Ph.D thesis

# Product-group Unification and its Extension to Higher Dimensional Spacetime

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Submitted to Department of Physics, University of Tokyo: *December 2002*

Accepted: *February 2003*

Revised version fixed: *June 2003*

- Ref. [47] containing Section 3.2–3.4 has been submitted to eprint arXive [hep-ph/0303123] in March 2003. It is being published from Physical Review D. Readers are advised to see it instead of the corresponding part in this thesis, since it is much better.
- Ref. [70]([hep-ph/0208107]), which corresponds to Section 4.2–4.6 in this thesis, will be revised considerablly in July–August, 2003, by the same authors. The revised article will have another eprint number in hep-ph. The readers are again advised to see the new article instead of the corresponding part in this thesis.

## Abstract

Supersymmetric unified theories (SUSY GUT's) is one of the most well-motivated candidates of new physics beyond the standard model. The most important mystery of the SUSY GUT's is the question why the mass of the two Higgs doublets is so tiny compared with the unification scale. It is quite natural to consider that there must be a symmetry behind this unexpectedly small mass parameter. This symmetry, if it exists, automatically suppresses dimension-five operators that would have induced too fast proton decay. Thus, it is also motivated by the totally independent experimental fact. Models of SUSY GUT's that have such a symmetry are discussed in this thesis. There are two models based on field theories in four-dimensional space-time; one uses $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(N)_{\mathrm{H}}$ ($N = 2, 3$) gauge group, and the other uses $\mathrm{SU}(5)_1 \times \mathrm{SU}(5)_2$. Brief review is given on both of these models. Main focus is on the former model in this thesis. The proton-decay amplitude of the model is analysed in detail. The estimate of the typical life-time is $(0.6 - 5) \times 10^{34}$ yrs., and hence there is an intriguing possibility of detecting the dimension-six proton decay in the next-generation water-Čerenkov detectors. We also derived theoretical upper bound of the life-time. It is also found that this class of models have partial $\mathcal{N} = 2$ supersymmetry. This extended supersymmetry suggests that there is a higher-dimensional space-time behind the model; a brane-world picture explains various features of the model, including the extended supersymmetry. The latter half of this thesis consists of an attempt to construct the brane-world model in string theories.

# Contents

# Chapter 1

# Introduction

The standard model of particle physics is consistent with enormous amount of experimental data. Moreover, it is not only successful from the experimental point of view, but also of theoretical beauties. First, spontaneous breaking of the electroweak symmetry $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ requires a doublet Higgs boson, which is also necessary and sufficient for quarks and charged leptons to obtain masses. That is, the particle contents required for the Higgs sector are minimal. Secondly, baryon- or lepton-number-violating operators are not allowed by the gauged symmetry of the standard model at the renormalizable level. Neutrino masses can be explained by dimension-five operators that involve lepton doublets and the Higgs doublet, where the smallness of the masses implies that the energy scale of a new physics is very high. Dimension-six operators that would have led to proton decay are suppressed when the energy scale that characterizes the coefficients of the operators is sufficiently high. Thus, generic interactions under the gauge symmetry are consistent with experiments, as long as an energy scale of a new physics beyond the standard model is much higher than the electroweak scale. That is, the interactions are generic in the standard model.

The large hierarchy between the energy scale of a new physics and the electroweak scale, which is sometimes called the grand desert, is one of the most important ingredients of the standard model. However, the hierarchy is not stable under radiative corrections. Masses of fermions and gauge bosons are determined by a single mass parameter in the Higgs-scalar potential, and this relation is maintained under the radiative corrections because of chiral symmetries and gauge symmetries. However, the mass parameter in the potential itself changes very much owing to the radiative corrections. Thus, it should be fine-tuned at a high energy scale so that it becomes the current value after radiative corrections are included. In other words, the standard model has minimal matter contents, generic interactions, but does not have natural parameters.

Introduction of supersymmetry is a big step toward the solution to the naturalness problem. It renders the fine-tuned parameter, i.e., the Higgs boson mass, stable under the renormalization group evolution. Thus, the minimal supersymmetric standard model (with R/matter parity) has minimal matter contents, generic interactions and technically natural parameters. Moreover, it supports gauge-coupling unification, as revealed by the LEP experiment.

Now a clear picture stands out: supersymmetric grand unified theories with a grand desert between the electroweak scale and the unification scale. However, there is one problem left; the Higgs mass parameter is technically natural, yet not natural. Why are the two Higgs doublets so light, despite the fact that their mass term is not forbidden by R parity or by any gauge symmetries of the standard model? In particular, in the context of the unified theories, the Higgs doublets should remain light whereas the mass of the Higgs triplets should be around the unification scale; otherwise, the gauge coupling constants would not have been unified. Why do the doublets have to be so light? This is the starting point of discussion in this thesis.

In chapter 2, we consider that the Higgs doublets are light because there is an unbroken symmetry. What is the implication of this symmetry to models of supersymmetric unified theories? The symmetry restricts models quite severely. In section 2.1, we begin with a brief look at a history of attempts in search of models, which sheds a light on possible directions of model building. This section ends with three possible classes of models that have such a symmetry. Section 2.2 and 2.3 are devoted to review of two of the three classes of models. The models in section 2.2 use $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(N)_{\mathrm{H}}$ ($N = 2, 3$) as the "unified gauge group", and those in section 2.3 use $\mathrm{SU}(5)_1\times\mathrm{SU}(5)_2$. We argue that the models in section 2.3 tend to predict long life-time of proton. The remaining one class of models requires a formulation in a higher-dimensional space-time.

Chapter 3 is devoted to analysis of the proton decay in models explained in section 2.2. It is clarified in section 3.1 that two models presented in section 2.2 tend to predict fast proton decay in contrast to the models in section 2.3. Simple spectrum and dynamical structure of these models allow us to make an estimate of the proton life-time. In section 3.2 – 3.4, we further derive predictions on the range of the life-time in the models, rather than a single estimate. In particular, we obtained theoretical upper bounds of the life-time in the models, which are predictions that can be tested in future experiments. Various sources of theoretical uncertainties in the predictions are also discussed.

Chapter 4 begins with a re-view of various features of the models explained in section 2.2 (and analyzed in chapter 3). It is argued in section 4.1 that those features are understood very well, when we assume that higher-dimensional brane-world structure is hidden behind the models. In order to give a firm base to the speculative idea of brane-world, we make an attempt to realize the models on D-branes of superstring theories, which is the main purpose of the following sections in this chapter. The D3–D7 system in Type IIB string theory is a promising candidate that accommodates the models. We only treat the D3–D7 system on Type IIB orientifolds. The GUT-symmetry-breaking sector is obtained in section 4.3 and 4.4 from the D3–D7 system on the $\mathbf{T}^6/(\mathbf{Z}_{12} \times \mathbf{Z}_2 \langle \Omega R_{89} \rangle)$ orientifold. R-charge assignment is re-obtained from the geometry, yet quarks and leptons are not obtained. Theoretical consistencies of the construction and implications on phenomenology are briefly discussed in section 4.5. Section 4.6 gives a summary on how the construction of the GUT-symmetry-breaking sector can be generalized so that the quarks and leptons are also obtained.

The final chapter (chapter 5) summarizes the conclusions obtained in this thesis, and discusses open problems.

# Chapter 2

# Supersymmetric Unified Theories

The energy scale of unified theories will be extremely high. Thus, experimental information on the unified theories is quite limited at this moment. What can we say about the theory of unification from our current knowledge (with a bit of theoretical prejudice)? It is argued in subsection 2.1.1 that it would be natural to consider that the Higgs doublets are light because there is an unbroken symmetry. Brief look at a history of model building in subsection 2.1.2 and 2.1.3 suggests possible directions of constructing models of the supersymmetric unified theories with such a symmetry. Section 2.1 ends up with three possible solutions. The first possibility can be realized as models only in higher-dimensional space-time. There are models for two other possibilities within field theories on four-dimensional space-time, which are explained in detail in section 2.2 and 2.3.

## 2.1 Natural Supersymmetric Unified Theories

### 2.1.1 R Symmetry and Peccei–Quinn Symmetry

Supersymmetric unified theories (SUSY GUT's) is one of the well-motivated candidates of physics beyond the standard model. Unified theories are interesting because of their conceptual beauties [1]. Supersymmetry (SUSY) above the electroweak scale, if it exists, not only solves the hierarchy problem [2] but also provides a good experimental support of the SU(5)-unification of the three gauge coupling constants of the standard model [3].

Introduction of the SUSY at the electroweak scale, however, would invalidate one of the most important achievements of the standard model: the stability of proton. The gauge symmetry of the standard model does not allow renormalizable operators that lead to proton decay. Leading effective operators in the standard model are dimension-six operators of

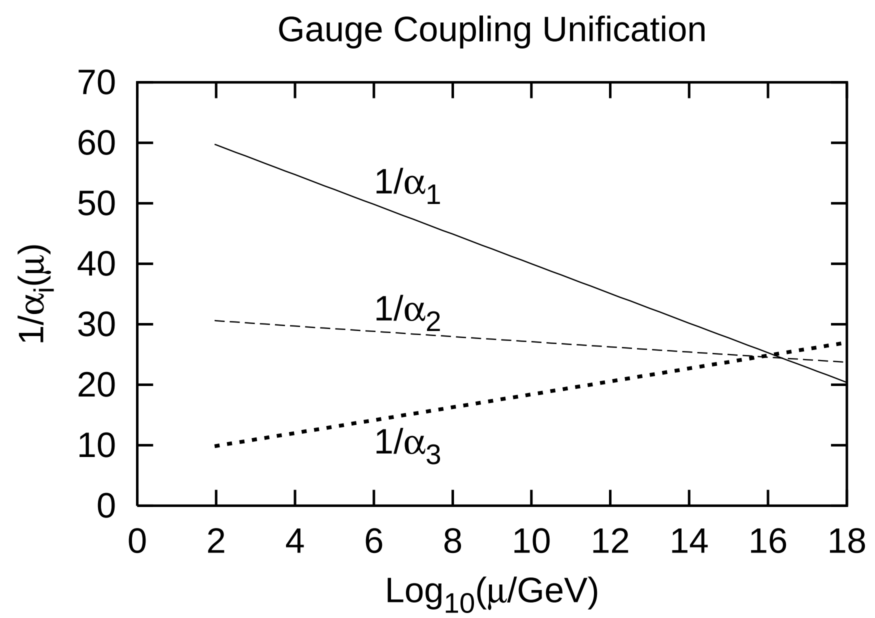


Figure 2.1: Unification of the three gauge coupling constants of the MSSM.

quarks and leptons. The coefficients of the operators, which are of mass-dimension minus two, will be given by inverse square of a scale of a new physics beyond the standard model. The amplitude of the proton decay is highly suppressed if the new scale is very high, e.g., $\sim 10^{15}$ GeV. New physics at the electroweak scale, such as low energy SUSY, should not induce the proton decay. The R parity forbids the dangerous dimension-four operators. Moreover, it predicts a stable particle, which may be a suitable candidate for the dark matter [4]. Therefore, the unbroken R parity is assumed in the following.

The three gauge coupling constants of the minimal supersymmetric standard model (MSSM) become approximately the same at around $10^{16}$ GeV (GUT scale), as shown in Figure 2.1. Requirements on models of SUSY GUT's are summarized as follows:

1. Natural explanation should be given to a question why the gauge coupling constants are unified.

2. Natural explanation should be given to a question why the $\mu$ term (mass term of the two Higgs doublets $W = \mu H_d H_u$) is highly suppressed. The $\mu$ parameter should be of the order of the electroweak scale, which is $10^{-14}$ times smaller than the GUT scale.

3. Dimension-five proton-decay operators [5],

$$W = \mathbf{10\ 10\ 10\ 5^*} \sim QQQL + \bar{U}\bar{E}\bar{U}\bar{D}, \tag{2.1}$$

   where $\mathbf{10}= (Q, \bar{U}, \bar{E})$ and $\mathbf{5}^* = (\bar{D}, L)$, should be suppressed sufficiently. Super-Kamiokande experiment places a limit $\tau(p \to K + \bar{\nu}) \gtrsim 6.7 \times 10^{32}$ yrs. (90 % C.L.) [6]. The dimension-five operators induced by coloured Higgsino exchange are consistent with the limit only when the mass of the coloured Higgs multiplets exceeds $2.0 \times 10^{17}$ GeV (for typical SUSY-breaking parameters) [7], which is one order of magnitude larger than the GUT scale. This implies that the effective operators in Eq. (2.1) need extra suppression in their coefficients by $10^{-1}$ in addition to the suppression owing to the Yukawa coupling constants of quarks and leptons.

4. Models should accommodate the fact that the running Yukawa coupling constants of strange quark and muon are different from each other around the GUT scale.

The last issue is not a difficult problem. The unified SU(5) symmetry is broken down spontaneously, and hence the effects of the symmetry breaking appear in the Yukawa couplings, e.g., through non-renormalizable operators.

The second and the third issues tell us that (exceedingly) small parameters are necessary. Then, it is a conventional wisdom that there will be a hidden symmetry when one finds small parameters. It is interesting that both suppressed parameters required in the second and the third issues, respectively, can be considered as consequences of a single symmetry as shown below. Let us consider a symmetry under which quarks and leptons in an SU(5)-unified multiplet have the same charges, i.e., $Q, \bar{U}$ and $\bar{E}$ in SU(5)-$\mathbf{10}$ representation have common charges, etc. The fact that the Yukawa couplings of quarks and charged leptons should be allowed by the symmetry determines how the fields transform under the symmetry:

$$H_d \to e^{-in\alpha} H_d, \qquad (\mathbf{10\ 5^*}) \to e^{-i(w-n)\alpha}(\mathbf{10\ 5^*}), \tag{2.2}$$

$$H_u \to e^{-im\alpha} H_u, \qquad (\mathbf{10\ 10}) \to e^{-i(w-m)\alpha}(\mathbf{10\ 10}), \tag{2.3}$$

where $\alpha$ is a transformation parameter of the symmetry, $n$ and $m$ the charges of two Higgs doublets $H_u$ and $H_d$, respectively, and $w = 2$ when an R symmetry is considered and $w = 0$ otherwise (pure Peccei–Quinn symmetry). It follows, then, that a symmetry which forbids the $\mu$ term (i.e., $n + m \neq w$) also forbids the dimension-five proton-decay operators (i.e., $(2w - (n + m)) \neq w$) and vice versa. Therefore, the two phenomena (the second and the third issues) are explained by the single symmetry. This thesis focuses on models of SUSY GUT's that have such a symmetry, i.e., the R symmetry or the Peccei–Quinn symmetry that is not broken at the GUT scale and under which quarks and leptons in an SU(5)-unified multiplet transform in the same way. We refer to this symmetry as the "S symmetry" in this chapter.

The $\mu$ parameter should not be of the order of the GUT scale, yet it should not be zero, either; otherwise, the SUSY spectrum would have contained a chargino lighter than the W boson, which contradicts with the LEP II experiment ($m_{\chi_1^\pm} > 104$ GeV (95 % C.L.)) [8]. Thus, the S symmetry, which forbids the $\mu$ term of the order of the GUT scale, should be broken down spontaneously to allow the $\mu$ term of the order of the electroweak scale. The $\mu$ parameter has the same order of magnitude as that of the soft-SUSY-breaking parameters in the MSSM. Two ways of generating the $\mu$ term are summarized in the following, in which the $\mu$ parameter is naturally, rather than accidentally, of the order of magnitude of the SUSY-breaking parameters.

First, the $\mu$ parameter can be obtained as a vacuum expectation value (VEV) of (a) massless field(s). Let us suppose that there are massless MSSM singlets and that the symmetry of the system (the MSSM with extra singlets), including the S symmetry, fixes the superpotential as

$$W = cM\left(\frac{\Phi}{M}\right)^k H_d H_u + c' M\left(\frac{\Phi}{M}\right)^{k+2} + \cdots, \tag{2.4}$$

where $M$ is an arbitrary mass scale, and $k = 1, 2, \ldots$. A chiral superfield $\Phi$ collectively denotes massless MSSM-singlet fields that are to develop VEV's. Note that $(\Phi/M)^k$ is not a singlet of the S symmetry. When the $\Phi$ has a negative soft-SUSY-breaking mass squared $\sim -m_{\rm SUSY}^2$, where $m_{\rm SUSY}$ is of the order of the electroweak scale, the S symmetry is broken spontaneously by $\Phi$'s VEV of the order of $(m_{\rm SUSY}M^{k-1})^{1/k}$. Effective $\mu$ parameter $(\langle\Phi\rangle/M)^k M$ is of the order of $m_{\rm SUSY}$.

This is nothing but the next-to-minimal SUSY standard model (NMSSM) [9] when $k=1$. The VEV of $\Phi$ can be relevant to the GUT-symmetry breaking when $M \sim M_{\rm pl} \simeq 2.4 \times 10^{18}$ GeV and $k = 4, 5, 6$, although one has to consider the cosmological moduli problem [10] in that case. Note also that this mechanism does not work well for higher $k$ an/or large $M$ in the case of gauge-mediated SUSY breaking; SUSY-breaking scalar potential changes its behaviour beyond the mass scale of the messenger particles.

When the S symmetry is broken before the end of primordial inflation, there is no cosmological problem of domain walls or cosmic strings. Even when it is broken after the inflation, there is no domain-wall problem either [11], if the soft-SUSY-breaking terms explicitly[1] break the symmetry, which is always the case if the S symmetry is an R symmetry — soft-SUSY-breaking parameters break R symmetry down to mod-2 R symmetry (i.e., R parity).

Secondly, the $\mu$ parameter can also be obtained as a VEV of a massive field. Let us consider a superpotential

$$W = \Phi H_d H_u + M\Phi\bar{\Phi} + \cdots, \tag{2.5}$$

where $\Phi$ and $\bar{\Phi}$ are messengers of the $\mu$ parameter whose supersymmetric masses are $M(\gg m_{\rm SUSY})$. The $\Phi$ field acquires a VEV when there are holomorphic scalar potential ($A$ term) linear in $\Phi$ :

$$W = \left(\frac{\bar{D}^2}{4}Z_1\right) M^2\Phi, \tag{2.6}$$

where $Z_1$ is a spurion that carries a SUSY-breaking parameter of the order of $m_{\rm SUSY}$ in its $\theta^2\bar{\theta}^2$ component. The effective $\mu$ parameter is obtained with a suitable order of magnitude after the messengers are integrated out;

$$W = -\left(\frac{\bar{D}^2 D^2}{16}Z_1^\dagger\right) H_d H_u. \tag{2.7}$$

When the S symmetry is an R symmetry, and when the R charge of $H_d H_u$ is zero, A term Eq. (2.6) is quite natural since $\Phi$ has R charge two[2]. This is nothing but the Giudice–Masiero mechanism [12] when the messenger scale $M$ is the Planck scale.

