\Lambda^*_{\mathcal{R},\mathrm{inv}}) = \sqrt{2}, \\
\sqrt{\det W^{\mathcal{K}}} &= \mathrm{Vol}(\Lambda_{-\mathcal{R},\mathrm{inv}}) = 4\sqrt{2}.
\end{aligned} \tag{5.4.26}$$

We also need the zero mode contributions with the other insertions $\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}$. Since these elements are given by $\mathcal{R}\theta^{k_1}\phi^{k_2} = (\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{b})$ for the DCC model, we have the same results as that of the $\mathcal{R}$ insertion.

We obtained all the ingredients to write down the Klein bottle amplitude for the **DCC** model, which is summarized as

$$\begin{aligned}\mathcal{K} = c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty \frac{dt}{t^3} \Big(&\mathcal{L}_{\mathcal{K}}^{(0,0)} \mathcal{K}^{(0,0)} + 2\mathcal{L}_{2,\frac{1}{2}} \mathcal{K}^{(1,0)} + 4\mathcal{L}_{2,\frac{1}{2}} \mathcal{K}^{(2,0)} + 2\mathcal{L}_{2\frac{1}{2}} \mathcal{K}^{(3,0)} \\ &+ \frac{1}{2}\big(8\mathcal{L}_{4,1} + 4\mathcal{L}_{2,\frac{1}{2}}\big)\mathcal{K}^{(0,1)} + 8\mathcal{K}^{(1,1)} \\ &+ \frac{1}{2}\big(8\mathcal{L}_{4,1} + 4\mathcal{L}_{2,\frac{1}{2}}\big)\mathcal{K}^{(2,1)} + 8\mathcal{K}^{(3,1)} \Big). \end{aligned} \tag{5.4.27}$$

Its modular transformation to the tree channel is

$$\begin{aligned}\tilde{\mathcal{K}} = 16c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty dl \Big(&\tilde{\mathcal{L}}_{\mathcal{K}}^{(0,0)} \tilde{\mathcal{K}}^{(0,0)} - 2\tilde{\mathcal{L}}_{2,8} \tilde{\mathcal{K}}^{(1,0)} - 4\tilde{\mathcal{L}}_{2,8} \tilde{\mathcal{K}}^{(2,0)} - 2\tilde{\mathcal{L}}_{2,8} \tilde{\mathcal{K}}^{(3,0)} \\ &- 2(\tilde{\mathcal{L}}_{1,4} + \tilde{\mathcal{L}}_{2,8})\mathcal{K}^{(0,1)} + 4\tilde{\mathcal{K}}^{(1,1)} \\ &- 2(\tilde{\mathcal{L}}_{1,4} + \tilde{\mathcal{L}}_{2,8})\tilde{\mathcal{K}}^{(2,1)} - 4\tilde{\mathcal{K}}^{(3,1)} \Big). \end{aligned} \tag{5.4.28}$$

Note that in the IR limit $l \to \infty$, the zero mode contributions $\tilde{\mathcal{L}}_{\mathcal{K}}^{(0,0)}$ and $\tilde{\mathcal{L}}_{\alpha,\beta}$ in the tree channel (5.4.28) go to unity, then we obtain $2(\tilde{\mathcal{L}}_{1,4} + \tilde{\mathcal{L}}_{2,8}) \to 4$. Then we observe that the prefactors are given by the complete projector [51]

$$\prod_{i=1,\ n_1 v_i + n_2 w_i \neq 0}^{3} \big(-2\sin(\pi n_1 v_i + \pi n_2 w_i) \big). \tag{5.4.29}$$

This relation implies that only the untwisted sector contributes to the RR-tadpole.

5.4.4 Annulus amplitude

In order to cancel the RR-tadpole we introduce D-branes parallel to O-planes. We attach a label (i_1, i_2) to a stack of D-branes which is invariant under the orientifold action $\mathcal{R}\theta^{i_1}\phi^{i_2}$, and define that $(0,0)$ denotes D-branes invariant under the action $\mathcal{R}$. The three-cycle wrapped by the brane $(0,0)$ is given by the $\mathcal{R}$ invariant lattice $\Lambda_{\mathcal{R},\mathrm{inv}}$ whose basis is given by

$$\big\{ \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{e}_3 - \mathbf{e}_5,\ \mathbf{e}_3 + \mathbf{e}_5 \big\}. \tag{5.4.30}$$

From (5.4.14) the brane $(1,0)$ is rotated by half of the angle θ with respect to the brane $(0,0)$. The three-cycle wrapped by the brane $(1,0)$ is given by the $\mathcal{R}\theta$ invariant lattice $\Lambda_{\mathcal{R}\theta,\mathrm{inv}}$ whose basis is given by

$$\big\{ \mathbf{e}_1 - \mathbf{e}_5,\ \mathbf{e}_3 + \mathbf{e}_4,\ \mathbf{e}_1 + \mathbf{e}_5 \big\}. \tag{5.4.31}$$

An open string stretching from the brane (i_1, i_2) to the brane $(i_1 - n_1, i_2 - n_2)$ is localized at intersection of D-branes. It is convenient to call such a state in the $\theta^{n_1}\phi^{n_2}$-twisted sector.

The three-cycles of the brane $(0,0)$ and the brane $(1,0)$ share a common direction, and the lattice vector in this direction is given by $2\mathbf{e}_5$. The momentum modes are obtained from the dual of the vector $2\mathbf{e}_5$ in such a way as

$$\mathbf{p} = \frac{n}{\sqrt{2}}\mathbf{e}_5, \tag{5.4.32}$$

where $n \in \mathbb{Z}$. The basis of the winding modes is related to the distances of the parallel D-branes. Because we put D-branes parallel to the O-planes, the shortest distance corresponds to the lattice vector projected by the actions $-\mathcal{R}$ and $-\mathcal{R}\theta$, i.e., $\Lambda_{\mathcal{R},\perp} \cap \Lambda_{\mathcal{R}\theta,\perp}$. Then the winding modes are

$$\mathbf{w} = \frac{n}{\sqrt{2}}\mathbf{e}_6. \tag{5.4.33}$$

Then zero mode contribution of the θ-twisted sector is expressed as $\mathcal{L}_{1,1}$.

Let us explain one more case of the ϕ-twisted sector. The winding modes are given as "half winding-like" modes, and are also given by the projected lattice $\Lambda_{\mathcal{R},\perp} \cap \Lambda_{\mathcal{R}\phi,\perp}$ whose basis is

$$\left\{\frac{1}{2\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)\right\}. \tag{5.4.34}$$

Then the zero modes of open string stretching between the brane $(0,0)$ and the brane $(0,1)$ are given by

$$\mathbf{p} = \frac{n}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2), \qquad \mathbf{w} = \frac{n}{2\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2). \tag{5.4.35}$$

The zero mode contribution in the annulus amplitude is expressed as $\mathcal{L}_{2,\frac{1}{2}}$. The other zero modes are calculated in a similar way.

Since in the $\theta^{n_1}\phi^{n_2}$-twisted sector the contributions from the oscillator modes do not depend on branes (i_1, i_2), they are given as $\mathcal{A}^{(n_1,k_1)(n_2,k_2)}$. The insertions of $\mathbb{1}$, θ^2, ϕ and $\theta^2\phi$ leave D-branes invariant, and perform non-trivial actions on the Chan-Paton factors described as $\gamma_{k_1,k_2}^{(i_1,i_2)}$, which appear in the amplitude as

$$\mathrm{tr}\left(\gamma_{k_1,k_2}^{(i_1-n_1,i_2-n_2)}\right)\mathrm{tr}\left(\gamma_{k_1,k_2}^{(i_1,i_2)}\right)^{-1} \tag{5.4.36}$$

in the $\theta^{n_1}\phi^{n_2}$-twisted sector. Sectors of $k_1 \neq 0$ or $k_2 \neq 0$ cannot be cancelled by the other diagrams. Therefore the $\mathbb{Z}_2$ twisted tadpole cancellation condition is required [43, 54]:

$$\mathrm{tr}\left(\gamma_{2,0}^{(i_1,i_2)}\right) = \mathrm{tr}\left(\gamma_{0,1}^{(i_1,i_2)}\right) = \mathrm{tr}\left(\gamma_{2,1}^{(i_1,i_2)}\right) = 0. \tag{5.4.37}$$

We should also evaluate the multiplicities $\chi_{\mathcal{M}}$ of the open string states, which are given by the intersection number of D-branes. The intersection numbers of two branes can be obtained by the determinant of vectors v_i and v_i' giving the three-cycles in respective D-branes [42]. These vectors can be expanded in terms of the lattice basis as $v_i = \sum v_{ij}\alpha_j$.

Then the intersection number is

$$I = \det \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{16} \\ v_{21} & v_{22} & \cdots & v_{26} \\ \vdots & & & \vdots \\ v'_{31} & v'_{32} & \cdots & v'_{36} \end{pmatrix}. \tag{5.4.38}$$

Owing to the above $\mathbb{Z}_2$ twisted tadpole condition, it is sufficient to consider the intersection numbers $\chi_{\mathcal{M}}$ for $k_1 = k_2 = 0$, which are given in Table 5.4.

$\chi_{\mathcal{A}}$	**CCC**	**CCD**	**CDD**	**DCC**
(i_1,i_2)–(i_1,i_2)	1	1	1	1
(i_1,i_2)–(i_1+1,i_2)	1	1	2	1
$(2i_1+1,i_2)$–$(2i_1+3,i_2)$	4	2	4	2
$(2i_1,i_2)$–$(2i_1+2,i_2)$	1	2	4	2
$(2i_1+1,i_2)$–$(2i_1+1,i_2+1)$	4	2	4	2
$(2i_1,i_2)$–$(2i_1,i_2+1)$	1	2	4	2
(i_1,i_2)–(i_1+1,i_2+1)	2	2	4	2
$(2i_1+1,i_2)$–$(2i_1+3,i_2+1)$	4	2	4	2
$(2i_1,i_2)$–$(2i_1+2,i_2+1)$	1	2	4	2

Table 5.4: *Intersection numbers in the annulus amplitudes in the* **DCC** *and* **CCC** *models.*

The contribution from the zero modes of the untwisted sector is obtained from (5.3.13b) and (5.3.13d). For the brane $(0,0)$, which is parallel to the $\mathcal{R}$-fixed O6-plane ,it is

$$\begin{aligned} \sqrt{\det M^{\mathcal{A}}} &= \mathrm{Vol}(\Lambda^*_{\mathcal{R},\perp}) = 4, \\ \sqrt{\det W^{\mathcal{A}}} &= \mathrm{Vol}(\Lambda_{-\mathcal{R},\perp}) = 4. \end{aligned} \tag{5.4.39}$$

The contributions from the other branes (i_1, i_2) give the same values. These values appear in prefactors of the amplitude after the modular transformation.

Summarizing the above, we obtain the annulus amplitude for the **DCC** model

$$\begin{aligned} \mathcal{A} = \frac{N^2 c}{4}(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty \frac{dt}{t^3} \Big(& \mathcal{L}^{(0,0)}_{\mathcal{A}} \mathcal{A}^{(0,0)} + \mathcal{L}_{1,1}\mathcal{A}^{(1,0)} + 2\mathcal{L}_{1,1}\mathcal{A}^{(2,0)} + \mathcal{L}_{1,1}\mathcal{A}^{(3,0)} \\ & + (\mathcal{L}_{2,\frac{1}{2}} + \mathcal{L}_{1,1})\mathcal{K}^{(0,1)} + 2\mathcal{A}^{(1,1)} \\ & + (\mathcal{L}_{1,1} + \mathcal{L}_{2,\frac{1}{2}})\mathcal{A}^{(2,1)} + 2\mathcal{A}^{(3,1)} \Big), \end{aligned} \tag{5.4.40}$$

where $\mathcal{A}^{(n_1,n_2)} \equiv \mathcal{A}^{(n_1,0)(n_2,0)}$. The modular transformation to the amplitude in the tree channel yields

$$\begin{aligned}\tilde{\mathcal{A}} = \frac{N^2 c}{4}(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty dl \Big(& \tilde{\mathcal{L}}_{\mathcal{A}}^{(0,0)} \tilde{\mathcal{A}}^{(0,0)} - 2\tilde{\mathcal{L}}_{2,2}\tilde{\mathcal{A}}^{(1,0)} - 4\tilde{\mathcal{L}}_{2,2}\tilde{\mathcal{A}}^{(2,0)} - 2\tilde{\mathcal{L}}_{2,2}\tilde{\mathcal{A}}^{(3,0)} \\ & - 2(\tilde{\mathcal{L}}_{1,4} + \tilde{\mathcal{L}}_{2,2})\mathcal{A}^{(0,1)} + 4\tilde{\mathcal{A}}^{(1,1)} \\ & - 2(\tilde{\mathcal{L}}_{2,2} + \tilde{\mathcal{L}}_{1,4})\tilde{\mathcal{A}}^{(2,1)} - 4\tilde{\mathcal{A}}^{(3,1)} \Big). \end{aligned} \tag{5.4.41}$$

We again observe the complete projector in the IR limit.

5.4.5 Möbius strip amplitude

The amplitude of the Möbius strip (5.2.6c) includes the insertion of $\Omega\mathcal{R}$, and string states should be invariant under these orientifold actions. In $\theta^{n_1}\phi^{n_2}$-twisted sector, the insertion $\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}$ acts on open strings stretching from the brane (i_1, i_2) to the brane (i_1-n_1, i_2-n_2) as

$$\Omega\mathcal{R}\theta^{k_1}\phi^{k_2} : [(i_1,i_2)(i_1-n_1,i_2-n_2)] \to [(-i_1+n_1-2k_1, -i_2+n_2-2k_2)(-i_1-2k_1, -i_2-2k_2)]. \tag{5.4.42}$$

Therefore in the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orbifold case the following conditions are required:

$$2(i_1 + k_1) - n_1 = 0 \pmod 4, \tag{5.4.43a}$$
$$2(i_2 + k_2) - n_2 = 0 \pmod 2. \tag{5.4.43b}$$

Then the sectors with $n_1 = 0, 2$ and $n_2 = 0$ contribute to the amplitude. The intersection number is obtained in the same way as in the case of annulus. In Table 5.4, we can see that $\chi_{\mathcal{M}} = 1$ for untwisted sectors and $\chi_{\mathcal{M}} = 2$ for θ^2-twisted sectors.

The momentum modes are evaluated in a similar way of section 5.4.3, however the winding modes are changed due to the insertions. In the untwisted sector with the $\Omega\mathcal{R}\theta$ insertion, from the condition (5.4.43) the open string states $[(1,0)(3,0)]$, $[(1,1)(3,1)]$, $[(1,1)(3,1)]$, $[(1,1)(3,1)]$, $[(3,0)(1,0)]$, $[(1,0)(3,0)]$ and $[(1,0)(3,0)]$ contribute to the amplitude. For instance, in the open string state $[(1,0)(3,0)]$, the momentum modes, which are generated by the dual lattice $\Lambda_{\mathcal{R}\theta,\mathrm{inv}} \cap \Lambda_{\mathcal{R}\theta^3,\mathrm{inv}}$ with its basis $\{\mathbf{2e_5}\}$, are given as

$$\mathbf{p} = \frac{n}{\sqrt{2}}\mathbf{e}_5. \tag{5.4.44}$$

The winding modes invariant under $-\Omega\mathcal{R}\theta$ are given by

$$\mathbf{w} = \frac{2n}{\sqrt{2}}\mathbf{e}_6. \tag{5.4.45}$$

This can also read from $\Lambda_{-\mathcal{R}\theta,\mathrm{inv}} \cap \Lambda_{-\mathcal{R}\theta^3,\mathrm{inv}}$. The zero mode contribution for this state is represented as $\mathcal{L}_{1,4}$.

We should take it account of the orientifold actions to the Chan-Paton factors. For the open strings $[(i_1,i_2)(i_1-n_1,i_2-n_2)]$, the $\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}$-insertion contributes in the amplitude as

$$\mathrm{tr}\Big[(\gamma^{(i_1,i_2)}_{\Omega\mathcal{R}k_1k_2})^{-1}(\gamma^{(i_1-n_1,i_2-n_2)}_{\Omega\mathcal{R}k_1k_2})^T\Big]. \tag{5.4.46}$$

Since only the sectors with $n_1=0,2$ and $n_2=0$ contribute to the amplitude, we abbreviate

$$a^{(n_1)}_{k_1k_2} \equiv \mathrm{tr}\Big[(\gamma^{(2i_1+1,i_2)}_{\Omega\mathcal{R}k_1k_2})^{-1}(\gamma^{(2i_1+1+n_1,i_2)}_{\Omega\mathcal{R}k_1k_2})^T\Big], \tag{5.4.47a}$$

$$b^{(n_1)}_{k_1k_2} \equiv \mathrm{tr}\Big[(\gamma^{(2i_1,i_2)}_{\Omega\mathcal{R}k_1k_2})^{-1}(\gamma^{(2i_1+n_1,i_2)}_{\Omega\mathcal{R}k_1k_2})^T\Big]. \tag{5.4.47b}$$

These assignments correspond to two different classes of the D-brane configurations in this model, and are sufficient to evaluate the tadpole cancellation conditions for $\mathbb{Z}_4\times\mathbb{Z}_2$ models. However we will need more independent variables for $\mathbb{Z}_2\times\mathbb{Z}_2$ models.

For the contributions from the untwisted sector, we can use the results from the (5.4.27) and (5.4.40) owing to the relations (5.3.13a)-(5.3.13d).

To summarize, we obtain the Möbius strip amplitude in the loop channel as

$$\begin{aligned}\mathcal{M} = -\frac{Nc}{4}(1_{\mathrm{RR}}-1_{\mathrm{NSNS}})\int_0^\infty \frac{dt}{t^3}\Big(&\frac{a^{(0)}_{0,0}+b^{(0)}_{0,0}}{2}\mathcal{L}^{(0,0)}_{\mathcal{M}}\mathcal{M}^{(0,0)(0,0)} + 2\frac{a^{(2)}_{3,0}+b^{(2)}_{3,0}}{2}\mathcal{L}_{1,4}\mathcal{M}^{(2,3)(0,0)}\\
&+\frac{a^{(0)}_{2,0}+b^{(0)}_{2,0}}{2}\mathcal{L}_{1,4}\mathcal{M}^{(0,2)(0,0)} + 2\frac{a^{(2)}_{1,0}+b^{(2)}_{1,0}}{2}\mathcal{L}_{1,4}\mathcal{M}^{(2,1)(0,0)}\\
&+\frac{a^{(0)}_{0,1}\mathcal{L}_{2,2}+b^{(0)}_{0,1}\mathcal{L}_{1,4}}{2}\mathcal{M}^{(0,0)(0,1)} + 2\frac{a^{(2)}_{3,1}+b^{(2)}_{3,1}}{2}\mathcal{M}^{(2,3)(0,1)}\\
&+\frac{a^{(0)}_{2,1}\mathcal{L}_{1,4}+b^{(0)}_{2,1}\mathcal{L}_{2,2}}{2}\mathcal{M}^{(0,2)(0,1)} + 2\frac{a^{(2)}_{1,1}+b^{(2)}_{1,1}}{2}\mathcal{M}^{(2,1)(0,1)}\Big).\end{aligned} \tag{5.4.48}$$

The modular transformation to the tree channel yields

$$\begin{aligned}\tilde{\mathcal{M}} = -4c(1_{\mathrm{RR}}-1_{\mathrm{NSNS}})\int_0^\infty dl\Big(&\frac{a^{(0)}_{0,0}+b^{(0)}_{0,0}}{2}\tilde{\mathcal{L}}^{(0,0)}_{\mathcal{M}}\tilde{\mathcal{M}}^{(0,0)} - (a^{(2)}_{3,0}+b^{(2)}_{3,0})\tilde{\mathcal{L}}_{8,2}\tilde{\mathcal{M}}^{(1,0)}\\
&+2(a^{(0)}_{2,0}+b^{(0)}_{2,0})\tilde{\mathcal{L}}_{8,2}\tilde{\mathcal{M}}^{(2,0)} - (a^{(2)}_{1,0}+b^{(2)}_{1,0})\tilde{\mathcal{L}}_{8,2}\tilde{\mathcal{M}}^{(3,0)}\\
&+2(a^{(0)}_{0,1}\tilde{\mathcal{L}}_{4,4}+b^{(0)}_{0,1}\tilde{\mathcal{L}}_{8,2})\mathcal{M}^{(0,1)} + 2(a^{(2)}_{3,1}+b^{(2)}_{3,1})\tilde{\mathcal{M}}^{(1,1)}\\
&+2(a^{(0)}_{2,1}\tilde{\mathcal{L}}_{8,2}+b^{(0)}_{2,1}\tilde{\mathcal{L}}_{4,4})\tilde{\mathcal{M}}^{(2,1)} + 2(a^{(2)}_{1,1}+b^{(2)}_{1,1})\tilde{\mathcal{M}}^{(3,1)}\Big).\end{aligned} \tag{5.4.49}$$

To obtain the complete projector and to cancel the tadpole [43], we set

$$a^{(0)}_{0,0}=a^{(2)}_{1,0}=-a^{(0)}_{2,0}=a^{(2)}_{3,0}=-a^{(0)}_{0,1}=-a^{(2)}_{1,1}=-a^{(0)}_{2,1}=a^{(2)}_{3,1}=N, \tag{5.4.50a}$$

$$b^{(0)}_{0,0}=b^{(2)}_{1,0}=-b^{(0)}_{2,0}=b^{(2)}_{3,0}=-b^{(0)}_{0,1}=-b^{(2)}_{1,1}=-b^{(0)}_{2,1}=b^{(2)}_{3,1}=N. \tag{5.4.50b}$$

Let us focus on the coefficients on the zero mode contributions in the Klein bottle amplitude (5.4.28), the annulus amplitude (5.4.41) and the Möbius strip amplitude (5.4.49). The RR-tadpole cancellation condition (5.3.3) leads to

$$0 = 16 + \frac{N^2}{4} - 4N = \frac{1}{4}(N-8)^2. \tag{5.4.51}$$

The number of one stack of the D-branes is $N = 8$ to cancel the RR-tadpole. Taking account of (5.4.37) and (5.4.50b), the gauge groups are determined as $(Sp(2))^4$ for the **DCC** model.

For the **CCC** model, one of its orientifold actions is given by

$$\mathcal{R} = (\mathbf{a}, \mathbf{a}, \mathbf{a}). \tag{5.4.52}$$

On the other hand, the element $\mathcal{R}\theta$ in the orientifold group is given by

$$\mathcal{R}\theta = (-\mathbf{b}, \mathbf{b}, \mathbf{a}). \tag{5.4.53}$$

As seen in Table 5.1, these two elements yield different numbers of O-planes. To show this, we evaluate the RR-tadpole amplitude in the following way: In the tree channel the Klein bottle amplitude is

$$\begin{aligned}\tilde{\mathcal{K}} = c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty dl \Big(& 20\tilde{\mathcal{L}}_{\mathcal{K}}^{(0,0)}\tilde{\mathcal{K}}^{(0,0)} - 32\tilde{\mathcal{L}}_{2,8}\tilde{\mathcal{K}}^{(1,0)} \\ & - 80\tilde{\mathcal{L}}_{2,8}\tilde{\mathcal{K}}^{(2,0)} - 32\tilde{\mathcal{L}}_{2,8}\tilde{\mathcal{K}}^{(3,0)} \\ & - 40(\tilde{\mathcal{L}}_{2,8} + \tilde{\mathcal{L}}_{1,4})\mathcal{K}^{(0,1)} + 64\tilde{\mathcal{K}}^{(1,1)} \\ & - 40(\tilde{\mathcal{L}}_{2,8} + \tilde{\mathcal{L}}_{1,4})\tilde{\mathcal{K}}^{(2,1)} - 64\tilde{\mathcal{K}}^{(3,1)}\Big).\end{aligned} \tag{5.4.54}$$

The prefactors do not correspond to that from the complete projector. The annulus and the Möbius strip amplitudes are also described as

$$\begin{aligned}\tilde{\mathcal{A}} = \frac{c}{16}(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty dl \Big(& (M^2 + 4N^2)\tilde{\mathcal{L}}_{\mathcal{A}}^{(0,0)}\tilde{\mathcal{A}}^{(0,0)} - 8MN\tilde{\mathcal{L}}_{2,2}\tilde{\mathcal{A}}^{(1,0)} \\ & - 4(M^2 + 4N^2)\tilde{\mathcal{L}}_{2,2}\tilde{\mathcal{A}}^{(2,0)} - 8MN\tilde{\mathcal{L}}_{2,2}\tilde{\mathcal{A}}^{(3,0)} \\ & - 4(M^2\tilde{\mathcal{L}}_{2,2} + 4N^2\tilde{\mathcal{L}}_{1,4})\mathcal{A}^{(0,1)} + 16MN\tilde{\mathcal{A}}^{(1,1)} \\ & - 4(M^2\tilde{\mathcal{L}}_{2,2} + 4N^2\tilde{\mathcal{L}}_{1,4})\tilde{\mathcal{A}}^{(2,1)} - 16MN\tilde{\mathcal{A}}^{(3,1)}\Big),\end{aligned} \tag{5.4.55a}$$

$$\begin{aligned}\tilde{\mathcal{M}} = -c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty dl \Big(& 2(M+N)\tilde{\mathcal{L}}_{\mathcal{M}}^{(0,0)}\tilde{\mathcal{M}}^{(0,0)} - 2(M+4N)\tilde{\mathcal{L}}_{8,2}\tilde{\mathcal{M}}^{(1,0)} \\ & - 8(M+N)\tilde{\mathcal{L}}_{8,2}\tilde{\mathcal{M}}^{(2,0)} - 2(M+4N)\tilde{\mathcal{L}}_{8,2}\tilde{\mathcal{M}}^{(3,0)} \\ & - 8(M\tilde{\mathcal{L}}_{8,2} + N\tilde{\mathcal{L}}_{4,4})\mathcal{M}^{(0,1)} + 4(M+4N)\tilde{\mathcal{M}}^{(1,1)} \\ & - 8(M\tilde{\mathcal{L}}_{8,2} + N\tilde{\mathcal{L}}_{4,4})\tilde{\mathcal{M}}^{(2,1)} - 4(M+4N)\tilde{\mathcal{M}}^{(3,1)}\Big),\end{aligned} \tag{5.4.55b}$$

where M and N are the numbers of D-branes which are invariant under the set of orientifold actions $\{\mathcal{R}, \mathcal{R}\theta^2, \mathcal{R}\phi, \mathcal{R}\theta^2\phi\}$, and under the other set of actions $\{\mathcal{R}\theta, \mathcal{R}\theta^3, \mathcal{R}\theta\phi, \mathcal{R}\theta^3\phi\}$, respectively. In the Möbius strip amplitude we have set

$$a_{0,0}^{(0)} = a_{1,0}^{(2)} = -a_{2,0}^{(0)} = a_{3,0}^{(2)} = -a_{0,1}^{(0)} = -a_{1,1}^{(2)} = -a_{2,1}^{(0)} = a_{3,1}^{(2)} = M, \tag{5.4.56a}$$

$$b_{0,0}^{(0)} = b_{1,0}^{(2)} = -b_{2,0}^{(0)} = b_{3,0}^{(2)} = -b_{0,1}^{(0)} = -b_{1,1}^{(2)} = -b_{2,1}^{(0)} = b_{3,1}^{(2)} = N. \tag{5.4.56b}$$

Focusing on the coefficient in (5.4.54), (5.4.55a) and (5.4.55b), we obtain the RR-tadpole cancellation conditions (5.3.3),

$$0 = 20 + \frac{1}{16}(M^2 + 4N^2) - 2(M+N) = \frac{1}{16}\Big((M-16)^2 + (N-4)^2\Big), \tag{5.4.57a}$$

$$0 = -32 - \frac{MN}{2} + 2(M+4N) = -\frac{1}{2}(M-16)(N-4), \tag{5.4.57b}$$

and find $M = 16$ and $N = 4$. This indicates that we should insert sets of different numbers of D-branes in an appropriate way in several kinds of non-factorizable tori.

The open string massless spectrum is given in Table 5.5. The multiplicities of twisted states spectra depend on the intersection numbers [43] (see Table 5.4). We see that the **CCD** and **DCC** models are distinct from the **CDD** model despite the same numbers of O-planes, and actually these four models have different spectra. For the closed string the numbers of massless states are considerably reduced due to their Hodge numbers in [46,47].

sectors	**CCC**	**CCD**	**CDD**	**DCC**	representations
untwisted	$1V$				$Sp[M/4]^2 \times Sp[N/4]^2$
	$3C$				$(\boxminus, 1; 1, 1) \oplus (1, \boxminus; 1, 1)$ $\oplus(1, 1; \boxminus, 1) \oplus (1, 1; 1, \boxminus)$
$\theta + \theta^3$	$2C$	$2C$	$4C$	$2C$	$(\square, \square; 1, 1) \oplus (1, 1; \square, \square)$
θ^2	$4C$	$2C$	$4C$	$2C$	$(\boxminus, 1; 1, 1) \oplus (1, \boxminus; 1, 1)$
	$1C$	$2C$	$4C$	$2C$	$(1, 1; \boxminus, 1) \oplus (1, 1; 1, \boxminus)$
ϕ	$4C$	$2C$	$4C$	$2C$	$(\square, 1; \square, 1)$
	$1C$	$2C$	$4C$	$2C$	$(1, \square; 1, \square)$
$\theta\phi + \theta^3\phi$	$2C$	$2C$	$4C$	$2C$	$(\square, 1; 1, \square) \oplus (1, \square; \square, 1)$
$\theta^2\phi$	$4C$	$2C$	$4C$	$2C$	$(\square, 1; \square, 1)$
	$1C$	$2C$	$4C$	$2C$	$(1, \square; 1, \square)$

Table 5.5: *Open string massless spectra of $\mathbb{Z}_4 \times \mathbb{Z}_2$ orbifold on the D_6 lattice. The symbols "V" and "C" denote the vector and chiral multiplets, respectively.*

5.4.6 $\mathbb{Z}_2 \times \mathbb{Z}_2$ model

Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$, the calculation is similar to the examples in the previous section. The new feature in $\mathbb{Z}_2 \times \mathbb{Z}_2$ is that we have more freedom to choose orbifold actions in comparison with the case of $\mathbb{Z}_4 \times \mathbb{Z}_2$.

For $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the D_6 lattice (5.3.27), all the point group elements can be given by the use of **a** and **b** in (3.1.5), see Appendix A. In the case of the **CCC** orientifold with $\mathcal{R} = (\mathbf{a}, \mathbf{a}, \mathbf{a})$, the point group elements θ and ϕ are

$$\theta: \ (-\mathbf{1}, -\mathbf{1}, \mathbf{1}), \qquad \phi: \ (\mathbf{1}, -\mathbf{1}, -\mathbf{1}). \tag{5.4.58}$$

The orientifold group elements including $\Omega\mathcal{R}$ are

$$\{\Omega\mathcal{R},\ \Omega\mathcal{R}\theta,\ \Omega\mathcal{R}\phi,\ \Omega\mathcal{R}\theta\phi\}, \tag{5.4.59}$$

and these elements generate O6-planes respectively. From Table 5.2 the numbers of O6-planes are read two for each elements.