### 2.1.2 Missing VEV Mechanism and Missing Partner Mechanism

The two Higgs doublets should remain light, while their SU(5) partners have to be heavy. It has been called the "doublet–triplet splitting problem" how such a spectrum is realized [13]. The missing VEV mechanism (Dimopoulos–Wilczek mechanism) [14] and the missing partner mechanism [15] are two basic ideas for the solution to this problem. The two mechanisms are explained in this subsection, but as one can see in the following, they cannot be reconciled in their original forms with the S symmetry discussed in subsection 2.1.1. Thus, the original models have to be extended keeping the original ideas. Models presented in section 2.2 and 2.3, which have the S symmetry, are based on these ideas.

[1]It can be understood that the symmetry is spontaneously broken in the SUSY-breaking sector. Domain walls can be created in that case, but only in much earlier stage of the universe. Thus, the primordial inflation can dilute such domain walls.

[2]An $A$ term linear in $\bar{\Phi}$ is quite dangerous, if it exists, since too large $\mu B$ parameter is generated in the effective theory.

|  | **1** |  | $\mathbf{10}^{ij}$ |  | $\mathbf{5}^*_i$ |  | $H(\mathbf{5})^i$ | $\bar{H}(\mathbf{5}^*)_i$ |  |
|---|---|---|---|---|---|---|---|---|---|
| Fields | $N$ | $Q$ | $\bar{U}$ | $\bar{E}$ | $\bar{D}$ | $L$ | $H_u$ | $H_d$ |  |
| $\mathrm{U(1)_Y}$ | 0 | $\frac{1}{6}$ | $-\frac{2}{3}$ | $+1$ | $\frac{1}{3}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |  |
| $\mathrm{U(1)_V}$ | $+5$ |  | $+1$ |  | $-3$ |  | $-2$ | $+2$ |  |
| $\mathrm{U(1)_{B-L}}$ | 0 | $\frac{1}{3}$ | $-\frac{1}{3}$ | $+1$ | $-\frac{1}{3}$ | $-1$ | 0 | 0 | $= \frac{1}{5}\,(\mathrm{U(1)_V} + 4\,\mathrm{U(1)_Y})$ |
| $\mathrm{U(1)_R}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $= \frac{-1}{10}\,(\mathrm{U(1)_V} - 6\,\mathrm{U(1)_Y})$ |
| $\mathrm{U(1)_{Y'}}$ | $+1$ | $\frac{1}{6}$ | $\frac{1}{3}$ | 0 | $-\frac{2}{3}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $= \frac{1}{5}\,(\mathrm{U(1)_V} - \mathrm{U(1)_Y})$ |
| $\mathrm{U(1)_{5'}}$ | 1 | 1 | $-3$ | 5 | 1 | $-3$ | 2 | $-2$ | $= \frac{1}{5}\,(\mathrm{U(1)_V} + 24\,\mathrm{U(1)_Y})$ |

Table 2.1: Various U(1) subgroups of SO(10) are listed. The first row is the standard $\mathrm{U(1)_Y}$, and the second row is the U(1) subgroup which commutes with and is orthogonal (in the Cartan metric) to the standard model gauge group. The $\mathrm{U(1)_{B-L}}$ belongs to $\mathrm{SU(4)_C}$ part of $\mathrm{SU(4)_C}\times\mathrm{SU(2)_L}\times\mathrm{SU(2)_R} \simeq \mathrm{SO(6)}\times\mathrm{SO(4)}\subset \mathrm{SO(10)}$. The $\mathrm{U(1)_R}$ is the maximal torus of the $\mathrm{SU(2)_R}$. The $\mathrm{U(1)_{B-L}}$ is orthogonal to the $\mathrm{U(1)_R}$. The fifth and the sixth rows, $\mathrm{U(1)_{Y'}}$ and $\mathrm{U(1)_{5'}}$, are those that are used in the flipped SU(5) model. They are orthogonal to each other. It follows that $\mathrm{U(1)_Y}=(1/5)(\mathrm{U(1)_{5'}} - \mathrm{U(1)_{Y'}})$. Note that the $H_u$ in the Yukawa coupling of up-type quarks is in the $\mathrm{SU(5)'}$-$\mathbf{5}'^*$ representation and the $H_d$ of the down-type quarks and charged leptons in the $\mathrm{SU(5)'}$-$\mathbf{5'}$ representation, respectively, in the flipped SU(5) model.

**missing VEV mechanism (Dimopoulos–Wilczek mechanism)**

The missing VEV mechanism [14] in its original form is based on an SO(10) unified gauge group. A chiral multiplet $A$ in the SO(10)-**adj.** representation is required to develop its VEV in the $B-L$ direction; the definition of the $\mathrm{U(1)_{B-L}}$ generator in the SO(10) is given in Table 2.1, along with those of other U(1) generators. Two Higgs doublets $H_d$ and $H_u$ are contained in a **10**(**vect.**) vector representation of the SO(10) along with newly introduced coloured triplets $H_c$ and $H_{\bar{c}}$, i.e., $\mathbf{10}=(H_c, H_u, H_{\bar{c}}, H_d)$. One more vector representation $\mathbf{10}'$ =("$\mathbf{3}^*$","$\mathbf{2}^*$","$\mathbf{3}$","$\mathbf{2}$") is necessary in order to form a mass term

$$W = \mathbf{10}\,\langle A\rangle\,\mathbf{10}'. \tag{2.8}$$

Only the triplets have mass terms while doublets not because the triplets in the vector representation are charged under the $\mathrm{U(1)_{B-L}}$, while doublets not (see Table 2.1). The model should be constructed so that the $A$ field does not develop its VEV in the $\mathrm{U(1)_R}$ direction (see Table 2.1)[3], and this missing VEV is the origin of the vanishing mass of the two Higgs doublets (missing VEV mechanism (Dimopoulos–Wilczek mechanism)). The mass

[3]This $\mathrm{U(1)_R}$ has nothing to do with the R symmetry of the SUSY.

partners of the triplets $H_c$ and $H_{\bar{c}}$ in **10** are triplets "$\mathbf{3}^*$" and "**3**" in $\mathbf{10}'$. Therefore, the dimension-five proton decay is not mediated by the coloured-Higgs fermions at this moment, since there is no mass term of the form $W = H_{\bar{c}}H_c$. The structure of the mass matrix is described schematically in Eq. (2.9):

$$\begin{pmatrix} H_c \\ H_u \end{pmatrix}^{(m)} \leftrightarrow \begin{pmatrix} \text{``}\mathbf{3}^*\text{''} \\ \text{``}\mathbf{2}^*\text{''} \end{pmatrix}^{(w-m)} \begin{matrix} --- \\ --- \end{matrix} \begin{pmatrix} \text{``}\mathbf{3}\text{''} \\ \text{``}\mathbf{2}\text{''} \end{pmatrix}^{(w-n)} \leftrightarrow \begin{pmatrix} H_{\bar{c}} \\ H_d \end{pmatrix}^{(n)}. \tag{2.9}$$

$\mathbf{10}'$ should have charge $(w-n)$ (or $(w-m)$) of the S symmetry in order that the mass term Eq. (2.8) is allowed by the symmetry. When it is assumed that the remaining doublets "$\mathbf{2}^*$" and "**2**" in the $\mathbf{10}'$ form their mass term within themselves (lower broken line in Eq. (2.9)), however, $(2w - n - m) \equiv w$ is required, and hence $W = H_d H_u$ is not forbidden by the S symmetry (i.e., $(n + m) \equiv w$). Thus, the original Dimopoulos–Wilczek mechanism does not work in the presence of the (unbroken) S symmetry.

Symmetries still exert control over low energy physics, even if it is broken at the GUT scale. We briefly mention two possibilities of constructing a model in this direction. First, a broken symmetry and combinatorics in group theory in certain models [16] do not allow non-renormalizable operators such as $W = M_{\rm pl}(\Phi/M_{\rm pl})^k H_d H_u (k = 0, 1, 2, 3, 4)$ with $\langle\Phi\rangle \sim$ (GUT scale), which would have led to too large $\mu$ parameter. The $\mu$ parameter is given by (GUT scale$/M_{\rm pl})^5 M_{\rm pl}$ in those models. Thus, the $\mu$ parameter is small and is of the order of the SUSY-breaking scale just as an accident. The other possibility is based on a fact that the small $\mu$ parameter requires a parameter $\lesssim 10^{-14}$, while the suppressed dimension-five proton-decay operators only require parameters $\lesssim 10^{-1}$. When a U(1) symmetry is broken exclusively by fields with positive U(1) charge, it happens (as long as $(w-n-m)$ is negative) that the $\mu$-term is not allowed by the U(1) symmetry and that the dimension-five operators are allowed by inserting the VEV's of the positively charged fields. One of generic consequences of this possibility is that the mass of the gauge boson that mediates the dimension-six proton decay is lighter than the scale at which the three gauge coupling constants appear to be unified, (i.e., $10^{15.8-16.3}$ GeV, see Figure 3.1) by typically one order of magnitude [17]. Thus, the proton decay is fairly fast. Explicit models based on the Dimopoulos–Wilczek mechanism and on this possibility are constructed in [18] and the proton decay amplitudes of those models are analyzed in [19].

**missing partner mechanism**

The missing partner mechanism [15] does not have to use the SO(10) gauge group. A chiral multiplet in the SU(5)-**75** representation develops a VEV so that the SU(5) gauge group is broken down to the standard-model gauge group. SU(5)-$\mathbf{50}^*$ and -$\mathbf{50}$ representations are introduced as mass partners of the triplets $H_c$ and $H_{\bar{c}}$ in $H(\mathbf{5}) = (H_c, H_u)$ and $\bar{H}(\mathbf{5}^*) = (H_{\bar{c}}, H_d)$. Superpotential

$$W = \bar{H}(\mathbf{5}^*)\ \mathbf{75}\ \mathbf{50} + \mathbf{50}^*\ \mathbf{75}\ H(\mathbf{5}) \tag{2.10}$$

contains mass terms of $H_{\bar{c}}$ and $H_c$, while not of two Higgs doublets because their mass partners are missing in $\mathbf{50}$ and $\mathbf{50}^*$. The structure of the mass matrix is described in Eq. (2.11):

$$\begin{pmatrix} H_c \\ H_u \\ - \end{pmatrix}^{(m)}_{H(\mathbf{5})} \longleftrightarrow \begin{pmatrix} \text{“}\mathbf{3}^*\text{”} \\ - \\ \mathbf{50}^*\backslash\mathbf{3}^* \end{pmatrix}^{(w-m)}_{\mathbf{50}^*} \longleftrightarrow \begin{pmatrix} \mathbf{3} \\ - \\ \mathbf{50}'\backslash\mathbf{3} \end{pmatrix}^{(m)}_{\mathbf{50}'} \longleftrightarrow \cdots . \tag{2.11}$$

$\mathbf{50}$ and $\mathbf{50}^*$ should have charge $(w-n)$ and $(w-m)$, respectively, under the S symmetry.

When remaining particles in $\mathbf{50}$ and $\mathbf{50}^*$ other than triplets have mass terms through $W = \mathbf{50}\ \mathbf{50}^*$, however, there is no symmetry that forbids the $\mu$ term, just as in the case of the missing VEV mechanism. When those particles have mass terms with particles in newly introduced multiplets $\mathbf{50}^{*'}$ and $\mathbf{50}^{'}$, whose charges are $n$ and $m$, respectively, triplets in the $\mathbf{50}^{*'}$ and $\mathbf{50}^{'}$ require another mass partners whose charges are $(w-n)$ and $(w-m)$, respectively (see Eq. (2.11)). Therefore, the fact that the doublet partners are missing in $\mathbf{50}$ and $\mathbf{50}^*$ does not help immediately the construction of a model with the S symmetry. The doublet–triplet splitting problem under the S symmetry was only translated into the splitting between triplets and forty-seven-plets.

### 2.1.3 Three Possibilities

What we call the S symmetry, which explains the small mass of the two Higgs doublets and the suppressed dimension-five proton-decay amplitude simultaneously, severely restricts the structure of the mass matrix of the Higgs particles around the GUT scale. Both original models of the missing VEV mechanism and the missing partner mechanism in subsection 2.1.2 were not compatible with the S symmetry, but one can learn lessons from the failure. Three possible structures of the mass matrix compatible with the symmetry are discussed in this subsection.

**First**, the mass matrix can easily be compatible with the symmetry if one allows to introduce infinite particles. The simplest mass matrix is described in Eq. (2.12):

$$\begin{pmatrix} H_c \\ H_u \end{pmatrix}^{(m)}_{H(\mathbf{5})} \longleftrightarrow \begin{pmatrix} \text{“}\mathbf{3}^*\text{”} \\ \text{“}\mathbf{2}^*\text{”} \end{pmatrix}^{(w-m)} \longleftrightarrow \begin{pmatrix} \mathbf{3} \\ \mathbf{2} \end{pmatrix}^{(m)} \longleftrightarrow \cdots$$

$$\left(\begin{array}{c} H_{\bar{c}} \\ H_d \end{array}\right)^{(n)}_{\bar{H}(\mathbf{5}^*)} \longleftrightarrow \left(\begin{array}{c} “\mathbf{3}” \\ “\mathbf{2}” \end{array}\right)^{(w-n)} \longleftrightarrow \left(\begin{array}{c} \mathbf{3}^* \\ \mathbf{2}^* \end{array}\right)^{(n)} \longleftrightarrow \cdots . \tag{2.12}$$

Models with infinite particles, however, cannot be treated within ordinary field theories on four-dimensional space-time. They can be treated as higher-dimensional field theories if those particles can be interpreted as a Kaluza–Klein tower [20, 21, 22].

**Secondly**, there will be a model with only finite number of particles if one succeeds in obtaining only triplets or only doublets in the spectrum without the others. This possibility is along the line of the missing partner mechanism. Indeed, this is not impossible; the flipped SU(5) model [23] is a good example which succeeds in obtaining only triplets.

**The flipped SU(5) model** [23] is based on $\mathrm{SU}(5)'\times\mathrm{U}(1)_{5'}$ gauge group. Quarks and leptons are classified into unified multiplets $\mathbf{1}'^{5}=\bar{E}$, $\mathbf{10}'^{1}=(Q,\bar{D},N)$ and $\mathbf{5}'^{*,-3}=(\bar{U},L)$, where the $\mathrm{U}(1)'_5$ charge of each multiplet is given by an integer on its shoulder. Two Higgs doublets are contained in $H(\mathbf{5}')^{-2}=(H_c,H_d)$ and $\bar{H}(\mathbf{5}'^{*})^{2}=(H_{\bar{c}},H_u)$. Two anti-symmetric tensors $H(\mathbf{10}')^{1}$ and $\bar{H}(\mathbf{10}'^{*})^{-1}$ develop VEV's so that the gauged symmetry breaks down to that of the standard model. The $\mathrm{U}(1)_{\mathrm{Y}}$ of the standard model is a linear combination of the $\mathrm{U}(1)_{\mathrm{Y}'}$ (a U(1) subgroup of the $\mathrm{SU}(5)'$ that commutes with the $\mathrm{SU}(3)_{\mathrm{C}}\times\mathrm{SU}(2)_{\mathrm{L}}$) and the $\mathrm{U}(1)_{5'}$. Irreducible decomposition of the $H(\mathbf{10})$ under the MSSM gauge group[4] is $(\mathbf{1},\mathbf{1})^{0}+(\mathbf{3},\mathbf{2})^{-\frac{5}{6}}+(\mathbf{3}^*,\mathbf{1})^{\frac{1}{3}}$, the first of which is the flat direction in which the GUT-breaking VEV develops, the second of which is absorbed through the Higgs mechanism due to the VEV, and finally the last of which is suitable for the mass partner of the $H_c$. The mass terms of the coloured Higgs particles are given by

$$W = H(\mathbf{10}')^{1}H(\mathbf{10}')^{1}H(\mathbf{5}')^{-2} + \bar{H}(\mathbf{10}'^{*})^{-1}\bar{H}(\mathbf{10}'^{*})^{-1}\bar{H}(\mathbf{5}'^{*})^{2}. \tag{2.13}$$

Yukawa couplings of quarks and leptons are

$$W = y_{u,n_D}\mathbf{5}'^{*,-3}\mathbf{10}'^{1}\bar{H}(\mathbf{5}'^{*})^{2} + y_d\mathbf{10}'^{1}\mathbf{10}'^{1}H(\mathbf{5}')^{-2} + y_e\mathbf{1}'^{5}\mathbf{5}'^{*,-3}H(\mathbf{5}')^{-2}. \tag{2.14}$$

The “unified gauge group” is not a simple group in this model. Thus, the gauge-coupling unification is not an inevitable prediction. The three gauge coupling constants of the MSSM are expressed (at tree level) in terms of gauge coupling constants of the GUT model as :

$$\frac{1}{\alpha_{\mathrm{C}}} = \frac{1}{\alpha_{\mathrm{L}}} = \frac{1}{\alpha_{\mathrm{SU(5)'}}}, \tag{2.15}$$

$$\frac{\frac{3}{5}}{\alpha_{\mathrm{Y}}} = \frac{1}{25}\left(\frac{1}{\alpha_{\mathrm{SU(5)'}}} + \frac{\frac{3}{5}}{\alpha_{5'}}\right), \tag{2.16}$$

[4] For example, $(\mathbf{3},\mathbf{2})^{-\frac{5}{6}}$ is triplets of the $\mathrm{SU}(3)_{\mathrm{C}}$, doublets of the $\mathrm{SU}(2)_{\mathrm{L}}$ and is charged $-5/6$ under the $\mathrm{U}(1)_{\mathrm{Y}}$ symmetry. This convention is used throughout this thesis.

where $\alpha_{\rm C}, \alpha_{\rm L}$ and $\alpha_{\rm Y}$ are fine structure constants of the $\mathrm{SU}(3)_{\rm C}$, $\mathrm{SU}(2)_{\rm L}$ and $\mathrm{U}(1)_{\rm Y}$ interactions of the MSSM, respectively. $\alpha_{\mathrm{SU}(5)'}$ and $\alpha_{5'}$ are those of the $\mathrm{SU}(5)'$ and $\mathrm{U}(1)_{5'}$ interactions; normalization of the charge (hence of the coupling constant) of the $\mathrm{U}(1)_{5'}$ is based on a convention adopted in Table 2.1. The approximate SU(5) relation $1/\alpha_{\rm C} \simeq 1/\alpha_{\rm L} \simeq (3/5)/\alpha_{\rm Y}$ is satisfied when $(3/5)/\alpha_{5'} \simeq 24/\alpha_{\mathrm{SU}(5)'}$, and this is the case when both $\mathrm{SU}(5)'$ and $\mathrm{U}(1)_{5'}$ come from a common SO(10) gauge group [24]. It is argued that a VEV of SO(10)-($\wedge^2$**vect**.) or -($\wedge^4$**vect**.) representations would break the SO(10) down to $\mathrm{SU}(5)' \times \mathrm{U}(1)_{5'}$ [24, 25], but a (super)potential that gives the VEV and a symmetry of the (super)potential have not been investigated very much.

**"Higgs as pseudo Nambu–Goldstone multiplet"** [26] is another idea for models that have only two doublets or two triplets. Let us assume an SU(6) global symmetry among which upper-left $5 \times 5$ corner is gauged as the SU(5) unified gauge group. When a chiral multiplet $\Sigma$ in SU(6)-**adj.** representation develops a VEV such as

$$\langle \Sigma \rangle = \begin{pmatrix} v & & & & & \\ & v & & & & \\ & & v & & & \\ & & & -2v & & \\ & & & & -2v & \\ & & & & & v \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} v & & & & & \\ & v & & & & \\ & & v & & & \\ & & & -v & & \\ & & & & -v & \\ & & & & & -v \end{pmatrix}, \tag{2.17}$$

the global symmetry SU(6) is broken down to SU(4)$\times$SU(2)$\times$U(1) or SU(3)$\times$SU(3)$\times$U(1), respectively. The first and the second factor in both cases contain the gauged $\mathrm{SU}(3)_{\rm C}$ and $\mathrm{SU}(2)_{\rm L}$ symmetry, respectively. The (pseudo) Nambu–Goldstone multiplet is ($\mathbf{4}$,$\mathbf{2}^*$) + h.c. or ($\mathbf{3}$,$\mathbf{3}^*$) + h.c. representations in terms of the unbroken global symmetry in each case. The ($\mathbf{3}$,$\mathbf{2}^*$) + h.c. part is absorbed through the Higgs mechanism in both cases, and as a result, only ($\mathbf{1}$,$\mathbf{2}^*$) + h.c. or ($\mathbf{3}$,$\mathbf{1}$) + h.c. remain as the (pseudo) Nambu–Goldstone multiplets.

It is discussed in [27] how an accidental (approximate) SU(6) global symmetry is obtained in the superpotential by assuming an SU(6) gauge group. Chiral multiplets $H$ and $\bar{H}$ in the SU(6)-**6** and -$\mathbf{6}^*$ representations are introduced. The superpotential are required to have a form

$$W = W_\Sigma(\Sigma) + W_H(H, \bar{H}), \tag{2.18}$$

say, because of an additional symmetry. Then, the superpotential has SU(6)$\times$SU(6) global symmetry accidentally, diagonal part of which is gauged. When the $H$ and $\bar{H}$ acquire VEV's (which are larger than the GUT scale) so that the gauged SU(6) symmetry is broken down to SU(5), the SU(6) global symmetry still exists in the superpotential of the effective theory (i.e., $W_\Sigma$) after $H$ and $\bar{H}$ are integrated out. Yukawa couplings of quarks and leptons in theories with the SU(6) gauge symmetry is discussed in [28].

It is difficult, however, to forbid all operators through which the accidental global symmetry SU(6)×SU(6) reduces to the diagonal SU(6). For examples, the Yukawa couplings given in [28] take the following form:

$$W = W_1(\Sigma, \text{matter fields}) + W_2(\bar{H}, \text{matter fields}). \tag{2.19}$$

It is true that these terms do not contribute to the $\mu$ term since matter fields do not have VEV's. However, it is shown in [29] that additional discrete symmetries are not sufficient in forbidding all non-renormalizable operators comprised of both $\Sigma$ and ($H$ or $\bar{H}$). Such operators do contribute to the $\mu$ term; such mixed terms become explicit breakings to the accidental global SU(6) symmetry in the effective theory, and the Nambu–Goldstone multiplets acquire masses.

Now that the explicit breaking to the SU(6) global symmetry turns out to be inevitable, the unbroken symmetry (referred to as S symmetry) discussed in subsection 2.1.1 is desirable to forbid too large $\mu$ parameter. Although the idea of the "Higgs as pseudo Nambu–Goldstone multiplet" may be plausible as an origin of the solution to the doublet–triplet splitting problem, it is not powerful enough by itself to guarantee the suppressed $\mu$ parameter. The S symmetry that commutes with SU(5) symmetry is given by a linear combination of an SU(6)-commuting symmetry and a subgroup of the SU(6) that commutes with the SU(5). Then, the product of the pseudo Nambu–Goldstone multiplets $(\mathbf{1},\mathbf{2}^*)\cdot(\mathbf{1},\mathbf{2})$ or $(\mathbf{3},\mathbf{1})\cdot(\mathbf{3}^*,\mathbf{1})$ in each case is necessarily singlet of the S symmetry[5]. This implies that the S symmetry should be an R symmetry rather than a pure Peccei–Quinn symmetry[6]. It also follows that the $(\mathbf{8}{\simeq}\mathbf{adj.},\mathbf{1})^0 + (\mathbf{1},\mathbf{3}{\simeq}\mathbf{adj.})^0$ part of the $\Sigma$ in both cases are singlets of the S symmetry. On the other hand, they should have mass terms consistent with the S symmetry. This means that their mass partners are additionally needed since the partners should have R charge two. An SU(5) multiplet that contains these mass partners, such as an extra **adj.** representation, would also contains more particles than are required as the mass partners. Thus, we still need mass partners that are not in an SU(5)-complete multiplet, and hence an almost similar

[5] The VEV is required to be a singlet of the symmetry.

[6] This is trivial for the former case where Higgs doublets are the pseudo Nambu–Goldstone multiplets. For the latter case where the mass partners of the coloured Higgs particles are the pseudo Nambu–Goldstone multiplets, if the S symmetry were a Peccei–Quinn symmetry, the product of the coloured Higgs would also be a singlet, and hence the product of two Higgs doublets would have been allowed in the superpotential by the S symmetry.

problem remains, even if the doublet–triplet splitting problem is solved in this way. However, a solution to this new problem is given by models explained in section 2.2.

**Finally**, we briefly mention about another possibility that was clearly pointed out quite recently in [30]. The discussion so far assumes that each MSSM-irreducible component of an SU(5)-unified multiplet has a common charge under the S symmetry. However, what is certain for us is that the S symmetry is desirable *in the MSSM*, and we do not have any knowledge about how the symmetry is realized at the GUT scale. In particular, each MSSM-irreducible component within an unified multiplet can have arbitrary charge in general. Although the charge assignment has meaning only up to addition of the $U(1)_Y$ charge, one can still think of a charge assignment in which a linear combination of the S symmetry and the $U(1)_Y$ symmetry cannot be chosen so that it commutes with the SU(5) unified symmetry. Thus, this is a possibility different from the previous ones. Section 2.3 explains a model that realizes this possibility.

## 2.2 Product-Group Unification Model I (Missing Partner Type)

The following two sections explain SUSY-GUT models based on field theories on four-dimensional space-time that have the S symmetry discussed in the previous section. Section 2.2 treats models in which the charge of the S symmetry is common in each SU(5)-unified multiplet, and section 2.3 in which the charges can be different among irreducible components of an SU(5)-unified Higgs multiplet.

### 2.2.1 SU(5)-based Models

Let us pursue the idea of the "Higgs as pseudo Nambu–Goldstone multiplet", where two Higgs doublets (or mass partners of the triplets) arise as pseudo Nambu–Goldstone multiplets in the spontaneous breaking of a global SU(6) symmetry. Only the spontaneous breaking due to the VEV of the SU(6)-**adj.** field $\Sigma$ is considered. We do not regard the difference between the two following possibilities as important for the time being[7]; the global SU(6) symmetry may be just an accidental one or it may originate from a gauged SU(6) symmetry. This difference was not a crucial problem as discussed in subsection 2.1.3. Whether the SU(6) symmetry is exact or only approximate is not an essential question, either. The crucial problem is to find

[7]Related discussion will be found in section 4.3, 4.4 and 4.6.

a way to reconcile the doublet–triplet splitting problem (and its cousin) with the S symmetry. SU(5)-unified multiplets that contain required mass partners tend to contain more particles than required, which require additional mass partners again. These chain requirements would never stop!

The crucial step was made in [31], where extra gauge groups were introduced and the SU(6)-**adj.** field $\Sigma$ was considered as a composite field of the extra gauge groups. The $\Sigma$ contains only one of $(\mathbf{8}\simeq\mathbf{adj.},\mathbf{1})^0$ and $(\mathbf{1},\mathbf{3}\simeq\mathbf{adj.})^0$ representations of the MSSM as one-particle degrees of freedom, as seen below. Based on this fact, Ref. [32] found a way to introduce only the required mass partner, and hence the chain requirements stop at finite order. Models with the S symmetry was discovered in [33], with a little modification to the superpotential given in [32].

**$\mathbf{SU(5)}_{\rm GUT}\times\mathbf{U(2)}_{\rm H}$ model**

Let us first explain a model in which two Higgs doublets are obtained as pseudo Nambu-Goldstone multiplets. Gauge group is extended into $\mathrm{SU(5)_{GUT}}\times\mathrm{U(2)_H}$, where $\mathrm{U(2)_H} \simeq \mathrm{SU(2)_H}\times\mathrm{U(1)_H}$. Chiral multiplets in the bi-fundamental representation $Q^\alpha{}_i + Q^\alpha{}_6(\mathbf{5}^*{+}\mathbf{1},\mathbf{2})$ and $\bar{Q}^i_\alpha + \bar{Q}^6_\alpha(\mathbf{5}{+}\mathbf{1},\mathbf{2}^*)$ ($\alpha = 4,5$ ; $i = 1,...,5$) are introduced as particles in the GUT-symmetry-breaking sector. Indices $i,j$ are used for the $\mathrm{SU(5)_{GUT}}$ and $\alpha,\beta$ for the $\mathrm{U(2)_H}$. When $\bar{Q}^k{}_\alpha$ and $Q^\alpha{}_l$ ($k,l = 1,...,6$) have VEV's of the form

$$\left\langle \bar{Q}^k{}_\alpha \right\rangle = \begin{pmatrix} & \\ & \\ & \\ v & \\ & v \\ & \end{pmatrix}, \qquad \langle Q^\alpha{}_l \rangle = \begin{pmatrix} & & & v & & \\ & & & & v & \end{pmatrix}, \tag{2.20}$$

which is the case for a superpotential Eq. (2.22) as shown later, a $\mathrm{U(2)_H}$-singlet composite field $(\bar{Q}Q)^k{}_l$ ($k,l = 1,...,6$) has a VEV of the form

$$\left\langle \left(\bar{Q}Q\right)^k{}_l \right\rangle = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & v^2 & & \\ & & & & v^2 & \\ & & & & & 0 \end{pmatrix}, \tag{2.21}$$

and effectively plays the role of $\Sigma$ in Eq (2.17).

Two Higgs doublets $H_u$ and $H_d$ are identified with the pseudo Nambu–Goldstone multiplets $(\bar{Q}Q)^{4,5}_{\ \ 6}$ and $(\bar{Q}Q)^6{}_{4,5}$. $(\bar{Q}Q)^{1,2,3}_{\ \ 4,5}{+}(\bar{Q}Q)^{4,5}_{\ \ 1,2,3}$ parts are absorbed by vector multiplets

| Fields | $\mathbf{10}^{ij}$ | $\mathbf{5}^*_i$ | $X$ | $Q_i, \bar{Q}^i$ | $Q_6, \bar{Q}^6$ |
|---|---|---|---|---|---|
| R charges | 1 | 1 | 2 | 0 | 0 |

Table 2.2: R charges of the fields in the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ model are given here.

in the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge group. $(\bar{Q}Q)^k{}_l$ with $k, l \neq 4, 5$ do not contain one-particle degrees of freedom, while $(\bar{Q}Q)^k{}_l$ with $k, l = 4, 5$ contains one-particle degrees of freedom that transforms as $(\mathbf{1}, \mathbf{2} \otimes \mathbf{2}^*)^0$ under the MSSM gauge group, requiring mass partners. The discussion in subsection 2.1.3 tells us that the unbroken symmetry should be an R symmetry and that the product $H_d H_u$ has R charge zero. The $(\bar{Q}Q)^k{}_l$ with $k, l = 4, 5$ part ($(\mathbf{1}, \mathbf{2} \otimes \mathbf{2}^*)^0$ of the MSSM) also has R charge zero, and hence its mass partner is additionally required, whose R charge is two. It suffices to introduce a chiral multiplet $X^\alpha{}_\beta$ ($\alpha, \beta = 4, 5$) in the $(\mathbf{1}, \mathbf{2} \otimes \mathbf{2}^* = \mathbf{3} + \mathbf{1})$ representation of the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ gauge group, because the $\mathrm{SU}(2)_{\mathrm{L}}$ gauge group of the MSSM is a diagonal subgroup of the $\mathrm{SU}(2)_{\mathrm{H}}$ and an SU(2) subgroup of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ (see Eq. (2.20)). The chiral superfield $X^\alpha{}_\beta$ is also written as $X^c (t_c)^\alpha{}_\beta (c = 0, 1, ..., 3)$, where $t_a (a = 1, ..., 3)$ are one half of the Pauli matrices[8] and $t_0 \equiv \mathbf{1}_{2\times 2}/2$. No further mass partners are necessary.

Three families of quarks and leptons are in the $(\mathbf{5}^* + \mathbf{10}, \mathbf{1})$ representations of the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ gauge group, as usual. R charge assignment of all the fields in this model is summarized in Table 2.2. This symmetry plays the role of the S symmetry in this model. The most generic superpotential under this R symmetry is given by

$$
\begin{aligned}
W \ = \ & \sqrt{2}\lambda_{2\mathrm{H}} \bar{Q}^i{}_\alpha X^a (t_a)^\alpha_\beta Q^\beta_i + \sqrt{2}\lambda'_{2\mathrm{H}} \bar{Q}^6{}_\alpha X^a (t_a)^\alpha_\beta Q^\beta_6 \\
& + \sqrt{2}\lambda_{1\mathrm{H}} \bar{Q}^i{}_\alpha X^0 (t_0)^\alpha_\beta Q^\beta_i + \sqrt{2}\lambda'_{1\mathrm{H}} \bar{Q}^6{}_\alpha X^0 (t_0)^\alpha_\beta Q^\beta_6 \\
& - \sqrt{2}\lambda_{1\mathrm{H}} v^2 X^\alpha_\alpha \\
& + c_{\mathbf{10}} \mathbf{10}^{i_1 i_2} \mathbf{10}^{i_3 i_4} (\bar{Q}Q)^{i_5}_6 + c_{\mathbf{5}^*} (\bar{Q}Q)^6{}_i \cdot \mathbf{10}^{ij} \cdot \mathbf{5}^*_j + \cdots ,
\end{aligned}
\tag{2.22}
$$

where the parameter $v$ is taken to be of the order of the GUT scale and $\lambda_{2\mathrm{H}}, \lambda'_{2\mathrm{H}}, \lambda_{1\mathrm{H}}$ and $\lambda'_{1\mathrm{H}}$ are dimensionless coupling constants. Ellipses stand for neutrino-mass terms and other non-renormalizable terms. The fields $Q^\alpha_i$ and $\bar{Q}^i{}_\alpha$ in the bi-fundamental representations acquire VEV's as in Eq. (2.20) because of the first through the third lines in Eq. (2.22). As a result, the gauge group $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ is broken down to that of the standard model. The first terms in both the first and the second lines in Eq. (2.22) provide mass terms of the

[8] The normalization condition $\mathrm{tr}(t_a t_b) = \delta_{ab}/2$ is understood. Note that the normalization of the following $t_0$ is determined in such a way that it also satisfies $\mathrm{tr}(t_0 t_0) = 1/2$.

unwanted particles in the $(\mathbf{1},\mathbf{2}\otimes\mathbf{2}^*)^0$ representations of the MSSM. No particle remains in the low-energy spectrum other than the MSSM fields, not even gauge singlets of the MSSM. This fact not only guarantees that the evolution of the gauge coupling constants are given up to the GUT scale by the well-known renormalization group (RG) of the MSSM, but that the vacuum is isolated. The R symmetry is not broken at the GUT scale, as required, and the $\mu$ term, dimension-four- and dimension-five- proton-decay operators are forbidden by this unbroken symmetry.

The fine structure constants of the MSSM are given (at tree level) by

$$\frac{1}{\alpha_{\rm C}} = \frac{1}{\alpha_{\rm GUT}}, \tag{2.23}$$

$$\frac{1}{\alpha_{\rm L}} = \frac{1}{\alpha_{\rm GUT}} + \frac{1}{\alpha_{\rm 2H}}, \tag{2.24}$$

and

$$\frac{\frac{3}{5}}{\alpha_{\rm Y}} = \frac{1}{\alpha_{\rm GUT}} + \frac{\frac{3}{5}}{\alpha_{\rm 1H}}, \tag{2.25}$$

where $\alpha_{\rm GUT}, \alpha_{\rm 2H}$ and $\alpha_{\rm 1H}$ are fine structure constants of $\mathrm{SU}(5)_{\rm GUT}$, $\mathrm{SU}(2)_{\rm H}$ and $\mathrm{U}(1)_{\rm H}$, respectively. Thus, the the approximate unification of $\alpha_{\rm C}, \alpha_{\rm L}$ and $5\alpha_{\rm Y}/3$ is maintained as long as

$$\frac{1}{\alpha_{\rm GUT}} \times 10^{-2} \gtrsim \frac{1}{\alpha_{\rm 2H}}, \frac{1}{\alpha_{\rm 1H}}. \tag{2.26}$$

In other words, the SU(5) unification of the MSSM gauge coupling constants follows from this model as long as gauge interactions other than that of the $\mathrm{SU}(5)_{\rm GUT}$ are relatively strong. It is true that the coupling unification is no longer a generic prediction of this model, but the unification is not just an accident, either. The coupling unification is understood, if only it is explained why other gauge interactions are relatively strong, or if only it is explained why only the $\mathrm{SU}(5)_{\rm GUT}$ gauge interaction is relatively weak.

Finally, one would notice that the cut-off scale of the model should be lower than the Planck scale. There are two reasons for that. First, the $(\bar{Q}Q)/M_*^2$ insertion with $M_* \simeq M_{\rm pl}$ is too small to account for the difference between running Yukawa coupling constants of strange quark and muon. Second, the gauge coupling constant of the $\mathrm{U}(1)_{\rm H}$ factor runs in an asymptotic non-free way, and it becomes large below the Planck scale.

**$\mathbf{SU(5)_{\rm GUT}\times U(3)_{\rm H}}$ model**

Let us now explain a model in which the mass partners of the two Higgs triplets $H_c$ and $H_{\bar{c}}$ are obtained as pseudo Nambu–Goldstone multiplets. Gauge group is extended into $\mathrm{SU}(5)_{\rm GUT}\times\mathrm{U}(3)_{\rm H}$, where $\mathrm{U}(3)_{\rm H} \simeq \mathrm{SU}(3)_{\rm H}\times\mathrm{U}(1)_{\rm H}$. Chiral multiplets in the bi-fundamental representation ${Q^\alpha}_i + {Q^\alpha}_6(\mathbf{5}^*+\mathbf{1},\mathbf{3})$ and $\bar{Q}^i_\alpha + \bar{Q}^6_\alpha(\mathbf{5}+\mathbf{1},\mathbf{3}^*)$ ($\alpha = 1,2,3$ ; $i = 1,...,5$) in the GUT-symmetry-breaking sector break the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ symmetry down to the gauged symmetry of the MSSM through VEV's of the form

$$\left\langle {\bar{Q}^k}_\alpha \right\rangle = \begin{pmatrix} v & & \\ & v & \\ & & v \\ & & \\ & & \\ & & \end{pmatrix}, \qquad \langle {Q^\alpha}_l \rangle = \begin{pmatrix} v & & & & & \\ & v & & & & \\ & & v & & & \end{pmatrix}, \tag{2.27}$$

which leads to

$$\left\langle {\left(\bar{Q}Q\right)^k}_l \right\rangle = \begin{pmatrix} v^2 & & & & & \\ & v^2 & & & & \\ & & v^2 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}. \tag{2.28}$$

Indices $i, j$ are used for the $\mathrm{SU}(5)_{\mathrm{GUT}}$, $k, l$ for the "SU(6) global symmetry"[9] and $\alpha, \beta$ for the $\mathrm{U}(3)_{\mathrm{H}}$. A $\mathrm{U}(3)_{\mathrm{H}}$-singlet ${(\bar{Q}Q)^k}_l$ effectively plays the role of $\Sigma$ in Eq (2.17).