In the **CCD** orientifold with $\mathcal{R} = (\mathbf{a}, \mathbf{a}, \mathbf{b})$, we have two distinct pairs of the point group elements:

$$\begin{cases} \theta\text{: } (\mathbf{1}, -\mathbf{1}, -\mathbf{1}) \\ \phi\text{: } (-\mathbf{1}, -\mathbf{1}, \mathbf{1}) \end{cases}, \qquad \begin{cases} \theta\text{: } (\mathbf{1}, -\mathbf{1}, -\mathbf{1}) \\ \phi\text{: } (-\mathbf{1}, -\mathbf{a}, \mathbf{b}) \end{cases}. \tag{5.4.60}$$

The numbers of O-planes generated by the former orbifold actions are also two. In the latter case, the $\Omega\mathcal{R}$ and $\Omega\mathcal{R}\theta$ ($\Omega\mathcal{R}\phi$ and $\Omega\mathcal{R}\theta\phi$) generate two (four) O6-planes, respectively. We can classify the distinct orientifold models on the Lie root lattices, and the other possible elements on the D_6 lattice are listed in Table 5.6. We should notice that even though the numbers of O6-planes are the same in any three-cycles in $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold models, those of non-factorizable models can be different.

Finally we check the RR-tadpole cancellation in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ **CCC** model on the D_6 lattice. The contribution from ϕ- and $\theta\phi$-twisted sectors are the same as θ-sector for the **CCC** model on the D_6 lattice. The RR-tadpole cancellation is satisfied with $N = 4$ as we can see the following amplitudes in the tree channel. The Klein bottle amplitude is given as

$$\tilde{\mathcal{K}} = 32c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty dl \Big(\tilde{\mathcal{L}}_{\mathcal{K}}^{(0,0)} \tilde{\mathcal{K}}^{(0,0)} - 4\tilde{\mathcal{L}}_{2,8} \tilde{\mathcal{K}}^{(1,0)} - 4\tilde{\mathcal{L}}_{2,8} \tilde{\mathcal{K}}^{(0,1)} - 4\tilde{\mathcal{L}}_{2,8} \tilde{\mathcal{K}}^{(1,1)} \Big). \tag{5.4.61}$$

The annulus and the Möbius amplitudes are also given as

$$\tilde{\mathcal{A}} = \frac{N^2 c}{8}(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty dl \Big(\tilde{\mathcal{L}}_{\mathcal{A}}^{(0,0)} \tilde{\mathcal{A}}^{(0,0)} - 4\tilde{\mathcal{L}}_{2,2} \tilde{\mathcal{A}}^{(1,0)} - 4\tilde{\mathcal{L}}_{2,2} \tilde{\mathcal{A}}^{(0,1)} - 4\tilde{\mathcal{L}}_{2,2} \tilde{\mathcal{A}}^{(1,1)} \Big), \tag{5.4.62a}$$

$$\tilde{\mathcal{M}} = -4Nc(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}}) \int_0^\infty dl \Big(\tilde{\mathcal{L}}_{\mathcal{M}}^{(0,0)} \tilde{\mathcal{M}}^{(0,0)} - 4\tilde{\mathcal{L}}_{8,2} \tilde{\mathcal{M}}^{(1,0)} - 4\tilde{\mathcal{L}}_{8,2} \tilde{\mathcal{M}}^{(0,1)} - 4\tilde{\mathcal{L}}_{8,2} \tilde{\mathcal{M}}^{(1,1)} \Big). \tag{5.4.62b}$$

Lattice	Label	reps. of $\mathcal{R}$	Orbifold		# of O6-planes			
			rep. of θ	rep. of ϕ	$\mathcal{R}$	$\mathcal{R}\theta$	$\mathcal{R}\phi$	$\mathcal{R}\theta\phi$
D_6	**CCC**	$(\mathbf{a},\mathbf{a},\mathbf{a})$	$(\mathbf{1},-\mathbf{1},-\mathbf{1})$	$(-\mathbf{1},-\mathbf{1},\mathbf{1})$	4	4	4	4
	CCD	$(\mathbf{a},\mathbf{a},\mathbf{b})$	$(\mathbf{1},-\mathbf{1},-\mathbf{1})$	$(-\mathbf{1},-\mathbf{1},\mathbf{1})$	2	2	2	2
			$(\mathbf{1},-\mathbf{1},-\mathbf{1})$	$(-\mathbf{1},-\mathbf{a},\mathbf{b})$	2	2	4	4
	CDD	$(\mathbf{a},\mathbf{b},\mathbf{b})$	$(\mathbf{1},-\mathbf{1},-\mathbf{1})$	$(-\mathbf{1},-\mathbf{1},\mathbf{1})$	1	1	1	1
			$(\mathbf{1},-\mathbf{1},-\mathbf{1})$	$(-\mathbf{1},\mathbf{b},-\mathbf{b})$	1	1	4	4
			$(-\mathbf{1},\mathbf{1},-\mathbf{1})$	$(\mathbf{a},-\mathbf{1},-\mathbf{b})$	1	1	2	2
			$(\mathbf{a},-\mathbf{1},-\mathbf{b})$	$(-\mathbf{a},\mathbf{b},-\mathbf{1})$	1	2	2	4
	DDD	$(\mathbf{b},\mathbf{b},\mathbf{b})$	$(-\mathbf{1},-\mathbf{1},\mathbf{1})$	$(\mathbf{1},-\mathbf{1},-\mathbf{1})$	2	2	2	2
			$(\mathbf{1},-\mathbf{1},-\mathbf{1})$	$(-\mathbf{1},-\mathbf{b},\mathbf{b})$	2	2	2	2
			$(-\mathbf{1},-\mathbf{b},\mathbf{b})$	$(\mathbf{b},-\mathbf{1},-\mathbf{b})$	2	2	2	2

Table 5.6: $\mathbb{Z}_2 \times \mathbb{Z}_2$ *orbifold models on the* D_6 *Lie root lattice.*

We observe that in any amplitude the prefactors are given by the complete projector (5.4.29).

There exists an exception in this classification for the $A_3 \times A_3$ lattice as mentioned before. We define the lattice $A_3 \times A_3$ by using the simple roots

$$\begin{aligned} \alpha_1 &= \mathbf{e}_1 - \mathbf{e}_2, & \alpha_4 &= \mathbf{e}_4 - \mathbf{e}_5, \\ \alpha_2 &= \mathbf{e}_2 - \mathbf{e}_3, & \alpha_5 &= \mathbf{e}_5 - \mathbf{e}_6, \\ \alpha_3 &= \mathbf{e}_2 + \mathbf{e}_3, & \alpha_6 &= \mathbf{e}_5 + \mathbf{e}_6. \end{aligned} \tag{5.4.63}$$

In this base $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds are obtained in a similar manner of the D_6 lattice[5]. Note that the action $\mathcal{R} = (*, \mathbf{b}, *)$, where $*$ is $\mathbf{b}$, $\mathbf{a}$ or $\mathbf{1}$, is forbidden due to the lattice structure. The outer automorphism between two A_3's generates an exceptional action

$$\mathcal{R}: \ \alpha_i \leftrightarrow \alpha_{i+3}, \quad i = 1, 2, 3. \tag{5.4.64}$$

If we redefine the base of $A_3 \times A_3$ as

$$\begin{aligned} \alpha_1 &= \mathbf{e}_1 - \mathbf{e}_3, & \alpha_4 &= \mathbf{e}_2 - \mathbf{e}_4, \\ \alpha_2 &= \mathbf{e}_3 - \mathbf{e}_5, & \alpha_5 &= \mathbf{e}_4 - \mathbf{e}_6, \\ \alpha_3 &= \mathbf{e}_3 + \mathbf{e}_5, & \alpha_6 &= \mathbf{e}_4 + \mathbf{e}_6, \end{aligned} \tag{5.4.65}$$

[5] It may seem that the classification with **b**,**a** and **1** elements is missing the action $\mathcal{R}: \ \alpha_i \to -\alpha_i$ with $i = 1, 2, 3$, however this action is included in orientifold groups, e.g. the $\mathcal{R}\theta\phi$ action of **DCD** model on Table 5.7.

the exceptional action is expressed by $(\mathbf{b}, \mathbf{b}, \mathbf{b})$ in the orthogonal $\mathbf{e}_i$ basis:

$$\mathcal{R}: \ \mathbf{e}_1 \leftrightarrow \mathbf{e}_2, \quad \mathbf{e}_3 \leftrightarrow \mathbf{e}_4, \quad \mathbf{e}_5 \leftrightarrow \mathbf{e}_6. \tag{5.4.66}$$

Actually this element gives only one inequivalent element including the outer automorphism, and we label it as $(\mathbf{DDD})'$.

Including this orientifold action we obtain all the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the $A_3 \times A_3$ lattice in Table 5.7.

Lattice	Label	rep. of $\mathcal{R}$	Orbifold		# of O6-planes			
			rep. of θ	rep. of ϕ	$\mathcal{R}$	$\mathcal{R}\theta$	$\mathcal{R}\phi$	$\mathcal{R}\theta\phi$
$A_3 \times A_3$	**CCC**	$(\mathbf{a}, \mathbf{a}, \mathbf{a})$	$(\mathbf{1}, -\mathbf{1}, -\mathbf{1})$	$(-\mathbf{1}, -\mathbf{1}, \mathbf{1})$	2	2	2	2
			$(-\mathbf{1}, \mathbf{1}, -\mathbf{1})$	$(\mathbf{a}, -\mathbf{1}, -\mathbf{a})$	2	2	2	8
	CCD	$(\mathbf{a}, \mathbf{a}, \mathbf{b})$	$(\mathbf{1}, -\mathbf{1}, -\mathbf{1})$	$(-\mathbf{1}, -\mathbf{1}, \mathbf{1})$	2	2	2	2
			$(\mathbf{1}, -\mathbf{1}, -\mathbf{1})$	$(-\mathbf{1}, \mathbf{a}, -\mathbf{b})$	2	2	2	2
			$(-\mathbf{1}, \mathbf{1}, -\mathbf{1})$	$(\mathbf{a}, -\mathbf{1}, -\mathbf{b})$	2	2	2	8
	DCD	$(\mathbf{b}, \mathbf{a}, \mathbf{b})$	$(\mathbf{1}, -\mathbf{1}, -\mathbf{1})$	$(-\mathbf{1}, -\mathbf{1}, \mathbf{1})$	2	2	2	2
			$(\mathbf{1}, -\mathbf{1}, -\mathbf{1})$	$(-\mathbf{1}, \mathbf{a}, -\mathbf{b})$	2	2	2	2
			$(-\mathbf{1}, \mathbf{1}, -\mathbf{1})$	$(\mathbf{b}, -\mathbf{1}, -\mathbf{b})$	2	2	2	8
			$(-\mathbf{1}, \mathbf{a}, -\mathbf{b})$	$(\mathbf{b}, -\mathbf{a}, -\mathbf{1})$	2	2	2	8
			$(-\mathbf{1}, -\mathbf{a}, \mathbf{b})$	$(-\mathbf{b}, \mathbf{a}, -\mathbf{1})$	2	2	2	2
	$(\mathbf{DDD})'$	$(\mathbf{b}, \mathbf{b}, \mathbf{b})$	$(\mathbf{1}, -\mathbf{1}, -\mathbf{1})$	$(-\mathbf{1}, -\mathbf{1}, \mathbf{1})$	1	1	1	1

Table 5.7: $\mathbb{Z}_2 \times \mathbb{Z}_2$ *orbifold models on the* $A_3 \times A_3$ *Lie root lattice.*

Chapter 6

Conclusions

In this work we have developed the orbifolds on non-factorizable tori T^6, and applied it to heterotic and Type IIA orientifolds. These investigations are motivated by the phenomenology beyond the Standard Model, and actually we constructed a few GUT-like models with three generations.

In chapter 3 we developed non-factorizable orbifolds on the Lie lattices, and explained the way to classify them. Our construction based on the Lie root lattices gives fairly complete classification for the orbifolds on six-tori T^6. This is because the tori on the Lie root lattices keep higher symmetries, and the other orbifolds would be obtained by continuous deformation of them. So far the orbifolds are constructed by means of the Coxeter elements. In our classification, we begin with the sixteen distinct Lie root lattice in six-dimensional space, and find out their automorphisms. Since the point group elements of orbifolds are defined as automorphisms of the lattices, our approach is rather easy and intuitive. The complete tables in our classification are listed in appendix C.

Standing on these basics of the orbifolds, in chapter 4 we constructed the $\mathbb{Z}_3 \times \mathbb{Z}_3$ heterotic orbifold on the E_6 torus. The reason why this orbifold is interesting is that it includes three fixed tori in the θ, ϕ and $\theta\phi$-twisted sectors respectively, and easily leads to three-family spectra. We presented the examples of $\mathcal{N} = 1$ three-family models from the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold on the E_6 torus. Our assumption is quite simple, i.e., a compactification on the orbifold with two gauge embedding in the models. As we see in table 4.3 and 4.2, the spectrum of these constructions are particularly simple as heterotic models, i.e small numbers of extra matters. Because the main part of extra matters are charged with only the hidden sector gauge groups, they are not unfavorable for the phenomenological motivation, but may be favorable as a candidate of the dark matter. The models will have strongly coupled sectors in the low energy and messenger-like states charged with both the hidden and visible sector gauge groups. Due to the complexity of the hidden sector spectrum it is difficult to analyze the strong dynamics, however we hope spontaneous supersymmetry breaking owing to it. In non-factorizable orbifolds the number of the twisted states can be smaller by factors of two or three than that of the factorizable orbifold. This makes it easier to obtain small number of generations and strong couplings of the hidden sector gauge group. Generally non-factorizable orbifolds have such favorable features. We also investigated the general properties of the three point interactions allowed by the space group selection rules. The twenty seven states in the $\theta\phi^2$-sector are divided to three flavors with respect to their interactions, and we observe three-flavor interactions with mixing. At the same time we have less freedom of the moduli space. These facts imply that this E_6 orbifold is quite symmetric from the viewpoint of the six dimensional space, and such a symmetric space would be natural for compact space. The phenomenological problem of the GUT-like models is that they do not include adjoint higgs which cause GUT group breaking. This is a notorious obstacle for the level $k = 1$ construction of the heterotic string. In this work we consider the models with no Wilson line, because it seems that Wilson lines break the structure of the degenerate three fixed tori. However the inclusion of Wilson lines may lead to other three-family model whose family is generated from different twisted sectors. If we introduce continuous Wilson lines [64,65,77,82], we can realize models with the rank-reduced gauge groups. Then it is interesting to explore these possibilities to realize more realistic models on the orbifold on the E_6 torus.

In chapter 5 we studied the RR-tadpole cancellation condition in Type II string models compactified on six dimensional tori and orbifolds. We obtained a simple derivation of RR-tadpole cancellation condition by the use of the Lefschetz fixed point theorem. As expected the RR-tadpole contributions are cancelled by adding an appropriate number of D-branes parallel to the O-planes. As explained in detail, the Lefschetz fixed point theorem provides an intuitive picture to non-factorizable models, and we easily showed a way to construct orientifold models on tori and orbifolds. In $D = 4$, $\mathcal{N} = 1$ $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds, mainly the factorizable models on $T^2 \times T^2 \times T^2$ have been constructed and investigated. We gave the classifications in Type IIA orientifold models with O6-planes, and obtained many new models. Since the condition derived in (5.3.15) is the necessary condition for orbifolds, we performed explicit calculations for $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models, and confirmed the RR-tadpole calculations. It is expected that even in other non-factorizable orbifold models the RR-tadpole cancellation should be checked in the same calculation. We further found many non-factorizable $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds in which the numbers of O-planes depend on the three-cycles left invariant under the orbifold projections in Table 5.6 and in Table 5.7. These features are not seen in factorizable models, and will provide new possibilities for model constructions. On the other hand, since the metric of non-factorizable tori is changed to B-field via T-duality, our consideration should be related to compactification with such backgrounds. Actually in heterotic orbifolds there are some coincidences between non-factorizable models and factorizable models with generalized discrete torsion [60]. Our results indicate that there would be a possibility to construct various class of $D = 4$, $\mathcal{N} = 1$ models with different set of chiral spectra from other well-known (non-)factorizable models.

An UV complete string model should contain various phenomenological features, such as moduli stabilization and spontaneous supersymmetry breaking in itself. The aim is to extend the string model constructions from these building blocks to more detailed phenomenology. I have been trying to investigate compactifications which give $N = 1$ models with appropriate Yukawa interactions. Then assuming supersymmetry breaking or including spontaneous supersymmetry breaking, we can calculate the mass spectra of the models. I expect these studies provide interesting candidate theories for new physics from the LHC experiment and the cosmology.

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Appendix A

Lie root lattices

In this appendix we give the definitions of the Lie root lattices, and explain that six-tori on them are classified to sixteen classes of lattices. We also comment on some relations of lattices under $\mathbb{Z}_2$ action $\mathcal{R}$, that are used to derive the formura for the tadpole cancellation condition (5.3.15).

A.1 Lie algebra and the lattices

We use the words of the Lie algebra in order to define the shape of tori defined in (5.3.8). The Lie algebras whose orders are within six are A_N, B_N, C_N, D_N, E_6, F_4 and G_2. The simple roots α_i of these Lie algebras can be given as follows:

$$\begin{array}{lllll}
A_N: & \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, & & i = 1, \dots, N & \\
B_N: & \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, & \alpha_N = \mathbf{e}_N, & i = 1, \dots, N-1 & \\
C_N: & \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, & \alpha_N = 2\mathbf{e}_N, & i = 1, \dots, N-1 & \\
D_N: & \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, & \alpha_N = \mathbf{e}_{N-1} + \mathbf{e}_N, & i = 1, \dots, N-1 & \\
E_6: & \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, & \alpha_6 = \mathbf{e}_4 + \mathbf{e}_5, & i = 1, \dots, 4 & \\
 & \multicolumn{4}{l}{\alpha_5 = \frac{1}{2}(-\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 + \mathbf{e}_5 + \sqrt{3}\mathbf{e}_6)} \\
F_4: & \alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, & \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \ \alpha_3 = 2\mathbf{e}_3, & \alpha_4 = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 & \\
G_2: & \alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, & \alpha_2 = -\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3, & &
\end{array} \tag{A.1.1}$$

where $\mathbf{e}_i$'s are unit vectors whose scalar product is defined as $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. The Dynkin diagrams are drawn in Figure A.1.

From these diagrams one can easily find a set of equivalence relations (isomorphism) among the simple Lie algebra,

$$A_1 \sim B_1 \sim C_1, \qquad B_2 \sim C_2, \qquad A_3 \sim D_3. \tag{A.1.2}$$

We further find the equivalence relations *from the Lie root lattice point of view*:

$$A_2 \sim G_2, \qquad B_2 \sim D_2 \sim (A_1)^2, \qquad D_4 \sim F_4, \tag{A.1.3a}$$

$$B_N \sim (A_1)^N, \qquad C_N \sim D_N, \tag{A.1.3b}$$

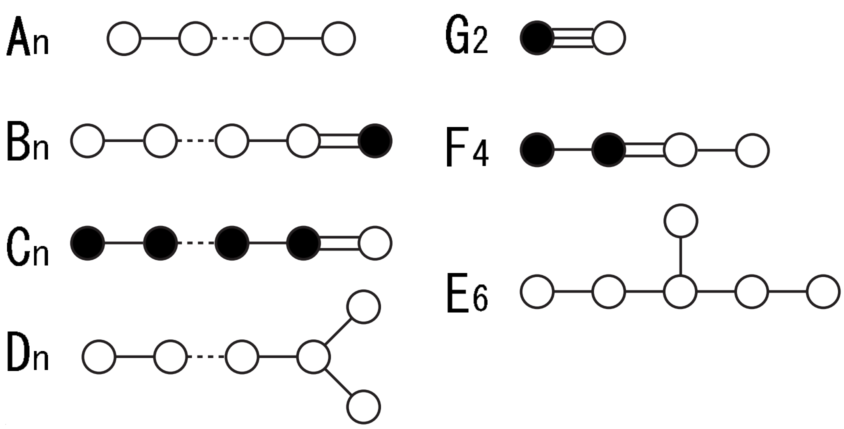

Figure A.1: *Dynkin diagrams of the simple Lie algebras.*

where $(A_1)^2 = A_1 \times A_1$. Here we assumed the most symmetric cases, where the lengths of the shortest roots are equal between the lattices given as direct products. Since we are interested in symmetries of the lattices, the assumption is rational. We often use these equivalence relations in the classification of the six-tori.

Taking the direct products of tori generated from these lattices, we obtain six-tori in terms of the Lie root lattices. We conclude that there are only twelve inequivalent non-factorizable six-tori and four factorizable ones[1] in such a way as in Table 3.1.

[non-factorizable tori]			
A_6	D_6	E_6	
$A_5 \times A_1$	$A_4 \times A_2$	$A_4 \times (A_1)^2$	
$D_5 \times A_1$	$D_4 \times A_2$	$D_4 \times (A_1)^2$	
$A_3 \times A_3$	$A_3 \times A_2 \times A_1$	$A_3 \times (A_1)^3$	
[factorizable tori]			
$(A_2)^3$	$(A_2)^2 \times (A_1)^2$	$A_2 \times (A_1)^4$	$(A_1)^6$

Table A.1: *All the Lie root lattices in six dimensions.*

[1]Most of other six-tori would be obtained by the continuous deformation of moduli of these tori [46].

A.2 Comments on lattices

In this appendix we briefly summarize conventions of the (sub-)lattice and its dual lattice space for a $\mathbb{Z}_2$ action $\mathcal{R}$ in the following way:

$$\begin{array}{rcl} \Lambda_{\mathcal{R},\perp} & : & \text{lattice projected out by the action } \mathcal{R},\ \Lambda_{\mathcal{R},\perp} \equiv \dfrac{1+\mathcal{R}}{2}\Lambda \\ \Lambda_{\mathcal{R},\mathrm{inv}} & : & \mathcal{R} \text{ invariant sublattice} \\ \Lambda^* & : & \text{dual lattice of } \Lambda, \text{ for its base } \alpha_j \cdot \alpha_i^* = \delta_{ji},\ \alpha_j \in \Lambda,\ \alpha_i^* \in \Lambda^* \end{array}$$

These three lattice spaces are closely related to one another. Introducing a lattice $\Lambda_{-\mathcal{R},\perp}$ which is projected out by the $-\mathcal{R}$ action on it, then we find the following non-trivial equations:

$$\begin{aligned} \Lambda^*_{\mathcal{R},\perp} &= (\Lambda_{\mathcal{R},\mathrm{inv}})^*, && \text{(A.2.1a)} \\ \mathrm{Vol}(\Lambda) &= \mathrm{Vol}(\Lambda_{\mathcal{R},\mathrm{inv}}) \cdot \mathrm{Vol}(\Lambda_{-\mathcal{R},\perp}), && \text{(A.2.1b)} \\ \mathrm{Vol}(\Lambda^*) &= \mathrm{Vol}(\Lambda)^{-1}. && \text{(A.2.1c)} \end{aligned}$$

Let us analyze in a more concrete way. For example, we consider the four-dimensional D_4 Lie root lattice Λ and its dual lattice Λ^* based on

$$\Lambda : \begin{cases} (\ 1,-1,\ \ 0,\ \ 0) \\ (\ 0,\ \ 1,-1,\ \ 0) \\ (\ 0,\ \ 0,\ \ 1,-1) \\ (\ 0,\ \ 0,\ \ 1,\ \ 1) \end{cases} \qquad \Lambda^* : \begin{cases} (\ 1, 0, 0,\ \ 0) \\ (\ 1, 1, 0,\ \ 0) \\ (\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \\ (\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2},\ \ \frac{1}{2}) \end{cases} \tag{A.2.2}$$

and we give a $\mathbb{Z}_2$ action $\mathcal{R}$ on the D_4 lattice as

$$\mathcal{R} = \mathrm{diag}(1, 1, -1, -1). \tag{A.2.3}$$

Then, we can obtain the basis vectors in the lattices $\Lambda_{\mathcal{R},\perp}$, $\Lambda_{\mathcal{R},\mathrm{inv}}$, $\Lambda^*_{\mathcal{R},\perp}$ and $\Lambda^*_{\mathcal{R},\mathrm{inv}}$ in the following forms:

$$\Lambda_{\mathcal{R},\perp} : \begin{cases} (1,-1,0,0) \\ (0,1,0,0) \end{cases} \qquad \Lambda_{\mathcal{R},\mathrm{inv}} : \begin{cases} (1,-1,0,0) \\ (0,2,0,0) \end{cases} \tag{A.2.4a}$$

$$\Lambda^*_{\mathcal{R},\perp} : \begin{cases} (1,0,0,0) \\ (\frac{1}{2},\frac{1}{2},0,0) \end{cases} \qquad \Lambda^*_{\mathcal{R},\mathrm{inv}} : \begin{cases} (1,0,0,0) \\ (1,1,0,0) \end{cases} \tag{A.2.4b}$$

Thus we easily see the relation among various lattice spaces:

$$\begin{array}{ccccc} \Lambda_{\mathcal{R},\perp} & \xleftarrow{\perp} & \Lambda & \xrightarrow{\mathrm{inv}} & \Lambda_{\mathcal{R},\mathrm{inv}} \\ \updownarrow * & & \updownarrow * & & \updownarrow * \\ \Lambda^*_{\mathcal{R},\mathrm{inv}} & \xleftarrow{\mathrm{inv}} & \Lambda^* & \xrightarrow{\perp} & \Lambda^*_{\mathcal{R},\perp} \end{array} \tag{A.2.5}$$

Appendix B

String one-loop amplitudes

The contribution from oscillators in the one-loop string amplitude are evaluated here. At first the definitions of the theta functions, the eta function and some useful equations are given.

B.1 Theta function and some useful formulae

For the modular transformation of the one-loop amplitudes, we often use the generalized Poisson resummation formula,

$$\sum_{m_i \in \mathbb{Z}} \exp[-\pi t (m-b)_i A_{ij} (m-b)_j] = \frac{1}{t^{\dim(A)} \sqrt{\det A}} \sum_{m_i \in \mathbb{Z}} \exp[-\frac{\pi}{t} n_i A^{-1}_{ij} n_j + 2i\pi b_i n_i] \quad \text{(B.1.1)}$$

In the case with n_i (for $i = 1$) and $b = 0$, it is simplified to the Poisson resummation formula

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2/t} = \sqrt{t} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}. \quad \text{(B.1.2)}$$

We also give basic definitions of the Jacobi theta function and Dedekind eta function which we frequently employ in the paper,

$$\begin{aligned} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (t) &= \sum_{n \in \mathbb{Z}} q^{(n+\alpha)^2/2} \, e^{2\pi i (n+\alpha)\beta}, \\ \eta(t) &= q^{1/24} \left(1 + \sum_{n=1}^{\infty} \beta (-1)^n \left[q^{n(3n-1)/2} + q^{n(3n+1)/2} \right] \right) . \end{aligned} \quad \text{(B.1.3)}$$

setting $q \equiv e^{-2\pi t}$, or in product forms:

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (t) = e^{2\pi i \alpha\beta} q^{\alpha^2/2} \prod_{n=1}^{\infty} (1 - q^n) \left(1 + e^{2\pi i \beta} q^{n-1/2+\alpha} \right) \left(1 + e^{-2\pi i \beta} q^{n-1/2-\alpha} \right), \quad \text{(B.1.4)}$$

$$\eta(t) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \text{(B.1.5)}$$

where α has to be chosen within the range $-1/2 < \alpha \leq 1/2$.

The theta functions have the following identities:

$$\begin{aligned}\vartheta\begin{bmatrix}\alpha\pm 1\\ \beta\end{bmatrix}(t) &= \vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(t),\\ \vartheta\begin{bmatrix}\alpha\\ \beta\pm 1\end{bmatrix}(t) &= e^{\pm 2\pi i\alpha}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(t),\end{aligned} \tag{B.1.6}$$

The modular T transformations of the theta and eta functions are

$$\begin{aligned}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(t) &= e^{\pi i(\alpha^2+\alpha)}\,\vartheta\begin{bmatrix}\alpha\\ \beta-\alpha-1/2\end{bmatrix}(t+1)\,,\\ \eta\,(t) &= e^{-\pi i/12}\,\eta(t+1).\end{aligned} \tag{B.1.7}$$

The modular S transformations are

$$\begin{aligned}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(t^{-1}\right) &= \sqrt{t}\,e^{2\pi i\alpha\beta}\vartheta\begin{bmatrix}-\beta\\ \alpha\end{bmatrix}(t)\,,\\ \eta\left(t^{-1}\right) &= \sqrt{t}\,\eta(t).\end{aligned} \tag{B.1.8}$$

The Jacobi's abstruse identity is

$$\vartheta^4\begin{bmatrix}0\\ 0\end{bmatrix}(t)-\vartheta^4\begin{bmatrix}0\\ 1/2\end{bmatrix}(t)-\vartheta^4\begin{bmatrix}1/2\\ 0\end{bmatrix}(t)=0. \tag{B.1.9}$$

In order to transform the loop channel Möbius strip amplitude to that of the tree channel, we use the identity

$$\frac{\vartheta\begin{bmatrix}\alpha+1/2\\ \beta\end{bmatrix}}{\vartheta\begin{bmatrix}\alpha+1/2\\ \beta+1/2\end{bmatrix}}(-q)=e^{-\pi i\alpha}\frac{\vartheta\begin{bmatrix}(\alpha+1)/2\\ \alpha/2+\beta\end{bmatrix}\vartheta\begin{bmatrix}\alpha/2\\ (\alpha+1)/2+\beta\end{bmatrix}}{\vartheta\begin{bmatrix}(\alpha+1)/2\\ (\alpha+1)/2+\beta\end{bmatrix}\vartheta\begin{bmatrix}\alpha/2\\ \alpha/2+\beta\end{bmatrix}}\left(q^2\right) \tag{B.1.10}$$

for $-1 < \alpha \leq 0$.

From the product expansions we also have the form

$$\frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}{\eta}(t)=e^{2\pi i\alpha\beta}\,q^{\alpha^2/2-1/24}\prod_{n=1}^{\infty}\left(\left(1+q^{n-1/2+\alpha}e^{2\pi i\beta}\right)\left(1+q^{n-1/2-\alpha}e^{-2\pi i\beta}\right)\right). \tag{B.1.11}$$

There is an useful relation for the modular transformation as follows,

$$\lim_{\alpha\to 0}\frac{2\sin(\pi\alpha)}{\vartheta\begin{bmatrix}1/2\\ 1/2+\alpha\end{bmatrix}(t)}=-\frac{1}{\eta^3(t)}. \tag{B.1.12}$$

B.2 Partition function and modular transformation

In order to evaluate the consistency conditions of string theory, we need often calculate the one-loop partition functions [71, 138]. We give some results for later use. There are two moduli associated with a world sheet torus, i.e. $\tau = \tau_1 + i\tau_2$. Here we use σ_1 and σ_2 for the world sheet coordinates in order to avoid confusing.