Discussion proceeds just as in the case of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model. The S symmetry should be an R symmetry. The product ${(\bar{Q}Q)^6}_{1,2,3}\cdot(\bar{Q}Q)^{1,2,3}_{6}$ has R charge zero, and hence the products $H_{\bar{c}}H_c$ and $H_dH_u$ have R charge four. Another chiral multiplet ${X^\alpha}_\beta$ ($\alpha,\beta = 1,2,3$) in the $(\mathbf{1},\mathbf{3}\otimes\mathbf{3}^*)^0$ representation of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ gauge group is introduced to form a mass term with $(\bar{Q}Q)^{1,2,3}_{1,2,3}$. The R charge of the ${X^\alpha}_\beta$ is two.

Particle contents of this model are ${\bar{Q}^k}_\alpha$, ${Q^\alpha}_k$ and ${X^\alpha}_\beta$ in addition to the ordinary three families of quarks and leptons $(\mathbf{5}^*+\mathbf{10},\mathbf{1})$ and Higgs multiplets $H(\mathbf{5})^i+\bar{H}(\mathbf{5}^*)_i$ $((\mathbf{5}+\mathbf{5}^*,\mathbf{1})$ of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ gauge group). Their R charges are summarized in Table 2.3. The most generic superpotential under the R symmetry is given [33] by

$$\begin{aligned} W \;=\;& \sqrt{2}\lambda_{3\mathrm{H}}{\bar{Q}^i}_\alpha X^a(t_a)^\alpha_\beta Q^\beta_i + \sqrt{2}\lambda'_{3\mathrm{H}}{\bar{Q}^6}_\alpha X^a(t_a)^\alpha_\beta Q^\beta_6 \\ & +\sqrt{2}\lambda_{1\mathrm{H}}{\bar{Q}^i}_\alpha X^0(t_0)^\alpha_\beta Q^\beta_i + \sqrt{2}\lambda'_{1\mathrm{H}}{\bar{Q}^6}_\alpha X^0(t_0)^\alpha_\beta Q^\beta_6 \\ & -\sqrt{2}\lambda_{1\mathrm{H}}v^2 X^\alpha_\alpha \\ & +h'\bar{H}_i{\bar{Q}^i}_\alpha Q^\alpha_6 + h{\bar{Q}^6}_\alpha Q^\alpha_i H^i \\ & +y_{\mathbf{10}}\mathbf{10}\cdot\mathbf{10}\cdot H + y_{\mathbf{5}^*}\mathbf{5}^*\cdot\mathbf{10}\cdot\bar{H} + \cdots, \end{aligned} \tag{2.29}$$

[9] There is no global SU(6) symmetry in the superpotential Eq. (2.29) because of its fourth line. Off-course this is why the pseudo Nambu–Goldstone multiplets have masses. There is global SU(6) symmetry in the absence of the fourth line.

| $\mathbf{10}^{ij}$ | $\mathbf{5}^*_i$ | $H(\mathbf{5})^i$ | $\bar{H}(\mathbf{5}^*)_i$ | $X^\alpha{}_\beta$ | $Q^\alpha{}_i$ | $\bar{Q}^i{}_\alpha$ | $Q^\alpha_6$ | $\bar{Q}^6{}_\alpha$ | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 2 | 2 | $0+a$ | $0-a$ | $0+a$ | $0-a$ | $+\left(\frac{1}{5}+\frac{N}{10}m\right)\times\mathrm{U}(1)_\mathrm{V}$ |
| 1 | 1 | 0 | 0 | 2 | $0+a'$ | $0-a'$ | $-2+a'$ | $2-a'$ | $+\frac{2}{5}m'\times\mathrm{U}(1)_\mathrm{V}$ $(N{=}4)$ |

Table 2.3: All the field contents of the $\mathrm{SU}(5)_\mathrm{GUT}\times\mathrm{U}(3)_\mathrm{H}$ model and the R charge assignment to them are summarized here. The R charge assignment in the second row is obtained by requiring the Yukawa couplings of quarks and leptons and the Majorana mass terms of neutrinos $W = \mathbf{5}^*_i H(\mathbf{5})^i \mathbf{5}^*_j H(\mathbf{5})^j$ are allowed by a mod-$N$ R symmetry. The charge assignment is not uniquely determined; arbitrariness is parametrized by a continuous parameters $a$ and by an integer $m$ in the table. Parameter $a$ corresponds to the $\mathrm{U}(1)_\mathrm{H}$ symmetry and $m$ to unbroken subgroup of the $\mathrm{U}(1)_\mathrm{V}$ symmetry. The normalization of the $\mathrm{U}(1)_\mathrm{V}$ charge is based on the convention adopted in Table 2.1 and it is further understood that the U(1)V charges are 2 and $-2$ for $Q^a{}_i$ and $\bar{Q}^i{}_\alpha$, and 0 for others in the GUT-symmetry-breaking sector, respectively. The third row shows an R charge assignment in the case $N=4$. $m = 2+m'$ and $a = -2+a'$.

where $t_a(a = 1, 2, ..., 8)$'s are Gell-mann matrices, $t_0 \equiv \mathbf{1}_{3\times 3}/\sqrt{6}$, $y_{\mathbf{10}}$ and $y_{\mathbf{5}^*}$ are Yukawa coupling constants of the quarks and leptons, and $\lambda_{3\mathrm{H}}, \lambda'_{3\mathrm{H}}, \lambda_{1\mathrm{H}}, \lambda'_{1\mathrm{H}}, h'$ and $h$ are dimensionless coupling constants. The first through the third lines of Eq. (2.29) lead to the VEV of the form given in Eq. (2.27). The mass terms of the coloured Higgs multiplets arise from the fourth line in Eq. (2.29) in the GUT-symmetry-breaking vacuum. No unwanted particle remains massless after the gauged symmetry is broken down to that of the standard model.

Fine structure constants of the $\mathrm{SU}(3)_\mathrm{H}\times\mathrm{U}(1)_\mathrm{H}$ must be larger than that of the $\mathrm{SU}(5)_\mathrm{GUT}$. This is because the gauge coupling constants of the MSSM are given by

$$\frac{1}{\alpha_\mathrm{C}} = \frac{1}{\alpha_\mathrm{GUT}} + \frac{1}{\alpha_{3\mathrm{H}}}, \tag{2.30}$$

$$\frac{1}{\alpha_\mathrm{L}} = \frac{1}{\alpha_\mathrm{GUT}}, \tag{2.31}$$

and

$$\frac{3/5}{\alpha_\mathrm{Y}} = \frac{1}{\alpha_\mathrm{GUT}} + \frac{2/5}{\alpha_{1\mathrm{H}}}, \tag{2.32}$$

where $\alpha_{3\mathrm{H}}$ and $\alpha_{1\mathrm{H}}$ are fine structure constants of $\mathrm{SU}(3)_\mathrm{H}$ and $\mathrm{U}(1)_\mathrm{H}$, respectively.

$$\frac{1}{\alpha_\mathrm{GUT}} \times 10^{-2} \gtrsim \frac{1}{\alpha_{3\mathrm{H}}}, \frac{1}{\alpha_{1\mathrm{H}}}, \tag{2.33}$$

is necessary at the GUT scale to reproduce the approximate unification of $\alpha_\mathrm{C}, \alpha_\mathrm{L}$ and $5\alpha_\mathrm{Y}/3$.

It was remarked that the cut-off scale of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model should be lower than the Planck scale. The same statement holds true also in this model because of the same reasons.

**$\mu$-term and anomaly-free condition**

The S symmetry that commutes with the SU(5)-unified gauge group should be an R symmetry if it is required to be a gauged symmetry[10]. This does not depend on any SUSY-GUT models, as shown below. The robustness of this statement originates from the fact that the anomaly can be considered as an infra-red (IR) effect, and hence can be calculated (to some extent) within IR effective theories, the MSSM. The S symmetry should be preserved at least at the accuracy of $10^{-14}$. On the other hand, it is argued that global symmetries are completely broken by quantum gravity effects, and only gauged symmetries are left unbroken [34]. In this respect, it is quite interesting that the S symmetry is an R symmetry both in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model and in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model.

Let us suppose that the S symmetry is a non-R (i.e., pure Peccei–Quinn) gauged symmetry, whose charge is defined mod $N$. Then, mixed triangle anomalies of the form (S $\cdot$[gauged non-Abelian]$^2$) should be cancelled mod $N$, where normalization of the anomaly is such that a fermion in the **fund.** representation of an SU($M$) gauge group contributes by (S charge of the fermion) [35, 36], and these are the only necessary conditions [35, 36]. $N$ is taken to be infinity when a continuous S symmetry is considered. Let us suppose $Q$, $\bar{U}$ and $\bar{E}$ have S charge $t$ (mod $N$), $\bar{D}$ and $L$ have S charge $f$, $H_u$ and $H_d$ have $h_u$ and $h_d$, respectively. Then, the anomaly-cancellation conditions require

$$\mathrm{S}\cdot[\mathrm{SU}(3)_{\mathrm{C}}]^2 \ : \ 3\times(3t+f)+ \qquad\qquad\qquad +\text{extra}\equiv 0 \pmod{N}, \tag{2.34}$$

$$\mathrm{S}\cdot[\mathrm{SU}(2)_{\mathrm{L}}]^2 \ : \ 3\times(3t+f)+(h_u+h_d)+\text{extra}\equiv 0 \pmod{N}, \tag{2.35}$$

where extra contributions come from extra particles that are massless unless the S symmetry is spontaneously broken down, if such particles exist. They should form SU(5)-complete multiplets, or otherwise, the approximate SU(5)-unification of the MSSM gauge coupling constants would no longer hold. Since the S symmetry forbids the $\mu$ term, $h_u+h_d\not\equiv 0$, and hence Eqs. (2.34) and (2.35) cannot be satisfied simultaneously.

When the S symmetry is a mod-$N$ R symmetry, the anomaly-cancellation conditions are

$$\mathrm{S}\cdot[\mathrm{SU}(3)_{\mathrm{C}}]^2 \ : \ 3\times(3t+f-4)+6+ \qquad\qquad\qquad\qquad +\text{extra}\equiv 0 \pmod{N}, \tag{2.36}$$

$$\mathrm{S}\cdot[\mathrm{SU}(2)_{\mathrm{L}}]^2 \ : \ 3\times(3t+f-4)+4+(h_u+h_d-2)+\text{extra}\equiv 0 \pmod{N}, \tag{2.37}$$

[10] "A discrete group is gauged" means that it is a subgroup of a continuous gauged symmetry in this thesis.

| Fields | $Q, \bar{U}, \bar{E}$ | $\bar{D}, L$ | $H_u$ | $\bar{H}_d$ |
|---|---|---|---|---|
| R charges (mod 4) | $1 + \frac{2}{5}m$ | $1 - 3\frac{2}{5}m$ | $0 - 2\frac{2}{5}m$ | $0 + 2\frac{2}{5}m$ |

Table 2.4: Mod-4 R charges of the fields in the MSSM are given here. $m$ is an arbitrary integer.

where the second contributions come from gluino and wino, respectively. $N$ should be 4 in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model [37], since R charges of $H_u = (\bar{Q}Q)^{4,5}_{\ \ 6}$ and $H_d = (\bar{Q}Q)^{6}_{\ 4,5}$ are both zero ($h_u = h_d = 0$). $N$ is arbitrary in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model, since $h_u + h_d \equiv 4$.

Extra SU(5)-complete multiplets are required in both models to cancel the anomaly by $-6$ units in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model and by 6 units in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model. Extra three pairs of chiral multiplets SU(5)-($\mathbf{5}$+$\mathbf{5}^*$) whose R charges are summed up to be 0 and 4, respectively, are sufficient in cancelling the mixed $\mathrm{S}\cdot[\mathrm{SU}(5)_{\mathrm{GUT}}]^2$ anomaly. Since the vector-like mass terms $W = \mathbf{5}^*\mathbf{5}$ of these extra multiplets have the same R charge as $W = H_u H_d$, those particles have SUSY masses of the order of the $\mu$ parameter[11], and hence their existence at low energy does not contradict with current experiments. For more detailed properties of the extra particles, see Ref. [37]. Only one pair is enough when $N = 4$.

One can also see by explicit calculation that mixed anomalies $\mathrm{S}\cdot[\mathrm{SU}(2)_{\mathrm{H}}]^2$ of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ $\times\mathrm{U}(2)_{\mathrm{H}}$ model and $\mathrm{S}\cdot[\mathrm{SU}(3)_{\mathrm{H}}]^2$ of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model vanish for $N = 4$ and for arbitrary $N$, respectively. This is not surprising because the mixed anomaly $\mathrm{S}\cdot[\mathrm{SU}(2)_{\mathrm{L}}]^2$ ($\equiv 0$) of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model, for example, is a sum of $\mathrm{S}\cdot[\mathrm{SU}(2)_{\mathrm{H}}]^2$ and $\mathrm{S}\cdot[\mathrm{SU}(2)\subset\mathrm{SU}(5)_{\mathrm{GUT}}]^2$ $= \mathrm{S}\cdot[\mathrm{SU}(3)\subset\mathrm{SU}(5)_{\mathrm{GUT}}]^2 = \mathrm{S}\cdot[\mathrm{SU}(3)_{\mathrm{C}}]^2 \equiv 0$.

It was discovered in [37] that the mod-4 subgroup is the only subgroup of the continuous R symmetry that can be gauged, when the Giudice–Masiero mechanism works, i.e., when $H_d H_u$ has R charge zero. In the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model, the Giudice–Masiero mechanism works when $N = 4$ (see Table 2.3). The mod-4 R charges of the MSSM fields determined in [37] is given in Table 2.4, which are the same as those given in Tables 2.2 and 2.3.

[11]The R symmetry is spontaneously broken down to R parity at the electroweak scale in the MSSM sector.

## 2.3 Product-Group Unification Model II (Missing VEV·Yukawa Type)

### 2.3.1 Models

We consider models of SUSY GUT's that have a symmetry we refer to as the S symmetry. The symmetry is supposed to be preserved even below the GUT scale, so that the $\mu$ term and the dimension-five proton-decay operators are forbidden to all orders in (GUT VEV/$M_{\rm pl}$). This does not mean, however, that the symmetry remains the same both above and below the GUT scale; the S symmetry at low energy, which we refer to as the S$'$ symmetry in this section, can be a combination of the S symmetry at high energy and a subgroup of the GUT symmetry. This is a point clearly stated recently in [30]. In particular, this implies that, say, $H_d$ and $H_{\bar{c}}$ do not have to have the same charges of the S$'$ symmetry.

The S$'$ symmetry will be a linear combination of an SU(5)-commuting S symmetry and the U(1)$_{\rm Y}$ symmetry from the low-energy point of view; although the lepton number and the baryon number symmetries are also preserved at low energy, they are not expected to be preserved at high energy. However, there are two problems: (i) the U(1)$_{\rm Y}$ symmetry is spontaneously broken at low energy if the S$'$ symmetry is the unbroken subgroup of the S symmetry and the U(1)$_{\rm Y}$ symmetry, and (ii) the charge assignment of the S$^{(\prime)}$ symmetry has a meaning up to adding the standard U(1)$_{\rm Y}$ charge; nothing would have changed if the S$'$ is a naive linear combination of the two symmetries. These problems are solved simultaneously when one introduces another SU(5) gauge group and assumes that $H(\mathbf{5})$ and $\bar{H}(\mathbf{5}^*)$ are quintets of different SU(5) gauge groups, as shown in the following. It is essential to adopt the product group again.

Reference [41] proposed a model based on SU(5)$_1\times$SU(5)$_2$ gauge group. A vector-like pair of bi-fundamental representations is introduced, where the missing VEV mechanism is possible since the traceless condition does not have to be imposed on them. Yukawa coupling constants and the mass matrix of Higgs particles in the model are schematically described in Eq. (2.38):

$$\left(\begin{array}{c} LQ \\ L\bar{E} \end{array}\right) \begin{array}{c} \times \\ \longleftrightarrow \end{array} \left(\begin{array}{c} H_{\bar{c}} \\ H_d \end{array}\right) \begin{array}{c} \longleftrightarrow \\ \times \end{array} \left(\begin{array}{c} H_c \\ H_u \end{array}\right) \begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \end{array} \left(\begin{array}{c} QQ \\ Q\bar{U} \end{array}\right), \tag{2.38}$$

where $\times$'s imply that the corresponding couplings are absent in the superpotential. Reference [42] (almost) found a symmetry in this model. The stability of the Dimopoulos–Wilczek form of the VEV is discussed there at all order in the non-renormalizable operators.

The following presentation of the model is based on [43, 44]. A $\mathbf{Z}_N$ symmetry is introduced as the S symmetry. Matter contents in the GUT-symmetry-breaking sector are chiral multiplets $(\Phi_n)^\alpha{}_i$+$(\bar{\Phi}_n)^i{}_\alpha$ ($n = 2, -3$) in the ($\mathbf{5}^*$,$\mathbf{5}$)+($\mathbf{5}$,$\mathbf{5}^*$) representations of the SU(5)$_1$×SU(5)$_2$ gauge group. The rest of the particle contents are $\bar{H}(\mathbf{1},\mathbf{5}^*)_\alpha$, $H(\mathbf{5},\mathbf{1})^i$, $\Psi(\mathbf{1},\mathbf{5})^\alpha$, $\bar{\Psi}(\mathbf{5}^*,\mathbf{1})_i$ and three families of quarks and leptons ($\mathbf{10}$+$\mathbf{5}^*$,$\mathbf{1}$). $\bar{H}$ and $H$ are identified with the ordinary Higgs multiplets. $i, j$ and $\alpha, \beta$ are used as indices of SU(5)$_1$ and SU(5)$_2$, respectively. The $\mathbf{Z}_N$ charges of $\Phi_n$ and $\bar{\Phi}_n$ are $n$ and $-n$, respectively;

$$\Phi_n \longmapsto \zeta^n \Phi_n, \quad \bar{\Phi}_n \longmapsto \zeta^{-n} \bar{\Phi}_n, \tag{2.39}$$

where $\zeta$ is an $N$-th root of unity. $N$ should not be 5, or otherwise, $\Phi_2$ and $\Phi_{-3}$ would have had the same charge. When the GUT-symmetry-breaking VEV's are of the form,

$$\langle \Phi_2 \rangle = \left\langle \bar{\Phi}_2 \right\rangle = v_2 \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \quad \langle \Phi_{-3} \rangle = \left\langle \bar{\Phi}_{-3} \right\rangle = v_{-3} \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \tag{2.40}$$

a combination of the S symmetry $\mathbf{Z}_N$ and the U(1)$_\mathrm{Y}$($\subset SU(5)_2$) transformation generated by diag$(\zeta^{-2}, \zeta^{-2}, \zeta^{-2}, \zeta^3, \zeta^3)$ in the SU(5)-**fund.** representation remains unbroken at low energy, which is the S′ symmetry. Particles charged under the SU(5)$_2$ have extra U(1)$_\mathrm{Y}$-twist under the S′ symmetry. One of the Higgs quintets $\bar{H}_\alpha$ is charged under the SU(5)$_2$ while the other $H^i$ is under the SU(5)$_1$, and the extra U(1)$_\mathrm{Y}$-twist only on $\bar{H}$ clearly distinguishes the structure of the symmetry and the mass matrix from the those of models in the previous section.