For a right-mover complex boson, the partition function is

$$\begin{aligned}
\mathrm{Tr}(q^{N_B - a}) &= q^{\frac{1}{24}-\frac{1}{8}(2v-1)^2} \prod_{n=0}^{\infty} \left(1-q^{n+v}\right)^{-1} \prod_{n=1}^{\infty} \left(1-q^{n-v}\right)^{-1} \\
&= q^{\frac{1}{24}-\frac{1}{2}(v-\frac{1}{2})^2} \prod_{n=1}^{\infty} \left(\left(1+e^{2\pi i \frac{1}{2}} q^{n-\frac{1}{2}+(v-\frac{1}{2})}\right) \left(1+e^{-2\pi i \frac{1}{2}} q^{n-\frac{1}{2}-(v-\frac{1}{2})}\right) \right)^{-1} \\
&= e^{\pi i (v-\frac{1}{2})} \frac{\eta}{\vartheta \begin{bmatrix} 1/2+v \\ 1/2 \end{bmatrix}}
\end{aligned} \tag{B.2.1}$$

Here $0 < v_i < 1$. Similarly we obtain the one for a right-mover complex fermion. From the relation

$$q^{-\frac{1}{24}+\frac{1}{8}(2\beta-1)^2} \prod_{n=0}^{\infty} \left(1+q^{n+\beta}\right) \prod_{n=1}^{\infty} \left(1+q^{n-\beta}\right) = \frac{\vartheta \begin{bmatrix} \beta \\ 0 \end{bmatrix}}{\eta} \tag{B.2.2}$$

we have

$$\begin{aligned}
\mathrm{Tr}(q^{N_F - a}) &= \frac{\vartheta \begin{bmatrix} \frac{1}{2}+v \\ 0 \end{bmatrix}}{\eta} \quad \text{For R} \\
&= \frac{\vartheta \begin{bmatrix} v \\ 0 \end{bmatrix}}{\eta} \quad \text{For NS}
\end{aligned} \tag{B.2.3}$$

Generally orbifold twists are differently imposed in the σ_1 and σ_2 direction. When the twist in the σ_1 direction is $h = e^{2\pi i u}$ and in the σ_2 direction is $g = e^{2\pi i v}$, they define the boundary condition by pairs (g, h). Then the boundary conditions of (B.2.1) and (B.2.3) are $(g, 1)$. The modular T and S transformations are

$$\tau \to \tau + 1, \tag{B.2.4a}$$

$$\tau \to -\frac{1}{\tau}, \tag{B.2.4b}$$

respectively, and constitute the modular group $SL(2, \mathbb{Z})$.

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1. \tag{B.2.5}$$

By this transformation a transformation of coordinates $(\sigma_1, \sigma_2) \to (a\sigma_1 - b\sigma_2, -c\sigma_1 + d\sigma_2)$ generate the transformation of the boundary condition $(g, h) \to (g^d h^c, g^b h^a)$. For example $\tau \to \tau + 1$ change the boundary condition as $(g, 1) \to (g, g)$. The corresponding partition functions are obtained by the modular transformation of the partition function of the g-twisted sector $\mathrm{Tr}(q^{L_0-a})$, or are equivalent to the insertion as $\mathrm{Tr}(g\,q^{L_0-a})$ up to the phase factor. g acts on the operators as $g\alpha_{n+v}g^{-1} = e^{2\pi i v}\alpha_{n+v}$. To sum up, for a general boundary condition (g,h), the partition functions in the R and NS sectors are given by

$$\mathrm{Tr}(g\,q^{L_0-a})_{\mathrm{R}} = e^{\pi i(v-\frac{1}{2})}\frac{\vartheta\begin{bmatrix}\frac{1}{2}+v\\ u\end{bmatrix}}{\vartheta\begin{bmatrix}1/2+v\\ 1/2+u\end{bmatrix}} \quad \text{For R} \tag{B.2.6a}$$

$$\mathrm{Tr}(g\,q^{L_0-a})_{\mathrm{NS}} = e^{\pi i(v-\frac{1}{2})}\frac{\vartheta\begin{bmatrix}v\\ u\end{bmatrix}}{\vartheta\begin{bmatrix}1/2+v\\ 1/2+u\end{bmatrix}} \quad \text{For NS} \tag{B.2.6b}$$

We also need to evaluate the partition functions with insertions $(-1)^F$ from the GSO projection. Since the insertion of $(-1)^F$ changes the sign of fermionic operators, it leads to shift u to $u+1/2$ in the theta functions of the corresponding numerators.

B.3 Tadpole amplitudes

In this appendix we summarize descriptions of the string one-loop amplitudes whose topologies are given by the Klein bottle, the annulus and the Möbius strip in the loop channel [43, 58]. These are applied to discuss the RR-tadpole amplitudes in the main part of this paper. Here we start from the forms[1] in which the zero mode and the oscillator modes are factorized:

$$\mathcal{K} = 4c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}})\int_0^\infty \frac{dt}{t^3}\left(\frac{1}{4NM}\sum_{n_1,k_1=0}^{N}\sum_{n_2,k_2=0}^{M}\mathcal{K}^{(n_1,k_1)(n_2,k_2)}\mathcal{L}_{\mathcal{K}}^{(n_1,k_1)(n_2,k_2)}\right), \tag{B.3.1a}$$

$$\begin{aligned}\mathcal{A} = c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}})\int_0^\infty \frac{dt}{t^3}\Bigg(&\frac{1}{4NM}\sum_{n_1,k_1=0}^{N}\sum_{n_2,k_2=0}^{M}\sum_{(i_1,i_2)=(0,0)}^{(N-1,M-1)}\mathrm{tr}\big(\gamma_{k_1k_2}^{(i_1,i_2)}\big)\mathrm{tr}\Big(\big(\gamma_{k_1k_2}^{(i_1-n_1,i_2-n_2)}\big)^{-1}\Big)\\ &\times\mathcal{A}^{(n_1,k_1)(n_2,k_2)}\mathcal{L}_{\mathcal{A}}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)}\Bigg),\end{aligned} \tag{B.3.1b}$$

$$\begin{aligned}\mathcal{M} = -c(1_{\mathrm{RR}} - 1_{\mathrm{NSNS}})\int_0^\infty \frac{dt}{t^3}\Bigg(&\frac{1}{4NM}\sum_{n_1,k_1=0}^{N}\sum_{n_2,k_2=0}^{M}\sum_{(i_1,i_2)=(0,0)}^{(N-1,M-1)}\mathrm{tr}\Big(\big(\gamma_{\Omega\mathcal{R}k_1k_2}^{(i_1,i_2)}\big)^{-1}\big(\gamma_{\Omega\mathcal{R}k_1k_2}^{(i_1-n_1,i_2-n_2)}\big)^{T}\Big)\\ &\times\mathcal{M}^{(n_1,k_1)(n_2,k_2)}\mathcal{L}_{\mathcal{M}}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)}\Bigg),\end{aligned} \tag{B.3.1c}$$

where the values $\mathcal{K}^{(n_1,k_1)(n_2,k_2)}$, $\mathcal{A}^{(n_1,k_1)(n_2,k_2)}$ and $\mathcal{M}^{(n_1,k_1)(n_2,k_2)}$ denote oscillator contributions, and $\mathcal{L}$ indicates the zero mode contributions in the amplitudes. They belong to the

[1] In this appendix we borrow quite useful conventions and equations in appendix A of [43].

$\theta^{n_1}\phi^{n_2}$-twisted sector with $\theta^{k_1}\phi^{k_2}$-insertion in the amplitudes. The $\gamma^{(i_1,i_2)}$'s are the matrix representations of the orientifold action on the Chan-Paton factors [54], whose superscript (i_1, i_2) labels the different types of D6-branes on which the open string attaches. The location of the brane (i_1, i_2) is defined by rotating brane $(0, 0)$ by the action $\theta^{-i_1/2}\phi^{-i_2/2}$.

B.3.1 Contributions from zero modes

The above one-loop amplitudes (B.3.1) contain the zero mode contributions $\mathcal{L}_{\mathcal{K},\mathcal{A},\mathcal{M}}$ from the sum of the Kaluza-Klein momentum modes and the winding modes, which are expressed in such a way as

$$\mathcal{L}_{\mathcal{K}}^{(n_1,k_1)(n_2,k_2)} = \chi_{\mathcal{K}}^{(n_1,k_1)(n_2,k_2)}\mathrm{Tr}_{\mathrm{KK+W}}^{(n_1,n_2)}\left(\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}e^{-2\pi t\left(L_0+\bar{L}_0\right)}\right), \tag{B.3.2a}$$

$$\mathcal{L}_{\mathcal{A}}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)} = \chi_{\mathcal{A}}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)}\mathrm{Tr}_{\mathrm{KK+W}}^{(i_1,i_2),(i_1-n_1,i_2-n_2)}\left(\theta^{k_1}\phi^{k_2}e^{-2\pi tL_0}\right), \tag{B.3.2b}$$

$$\mathcal{L}_{\mathcal{M}}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)} = \chi_{\mathcal{M}}^{(n_1,k_1)(n_2,k_2)(i_1,i_2)}\mathrm{Tr}_{\mathrm{KK+W}}^{(i_1,i_2),(i_1-n_1,i_2-n_2)}\left(\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}e^{-2\pi tL_0}\right). \tag{B.3.2c}$$

Note that in the Klein bottle amplitude $\chi_{\mathcal{K}}$ denotes the number of the corresponding fixed points which are left invariant under orientifold group actions $\mathcal{R}\theta^{k_1}\phi^{k_2}$. In the open string amplitudes $\chi_{\mathcal{A}}$ gives the intersection number of the D-branes involved.

When we consider string propagating in the torus $T^6 = \mathbb{R}^6/\Lambda$, the zero modes contributions $\mathcal{L}$ from the momentum modes $\mathbf{p} = \sum_i n_i\mathbf{p}_i$ and the winding modes $\mathbf{w} = m_i\mathbf{w}_i$ are given by

$$\mathcal{L} \equiv \sum_{n_i} \exp\Big(-\delta\pi t n_i M_{ij} n_j\Big) \sum_{m_i} \exp\Big(-\delta\pi t m_i W_{ij} m_j\Big), \tag{B.3.3}$$

where t is the modulus in the loop channel and n_i, $m_i \in \mathbb{Z}$ are the quanta in the momentum modes and the winding modes [42]. Note that the matrices M_{ij} and W_{ij} are given by the products of $\mathbf{p}_i$ and of $\mathbf{w}_i$ in such a way as $M_{ij} = \mathbf{p}_i \cdot \mathbf{p}_j$, $W_{ij} = \mathbf{w}_i \cdot \mathbf{w}_j$; we set $\delta = 1$ (the Klein bottle), $\delta = 2$ (the annulus and the Möbius strip). Due to this, in two-dimensional torus $T^2 \subset T^6$, we can rewrite the above equations (B.3.2) in the following form:

$$\mathcal{L}_{\alpha,\beta} \equiv \sum_{m\in\mathbb{Z}} \exp\Big(-\frac{\alpha\pi t m^2}{\rho}\Big) \sum_{n\in\mathbb{Z}} \exp\Big(-\beta\pi t n^2\rho\Big), \tag{B.3.4}$$

where $\rho = r^2/\alpha'$. It is worth rewriting this to the one in the tree channel. According to the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} = \sqrt{t}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, \tag{B.3.5}$$

we find that the zero mode contribution in the tree channel is given as

$$\tilde{\mathcal{L}}_{\alpha,\beta} \equiv \sum_{m\in\mathbb{Z}} \exp\Big(-\alpha\pi l m^2\rho\Big) \sum_{n\in\mathbb{Z}} \exp\Big(-\frac{\beta\pi l n^2}{\rho}\Big). \tag{B.3.6}$$

This formulation is quite useful not only for factorizable torus $T^2 \times T^2 \times T^2$ but also for non-factorizable tori in the main text via a suitable arrangement.

B.3.2 Contributions from oscillator modes

Here we move to the discussion on the oscillator modes. These contributions into the one-loop amplitudes (B.3.1) are given by

$$\mathcal{K}^{(n_1,k_1)(n_2,k_2)} = \mathrm{Tr}^{(n_1,n_2)}_{\mathrm{NSNS}}\Big(\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}(-1)^F e^{-2\pi t(L_0+\bar{L}_0)}\Big), \tag{B.3.7a}$$

$$\mathcal{A}^{(n_1,k_1)(n_2,k_2)} = \mathrm{Tr}^{(0,0)(-n_1,-n_2)}_{\mathrm{NS}}\Big(\theta^{k_1}\phi^{k_2}(-1)^F e^{-2\pi t L_0}\Big), \tag{B.3.7b}$$

$$\mathcal{M}^{(n_1,k_1)(n_2,k_2)} = \mathrm{Tr}^{(0,0)(-n_1,-n_2)}_{\mathrm{R}}\left(\Omega\mathcal{R}\theta^{k_1}\phi^{k_2}e^{-2\pi t L_0}\right). \tag{B.3.7c}$$

The superscript $(0,0)(-n_1,-n_2)$ on the trace $\mathrm{Tr}^{(0,0)(-n_1,-n_2)}_{\mathrm{NS}}$ in (B.3.7b) indicates open string states stretching between two distinct branes $(0,0)$ and $(-n_1,-n_2)$, or equivalently, between the brane (i_1,i_2) and the brane (i_1-n_1,i_2-n_2). The oscillator contributions (B.3.7) can be expressed by the use of Jacobi theta functions $\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(t)$ and the Dedekind eta function $\eta(t)$:

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(t) = \sum_{n\in\mathbb{Z}} q^{\frac{(n+\alpha)^2}{2}} e^{2\pi i(n+\alpha)\beta}, \qquad \eta\left(t\right) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}\left(1-q^n\right), \tag{B.3.8}$$

with $q=e^{-2\pi t}$. Then the amplitudes are expressed as

$$\mathcal{K}^{(n_1,n_2)} = \frac{\vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}}{\eta^3}\prod_{n_1v_i+n_2w_i\notin\mathbb{Z}}\left(\frac{\vartheta\begin{bmatrix}n_1v_i+n_2w_i\\ 1/2\end{bmatrix}}{\vartheta\begin{bmatrix}1/2+n_1v_i+n_2w_i\\ 1/2\end{bmatrix}}e^{\pi i\langle n_1v_i+n_2w_i\rangle}\right) \times\prod_{n_1v_i+n_2w_i\in\mathbb{Z}}\left(\frac{\vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}}{\eta^3}\right), \tag{B.3.9a}$$

$$\mathcal{A}^{(n_1,k_1)(n_2,k_2)} = \frac{\vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}}{\eta^3}\prod_{(n_1v_i+n_2w_i,k_1v_i+k_2w_i)\notin\mathbb{Z}^2}\left(\frac{(-2i)^\delta\vartheta\begin{bmatrix}n_1v_i+n_2w_i\\ 1/2+k_1v_i+k_2w_i\end{bmatrix}}{\vartheta\begin{bmatrix}1/2+n_1v_i+n_2w_i\\ 1/2+k_1v_i+k_2w_i\end{bmatrix}}e^{\pi i\langle n_1v_i+n_2w_i\rangle}\right) \times\prod_{(n_1v_i+n_2w_i,k_1v_i+k_2w_i)\in\mathbb{Z}^2}\left(\frac{\vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}}{\eta^3}\right), \tag{B.3.9b}$$

$$\mathcal{M}^{(n_1,k_1)(n_2,k_2)} = \frac{\vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}}{\eta^3}\prod_{(n_1v_i+n_2w_i,k_1v_i+k_2w_i)\notin\mathbb{Z}^2}\left(\frac{(-2i)^\delta\vartheta\begin{bmatrix}1/2+n_1v_i+n_2w_i\\ k_1v_i+k_2w_i\end{bmatrix}}{\vartheta\begin{bmatrix}1/2+n_1v_i+n_2w_i\\ 1/2+k_1v_i+k_2w_i\end{bmatrix}}e^{\pi i\langle n_1v_i+n_2w_i\rangle}\right) \times\prod_{(n_1v_i+n_2w_i,k_1v_i+k_2w_i)\in\mathbb{Z}^2}\left(\frac{\vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}}{\eta^3}\right). \tag{B.3.9c}$$

Notice that except for the $\mathbb{Z}_6'$ orbifold the values $\mathcal{K}^{(n_1,k_1)(n_2,k_2)}$ are equal for any insertion of $\theta^{k_1}\phi^{k_2}$, even though the lattice contributions differ [58]. Then we omit the label k_i in

(B.3.9a). The arguments in the theta and eta functions are $2t$ in the Klein bottle, $t+\frac{i}{2}$ in the Möbius strip, and t in the annulus. Further, we used the notation [58], $\langle x\rangle \equiv x-[x]-\frac{1}{2}$, where the brackets on the rhs denote the integer part and

$$\delta = \begin{cases} 1 & \text{if} \quad (n_1 v_i + n_2 w_i, k_1 v_i + k_2 w_i) \in \mathbb{Z}\times\mathbb{Z}+\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{B.3.10}$$

The tree channel expressions $\tilde{\mathcal{K}}$, $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{M}}$ can be evaluated with the help of the modular transformation of (B.3.8).

Appendix C

Classification of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds

C.1 Coxeter orbifolds

We shortly explain the Coxeter elements and the generalized Coxeter elements[1]. The Coxeter element of the Lie root lattice is defined by product of all the Weyl reflections which associate with simple roots,

$$C \equiv r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_N}. \tag{C.1.1}$$

We label the Coxeter element of a lattice by the label of the lattice itself. The other Coxeter elements, which are generated by different ordering of product, are conjugate to one another, and lead to the same class of orbifolds. There are other elements generated by the Weyl reflections. These orbifolds can be classified by the Carter diagrams [59]. For example the Coxeter element of the D_4 lattice is D_4, and we also have other element from the Carter diagram $D_4(a1)$ as follows,

$$D_4 = r_{\alpha_1} r_{\alpha_2} r_{\alpha_3} r_{\alpha_4} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{C.1.2a}$$

$$D_4(a1) = r_{\alpha_1} r_{\alpha_2} r_{\alpha_3} r_{\alpha_2+\alpha_3+\alpha_4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{C.1.2b}$$

where $r_{\alpha_2+\alpha_3+\alpha_4}$ is a Weyl reflection associated with the sum of simple roots $\alpha_2 + \alpha_3 + \alpha_4$. Then the order of D_4 is six, and that of $D_4(a1)$ is four.

[1] From the definition of the (generalized) Coxeter elements, we can see that the elements do not left any directions invariant for corresponding sub-space. Then it is apparent that for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold they lead to factorizable models on $T^2 \times T^2 \times T^2$.

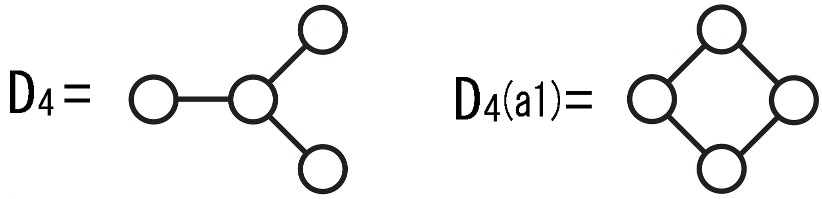

Figure C.1: Carter diagrams

However these elements do not include the outer automorphisms[2]. The generalized Coxeter elements are defined by adding outer automorphisms to the Coxeter elements. For example the D_N Lie root lattice has a graph automorphism g which exchanges the simple root α_{N-1} and α_N. The generalized Coxeter element is defined by

$$C^{[2]} \equiv r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_{N-2}} g. \tag{C.1.3}$$

For instance the generalized Coxeter element of D_4 is

$$D_4^{[2]} = r_{\alpha_1} r_{\alpha_2} r_{\alpha_3} g = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{C.1.4}$$

and the order of this element is eight.

These (generalized) Coxeter elements and elements from the Cater diagrams are included in the classification by the use of $\mathbf{e}_i$ in Section 2. An exception occurs in the D_4 lattice, which has another outer automorphism g',

$$g' : \ \alpha_1 \to \alpha_3 \to \alpha_4 \to \alpha_1. \tag{C.1.5}$$

The generalized Coxeter element of this outer automorphism is defined by

$$D_4^{[3]} \equiv r_{\alpha_1} r_{\alpha_2} g' = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}. \tag{C.1.6}$$

This action corresponds to a rotation of $(e^{1\pi i/6}, e^{5\pi i/6})$. For this element the classification in the $\mathbf{e}_i$ basis is inconvenient (since for example it acts as $g' : \ \mathbf{e}_1 \to (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)/2$). We comment that among $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds this element generates new orbifold only for $\mathbb{Z}_3 \times \mathbb{Z}_3$, e.g. $(C^{[3]})^4$ is rotation of $(e^{2\pi i/3}, e^{2\pi i/3})$ and that of $r_{\alpha_3} g'$ is $(1, e^{2\pi i/3})$. Then a torus on the $D_4 \times A_2$ lattice allows a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold.

[2]There would be complete classifications including the outer automorphisms by mathematicians. However the authors do not know it. Alternatively our approach provides a complete classification and useful formula for the six-dimensional Lie root lattices, except for E_6.

C.2 Table of the orbifolds on the Lie lattices

In this appendix we list all the point group elements, the Euler numbers and hodge numbers on the orbifolds on the sixteen Lie root lattices in (A.1). Most of the $\mathbb{Z}_N$ orbifolds are given in the papers [70,72–74,106,116,117,134], and $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifolds are in [44–46]. We classified the other $\mathbb{Z}_N\times\mathbb{Z}_M$ orbifolds in [47,48]. In the way of chapter 3, the following lists include new results of the $\mathbb{Z}_N\times\mathbb{Z}_M$ orbifolds in addition to these results. The correspondences to the notations of the references are also referred, especially in the word of the (generalized) Coxeter elements. Table 3.3 and 3.4 are the tables of the following tables that indicate the allowed orbifolds on the lattices.

At first we comment on our notations. The point group elements are given both in the simple root basis α_i and the orthogonal basis $\mathbf{e}_i$, and expressed in the matrix representations. For example the Weyl reflection r_{α_1} on the simple roots of A_2 which is given by

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \quad \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \tag{C.2.1}$$

act on the lattice basis as

$$r_{\alpha_1} \;:\; \alpha_1 \to -\alpha_1, \quad \alpha_2 \to \alpha_1 + \alpha_2, \tag{C.2.2a}$$
$$r_{\alpha_1} \;:\; \mathbf{e}_1 \leftrightarrow \mathbf{e}_2, \quad \mathbf{e}_3 \to \mathbf{e}_3, \tag{C.2.2b}$$

We express these actions by the matrices in the simple root basis α_i as

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}_\alpha . \tag{C.2.3}$$

with the subscript α, and in the the orthogonal basis $\mathbf{e}_i$ as

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\mathbf{e}} , \tag{C.2.4}$$

with the subscript $\mathbf{e}$. Note that for the A_N lattice one direction is extra in this representation, and we remove it when we refer to the eigenvalues of the actions. In the $\mathbf{e}_i$ basis the geometrical meaning would be clear, as explained in chapter 3. On the other hand the α_i basis do not depend on the coordinate system.

In some cases we also use the abbreviations (3.1.5) for six-dimensional matrices, i.e.

$$(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3) \equiv \begin{pmatrix} \mathbf{m}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{m}_3 \end{pmatrix} \text{ with } \mathbf{m}_i \in \{\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{1}\} \tag{C.2.5}$$

and

$$\mathbf{a} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; \mathbf{b} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \mathbf{4} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\mathbf{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \;\; \mathbf{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{C.2.6}$$

for the matrix representation of the point group element in the $\mathbf{e}_i$ basis.

The Lie algebras generate the Lie groups as follows,

$$\begin{aligned} A_N &\rightarrow SU(N+1), \\ B_N &\rightarrow SO(2N+1), \\ C_N &\rightarrow Sp(2N), \\ D_N &\rightarrow SO(2N). \end{aligned} \tag{C.2.7}$$

In order to avoid confusing with the gauge groups, we use the notations of the Lie algebras for the definition of the lattices, not that of the Lie groups.

C.2.1 A_6

The simple roots of the A_6 lattice are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad i = 1, \dots, 6. \tag{C.2.8}$$

The allowed orbifold on the A_6 lattice is

$$\mathbb{Z}_7, \tag{C.2.9}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

- $\mathbb{Z}_7 : \ v = (1/7, 2/7, -3/7)$,

  The point group element is given by the Coxeter element A_6,

$$\theta \equiv A_6 = \begin{pmatrix} 0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \end{pmatrix}_e = \begin{pmatrix} 0&0&0&0&0&-1 \\ 1&0&0&0&0&-1 \\ 0&1&0&0&0&-1 \\ 0&0&1&0&0&-1 \\ 0&0&0&1&0&-1 \\ 0&0&0&0&1&-1 \end{pmatrix}_\alpha, \tag{C.2.10}$$

  The Euler number and the hodge numbers are listed on table C.1.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_7$	48	3	0	21	0

Table C.1: *The Euler number and the hodge numbers of the $\mathbb{Z}_7$ orbifold on the A_6 lattice.*

C.2.2 D_6

The simple roots of the D_6 lattice are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_6 = \mathbf{e}_5 + \mathbf{e}_6, \quad i = 1, \ldots, 5, \tag{C.2.11}$$

The allowed orbifold on the D_6 lattice is

$$\begin{gathered} \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_4 \times \mathbb{Z}_4, \\ \mathbb{Z}_4, \quad \mathbb{Z}_8\text{-I}, \quad \mathbb{Z}_8\text{-II}, \end{gathered} \tag{C.2.12}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the D_6 lattice are listed on table C.2.

orbifold	θ	ϕ	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\mathbf{-1,-1,1})$	$(\mathbf{1,-1,-1})$	48	3	3	24	0
	$(\mathbf{b,-1,a})$	$(\mathbf{-1,1,-1})$	24	3	3	14	2
	$(\mathbf{b,-1,a})$	$(\mathbf{-1,b,-a})$	12	3	3	9	3
	$(\mathbf{b,b,-1})$	$(\mathbf{-1,-1,1})$	24	3	3	14	2
	$(\mathbf{b,b,-1})$	$(\mathbf{-b,-1,b})$	24	3	3	12	0
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(\mathbf{4,-4,1})$	$(\mathbf{1,-1,-1})$	72	3	1	34	0
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(\mathbf{4,-4,1})$	$(\mathbf{1,4,-4})$	108	3	0	51	0

Table C.2: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the D_6 lattice.*

- $\mathbb{Z}_4$: $v = (1, 1, -2)/4$,

$$\theta = (\mathbf{4,4,-1})_{\mathbf{e}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & -1 & 0 & 0 \\ 2 & -2 & 2 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & -1 \end{pmatrix}_\alpha, \tag{C.2.13}$$

- $\mathbb{Z}_8$-I: $v = (1, 2, -3)/8$,

$$\theta = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}_\alpha, \tag{C.2.14}$$

- $\mathbb{Z}_8$-II: $v = (1, 3, -4)/8$,

$$\theta = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.15}$$

This lattice can be deformed to the $D_5 A_1$ lattice.

The Euler numbers and the hodge numbers are listed on table C.3.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_4$	48	5	1	22	2
$\mathbb{Z}_8$-I	48	3	0	21	0
$\mathbb{Z}_8$-II	48	3	1	24	2

Table C.3: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the D_6 lattice.*

C.2.3 E_6

The simple roots of the E_6 lattice are given in (A.1.1),

$$\begin{aligned} \alpha_1 &= (1, -1, 0, 0, 0, 0)\,, \\ \alpha_2 &= (0, 1, -1, 0, 0, 0)\,, \\ \alpha_3 &= (0, 0, 1, -1, 0, 0)\,, \\ \alpha_4 &= (0, 0, 0, 1, -1, 0)\,, \\ \alpha_5 &= (0, 0, 0, 1, 1, 0)\,, \\ \alpha_6 &= \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \end{aligned} \tag{C.2.16}$$

We also define the lowest root,

$$\alpha_0 \equiv -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5 - 2\alpha_6 \tag{C.2.17}$$

For some cases, this basis leads to complicated representations, and other choices of basis will be more convenient. Therefore we use this e_i basis as the case may be.

The allowed orbifold on the E_6 lattice is

$$\begin{aligned} &\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \\ &\mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_6\text{-I}, \quad \mathbb{Z}_6\text{II}, \quad \mathbb{Z}_{12}\text{-I}, \end{aligned} \tag{C.2.18}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

- $\mathbb{Z}_2 \times \mathbb{Z}_2$: $\theta = -r_1 r_3, \quad \phi = -r_5 r_0,$

$$\theta = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha, \phi = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_\alpha, \tag{C.2.19}$$

- $\mathbb{Z}_2 \times \mathbb{Z}_4$:

$$\theta = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}_\alpha, \tag{C.2.20}$$

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 \end{pmatrix}_\alpha, \tag{C.2.21}$$

- $\mathbb{Z}_3 \times \mathbb{Z}_3$: $\theta = r_1 r_2 r_4 r_5, \quad , \phi = r_5 r_4 r_6 r_0,$

$$\theta = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_\alpha, \phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}_\alpha, \tag{C.2.22}$$

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the E_6 lattice are listed on table C.4.