Gauged SU(5)$_1$×SU(5)$_2$ symmetry is broken down to that of the standard model. All three MSSM gauge coupling constants are given (at tree level) by

$$\frac{1}{\alpha_\mathrm{C}} = \frac{1}{\alpha_\mathrm{L}} = \frac{3/5}{\alpha_\mathrm{Y}} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}, \tag{2.41}$$

where $\alpha_1$ and $\alpha_2$ are fine structure constants of the SU(5)$_1$ and SU(5)$_2$ gauge interactions, respectively, and hence the gauge-coupling unification is a natural prediction, irrespective of the ratio between these two couplings.

**various consequences of the symmetry**

The mass term of the coloured Higgs particles are obtained through

$$W = \bar{H}_\alpha \left\langle (\Phi_2)^\alpha{}_i \right\rangle H^i. \tag{2.42}$$

| Fields | $\mathbf{10}^{ij}$ | $\mathbf{5}^*_i$ | $H^i$ | $\bar{H}_\alpha$ | $\Psi^\alpha$ | $\bar{\Psi}_i$ | |
|---|---|---|---|---|---|---|---|
| mod-N PQ charges | 1 | 2 | $-2$ | 0 | $a$ | $b$ | $+\left(\frac{N}{10}m\right) \times \mathrm{U(1)_V}$ |
| mod-N R charges | $\frac{w}{2}$ | $5-\frac{w}{2}$ | 0 | $w-2$ | $a$ | $b$ | $+\left(\frac{10-2w}{10}+\frac{N}{10}m\right) \times \mathrm{U(1)_V}$ |

Table 2.5: $\mathbf{Z}_N$ charges of the fields in the low energy spectrum are given here. $m$ is an arbitrary integer, and $w$ is the R charge of the superpotential.

The total $\mathbf{Z}_N$ charge of $\bar{H}$ and $H$ is required to be $w-2$. Note that the mass term of two Higgs doublets are not contained because of the missing VEV of $\Phi_2$. Yukawa couplings of quarks and leptons and neutrino-mass terms are

$$W = H^{i_1}\mathbf{10}^{i_2 i_3}\mathbf{10}^{i_4 i_5}\epsilon_{i_1 i_2 i_3 i_4 i_5} + \bar{H}_\alpha \left\langle (\Phi_{-3})^\alpha{}_i \right\rangle \mathbf{10}^{ij}\mathbf{5}^*_j + \mathbf{5}^*_i H^i \mathbf{5}^*_j H^j. \tag{2.43}$$

Note also that the Yukawa couplings of $LQH_{\bar{c}}+\bar{D}\bar{U}H_{\bar{c}}$ are missing because of the missing VEV of $\Phi_{-3}$. The $\mathbf{Z}_N$ charge of quarks and leptons are determined so that all terms in Eq. (2.43) are allowed; charge assignment is given in Table 2.5. Then, it follows that the $\mathbf{10}\ \mathbf{10}\ \mathbf{10}\ \mathbf{5}^*$ have $\mathbf{Z}_N$ charge $5+w$, which is never congruent to $w$ since $N\not\equiv 5$. $(\bar{\Phi}_n\Phi_n)^i{}_j$ $(n=2,-3)$ insertion do not give rise to the proton-decay operators, since $(\bar{\Phi}_n\Phi_n)^i{}_j$ $(n=2,-3)$ are singlets of the $\mathbf{Z}_N$ symmetry. $(\bar{\Phi}_{-3}\Phi_2)^i{}_j$ and $(\bar{\Phi}_2\Phi_{-3})^i{}_j$ have non-zero $\mathbf{Z}_N$ charges, but their VEV's vanish. Thus, the dimension-five proton decay is always forbidden (for $^\forall N$), as discussed in subsection 2.1.1.

Difference between the Yukawa couplings of strange quark and muon would come from insertion of $\left\langle (\bar{\Phi}_n\Phi_n)^i{}_j \right\rangle / M_*^2$, where $M_*$ is the cut-off scale of the model. This would imply that the $M_*$ is lower than Planck scale $M_{\rm pl}$. This would change various power-counting arguments, in particular in evaluation of how much the higher non-renormalizable operators lift flat directions of the model.

If either operator

$$W = \bar{\Psi}_i(\bar{\Phi}_2)^i{}_\alpha\Psi^\alpha \quad \text{or} \quad \bar{\Psi}_i(\bar{\Phi}_{-3})^i{}_\alpha\Psi^\alpha \tag{2.44}$$

were allowed by the $\mathrm{S}^{(\prime)}$ symmetry, the other one would never be allowed. Thus, extra two doublets or extra two triplets would remain in the low energy spectrum, which would invalidate the gauge-coupling unification. Therefore, we should expect that neither operator is allowed by the symmetry. Now, if an operator

$$W = \bar{\Psi}_i H^i \tag{2.45}$$

were allowed by the symmetry, the coloured Higgs mass term would come from

$$W = \bar{H}_\alpha (\Phi_2 \bar{\Phi}_2)^\alpha{}_\beta \Psi^\beta, \tag{2.46}$$

but then, an operator

$$W = \bar{H}_\alpha \Psi^\alpha \tag{2.47}$$

should have also been allowed by the symmetry, and there would be no Higgs doublets. Therefore, one cannot introduce mass terms of $\Psi$ or $\bar{\Psi}$, and hence this framework predicts a pair of complete SU(5) multiplets $\Psi^a + \bar{\Psi}_\alpha$, in $\mathbf{5}$+$\mathbf{5}^*$ representations.

**anomaly-free symmetry**

Before we discuss the spectrum around the GUT scale of this model, let us examine the anomaly-free conditions of this discrete symmetry. Anomaly-free conditions can be examined at low energy; we calculate the mixed anomalies of the S′ symmetry and the $\mathrm{SU}(3)_\mathrm{C}\times\mathrm{SU}(2)_\mathrm{L}$ gauge group of the MSSM.

When the $\mathrm{S}^{(\prime)}$ symmetry is a pure Peccei–Quinn symmetry, i.e., non-R symmetry, then the necessary anomaly-free conditions are (see Table 2.5)

$$\begin{aligned} \mathrm{S}' \cdot [\mathrm{SU}(3)_\mathrm{C}]^2 \quad &: \quad 3\times(3+2) + \qquad\qquad\quad +(a+b+\ -\mathit{2}) \equiv 0 \pmod N, \quad (2.48)\\ \mathrm{S}' \cdot [\mathrm{SU}(2)_\mathrm{L}]^2 \quad &: \quad 3\times(3+2) + (-2+-\mathit{3}) + (a+b++\mathit{3}) \equiv 0 \pmod N, \quad (2.49) \end{aligned}$$

where the contributions in *Italic* form comes from the twist by $\mathrm{U}(1)_\mathrm{Y}$. One can easily see that the $\mathrm{S}'\cdot[\mathrm{SU}(3)_\mathrm{C}]^2$- and $\mathrm{S}'\cdot[\mathrm{SU}(2)_\mathrm{L}]^2$-mixed anomalies are the same for any $N$. The only necessary condition from the anomaly cancellation is ($a+b \equiv -13 (\mathrm{mod}\ N)$), and $N$ is almost completely arbitrary. Since operators in Eq. (2.44) should be forbidden by the symmetry, it follows that $(a+b) \not\equiv -3, 2$, and hence $N \neq 5, 10, 15$ is required for $N$.

When the $\mathrm{S}^{(\prime)}$ symmetry is an R symmetry under which the superpotential is of charge $w$, then the anomaly cancellation conditions for the discrete symmetry are

$$\begin{aligned} \mathrm{S}' \cdot [\mathrm{SU}(3)_\mathrm{C}]^2 \quad &: \quad 3\times(5-w) + 3w + \qquad\qquad +(a+b-w-\mathit{2}) \equiv 0 \pmod N, \quad (2.50)\\ \mathrm{S}' \cdot [\mathrm{SU}(2)_\mathrm{L}]^2 \quad &: \quad 3\times(5-w) + 2w + (-2-\mathit{3}) + (a+b-w+\mathit{3}) \equiv 0 \pmod N. \quad (2.51) \end{aligned}$$

One can see by taking the difference of these two conditions that $w \equiv 0 \pmod N$. Thus, the discrete symmetry $\mathbf{Z}_N$ cannot be an R symmetry if it is gauged. In what follows, we consider that the $\mathbf{Z}_N$ $\mathrm{S}^{(\prime)}$ symmetry is a pure Peccei–Quinn symmetry.

**superpotential and spectrum**

Reference [42] wrote down a superpotential that determines the GUT-symmetry breaking VEV:

$$
\begin{aligned}
W &= M\left(\mathrm{tr}(\bar{\Phi}_2\Phi_2)+\mathrm{tr}(\bar{\Phi}_{-3}\Phi_{-3})\right) \\
&\quad + \frac{1}{M'_X}\left((\mathrm{tr}(\bar{\Phi}_2\Phi_2))^2+(\mathrm{tr}(\bar{\Phi}_{-3}\Phi_{-3}))^2+\mathrm{tr}(\bar{\Phi}_2\Phi_2)\,\mathrm{tr}(\bar{\Phi}_{-3}\Phi_{-3})\right) \\
&\quad + \frac{1}{M'_X}\,\mathrm{tr}(\bar{\Phi}_2\Phi_{-3})\,\mathrm{tr}(\bar{\Phi}_{-3}\Phi_2) \\
&\quad + \frac{1}{M_X}\left(\mathrm{tr}(\bar{\Phi}_2\Phi_2\bar{\Phi}_2\Phi_2)+\mathrm{tr}(\bar{\Phi}_2\Phi_2\bar{\Phi}_{-3}\Phi_{-3})+\mathrm{tr}(\bar{\Phi}_{-3}\Phi_{-3}\bar{\Phi}_{-3}\Phi_{-3})\right) \\
&\quad + \frac{1}{M_X}\,\mathrm{tr}(\bar{\Phi}_2\Phi_{-3}\bar{\Phi}_{-3}\Phi_2)+\cdots,
\end{aligned}
\tag{2.52}
$$

which is consistent with the $\mathbf{Z}_N$ S symmetry. Dimensionless coefficients are omitted. One can see that all the MSSM-charged particles become massive. Absorbed by vector multiplets through the Higgs mechanism are chiral multiplets in the $(\mathbf{8},\mathbf{1})^0+(\mathbf{1},\mathbf{3})^0+2\times((\mathbf{3},\mathbf{2})^{-\frac{5}{6}}+\text{h.c.})$ representations of the MSSM, and all others, namely, $3\times((\mathbf{8},\mathbf{1})^0+(\mathbf{1},\mathbf{3}))^0+2\times((\mathbf{3},\mathbf{2})^{-\frac{5}{6}}+\text{h.c.})$ representations acquire mass terms from the non-renormalizable operators in (2.53). Higgsed particles have masses of the order of GUT-symmetry-breaking VEV, while other particles are lighter because their masses are of the order of $\sim$(GUT-scale VEV)$^2/M_{X^{(\prime)}}$. The lighter particles can be regarded as two sets of SU(5)-**adj.** and extra $(\mathbf{8},\mathbf{1})^0+(\mathbf{1},\mathbf{3})^0$. It is known that the threshold corrections from the extra octet and triplet push the actual unification scale upward from the apparent energy scale of the gauge coupling unification [45]. Therefore, the life-time of proton tends to be longer in this model; as the energy scale of the denominator $M_X$ and $M_{X'}$ becomes larger, the life-time becomes longer. In other words, there will be a lower bound on the life-time of proton in this model. It might be a difficult task to obtain it, however, since there are many coefficients in the superpotential, many particles around the GUT scale, and moreover, one has to deal with non-renormalizable operators.

Reference [44] proposed a variation of the above model. Chiral multiplets $(X_n)^\alpha{}_\beta$ ($n=0,\pm 5$) in the $(\mathbf{1},\mathbf{adj.})$ representation of the $\mathrm{SU}(5)_1\times\mathrm{SU}(5)_2$ gauge group are introduced. One more chiral multiplet $S$, which is a singlet, is also introduced. The superpotential is

$$
\begin{aligned}
W &= \bar{\Phi}_2X_5\Phi_{-3}+\bar{\Phi}_{-3}X_{-5}\Phi_2+\bar{\Phi}_{-3}X_0\Phi_{-3}+\bar{\Phi}_2X_0\Phi_2 \\
&+ SX_{-5}X_5+SX_0X_0.
\end{aligned}
\tag{2.53}
$$

$X_n$ has $\mathbf{Z}_N$ charge $n$. For other symmetry of this model, see [44]. $S$ is a flat direction that coexists with the GUT-symmetry-breaking VEV in Eq. (2.40). When the VEV of $S$ is large,

$X_{0,\pm 5}$ are integrated out and quartic superpotential is generated for $\Phi_n, \bar{\Phi}_n$. The generated superpotential is almost like the fourth and the fifth line of (2.53); $M_X$ in (2.53) is given by $\langle S \rangle$. Spectrum around the GUT scale of MSSM-charged particles are essentially the same and the life-time of proton tends to be long. The difference from the model in [42] is that VEV's of the order of the GUT scale is *not* determined by the hard superpotential, but rather by a SUSY-breaking potential. There are a number of MSSM-singlet massless fields in the low energy spectrum, some of which develop VEV's of the order of the GUT scale or so.

The $\mu$ parameter are obtained either as a VEV of massive field or massless field(s). It might be difficult, however, to construct a realistic model in which both $\mathrm{SU}(3)_{\mathrm{C}}$-triplets and $\mathrm{SU}(2)_{\mathrm{L}}$-doublets in $\Psi$ and $\bar{\Psi}$ have SUSY masses of the same order of magnitude.

# Chapter 3

# Proton Decay in the Product-Group Unification Model I

Two classes of models of SUSY GUT's were reviewed in chapter 2. Both classes have symmetries that explain the smallness of the $\mu$ parameter. Proton decay through the dimension-five operators are forbidden by the same symmetry. There are good reasons to consider that the low energy spectrum contains an extra pair of complete multiplets of SU(5), $\mathbf{5}+\mathbf{5}^*$, in both classes. How can we distinguish one from the other based on our current and future knowledge from experiments?

Proton decay through a dimension-six operator (through gauge-boson exchange) is also a prediction common to both classes. However, even a small difference in the mass of the gauge boson that mediates the proton decay (we call it GUT gauge boson, hereafter) leads to large difference in the predicted life-time of proton; the life-time is proportional to the fourth power of the GUT-gauge-boson mass.

Different models have different spectra around the GUT scale, and the different spectra yield threshold corrections in different ways to the gauge coupling constants. Thus, one can obtain model-dependent constraints on the GUT-gauge-boson mass (and hence on the life-time of proton) by examining the threshold corrections to the gauge coupling constants.

We briefly mentioned that the models in section 2.3 (product-group unification II) have a tendency to predict longer life-time of proton; the GUT-gauge-boson mass tends to be heavier than the energy scale at which the gauge coupling constants of the MSSM appear to be unified. On the other hand, it is shown in this chapter that the class of models discussed in section 2.2 (product-group unification I) predicts lighter GUT-gauge-boson mass, and hence faster proton decay. Rough estimate for the life-time is obtained in section 3.1, where we clarify the reason why protons decay fast in this class of models [46]. Section 3.2 is devoted to an extensive study of the parameter region of the $SU(5)_{GUT} \times U(2)_H$ model, where we determine the range, rather than an estimate, of the GUT-gauge-boson mass. Section 3.3 is for the $SU(5)_{GUT} \times U(3)_H$ model. Upper bound of the life-time is obtained for both models in section 3.4 as a function of SUSY-breaking parameters [47]. Uncertainties in these predictions are discussed at the end of section 3.1 and 3.4.

## 3.1 Rough Estimate of the Proton Life-Time

**naive estimate**

Figure 3.1 shows renormalization-group (RG) evolution of the three gauge coupling constants of the MSSM. Tree-level matching equations (2.23)–(2.25) and (2.30)–(2.32) suggest that the "GUT scale" is below the $M_{2-3}$ in the $SU(5)_{GUT} \times U(2)_H$ model, and is in between $M_{2-3}$ and $M_{1-2}$ in the $SU(5)_{GUT} \times U(3)_H$ model. Here, $M_{2-3}$ denotes the energy scale at which coupling constants of the $SU(2)_L$ and the $SU(3)_C$ become equal (see Figure 3.1), and $M_{1-2}$ of the $SU(2)_L$ and the $U(1)_Y$. In particular, the "GUT scale" is lower than the scale $M_{1-2}$, which is the conventional definition of the unification scale. Thus, the decay rate of proton is enhanced compared with the conventional estimate, which uses $M_{1-2} \sim 2 \times 10^{16}$ GeV as the GUT-gauge-boson mass.