- $\mathbb{Z}_3$: $v = (1, 1, -2)/3, \quad \theta = r_1 r_2 r_5 r_4 r_6 r_0,$

$$\theta = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}_\alpha, \tag{C.2.23}$$

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	24	3	3	12	0
$\mathbb{Z}_2 \times \mathbb{Z}_4$	48	3	1	24	2
$\mathbb{Z}_3 \times \mathbb{Z}_3$	72	3	0	33	0

Table C.4: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the E_6 lattice.*

- $\mathbb{Z}_4$: $v = (1, 1, -2)/4$,

$$\theta = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & -1 & 0 \\ 2 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & -1 \end{pmatrix}_\alpha , \quad \text{(C.2.24)}$$

- $\mathbb{Z}_6$-I: $v = (1, 1, -2)/6$,
  The point group element is given by the Cater diagram $E_6(a2) \sim (E_6)^2$ [59],

$$\theta \equiv (E_6)^2 = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}_\alpha , \quad \text{(C.2.25)}$$

- $\mathbb{Z}_6$-II : $v = (1, 2, -3)/6$, $\quad \theta = r_1 r_2 r_3 r_4 r_5 r_0$,

$$\theta = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \quad \text{(C.2.26)}$$

- $\mathbb{Z}_{12}$-I: $v = (1, 4, -5)/12$, The point group element is given by the Coxeter element, $E_6 = r_1 r_2 r_3 r_4 r_5 r_6$,

$$\theta \equiv E_6 = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.27}$$

The Euler numbers and the hodge numbers are listed on table C.5.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_3$	48	9	0	27	0
$\mathbb{Z}_4$	48	5	1	20	0
$\mathbb{Z}_6$-I	48	5	0	20	1
$\mathbb{Z}_6$-II	48	5	0	26	7
$\mathbb{Z}_{12}$-I	48	3	0	22	1

Table C.5: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the E_6 lattice.*

C.2.4 $A_5 \times A_1$

The simple roots of the $A_5 \times A_1$ lattice are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_6 = \mathbf{e}_7, \quad i = 1, \dots, 5, \tag{C.2.28}$$

The allowed orbifolds on the $A_5 \times A_1$ lattice are

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_6\text{-II}, \tag{C.2.29}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

- $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\theta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \qquad \text{(C.2.30)}$$

$$\phi = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_e = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_\alpha ,$$

- $\mathbb{Z}_6$-II : $v = (1, 2, -3)/6$,

$$\theta \equiv A_5\, A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \qquad \text{(C.2.31)}$$

The Euler numbers and the hodge numbers are listed on table C.6.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	24	3	3	14	2
$\mathbb{Z}_6$-II	48	3	1	22	0

Table C.6: *The Euler numbers and the hodge numbers of the orbifolds on the $A_5 \times A_1$ lattice.*

C.2.5 $D_5 \times A_1$

The simple roots of the $D_5 \times A_1$ lattice are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_5 = \mathbf{e}_4 + \mathbf{e}_5, \quad \alpha_6 = \mathbf{e}_6, \quad i = 1, \ldots, 4,$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

The allowed orbifolds on the $D_5 \times A_1$ lattice are

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \tag{C.2.32}$$

$$\mathbb{Z}_4, \quad \mathbb{Z}_8\text{-II}, \tag{C.2.33}$$

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $D_5 \times A_1$ lattice are listed on table C.7.

orbifold	θ	ϕ	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\mathbf{-1}, \mathbf{-1}, \mathbf{1})$	$(\mathbf{1}, \mathbf{-1}, \mathbf{-1})$	48	3	3	24	0
	$(\mathbf{b}, \mathbf{-1}, \mathbf{-a})$	$(\mathbf{-1}, \mathbf{1}, \mathbf{-1})$	24	3	3	16	4
	$(\mathbf{b}, \mathbf{-1}, \mathbf{a})$	$(\mathbf{-1}, \mathbf{a}, \mathbf{-a})$	24	3	3	14	2
	$(\mathbf{b}, \mathbf{-1}, \mathbf{-a})$	$(\mathbf{-1}, \mathbf{b}, \mathbf{a})$	24	3	3	14	2
	$(\mathbf{b}, \mathbf{b}, \mathbf{-1})$	$(\mathbf{-1}, \mathbf{-1}, \mathbf{1})$	24	3	3	14	2
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(\mathbf{4}, \mathbf{-4}, \mathbf{1})$	$(\mathbf{1}, \mathbf{-1}, \mathbf{-1})$	72	3	1	34	0

Table C.7: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $D_5 \times A_1$ lattice.*

- $\mathbb{Z}_4$: $v = (1, 1, -2)/4$,

$$\theta = (\mathbf{4}, \mathbf{4}, \mathbf{-1})_{\mathbf{e}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha, \tag{C.2.34}$$

- $\mathbb{Z}_8$-II: $v = (1, 3, -4)/8$
  The point group element is given by the Coxeter element,

$$\theta \equiv D_5 A_1 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha, \tag{C.2.35}$$

The Euler numbers and the hodge numbers are listed on table C.6.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_4$	48	5	1	22	2
$\mathbb{Z}_8$-II	48	3	1	24	2

Table C.8: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the $D_5 \times A_1$ lattice.*

C.2.6 $A_4 \times A_2$

The simple roots of the $A_4 \times A_2$ lattice are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \qquad \alpha_5 = \mathbf{e}_6, \qquad \alpha_6 = -\frac{1}{2}\mathbf{e}_6 + \frac{\sqrt{3}}{2}\mathbf{e}_7, \qquad i = 1, \ldots, 4, \tag{C.2.36}$$

The allowed orbifolds on the $A_4 \times A_2$ lattice are

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_6\text{-II}. \tag{C.2.37}$$

For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, the $A_4 \times A_2$ lattice is deformed to the $A_4 \times (A_1)^2$ lattice, and we omit it [3]. The other point group element, the Euler number and hodge numbers are listed in the following.

- $\mathbb{Z}_6$-II: $v = (1, 2, -3)/6$,

$$\theta = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}_e = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}_\alpha , \tag{C.2.38}$$

The Euler numbers and the hodge numbers are listed on table C.9.

[3] The orbifold which acts on the A_2 lattice as $\theta and \phi : \alpha_{1,2} \to \pm\alpha_{1,2}$ is deformed to the $(A_1)^2$ lattice, and we would omit the case of A_2 in the following.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_6$-II	48	3	1	26	4

Table C.9: *The Euler number and the hodge numbers of the $\mathbb{Z}_6-II$ orbifold on the $A_4 \times A_2$ lattice.*

C.2.7 $A_4 \times (A_1)^2$

The simple roots of the $A_4 \times (A_1)^2$ lattice are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_5 = \mathbf{e}_6, \quad \alpha_6 = \mathbf{e}_7, \quad i = 1, \ldots, 4, \tag{C.2.39}$$

The allowed orbifolds on the $A_4 \times (A_1)^2$ lattice are

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \tag{C.2.40}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

- $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\theta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha, \tag{C.2.41}$$

$$\phi = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_e = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_\alpha,$$

There are two other choices for the element ϕ,

$$\phi = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha, \tag{C.2.42}$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_\alpha , \quad \text{(C.2.43)}$$

The Euler numbers and the hodge numbers are listed on table C.10.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	24	3	3	16	4
	12	3	3	9	3
	24	3	3	18	6

Table C.10: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds on the $A_4 \times (A_1)^2$ lattice.*

C.2.8 $D_4 \times A_2$

The simple roots of the $D_4 \times A_2$ lattice are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_4 = \mathbf{e}_3 + \mathbf{e}_4, \quad \alpha_5 = \mathbf{e}_5, \quad \alpha_6 = -\frac{1}{2}\mathbf{e}_5 + \frac{\sqrt{3}}{2}\mathbf{e}_6, \quad i = 1, \ldots, 3, \quad \text{(C.2.44)}$$

The allowed orbifolds on the $D_4 \times A_2$ lattice are

$$\begin{gathered} \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_3 \times \mathbb{Z}_3, \\ \mathbb{Z}_4, \quad \mathbb{Z}_6\text{-II}, \quad \mathbb{Z}_8\text{-II}, \quad \mathbb{Z}_{12}\text{-I}, \quad \mathbb{Z}_{12}\text{-II}, \end{gathered} \quad \text{(C.2.45)}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

- $\mathbb{Z}_2 \times \mathbb{Z}_4$:

$$\theta = (\mathbf{4}, -\mathbf{4}, \mathbf{1})_{\mathbf{e}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{\alpha}, \tag{C.2.46}$$

$$\phi = (\mathbf{1}, -\mathbf{1}, -\mathbf{1})_{\mathbf{e}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{\alpha}, \tag{C.2.47}$$

- $\mathbb{Z}_3 \times \mathbb{Z}_3$: $\theta = (D_4^{[3]})^4$, $\phi = \theta r_{\alpha_3} g' r_{\alpha_3} A_2$

$$\theta = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{e} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{\alpha}, \tag{C.2.48}$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}_{e} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}_{\alpha}, \tag{C.2.49}$$

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $D_4 \times A_2$ lattice are listed on table C.11. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is deformed to that on the $D_4 \times (A_1)^2$ lattice, and omitted.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	72	3	1	36	2
$\mathbb{Z}_3 \times \mathbb{Z}_3$	72	3	0	37	4

Table C.11: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $D_4 \times A_2$ lattice.*

- $\mathbb{Z}_4$: $v = (1, 1, -2)/4$,

$$\theta = (\mathbf{4}, \mathbf{4}, -\mathbf{1})_{\mathbf{e}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.50}$$

- $\mathbb{Z}_6$-II: $v = (1, 2, -3)/6$,
  The point group element is given by the Coxeter element,

$$\theta \equiv D_4 A_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}_e = \begin{pmatrix} 0 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}_\alpha , \tag{C.2.51}$$

- $\mathbb{Z}_8$-II: $v = (1, 3, -4)/8$,

$$\theta = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.52}$$

- $\mathbb{Z}_{12}$-I: $v = (1, 4, -5)/12$,
  The point group element is given by the generalized Coxeter element,

$$\theta \equiv D_4^{[3]} A_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}_\alpha , \tag{C.2.53}$$

- $\mathbb{Z}_{12}$-II: $v = (1, 5, -6)/12$,

$$\theta = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha, \quad \text{(C.2.54)}$$

The Euler numbers and the hodge numbers are listed on table C.12.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_4$	48	5	1	26	6
$\mathbb{Z}_6$-II	48	3	1	26	4
$\mathbb{Z}_8$-II	48	3	1	28	6
$\mathbb{Z}_{12}$-I	48	3	0	26	5
$\mathbb{Z}_{12}$-II	48	3	1	28	6

Table C.12: *The Euler number and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the $D_4 \times A_2$ lattice.*

C.2.9 $D_4 \times (A_1)^2$

The simple roots of the $D_4 \times (A_1)^2$ lattice are given by

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad \alpha_4 = \mathbf{e}_3 + \mathbf{e}_4, \quad \alpha_5 = \mathbf{e}_5, \quad \alpha_6 = \mathbf{e}_6, \quad i = 1, \dots, 3, \quad \text{(C.2.55)}$$

The allowed orbifolds on the $D_4 \times (A_1)^2$ lattice are

$$\begin{gathered} \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_4 \times \mathbb{Z}_4, \\ \mathbb{Z}_4, \quad \mathbb{Z}_8\text{-I}, \quad \mathbb{Z}_8\text{-II}, \quad \mathbb{Z}_{12}\text{-II}, \end{gathered} \quad \text{(C.2.56)}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $D_4 \times (A_1)^2$ lattice are listed on table C.13.

- $\mathbb{Z}_4$: $v = (1, 1, -2)/4$,

$$\theta = (\mathbf{4}, -\mathbf{1}, \mathbf{4})_{\mathbf{e}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_\alpha, \quad \text{(C.2.57)}$$

orbifold	θ	ϕ	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\mathbf{-1},\mathbf{-1},\mathbf{1})$	$(\mathbf{1},\mathbf{-1},\mathbf{-1})$	48	3	3	28	4
	$(\mathbf{-1},\mathbf{a},\mathbf{a})$	$(\mathbf{1},\mathbf{-1},\mathbf{-1})$	48	3	3	24	0
	$(\mathbf{b},\mathbf{-1},\mathbf{a})$	$(\mathbf{-1},\mathbf{a},\mathbf{-a})$	24	3	3	16	4
	$(\mathbf{b},\mathbf{-1},\mathbf{a})$	$(\mathbf{-1},\mathbf{b},\mathbf{-1})$	48	3	3	24	0
	$(\mathbf{b},\mathbf{a},\mathbf{-1})$	$(\mathbf{-1},\mathbf{-1},\mathbf{1})$	24	3	3	18	6
	$(\mathbf{-1},\mathbf{a},\mathbf{b})$	$(\mathbf{1},\mathbf{-1},\mathbf{-1})$	24	3	3	14	2
	$(\mathbf{b},\mathbf{-1},\mathbf{a})$	$(\mathbf{-1},\mathbf{b},\mathbf{-a})$	24	3	3	24	0
	$(\mathbf{b},\mathbf{a},\mathbf{-1})$	$(\mathbf{-1},\mathbf{-a},\mathbf{b})$	12	3	3	9	3
	$(\mathbf{b},\mathbf{b},\mathbf{-1})$	$(\mathbf{-1},\mathbf{-1},\mathbf{1})$	48	3	3	28	4
	$(\mathbf{b},\mathbf{-1},\mathbf{b})$	$(\mathbf{-1},\mathbf{1},\mathbf{-1})$	24	3	3	14	2
	$(\mathbf{b},\mathbf{b},\mathbf{-1})$	$(\mathbf{-b},\mathbf{-1},\mathbf{b})$	24	3	3	14	2
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(\mathbf{4},\mathbf{-4},\mathbf{1})$	$(\mathbf{1},\mathbf{-1},\mathbf{-1})$	96	3	1	48	2
	$(\mathbf{4},\mathbf{1},\mathbf{-4})$	$(\mathbf{1},\mathbf{-1},\mathbf{-1})$	72	3	1	36	2
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(\mathbf{4},\mathbf{-4},\mathbf{1})$	$(\mathbf{1},\mathbf{4},\mathbf{-4})$	120	3	0	58	1

Table C.13: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $D_4 \times (A_1)^2$ lattice.*

- $\mathbb{Z}_4$: $v = (1, 1, -2)/4$,

$$\theta = (\mathbf{4}, \mathbf{4}, \mathbf{-1})_{\mathbf{e}} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{\alpha}, \qquad \text{(C.2.58)}$$

- $\mathbb{Z}_8$-I: $v = (1, 2, -3)/8$,
  The point group element is given by the Coxeter element,

$$\theta \equiv D_4^{[2]} B_2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{e} = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{\alpha}, \qquad \text{(C.2.59)}$$

- $\mathbb{Z}_8$-II: $v = (1,3,-4)/8$,
  The point group element is given by the generalized Coxeter element,

$$\theta \equiv D_4^{[2]}(A_1)^2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.60}$$

- $\mathbb{Z}_{12}$-II: $v = (1,5,-6)/12$, $\quad F_4 D_2$,
  The point group element is given by the generalized Coxeter element,

$$\theta \equiv D_4^{[3]}(A_1)^2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.61}$$

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_4$	48	5	1	22	2
	48	5	1	26	6
$\mathbb{Z}_8$-I	48	3	0	24	3
$\mathbb{Z}_8$-II	48	3	1	28	6
$\mathbb{Z}_{12}$-II	48	3	1	28	6

Table C.14: *The Euler number and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the $D_4 \times (A_1)^2$ lattice.*

C.2.10 $A_3 \times A_3$

The simple roots of the $A_3 \times A_3$ lattice are equivalent to that of the $D_3 \times D_3$ lattice, and given by

$$\begin{aligned} &\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, && \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, && \alpha_3 = \mathbf{e}_2 + \mathbf{e}_3, \\ &\alpha_4 = \mathbf{e}_4 - \mathbf{e}_5, && \alpha_5 = \mathbf{e}_5 - \mathbf{e}_6, && \alpha_6 = \mathbf{e}_5 + \mathbf{e}_6, \end{aligned} \tag{C.2.62}$$

The allowed orbifolds on the $A_3 \times A_3$ lattice are

$$\begin{gathered} \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \\ \mathbb{Z}_4, \quad \mathbb{Z}_8\text{-I}, \end{gathered} \tag{C.2.63}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $A_3 \times A_3$ lattice are listed on table C.15.

orbifold	θ	ϕ	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\mathbf{-1,-1,1})$	$(\mathbf{1,-1,-1})$	24	3	3	12	0
	$(\mathbf{-1,1,-1})$	$(\mathbf{a,-1,-a})$	24	3	3	12	0
	$(\mathbf{b,-1,a})$	$(\mathbf{-1,1,-1})$	24	3	3	12	0
	$(\mathbf{b,-1,a})$	$(\mathbf{-1,1,-1})$	24	3	3	12	0
	$(\mathbf{b,-a,-a})$	$(\mathbf{-1,a,b})$	24	3	3	12	0
	$(\mathbf{b,a,-1})$	$(\mathbf{-1,-a,b})$	24	3	3	16	4
	$(\mathbf{b,-1,b})$	$(\mathbf{-1,1,-1})$	24	3	3	12	0
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(\mathbf{4,1,-4})$	$(\mathbf{1,-1,-1})$	48	3	1	24	2

Table C.15: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $A_3 \times A_3$ lattice.*

- $\mathbb{Z}_4$: $v = (1,1,-2)/4$,
  The point group element is given by the Coxeter element,

$$\theta \equiv (A_3)^2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.64}$$

- $\mathbb{Z}_8$-I: $v = (1,2,-3)/8$,

$$\theta \equiv (A_3)^{2*} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}_\alpha , \quad \text{(C.2.65)}$$

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_4$	48	5	1	20	0
$\mathbb{Z}_8$-I	48	3	0	21	0

Table C.16: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the $A_3 \times A_3$ lattice.*

C.2.11 $A_3 \times A_2 \times A_1$

The simple roots of the $A_3 \times A_2 \times A_1$ lattice are given by

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \qquad \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \qquad \alpha_3 = \mathbf{e}_2 + \mathbf{e}_3, \quad \text{(C.2.66)}$$

$$\alpha_4 = \mathbf{e}_4, \qquad \alpha_5 = -\frac{1}{2}\mathbf{e}_4 + \frac{\sqrt{3}}{2}\mathbf{e}_5, \qquad \alpha_6 = \mathbf{e}_6, \quad \text{(C.2.67)}$$

The allowed orbifolds on the $A_3 \times A_2 \times A_1$ lattice are

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_6\text{-II}. \quad \text{(C.2.68)}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is deformed to the orbifold on the $A_3 \times (A_1)^3$ lattice. The Euler number and the hodge numbers of the $\mathbb{Z}_6$-II orbifolds on the $A_3 \times A_2 \times A_1$ lattice are listed on table C.17.

- $\mathbb{Z}_6$-II: $v = (1,2,-3)/6$,

$$\theta \equiv A_3^{[2]} A_2 A_1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \quad \text{(C.2.69)}$$

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_6$-II	48	3	1	26	4

Table C.17: *The Euler number and the hodge numbers of the $\mathbb{Z}_6$-II orbifold on the $A_3 \times A_2 \times A_1$ lattice.*

C.2.12 $A_3 \times (A_1)^3$

The simple roots of the $A_3 \times (A_1)^3$ lattice are given by

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \quad \alpha_2 = \mathbf{e}_2 - \mathbf{e}_3, \quad \alpha_3 = \mathbf{e}_2 + \mathbf{e}_3, \quad \alpha_i = \mathbf{e}_i, \quad i = 4, 5, 6, \tag{C.2.70}$$

The allowed orbifolds on the $A_3 \times (A_1)^3$ lattice are

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_4, \tag{C.2.71}$$

and the point group elements, the Euler numbers and hodge numbers are listed in the following.

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $A_3 \times (A_1)^3$ lattice are listed on table C.18.

orbifold	θ	ϕ	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\mathbf{-1}, \mathbf{-1}, \mathbf{1})$	$(\mathbf{-1}, \mathbf{1}, \mathbf{-1})$	48	3	3	28	4
	$(\mathbf{-1}, \mathbf{1}, \mathbf{-1})$	$(\mathbf{a}, \mathbf{-1}, \mathbf{-a})$	48	3	3	24	0
	$(\mathbf{b}, \mathbf{-1}, \mathbf{a})$	$(\mathbf{-1}, \mathbf{-a}, \mathbf{-a})$	48	3	3	28	4
	$(\mathbf{-1}, \mathbf{a}, \mathbf{b})$	$(\mathbf{1}, \mathbf{-1}, \mathbf{-1})$	24	3	3	16	4
	$(\mathbf{-1}, \mathbf{-a}, \mathbf{-a})$	$(\mathbf{b}, \mathbf{-1}, \mathbf{a})$	48	3	3	24	0
	$(\mathbf{-1}, \mathbf{a}, \mathbf{b})$	$(\mathbf{-a}, \mathbf{-a}, \mathbf{-1})$	24	3	3	14	2
	$(\mathbf{b}, \mathbf{-a}, \mathbf{-1})$	$(\mathbf{-1}, \mathbf{a}, \mathbf{b})$	24	3	3	14	2
	$(\mathbf{b}, \mathbf{a}, \mathbf{-1})$	$(\mathbf{-1}, \mathbf{-a}, \mathbf{b})$	24	3	3	16	4
	$(\mathbf{b}, \mathbf{-1}, \mathbf{b})$	$(\mathbf{-1}, \mathbf{1}, \mathbf{-1})$	24	3	3	14	2
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(\mathbf{4}, \mathbf{1}, \mathbf{-4})$	$(\mathbf{1}, \mathbf{-1}, \mathbf{-1})$	72	3	1	36	2

Table C.18: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $A_3 \times (A_1)^3$ lattice.*

- $\mathbb{Z}_4$: $v = (1, 1, -2)/4$,

$$\theta \equiv A_3 B_2 A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.72}$$

The Euler numbers and the hodge numbers are listed on table C.19.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_4$	48	5	1	22	2

Table C.19: *The Euler number and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the $A_3 \times (A_1)^3$ lattice.*

C.2.13 $(A_2)^3$

The simple roots of the $(A_2)^3$ lattice are given by

$$\alpha_{2i-1} = \mathbf{e}_{2i-1}, \qquad \alpha_{2i} = -\frac{1}{2}\mathbf{e}_{2i-1} + \frac{\sqrt{3}}{2}\mathbf{e}_{2i}, \qquad i = 1, 2, 3, \tag{C.2.73}$$

The allowed orbifolds on the $(A_2)^3$ lattice are

$$\begin{gathered} \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_6', \\ \mathbb{Z}_3 \times \mathbb{Z}_3, \quad \mathbb{Z}_3 \times \mathbb{Z}_6, \quad \mathbb{Z}_6 \times \mathbb{Z}_6, \\ \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_6\text{-I}, \quad \mathbb{Z}_6\text{-II}. \end{gathered} \tag{C.2.74}$$

Among these orbifolds, the $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$-II orbifolds on the $(A_2)^3$ lattice are deformed to the orbifolds on the $(A_1)^6$ or $(A_2)^2(A_1)^2$ lattices. The other point group elements, the Euler numbers and hodge numbers are listed in the following.

The following $\mathbb{Z}_N \times \mathbb{Z}_M$ elements are given by the Coxeter elements, and are factorizable orbifolds.

- $\mathbb{Z}_2 \times \mathbb{Z}_6'$: $v = (1, 0, -1)/2$, $\quad w = (1, 1, -2)/6$, $\quad A_2(G_2)^2$,
- $\mathbb{Z}_3 \times \mathbb{Z}_3$: $v = (1, -1, 0)/3$, $\quad w = (0, 1, -1)/3$, $\quad (A_2)^3$,
- $\mathbb{Z}_3 \times \mathbb{Z}_6$: $v = (1, -1, 0)/3$, $\quad w = (0, 1, -1)/6$, $\quad A_2(G_2)^2$

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2 \times \mathbb{Z}_6'$	72	3	0	33	0
$\mathbb{Z}_3 \times \mathbb{Z}_3$	168	3	0	81	0
$\mathbb{Z}_3 \times \mathbb{Z}_6$	144	3	0	70	1
$\mathbb{Z}_6 \times \mathbb{Z}_6$	168	3	0	81	0

Table C.20: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $(A_2)^3$ lattice.*

- $\mathbb{Z}_6 \times \mathbb{Z}_6$: $v = (1, -1, 0)/6, \quad w = (0, 1, -1)/6, \quad (G_2)^3$

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds on the $(A_2)^3$ lattice are listed on table C.20.

- $\mathbb{Z}_3 : v = (1, 1, -2)/3$,
  The point group element is given by the Coxeter element $(A_2)^3$, and this gives the factorizable orbifold.

- $\mathbb{Z}_6$-I : $v = (1, 1, -2)/6$,
  The point group element is given by the Coxeter element, which leads to factorizable orbifold,

$$\theta \equiv A_2(G_2)^2 = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}_e = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}_\alpha , \tag{C.2.75}$$

- $\mathbb{Z}_6$-I : $v = (1, 1, -2)/6$,

$$\theta = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}_\alpha , \tag{C.2.76}$$

The Euler numbers and the hodge numbers are listed on table C.21.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_3$	72	9	0	27	0
$\mathbb{Z}_6$-I	48	5	0	24	5
$\mathbb{Z}_6$-I	48	5	0	20	1

Table C.21: *The Euler number and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the $(A_2)^3$ lattice.*

C.2.14 $(A_2)^2 \times (A_1)^2$

The simple roots of the $(A_2)^2 \times (A_1)^2$ lattice are given by

$$\alpha_2 = -\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2, \qquad \alpha_4 = -\frac{1}{2}\mathbf{e}_3 + \frac{\sqrt{3}}{2}\mathbf{e}_4, \qquad \alpha_i = \mathbf{e}_i, \quad i = 1, 3, 5, 6, \tag{C.2.77}$$

The allowed orbifolds on the $(A_2)^2 \times (A_1)^2$ lattice are

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_4, \quad \mathbb{Z}_6\text{-II}. \tag{C.2.78}$$

Except the $\mathbb{Z}_6$-II orbifolds, these orbifolds on the $(A_2)^2 \times (A_1)^2$ lattice can be deformed to the orbifolds on the $(A_1)^6$ lattice, and we give only the result of the $\mathbb{Z}_6$-II orbifolds.

- $\mathbb{Z}_6$-II: $v = (1, 2, -3)/6$,
  The point group element is given by the Coxeter element,

$$\theta \equiv A_2 G_2 (A_1)^2 = \tag{C.2.79}$$

$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha,$$

- $\mathbb{Z}_6$-II: $v = (1, 2, -3)/6$,

$$\theta = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha, \tag{C.2.80}$$

The Euler numbers and the hodge numbers are listed on table C.22.

C.2.15 $A_2 \times (A_1)^4$

The simple roots of the $A_2 \times (A_1)^4$ lattice are given by

$$\alpha_1 = \mathbf{e}_1, \qquad \alpha_2 = -\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2, \qquad \alpha_i = \mathbf{e}_i, \quad i = 3, \ldots, 6, \tag{C.2.81}$$

The allowed orbifolds on the $A_2 \times (A_1)^4$ lattice are

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_4, \quad \mathbb{Z}_8\text{-II}, \tag{C.2.82}$$

The orbifolds on the $A_2 \times (A_1)^4$ lattice can be deformed to the orbifolds on the $(A_1)^6$ lattice, and we omit them.

C.2.16 $(A_1)^6$

The simple roots of the $(A_1)^6$ lattice are given by

$$\alpha_i = \mathbf{e}_i, \quad i = 1, \ldots, 6, \tag{C.2.83}$$

The allowed orbifolds on the $(A_1)^6$ lattice are

$$\begin{gathered} \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_4 \times \mathbb{Z}_4, \\ \mathbb{Z}_4, \quad \mathbb{Z}_8\text{-I}, \quad \mathbb{Z}_8\text{-II}. \end{gathered} \tag{C.2.84}$$

The following $\mathbb{Z}_N \times \mathbb{Z}_M$ elements are given by the Coxeter elements.

- $\mathbb{Z}_2 \times \mathbb{Z}_2$: $v = (1, -1, 0)/2, \quad w = (0, 1, -1)/2$,

$$(A_1)^6 \to \theta = (-\mathbf{1}, -\mathbf{1}, \mathbf{1}), \quad \phi = (\mathbf{1}, -\mathbf{1}, -\mathbf{1}), \tag{C.2.85}$$

- $\mathbb{Z}_2 \times \mathbb{Z}_4$: $v = (1, -1, 0)/4, \quad w = (0, 1, -1)/2$,

$$(B_2)^2(A_1)^2 \to \theta = (\mathbf{4}, -\mathbf{4}, \mathbf{1}), \quad \phi = (\mathbf{1}, -\mathbf{1}, -\mathbf{1}), \tag{C.2.86}$$

- $\mathbb{Z}_4 \times \mathbb{Z}_4$: $v = (1, -1, 0)/4, \quad w = (0, 1, -1)/4$,

$$(B_2)^3 \to \theta = (\mathbf{4}, -\mathbf{4}, \mathbf{1}), \quad \phi = (\mathbf{1}, \mathbf{4}, -\mathbf{4}), \tag{C.2.87}$$

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_6$-II	48	3	1	28	6
$\mathbb{Z}_6$-II	48	3	1	32	10

Table C.22: *The Euler number and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the $(A_2)^2 \times (A_1)^2$ lattice.*

orbifold	θ	ϕ	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_2\times\mathbb{Z}_2$	$(\mathbf{-1,-1,1})$	$(\mathbf{1,-1,-1})$	96	3	3	48	0
	$(\mathbf{b,-1,a})$	$(\mathbf{-1,1,-1})$	48	3	3	28	4
	$(\mathbf{b,-1,a})$	$(\mathbf{-1,b,-a})$	24	3	3	16	4
	$(\mathbf{b,b,-1})$	$(\mathbf{-1,-1,1})$	24	3	3	18	6
	$(\mathbf{b,b,-1})$	$(\mathbf{-b,-1,b})$	24	3	3	12	0
$\mathbb{Z}_2\times\mathbb{Z}_4$	$(\mathbf{4,-4,1})$	$(\mathbf{1,-1,-1})$	120	3	1	58	0
$\mathbb{Z}_4\times\mathbb{Z}_4$	$(\mathbf{4,-4,1})$	$(\mathbf{1,4,-4})$	180	3	0	87	0

Table C.23: *The Euler number and the hodge numbers of the $\mathbb{Z}_N\times\mathbb{Z}_M$ orbifolds on the $(A_1)^6$ lattice.*

The Euler numbers and the hodge numbers of the $\mathbb{Z}_N\times\mathbb{Z}_M$ orbifolds on the $(A_1)^6$ lattice are listed on table C.23.

- $\mathbb{Z}_4$: $v=(1,1,-2)/4$,
  The point group element is given by the Coxeter element,

$$\theta\equiv(B_2)^2(A_1)^2=(\mathbf{4,4,-1})_{\mathbf{e}}=\begin{pmatrix}0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{pmatrix}_\alpha, \tag{C.2.88}$$

- $\mathbb{Z}_8$-I: $v=(1,2,-3)/8$
  The point group element is given by the Coxeter element,

$$\theta\equiv B_4B_2=\begin{pmatrix}0&0&0&-1&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\end{pmatrix}_e=\begin{pmatrix}0&0&0&-1&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\end{pmatrix}_\alpha, \tag{C.2.89}$$

- $\mathbb{Z}_8$-II: $v=(1,3,-4)/8$

The point group element is given by the Coxeter element,

$$\theta \equiv B_4 D_2 = B_4(A_1)^2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_e = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_\alpha , \tag{C.2.90}$$

The Euler numbers and the hodge numbers are listed on table C.24.

orbifold	χ	$h^{1,1}_{untwisted}$	$h^{2,1}_{untwisted}$	$h^{1,1}_{twisted}$	$h^{2,1}_{twisted}$
$\mathbb{Z}_4$	48	5	1	26	6
$\mathbb{Z}_8$-I	48	3	0	24	3
$\mathbb{Z}_8$-II	48	3	1	28	6

Table C.24: *The Euler numbers and the hodge numbers of the $\mathbb{Z}_N$ orbifolds on the $(A_1)^6$ lattice.*

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