At one-loop level, further information is obtained for the $SU(5)_{GUT} \times U(3)_H$ model. The gauge coupling of the $U(1)_H$ runs asymptotic non-free, whose one-loop RG equation is

$$\frac{\partial}{\partial \ln \mu} \left( \frac{1}{\alpha_{1H}}(\mu) \right) = -\frac{6}{2\pi}. \tag{3.1}$$

The $SU(3)_H$ coupling constant is not renormalized at one-loop level. Now we have two remarks. First, the cut-off scale $M_*$ of these models exists below the Planck scale; it should be lower than the energy scale at which the $U(1)_H$ coupling constant becomes very large. One can expect from the RG coefficient $(-6/(2\pi))$ and an expected order of magnitude of the $1/\alpha_{1H}$ at the "GUT scale" that the $M_*$ lies around $10^{17}$GeV. Secondly, the IR-free (asymptotic non-free) behaviour of the $U(1)_H$ coupling leads to

$$\frac{1}{\alpha_{3H}} \ll \frac{1}{\alpha_{1H}} \tag{3.2}$$

at the "GUT scale" under an assumption that

$$\frac{1}{\alpha_{3H}}(M_*) \simeq \frac{1}{\alpha_{1H}}(M_*). \tag{3.3}$$

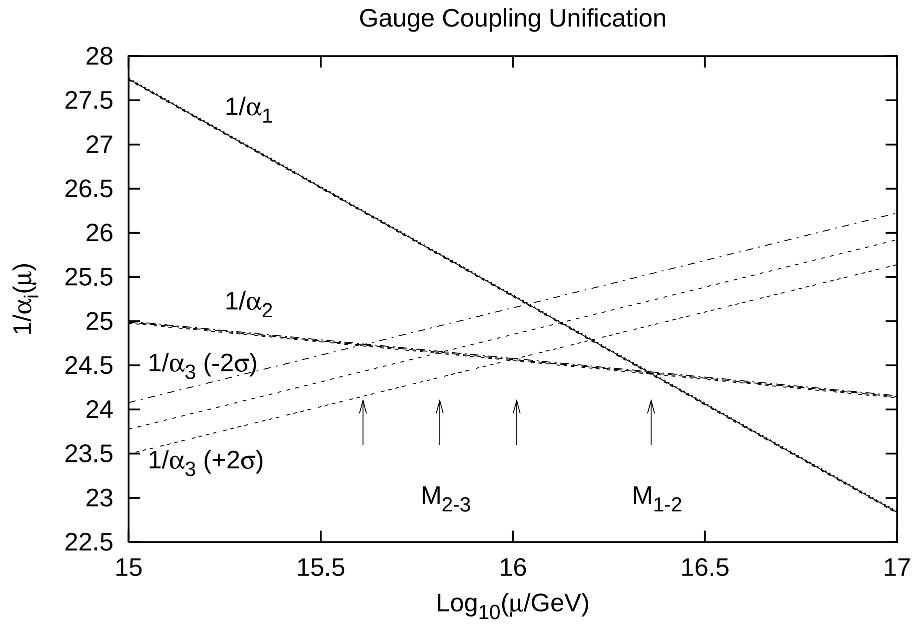


Figure 3.1: Close-up view of the unification of the three gauge coupling constants of the MSSM. $\alpha_{1,2,3}$ are fine structure constants in the $\overline{\mathrm{DR}}$ scheme of the $\mathrm{U}(1)_{\mathrm{Y}}$, $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(3)_{\mathrm{C}}$, respectively. Three lines of $\alpha_3$ correspond to three different experimental inputs. The QCD coupling constants $\alpha_s^{\overline{\mathrm{MS}},(5)}(M_Z) = 0.1132$ $(-2\sigma)$, $\alpha_s^{\overline{\mathrm{MS}},(5)}(M_Z) = 0.1172$ (centre value) and $\alpha_s^{\overline{\mathrm{MS}},(5)}(M_Z) = 0.1212$ $(+2\sigma)$ are used [48]. Two-loop RG effects of the MSSM and the one-looop threshold corrections from the SUSY-particle spectrum are taken into account. We used in the threshold correction the SUSY-particle spectrum determined by the mSUGRA boundary condition with $\tan\beta = 10$, $A_0 = 0$ GeV, $(m_0, m_{1/2})$=(400 GeV,300 GeV) and $\mu > 0$ (see captions of Figures 3.2 and 3.3 for the convention of the sign of $\mu$).

This relation at the cut-off scale, which we call the $\mathrm{U}(3)_{\mathrm{H}}$ relation, is quite natural when $\mathrm{U}(3)_{\mathrm{H}}$ gauge group arises just as the U(3) from a fundamental theory, rather than as an SU(3) times an independent U(1) factor. At least, representations of the matter contents allow such a possibility. Then, as a consequence of the relation Eq. (3.2), the "GUT scale" will be closer to $M_{2-3}$ than to $M_{1-2}$ in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model (see Figure 3.1). Therefore, the proton decay is expected to be fairly fast in both models.

**threshold corrections at the GUT scale**

Let us now take account of the one-loop threshold corrections from the GUT models. The three MSSM gauge coupling constants just below the GUT scale are expressed in terms of parameters of the GUT models: namely the gauge coupling constants and various masses

| $(\mathbf{3},\mathbf{2})^{-\frac{5}{6}}$ | $(\mathbf{3},\mathbf{1})^{-\frac{1}{3}}$ | $(\mathbf{3},\mathbf{1})^{-\frac{1}{3}}$ | $(\mathbf{1},\mathbf{1})^{0}$ | $(\mathbf{1},\mathbf{1})^{0}$ | $(\mathbf{adj.},\mathbf{1})^{0}$ | $(\mathbf{adj.},\mathbf{1})^{0}$ |
|---|---|---|---|---|---|---|
| m.vect. | $\chi+\chi^\dagger$ | $\chi+\chi^\dagger$ | m.vect. | $\chi+\chi^\dagger$ | m.vect. | $\chi+\chi^\dagger$ |
| $M_G =$ | $M_{H_c} =$ | $M_{H_{\bar{c}}} =$ | $M_{1V} =$ | $M_{1C} =$ | $M_{8V} =$ | $M_{8C} =$ |
| $\sqrt{2}gv$ | $hv$ | $h'v$ | $\sqrt{2(g_{1\mathrm{H}}^2+2g^2/5)}v$ | $\sqrt{2}\lambda_{1\mathrm{H}}v$ | $\sqrt{2(g_{3\mathrm{H}}^2+g^2)}v$ | $\sqrt{2}\lambda_{3\mathrm{H}}v$ |

Table 3.1: Summary of the particle spectrum around the GUT scale of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model. The first line shows representations of all the particles in the spectrum under the MSSM gauge group. In the second line, m.vect. denotes $\mathcal{N}=1$ SUSY massive vector multiplet and $\chi+\chi^\dagger$ a pair of $\mathcal{N}=1$ SUSY chiral and anti-chiral multiplet. Mass of each multiplet, whose notation used in the text is in the third line, is given in terms of gauge coupling constants and parameters in the superpotential (2.29) in the last line. $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge coupling constant $g_{\mathrm{GUT}}$ is abbreviated as $g$ in this table.

in the spectrum around the GUT scale, such as the GUT-gauge-boson mass $M_G$. Those expressions are compared with their numerical values determined from experiments to put constraints on the parameters of the GUT models.

We take the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model as an example in this section, and derive a rough estimate of the GUT-gauge-boson mass. One would see that the discussion in the following holds exactly the same also in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model. Thus, we only show the final expression of the GUT-gauge-boson mass $M_G$ for the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model.

The particle spectrum around the GUT scale, which comes into the threshold corrections, is summarized for the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model in Table 3.1. The MSSM gauge coupling constants are given as follows :

$$\frac{1}{\alpha_3}(\mu) = \frac{1}{\alpha_{\mathrm{GUT}}}(M_*) + \frac{1}{\alpha_{3\mathrm{H}}}(M_*) + \frac{3}{2\pi}\ln\left(\frac{\mu}{M_*}\right) + \frac{4}{2\pi}\ln\left(\frac{M_G}{M_*}\right) + \frac{6}{2\pi}\ln\left(\frac{M_{8V}}{M_{8C}}\right) - \frac{1}{2\pi}\ln\left(\frac{M_{H_{\bar{c}}}M_{H_c}}{M_*^2}\right), \tag{3.4}$$

$$\frac{1}{\alpha_2}(\mu) = \frac{1}{\alpha_{\mathrm{GUT}}}(M_*) + \frac{-1}{2\pi}\ln\left(\frac{\mu}{M_*}\right) + \frac{6}{2\pi}\ln\left(\frac{M_G}{M_*}\right), \tag{3.5}$$

$$\frac{1}{\alpha_1}(\mu) = \frac{1}{\alpha_{\mathrm{GUT}}}(M_*) + \frac{\frac{2}{5}}{\alpha_{1\mathrm{H}}}(M_*) + \frac{-\frac{33}{5}}{2\pi}\ln\left(\frac{\mu}{M_*}\right) + \frac{10}{2\pi}\ln\left(\frac{M_G}{M_*}\right) - \frac{\frac{2}{5}}{2\pi}\ln\left(\frac{M_{H_{\bar{c}}}M_{H_c}}{M_*^2}\right), \tag{3.6}$$

where $\mu$ is a renormalization point of the MSSM, which is taken to be just below the GUT scale, $M_G$, $M_{H_c}$, $M_{H_{\bar{c}}}$, $M_{8V}$, $M_{8C}$ are masses of particles around the GUT scale (see Table

3.1) and $\alpha_{\mathrm{GUT,3H,1H}}(M_*)$ are fine structure constants of the gauge groups $\mathrm{SU}(5)_{\mathrm{GUT}}$, $\mathrm{SU}(3)_{\mathrm{H}}$ and $\mathrm{U}(1)_{\mathrm{H}}$, respectively, at the cut-off scale $M_*$. Note also that $\alpha_3 \equiv \alpha_{\mathrm{C}}$, $\alpha_2 \equiv \alpha_{\mathrm{L}}$ and $\alpha_1 \equiv \alpha_{\mathrm{Y}}/(3/5)$. The gauge coupling constants in these expressions are those in the $\overline{\mathrm{DR}}$ scheme, where SUSY is preserved at least at two-loop level [49]. The step-function approximation for the one-loop threshold corrections, which is adopted in Eqs. (3.4)–(3.6), is exact in the $\overline{\mathrm{DR}}$ scheme [50].

In general, it is impossible to determine the $M_G$ if GUT models have more than three parameters. However, it is not necessarily the case in the model. Threshold corrections in Eqs. (3.4)–(3.6) are simplified considerably under two following assumptions. One is the $\mathrm{U}(3)_{\mathrm{H}}$ relation Eq. (3.3) and the other is

$$M_{8V} \simeq M_{8C}, \tag{3.7}$$

which we call ($\mathcal{N}=2$)-SUSY relation. Analysis in section 3.2 shows that the latter relation holds fairly well. The reason why we call this relation an "($\mathcal{N}=2$)-SUSY relation" is also explained there. Under the latter condition, a large threshold correction from the massive $\mathrm{SU}(3)_{\mathrm{H}}$ vector multiplet[1] is almost cancelled by that from the $\mathrm{SU}(3)_{\mathrm{C}}$-**adj**. chiral multiplets. Now that the threshold corrections form the $\mathrm{SU}(3)_{\mathrm{C}}$-**adj**. multiplets decouple from Eqs.(3.4)–(3.6), we are left only with two threshold corrections: one from the massive vector multiplet of the GUT gauge boson and the other from coloured Higgs chiral multiplets. Therefore, three combinations,

$$\frac{1}{\alpha_{\mathrm{GUT}}}(M_*), \quad \ln\left(\frac{M_*}{M_G}\right) \quad \text{and} \quad \frac{1}{\alpha_{3\mathrm{H}}}(M_*) + \frac{1}{2\pi}\ln\left(\frac{M_*^2}{M_{H_{\bar{c}}}M_{H_c}}\right), \tag{3.8}$$

are determined in terms of the values to be put in the left-hand sides of Eqs. (3.4)–(3.6).

In particular, the GUT-gauge-boson mass is given by

$$M_G = \sqrt{\frac{\mu^3}{M_*}}\exp\left(-\frac{2\pi}{24}\left(\frac{2}{\alpha_3}+\frac{3}{\alpha_2}-\frac{5}{\alpha_1}\right)(\mu)\right)\sqrt{\frac{M_{8V}}{M_{8C}}}\exp\left(\frac{2\pi}{12}\left(\frac{1}{\alpha_{3\mathrm{H}}}-\frac{1}{\alpha_{2\mathrm{H}}}\right)(M_*)\right). \tag{3.9}$$

This expression does not depend on the renormalization point $\mu$ at one-loop level. $M_*^{-1/2}$-dependence is a direct consequence of the one-loop running of the $\alpha_{1\mathrm{H}}$ (see Eq. (3.1)), and this negative-power dependence on the cut-off scale $M_*$ implies that the gauge-boson mass is generically light. The last two factors show how the result is changed due to the deviation

[1] Note that $1/\alpha_{\mathrm{GUT}} \gtrsim 100/\alpha_{3\mathrm{H}}$ implies $M_{8V} \sim 10 \cdot M_G$, and hence the threshold correction is large.

from the assumptions we made. The life-time of proton through the GUT-gauge-boson exchange is given in terms of $M_G$ as [51]

$$\tau(p \to \pi^0 e^+) \simeq 1.0 \times 10^{35} \times \left(\frac{M_G}{10^{16}\text{GeV}}\right)^4 \left(\frac{1}{25\alpha_{\text{GUT}}(M_G)}\right)^2 \left(\frac{0.15(\text{GeV})^2}{|W|}\right)^2 \text{yrs.}, \quad (3.10)$$

where $W$ is a hadron matrix element calculated with lattice quenched QCD [52].

The GUT-gauge-boson mass in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model is determined in the same way as

$$M_G = \sqrt{\frac{\mu^3}{M_*}} \exp\left(-\frac{2\pi}{24}\left(\frac{2}{\alpha_3}+\frac{3}{\alpha_2}-\frac{5}{\alpha_1}\right)(\mu)\right)\sqrt{\frac{M_{3V}}{M_{3C}}}\exp\left(\frac{2\pi}{8}\left(\frac{1}{\alpha_{2\mathrm{H}}}-\frac{1}{\alpha_{1\mathrm{H}}}\right)(M_*)\right). \quad (3.11)$$

One would notice that this expression is the same as that of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model, when the $\mathrm{U}(2)_{\mathrm{H}}$ relation ($\alpha_{2\mathrm{H}}(M_*) \simeq \alpha_{1\mathrm{H}}(M_*)$) and the ($\mathcal{N}=2$)-SUSY relation ($M_{3V} \simeq M_{3C}$) are satisfied.

**results**

Threshold corrections to the gauge coupling constants from the SUSY-particle spectrum are of the same order of magnitude as those from particles around the GUT scale. Two-loop effects in the RG evolution between the electroweak scale and the GUT scale also affect the MSSM gauge coupling constants by the same order of magnitude. Thus, those contributions should be taken into account in calculating the numerical values of the MSSM gauge coupling constants used in Eqs. (3.9) and (3.11) to obtain the required precision. The renormalization point $\mu$ should be taken to be around the GUT scale.

Now, we can estimate the proton life-time for various SUSY-particle spectra. Values of the MSSM gauge coupling constants just below the GUT scale are calculated by the *SOFTSUSY1.7* code [53], where SUSY threshold corrections to the MSSM coupling constants in the $\overline{\mathrm{DR}}$ scheme are included just as in [54] along with the two-loop effects in the RG evolution. We obtain an estimate by neglecting possible two uncertainties expressed by the last two factors in Eqs. (3.9) and (3.11) coming from the deviation from the ($\mathcal{N}=2$)-SUSY relation and the $\mathrm{U}(N)_{\mathrm{H}}$ ($N=2,3$) relation. Effects of such deviations are discussed later. Here, we also set the cut-off scale to be $10^{17}$ GeV; in most part of SUSY-breaking-parameter space, the three gauge coupling constants are unified approximately at around $10^{16}$ GeV and hence the cut-off scale $M_*$ is expected to be around $10^{17}$ GeV.

Figure 3.2 shows how the estimated life-time changes over the parameter space of the mSUGRA SUSY breaking. Since the estimate for the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model differs only through the value of the gauge coupling constant $1/\alpha_{\mathrm{GUT}}(M_G)$ from that of the $\mathrm{SU}(5)_{\mathrm{GUT}}$

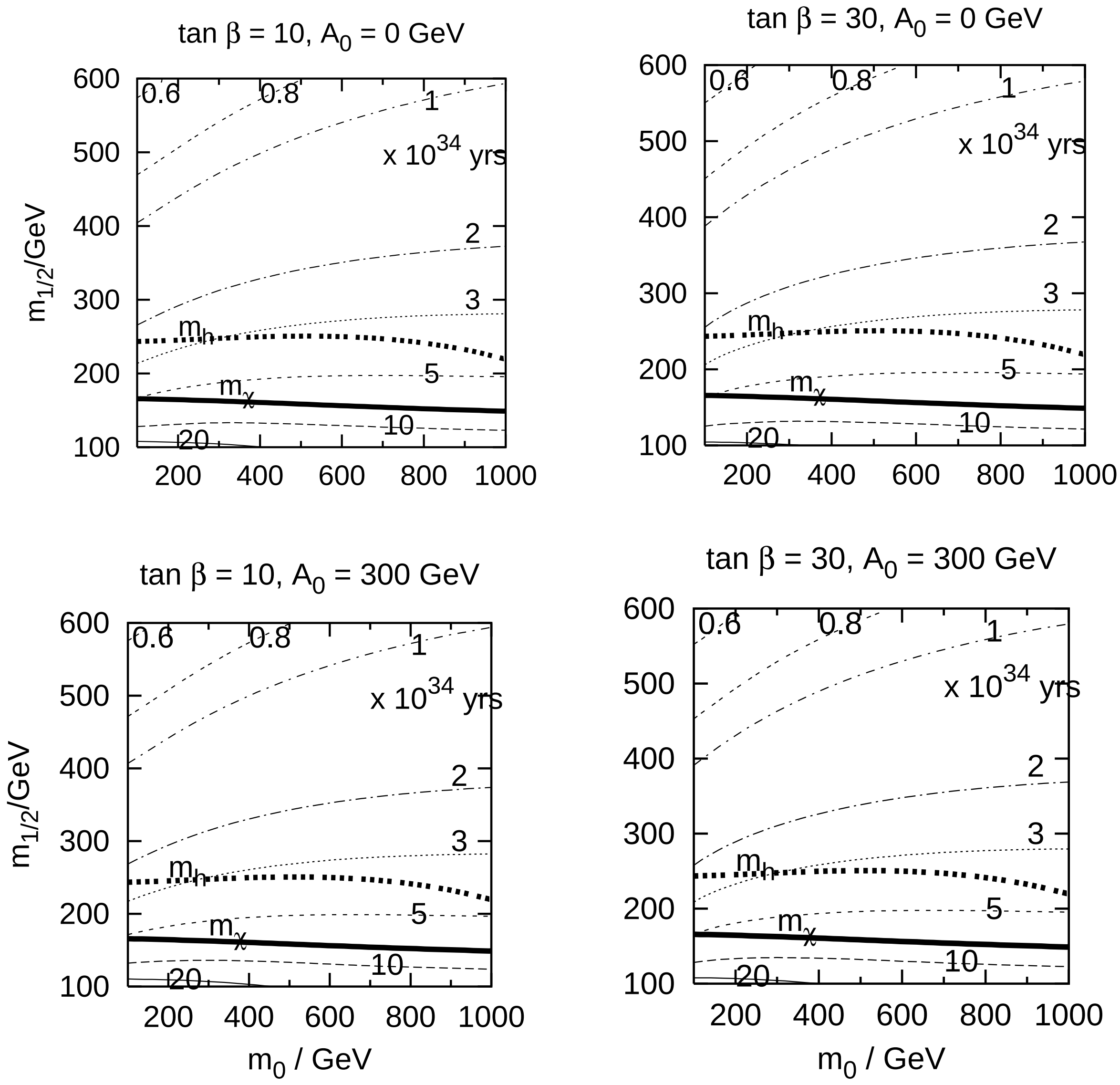


Figure 3.2: Contour plots of the estimate of proton life-time on a parameter space of the mSUGRA boundary condition. Universal scalar mass $m_0$ and gaugino mass $m_{1/2}$ are varied, and each plot corresponds to various choices of $\tan\beta$ and $A_0$. The $\mu$ parameter is chosen to be positive, when the constraint from the branching ratio of the $b \to s\gamma$ process is less severe. Thick broken lines labeled by $m_h$ and $m_\chi$ are the bounds on the parameter space from the LEP II experiment in search of the lightest Higgs mass ($m_h \geq 114$ GeV,95 % C.L. ) [55] and the lightest chargino mass ($m_\chi \geq 103.5$ GeV, 95 % C.L.) [8]. The parameter region below either one of these lines is excluded.

$\times$U(3)$_{\rm H}$ model, we only show the result of the SU(5)$_{\rm GUT}\times$U(3)$_{\rm H}$ model. We confirmed that the results differ between the two models by only a few per-cent.

As one can see from the contour plots in Figure 3.2 and 3.3, the proton life-time is typically in the range (0.6–20)$\times 10^{34}$ yrs. The results do not depend on choices of $\tan\beta$, $A_0$ and sign of the $\mu$ parameter. This estimate is above the current experimental limit by Super-

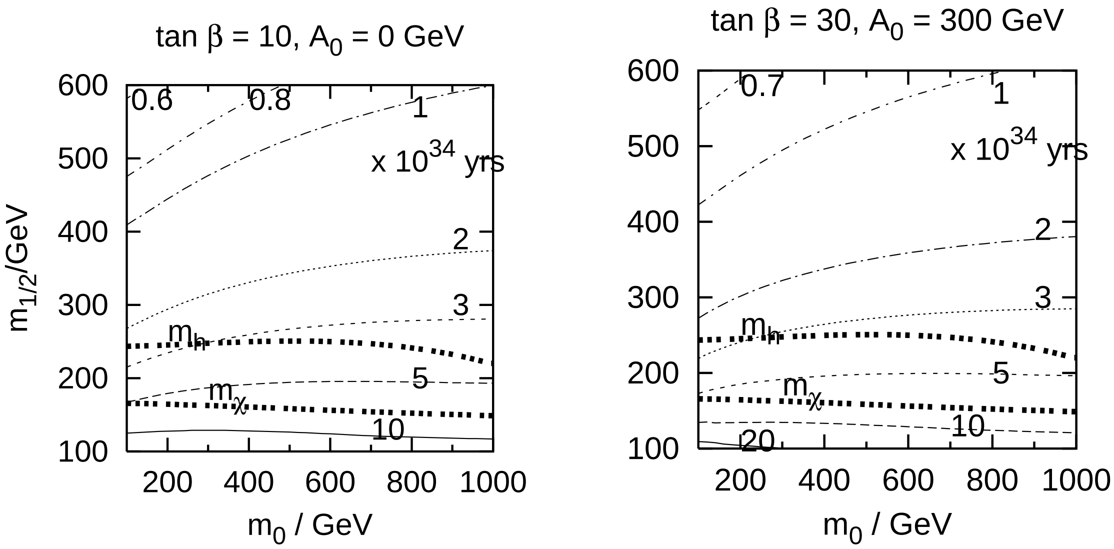


Figure 3.3: Contour plots of the estimate of the proton life-time. The sign of the $\mu$ parameter is opposite to that of the of the $\mu$ parameter used in Figure 3.2. The branching ratio of $b \to s\gamma$ [56], which is consistent with the standard-model calculation, excludes a parameter region with lighter $m_{1/2}$, say $\lesssim 300$ GeV [57]. This bound might be severer than that from the lightest Higgs mass for some range of values of $m_0$.

Kamiokande, $\tau(p \to \pi^0 e^+) \geq 4.4 \times 10^{33}$ yrs. (90 % C.L.) [58][2], in most part of the parameter space of the mSUGRA boundary condition. Within the parameter space[3] that is not excluded

[2]Reference [59] says the limit is $5.0 \times 10^{33}$ yrs. (90 % C.L.).

[3] *Theoretical* computation of the Higgs scalar potential and the mass of the lightest Higgs particle are quite subtle. Various loop corrections play quite important roles. Several codes are available that calculate the Higgs boson mass, but their calculation of the lightest Higgs mass differ by a few per-cent (i.e., a few GeV) [60, 61]. This difference in the calculated pole mass results in lower bounds of $m_{1/2}$ that are different by about 100 GeV from codes to codes. The *SOFTSUSY1.7* [53] takes into account as many loop corrections as possible. Moreover, the lightest Higgs mass it calculates is the heaviest among all the codes discussed in [61]. Thus, the least parameter region is excluded since we used it, and hence the upper bound of the estimate $5 \times 10^{34}$ yrs. (see the text) is quite a conservative one.

by the Higgs mass bound from LEP II experiment [55], the typical life-time is estimated not to be longer than $5 \times 10^{34}$ yrs.

Therefore, in the models analyzed in this section, there is an intriguing possibility that the dimension-six proton decay is confirmed in the next-generation water Čerenkov detectors, such as Hyper-Kamiokande [62] and UNO [63] experiments ($\sim$ 500kt fiducial volume) and TITAND [59] (Mega-ton fiducial volume) experiment. The $3\sigma$-discovery limit for ten-years running is $3 \times 10^{34}$ yrs. in the 500 kt detectors, and $7 \times 10^{34}$ yrs. in the Mega-ton detectors [59].

Let us now discuss various uncertainties in the estimate obtained above. First, we summarize the uncertainties that do not arise from the GUT models. The first uncertainty comes from an error bar of the experimental values of the QCD coupling constant. The estimate in Figure 3.2 and 3.3 is based on the current centre value $\alpha_s(M_Z)^{\overline{\mathrm{MS}},(5)} = 0.1172$ [48]. The $1\sigma$ error $\alpha_s(M_Z)^{\overline{\mathrm{MS}},(5)} = 0.1172 \pm 0.0020$ results in uncertainties in the life-time of factor $\times(0.4 - 2.)$. The calculation of hadron matrix element in [52] has a statistic error $W = -0.153(19)\mathrm{GeV}^2$, which leads to a factor $\times(0.8 - 1.3)$. However, this does not include systematic errors, (e.g., an error owing to the quenched approximation). Ref. [64] estimates that the systematic error in $W$ is about 50 %, leading to uncertainties in the life-time by factor two.

The decay rate of the proton is enhanced when there are $\mathrm{SU}(5)_{\mathrm{GUT}}$-charged particles at low energies. Existence of such particles are highly motivated in the model; $\mathbf{5}+\mathbf{5}^*$ representations are required at the electroweak scale when the mod-4 R symmetry is gauged (see section 2.2). In this case, the gauge coupling constant $\alpha_{\mathrm{GUT}}$ is stronger as a result of the RG evolution with new matter particles, and the decay rate is enhanced[4] through $(\alpha_{\mathrm{GUT}})^2$ by $\sim \times 1.6$.

Now let us turn to the uncertainties that originate from the GUT models. The first uncertainty comes from possible violation of the $\mathrm{U}(N)_{\mathrm{H}}$ ($N = 2, 3$) relation. It would affect the life-time by $\times(0.35 - 2.9)$ in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model ($|(1/\alpha_{3\mathrm{H}} - 1/\alpha_{1\mathrm{H}})(M_*)| \leq 1/2$ is assumed), and by $\times(0.21 - 4.8)$ in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model (where $|(1/\alpha_{2\mathrm{H}} - 1/\alpha_{1\mathrm{H}})(M_*)| \leq 1/2$ is assumed). Thus, this dominates over all other uncertainties above, especially in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model. In the meanwhile, deviation from the ($\mathcal{N} = 2$)-SUSY relation is not expected to be large, as discussed in section 3.2 and 3.3. Finally, the actual cut-off scale may be different from $10^{17}$ GeV, although we set $M_* = 10^{17}$ GeV in the

[4] Although one might suspect that there is an extra one-loop threshold correction from a possible mass splitting between triplets and doublets in the $\mathbf{5} + \mathbf{5}^*$, and that the GUT-gauge-boson mass would also be affected, yet the GUT-gauge-boson mass is actually stable against this correction. This is because Eqs. (3.9) and (3.11) are expressions from which the threshold corrections from the coloured Higgs multiplets decouple.

estimate in Figure 3.2 and 3.3. Then, the life-time changes by $\times(M_*/10^{17}\text{GeV})^2$, which can also be of the same order as the uncertainties from the violation of the $\text{U}(N)_\text{H}$ ($N = 2, 3$) relation.

## 3.2 Parameter Region of the $\mathbf{SU(5)_{\text{GUT}}\times U(2)_{\text{H}}}$ Model

We obtained the estimate of the typical life-time of proton. Uncertainties in the estimate, however, are dominated by those that originate from the GUT models. We derive, in the following sections, predictions of the *range* of the proton life-time of the models, rather than a *single estimate* with uncertainties. Whole parameter regions of the GUT models are exploited. We do not single out only one point from the regions by imposing ad-hoc assumptions. Such analysis enables us to obtain, in particular, upper bounds of the life-time for both models, as we see later. The theoretical upper bounds are important in distinguishing the models from others experimentally.

Only three parameters of a GUT model are determined from the three gauge coupling constants of the MSSM. In the minimal SU(5) SUSY-GUT model, for example, coloured Higgs mass is determined, and we have two other independent constraints between three parameters of the model: namely the unified gauge coupling constant, the GUT-gauge-boson mass, and a coefficient of the cubic coupling of the $\text{SU}(5)_\text{GUT}$-**adj.** chiral multiplet in the superpotential [51]. Which parameter is determined and which are only under constraints completely depend on models.

We reduced the number of parameters of the models down to three in section 3.1 by introducing two assumptions, and hence we could determine the GUT-gauge-boson mass. However, the reliability of the assumptions was the major source of the uncertainties in the estimate of the life-time so obtained. In this section and in what follows, we do not rely on the assumptions, and we leave the parameters undetermined as free parameters of the model. The free parameters, nevertheless, cannot take completely arbitrary values. One can remember that the cubic-coupling coefficient in the minimal SU(5) model, which can be chosen as the free parameter of the model, cannot be too large because it makes itself extremely large immediately in the RG evolution toward ultraviolate (UV) when it is large. The upper bound on this free parameter leads to the lower bound of the GUT-gauge-boson mass of the minimal SU(5) model [51]. What sort of limits there are on the space of free parameters completely depend of GUT models, again.

| $(\mathbf{3},\mathbf{2})^{-\frac{5}{6}}$ | $(\mathbf{1},\mathbf{1})^0$ | $(\mathbf{1},\mathbf{1})^0$ | $(\mathbf{1},\mathbf{adj.})^0$ | $(\mathbf{1},\mathbf{adj.})^0$ |
|---|---|---|---|---|
| m.vect. | m.vect. | $\chi + \chi^\dagger$ | m.vect. | $\chi + \chi^\dagger$ |
| $M_G =$ | $M_{1V} =$ | $M_{1C} =$ | $M_{3V} =$ | $M_{3C} =$ |
| $\sqrt{2} g_{\mathrm{GUT}} v$ | $\sqrt{2(g_{1\mathrm{H}}^2 + 3g_{\mathrm{GUT}}^2/5)} v$ | $\sqrt{2}\lambda_{1\mathrm{H}} v$ | $\sqrt{2(g_{2\mathrm{H}}^2 + g_{\mathrm{GUT}}^2)} v$ | $\sqrt{2}\lambda_{2\mathrm{H}} v$ |

Table 3.2: Summary of the particle spectrum around the GUT scale of the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ model. See caption of Table 3.1. $\lambda_{1\mathrm{H}}$ and $\lambda_{2\mathrm{H}}$ in the last row of this table are the parameters that appear in the superpotential (2.22).

### 3.2.1 One-loop Analysis

Let us perform such an analysis for the the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ model. The result of the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(3)_{\mathrm{H}}$ model is described in section 3.3. Matching conditions of the gauge coupling constants between the GUT model and the MSSM are given as follows at one-loop level:

$$\frac{1}{\alpha_{\mathrm{C}}}(\mu) = \frac{1}{\alpha_{\mathrm{GUT}}}(M) + \frac{3}{2\pi}\ln\left(\frac{\mu}{M}\right) + \frac{4}{2\pi}\ln\left(\frac{M_G}{M}\right), \tag{3.12}$$

$$\frac{1}{\alpha_{\mathrm{L}}}(\mu) = \frac{1}{\alpha_{\mathrm{GUT}}}(M) + \frac{1}{\alpha_{2\mathrm{H}}}(M) + \frac{-1}{2\pi}\ln\left(\frac{\mu}{M}\right) + \frac{6}{2\pi}\ln\left(\frac{M_G}{M}\right) + \frac{4}{2\pi}\ln\left(\frac{M_{3V}}{M_{3C}}\right), \tag{3.13}$$

$$\frac{\frac{3}{5}}{\alpha_{\mathrm{Y}}}(\mu) = \frac{1}{\alpha_{\mathrm{GUT}}}(M) + \frac{\frac{3}{5}}{\alpha_{1\mathrm{H}}}(M) + \frac{-\frac{33}{5}}{2\pi}\ln\left(\frac{\mu}{M}\right) + \frac{10}{2\pi}\ln\left(\frac{M_G}{M}\right), \tag{3.14}$$

where $M$ is an arbitrary energy scale above the GUT scale where coupling constants of the GUT model can be defined, while the renormalization point $\mu$ is an energy scale slightly lower than the GUT scale, where the MSSM gives better description. The gauge coupling constants of the MSSM are given by the tree level contributions (the first and the second term) and renormalization and threshold corrections at one-loop (the remaining terms). Gauge coupling constants are considered to be defined in the $\overline{\mathrm{DR}}$ scheme, and hence the step-function approximation is valid in the one-loop threshold corrections [50] as in Eqs. (3.4)–(3.6). Various mass parameters of the model enter into the equations through these corrections. $M_G$ is the gauge boson mass, $M_{3V}$ and $M_{3C}$ are masses of the $\mathrm{SU}(2)_{\mathrm{L}}$-**adj.** vector multiplet and chiral multiplets, respectively. These mass parameters are given in terms of parameters of the Lagrangian (at tree level) as in Table 3.2.

There are five parameters of the GUT model in the above equations: $M_G$, $M_{3V}/M_{3C}$, $1/\alpha_{2\mathrm{H}}$, $1/\alpha_{1\mathrm{H}}$ and $1/\alpha_{\mathrm{GUT}}$. The $\mathrm{U}(2)_{\mathrm{H}}$ relation $1/\alpha_{2\mathrm{H}} \simeq 1/\alpha_{1\mathrm{H}}$ at a certain renormalization point and the ($\mathcal{N} = 2$)-SUSY relation $M_{3V} \simeq M_{3C}$ reduced the number of parameters down to three in section 3.1, and hence $M_G$ was determined. In this section, however, we leave two free parameters, and three others are solved in terms of the two. We do not rely on these two assumptions. Two parameters $M_G$ and $M_{3V}/M_{3C}$ are taken as the independent degrees of freedom that are not fixed by the matching equations. Three others, namely $\alpha_{\rm GUT}(M_G)$, $\alpha_{\rm 2H}(M_G)$ and $\alpha_{\rm 1H}(M_G)$, are determined through Eqs. (3.12)–(3.14) by setting $\mu = M = M_G$. More explicitly,

$$\frac{1}{\alpha_{\rm GUT}}(M_G) = \frac{1}{\alpha_{\rm C}}(M_G), \tag{3.15}$$

$$\frac{1}{\alpha_{\rm 2H}}(M_G) = \left(\frac{1}{\alpha_{\rm L}} - \frac{1}{\alpha_{\rm C}}\right)(M_G) - \frac{4}{2\pi}\ln\left(\frac{M_{3V}}{M_{3C}}\right), \tag{3.16}$$

$$\frac{1}{\alpha_{\rm 1H}}(M_G) = \frac{5}{3}\left(\frac{\frac{3}{5}}{\alpha_{\rm Y}} - \frac{1}{\alpha_{\rm C}}\right)(M_G). \tag{3.17}$$

Another parameter of the model, $\alpha^{\lambda}_{\rm 2H} \equiv (\lambda_{\rm 2H})^2/(4\pi)(M_G)$ is also expressed in terms of $\alpha_{\rm 2H}(M_G)$, $\alpha_{\rm GUT}(M_G)$ and $M_{3V}/M_{3C}$:

$$\frac{1}{\alpha^{\lambda}_{\rm 2H}}(M_G) = \left(\frac{1}{\alpha_{\rm 2H}(M_G) + \alpha_{\rm GUT}(M_G)}\right)\left(\frac{M_{3V}}{M_{3C}}\right)^2. \tag{3.18}$$

We require that all the coupling constants in the model should remain finite in the course of the RG evolution toward UV at least within the range of spectrum of the model. More explicitly, the coupling constants $\alpha_{\rm 2H}(M)$, $\alpha^{\lambda}_{\rm 2H}(M)$ and $\alpha_{\rm 1H}(M)$ of the model are required to be finite under the one-loop RG evolution at least until the renormalization point $M$ reaches the heaviest particle of the model. This requirement sets limits on the two free parameters. The lower-left panel of Figure 3.4 shows how the parameter space spanned by the two independent parameters $M_G$ and $M_{3V}/M_{3C}$ are restricted owing to the requirement, where one-loop RG equations are used. The RG equations of those couplings are as follows:

$$\frac{\partial}{\partial \ln M}\left(\frac{1}{\alpha_{\rm 2H}}\right)(M) = \frac{-2}{2\pi}, \qquad \frac{\partial}{\partial \ln M}\left(\frac{1}{\alpha_{\rm 1H}}\right)(M) = \frac{-6}{2\pi}, \tag{3.19}$$

and that of the $\alpha^{\lambda}_{\rm 2H}$ is given in Eq. (A.2). The lower-left panel of Figure 3.4 is based on the MSSM gauge coupling constants obtained in the upper panel of Figure 3.4, which is the same as Figure 3.1.

Lower left panel of Figure 3.4 is understood intuitively as follows. First, the parameter $M_G$ is bounded from above (bounded from the right in the figure). Roughly $M_G \lesssim 10^{15.6}$ GeV. This is quite a natural consequence, since it is consistent with the discussion in the

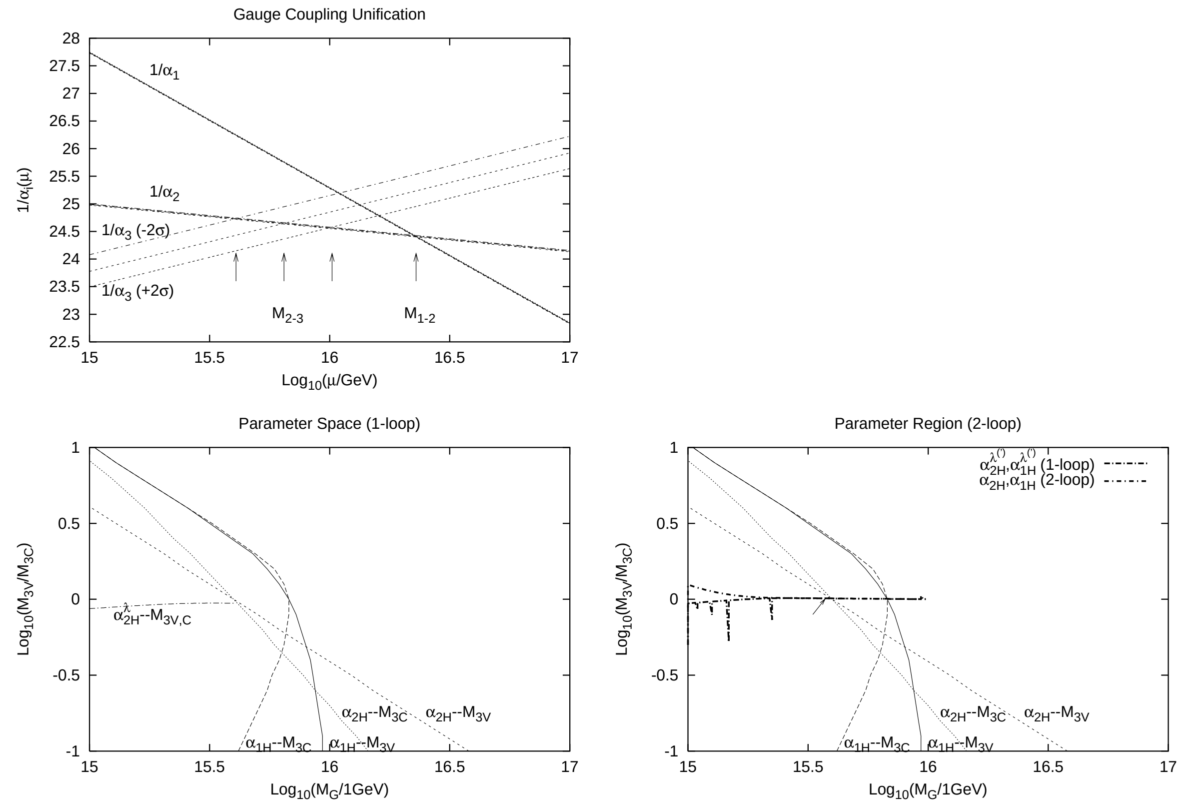


Figure 3.4: The parameter Region of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model. The lower-left panel is the result of the one-loop analysis in subsection 3.2.1. The parameter space of the model spanned by two free parameters $M_G$ and $M_{3V}/M_{3C}$ are restricted by requiring that all the running coupling constants of the model should stay finite below the heaviest particle in the model. Right-hand-side regions of four curves labeled by "(gauge-coupling)-mass" are excluded. Region below a curve labeled by "$\alpha_{2\mathrm{H}}^{\lambda} - M_{3V,C}$" is also excluded. Thus, the parameter space of the model is restricted to a triangular region. The lower-right panel shows the result of the analysis in subsection 3.2.2, which includes the two-loop effects in the RG equations of gauge coupling constants. Two thick dash-doted curves are the boundary of the region of viable parameters. Four other courves are drawn so that the comparison with the panel in the left-hand-side is easy. Most part of the triangular region in the lower-left panel is further excluded because of the two-loop effects, and only a small region near $M_{3V} \simeq M_{3C}$ survives. The upper bound of the $M_G$ is indicated by an arrow. The upper panel is the same as Figure 3.1.

naive estimate at the beginning of section 3.1. The parameter space where $M_G$ is just below the $M_{2-3}$ is excluded, because both gauge coupling constants of the model $\alpha_{2\mathrm{H}}$ and $\alpha_{1\mathrm{H}}$ are already large at the GUT scale and they become infinity quite soon, far below the mass scale of $M_{3V}$ or $M_{3C}$. Secondly, the parameter space is bounded also from below in the figure. The beta function of the superpotential coupling $\alpha_{2\mathrm{H}}^{\lambda}$ (Eq. (A.2)) implies that this coupling constant becomes infinity at UV unless renormalization effect from itself is overcome by those from gauge interactions, i.e., parameter space with $\alpha_{2\mathrm{H}} \ll \alpha_{2\mathrm{H}}^{\lambda}$ (which is almost equivalent to $M_{3C} \gg M_{3V}$) is excluded.

The beta function of the $\alpha_{2\mathrm{H}}^{\lambda}(M)$ depends also on $\alpha_{2\mathrm{H}}^{\lambda'}(M)$, $\alpha_{1\mathrm{H}}^{\lambda}(M)$ and $\alpha_{1\mathrm{H}}^{\lambda'}(M)$. Those coupling constants are not determined through the one-loop matching equations (3.12)–(3.14). We set these coupling constants in the beta function as 0, so that the $\alpha_{2\mathrm{H}}^{\lambda}(M)$ becomes larger as slowly as possible in the evolution to UV. This makes the excluded parameter space smaller and makes our analysis more conservative. Figure 3.4 is based on the value of $\alpha_3(\mu)$ that is calculated from $\alpha_s^{\overline{\mathrm{MS}},(5)}(M_Z) = 0.1212$, i.e., the value larger by $2\sigma$ from the centre value. Therefore, the excluded region becomes smaller under this choice, and our result on the upper bound of the GUT-gauge-boson mass becomes more conservative.

### 3.2.2 Two-loop Effects

Now we go on to include the two-loop effects in the RG evolution of the parameters. Reference [46] pointed out, based on the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model, that the two-loop effects become important at generic points of the parameter space, because the beta function of the $\alpha_{3\mathrm{H}}$ is accidentally small ($= 0$) at one-loop level, and also because the coupling constant $\alpha_{3\mathrm{H}}$ is not small. The two-loop effects have considerable effects also in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model because of the same reasons.

RG equations of the model are given in the appendix A.1. Beta functions are given up to two-loop for gauge coupling constants $\alpha_{2\mathrm{H}}$ and $\alpha_{1\mathrm{H}}$, while only up to one-loop for coupling constants that appear in the superpotential. This is because the beta function is scheme-independent up to two-loop for gauge coupling constants while only up to one-loop for superpotential coupling constants in $\mathcal{N} = 1$ SUSY gauge theories. In what follows, we use a following set of notations:

$$\alpha_{2\mathrm{H}}^{\lambda} \equiv \frac{(\lambda_{2\mathrm{H}})^2}{4\pi}, \quad \alpha_{2\mathrm{H}}^{\lambda'} \equiv \frac{(\lambda'_{2\mathrm{H}})^2}{4\pi}, \quad \alpha_{1\mathrm{H}}^{\lambda} \equiv \frac{(\lambda_{1\mathrm{H}})^2}{4\pi}, \quad \alpha_{1\mathrm{H}}^{\lambda'} \equiv \frac{(\lambda'_{1\mathrm{H}})^2}{4\pi}. \tag{3.20}$$

Coupling constants of the model, namely $\alpha_{2\mathrm{H}}, \alpha_{1H}, \alpha_{2\mathrm{H}}^{\lambda(')}$ and $\alpha_{1\mathrm{H}}^{\lambda(')}$ are required to stay finite under the RG evolution, at least until the renormalization point reaches the heaviest particle in the spectrum. Initial values, i.e., values at the matching scale $M = M_G$, are not determined for $\alpha^{\lambda'}_{2\mathrm{H}}$, $\alpha^{\lambda}_{1\mathrm{H}}$ and $\alpha^{\lambda'}_{1\mathrm{H}}$ from the matching equations (3.12)–(3.14). Thus, we set their values as

$$\alpha^{\lambda'}_{2\mathrm{H}} = \alpha^{\lambda}_{2\mathrm{H}}, \qquad \alpha^{\lambda}_{1\mathrm{H}} = \alpha^{\lambda'}_{1\mathrm{H}} = \alpha_{1\mathrm{H}}, \tag{3.21}$$

when the renormalization point $M$ is $M_G$. Although we should have varied these values to search for wider parameter region in the $M_G$–$(M_{3V}/M_{3C})$ plane, we believe that it would not change the result very much; the reason is explained later. The lower-right panel of Figure 3.4 shows that the parameter space with $M_{3V} \gg M_{3C}$ (i.e., $\alpha_{2\mathrm{H}} \gg \alpha^{\lambda}_{2\mathrm{H}}$) is further excluded by the above requirement, and the parameter space survives only around the line of $M_{3V} \simeq M_{3C}$. This is nothing but the ($\mathcal{N} = 2$)-SUSY relation we assumed in the analysis in section 3.1. The one-loop threshold correction $+\frac{4}{2\pi} \ln{(M_{3V}/M_{3C})}$ to the $\mathrm{SU(2)_L}$ coupling constant in Eq. (3.13) vanishes in the region.

It is clear why this region, and only this region survives. Let us neglect, for the moment, the renormalization effects from the $\mathrm{SU(5)_{GUT}}$ gauge interactions; $\mathrm{SU(5)_{GUT}}$ gauge coupling constant is small compared with those of the $\mathrm{SU(2)_H}$ and the $\mathrm{U(1)_H}$ interactions. Then, one can see that the two-loop part of the beta functions of $\alpha_{2\mathrm{H}}$ and $\alpha_{1\mathrm{H}}$ are proportional to $(\alpha_{2\mathrm{H}} - \alpha^{\lambda(')}_{2\mathrm{H}})$ and $(\alpha_{1\mathrm{H}} - \alpha^{\lambda(')}_{1\mathrm{H}})$. Thus, the renormalization effects from $\alpha_{2\mathrm{H}}$ and $\alpha_{1\mathrm{H}}$ are completely cancelled by $\alpha^{\lambda(')}_{2\mathrm{H}}$ and $\alpha^{\lambda(')}_{1\mathrm{H}}$ in that region (provided the relations Eqs (3.21) are satisfied).

We call the $M_{3V} \simeq M_{3C}$ as the ($\mathcal{N} = 2$)-SUSY relation, because there is an $\mathcal{N} = 2$ SUSY in the GUT-symmetry-breaking sector ($\mathrm{SU(5)_{GUT}}$ gauge interactions are still neglected. Non-renormalizable operators that lead to the Yukawa couplings of quarks and leptons in the fourth line of (2.22) are irrelevant in the RG evolution.). Particle contents in the sector can be regarded as multiplets of the $\mathcal{N} = 2$ SUSY [32]; the $\mathrm{U(2)_H}$ vector multiplet of the $\mathcal{N} = 1$ SUSY and the chiral multiplet $X^{\alpha}{}_{\beta}$ in the $\mathrm{U(2)_H}$-($\mathbf{2}\otimes\mathbf{2}^*$) representation form an ($\mathcal{N} = 2$)-SUSY vector multiplet, and chiral multiplets in the bi-fundamental representations $Q^{\alpha}{}_{k} + \bar{Q}^{k}{}_{\alpha}$ ($k = 1, ..., 6$) form $\mathcal{N} = 2$ hypermultiplets. When coupling constants satisfy ($\mathcal{N} = 2$)-SUSY relation

$$g_{1\mathrm{H}} \simeq \lambda_{1\mathrm{H}} (\sim \lambda'_{1\mathrm{H}}), \qquad g_{2\mathrm{H}} \simeq \lambda_{2\mathrm{H}} (\sim \lambda'_{2\mathrm{H}}), \tag{3.22}$$

the first, second and the third lines of the superpotential (2.22) is consistent with the $\mathcal{N} = 2$ SUSY [65]. Thus, the spontaneous breakdown of the $\mathrm{U(2)_H}$ symmetry caused by the second and the third lines preserves the $\mathcal{N} = 2$ SUSY, and all the particles in the sector, except for those that are absorbed by the GUT gauge boson, are gathered into massive vector multiplets of the $\mathcal{N} = 2$ SUSY (which are in the $\mathrm{SU(2)_L}$-(**adj.**+$\mathbf{1}\simeq\mathbf{2}\otimes\mathbf{2}^*$) representations).

$M_{3V} \simeq M_{3C}$ and $M_{1V} \simeq M_{1C}$ (see Table 3.2) are the masses common within these ($\mathcal{N} = 2$)-SUSY multiplets. Thus, the decoupling of the threshold corrections from the massive vector multiplets is a manifestation of the $\mathcal{N} = 2$ SUSY in the sector. Furthermore, in any gauge theories with $\mathcal{N} = 2$ SUSY, gauge coupling constants are renormalized only at one-loop level [66]. RG evolution of $\alpha_{2\mathrm{H}}$ and $\alpha_{1\mathrm{H}}$ are exact at one-loop level; two-loop and higher-loop effects do not change the one-loop evolution. Anomalous dimensions of hypermultiplets vanish [67], and the renormalization of $\alpha_{2\mathrm{H}}^{\lambda(')}$ and $\alpha_{1\mathrm{H}}^{\lambda(')}$ coupling constants only come from the anomalous dimension of the chiral multiplets $X^{\alpha}{}_{\beta}$, which belong to the $\mathcal{N} = 2$ vector multiplets. As a result, the RG equations of these coupling constants are the same as those of $\alpha_{2\mathrm{H}}$ and $\alpha_{1\mathrm{H}}$, respectively, as one can see in Eqs. (A.1)–(A.6). Thus, the ($\mathcal{N} = 2$)-SUSY relation Eq. (3.22) is RG invariant, and hence the evolution of coupling constants is exact at one-loop level, and the parameter allowed in the one-loop analysis is still allowed when the two-loop effects are also taken into account.

The region of viable parameter forms a band along the ($\mathcal{N} = 2$)-SUSY line $M_{3V} \simeq M_{3C}$. The width of the band becomes thiner until the band becomes a line as the $M_G$ becomes larger. $\alpha_{2\mathrm{H}}$ becomes large at a renormalization point lower than $M_{3V}$ for parameters above the line, and $\alpha_{2\mathrm{H}}^{\lambda}$ becomes large at a renormalization point lower than $M_{3C}$ for parameters below the line; viable parameter was not found even *on* the line in our numerical calculation. It does not mean, however, that the parameter does not exist on the line for large $M_G$, as seen below. Indeed, $\alpha_{2\mathrm{H}}^{\lambda} = \alpha_{2\mathrm{H}}^{\lambda(')}$ and $\alpha_{1\mathrm{H}}^{\lambda} = \alpha_{1\mathrm{H}}^{\lambda(')}$ are preserved by the RG equations Eqs. (A.1)–(A.6) in the absence of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ interaction, and under these conditions,

$$\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha_{2\mathrm{H}}} - \frac{1}{\alpha_{2\mathrm{H}}^{\lambda}}\right)(\mu) = \left(\frac{21}{2\pi} + \frac{7}{\alpha_{2\mathrm{H}}^{\lambda}}\right)\frac{(\alpha_{2\mathrm{H}}^{\lambda} - \alpha_{2\mathrm{H}})}{2\pi} + \left(\frac{6}{2\pi} + \frac{2}{\alpha_{2\mathrm{H}}^{\lambda}}\right)\frac{(\alpha_{1\mathrm{H}}^{\lambda} - \alpha_{1\mathrm{H}})}{4\pi} \tag{3.23}$$

$$\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha_{1\mathrm{H}}} - \frac{1}{\alpha_{1\mathrm{H}}^{\lambda}}\right)(\mu) = \left(\frac{6}{2\pi} + \frac{2}{\alpha_{1\mathrm{H}}^{\lambda}}\right)\frac{3(\alpha_{2\mathrm{H}}^{\lambda} - \alpha_{2\mathrm{H}}) + (\alpha_{1\mathrm{H}}^{\lambda} - \alpha_{1\mathrm{H}})}{4\pi}. \tag{3.24}$$

Thus, the ($\mathcal{N} = 2$)-SUSY relation Eq. (3.22) is an infra-red(IR)-fixed relation of the RG equations (3.23) and (3.24). Even generic values of those coupling constants at an energy scale higher than the $M_G$ tend to flow into the IR-fixed relation during the RG evolution down to $M = M_G$. This also implies that even a set of parameters that is slightly away from the IR-fixed relation cannot be evolved toward UV for a long energy interval[5]. This is the reason why the parameter $M_{3V}/M_{3C}$ of Figure 3.4, which is $\sqrt{(\alpha_{2\mathrm{H}}(+\alpha_{\mathrm{GUT}}))/\alpha_{2\mathrm{H}}^{\lambda}}$ evaluated

[5]This is the reason why we believe that it would not help in finding wider parameter region to set the values of undetermined parameters $\alpha_{2\mathrm{H}}^{\lambda'}$ and $\alpha_{1\mathrm{H}}^{\lambda(')}$ differently from those in Eq. (3.21). Deviation from the ($\mathcal{N} = 2$)-SUSY relation at $M = M_G$ would immediately lead to UV-instable behaviour in the RG evolution.

at $M = M_G$, should be fine-tuned to the IR-fixed relation $M_{3V}/M_{3C} \simeq 1$. Therefore, for a set of parameters that exactly satisfy the $(\mathcal{N}=2)$-SUSY relation at $M = M_G$, RG evolution is exactly that of the one-loop running to all orders in perturbation theory, and hence the $M_{3V} \simeq M_{3C}$ line within the region allowed by the one-loop analysis *does* contain viable parameter of the model (as long as the $\mathrm{SU}(5)_{\mathrm{GUT}}$ interactions are neglected). Thus, the upper bound of the parameter $M_G$ is given by the point in the lower-right panel of Figure 3.4 indicated by an arrow.

The above argument is correct only when the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge interaction is neglected. Therefore, let us now discuss the effects of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge interaction. It breaks the $\mathcal{N}=2$ SUSY in the sector, and the $(\mathcal{N}=2)$-SUSY relations Eqs. (3.22) are no longer RG invariant. However, the $\mathrm{SU}(5)_{\mathrm{GUT}}$ interaction is much weaker than the $\mathrm{U}(2)_{\mathrm{H}}$ interactions and its effect is small. One can see this from the fact that the allowed parameter region is still almost on the line of $\mathcal{N}=2$ SUSY, i.e., $M_{3V} \simeq M_{3C}$, in the lower right panel of Figure 3.4. Thus, the $\mathrm{SU}(5)_{\mathrm{GUT}}$ effects on the RG evolution of couplings can be treated as small perturbation to the $(\mathcal{N}=2)$-SUSY RG flow. In particular, the IR-fixed property of the RG equations Eqs. (A.1)–(A.6) is not changed[6], except that the IR-fixed relations are slightly modified to become

$$(\alpha_{2\mathrm{H}} - \alpha^{\lambda}_{2\mathrm{H}}),\ \ (\alpha_{2\mathrm{H}} - \alpha^{\lambda'}_{2\mathrm{H}}),\ \ (\alpha_{1\mathrm{H}} - \alpha^{\lambda}_{1\mathrm{H}}),\ \ (\alpha_{1\mathrm{H}} - \alpha^{\lambda'}_{1\mathrm{H}}) \simeq \mathcal{O}(\alpha_{\mathrm{GUT}}). \tag{3.25}$$

RG evolution down to IR would flow into the modified IR-fixed relation, and follow the fixed relation. In particular, coupling constants $\alpha_{2\mathrm{H}}$, $\alpha^{\lambda(')}_{2\mathrm{H}}$, $\alpha_{1\mathrm{H}}$ and $\alpha^{\lambda(')}_{1\mathrm{H}}$ partially absorb the $\mathrm{SU}(5)_{\mathrm{GUT}}$ effects and the beta functions are re-minimized on the fixed relations.

Let us now focus on the evolution of $\alpha_{2\mathrm{H}}$ coupling constant[7] and evaluate the effects of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ coupling. We focus on $\alpha_{2\mathrm{H}}$ because the evolution of $\alpha_{2\mathrm{H}}$ provides more stringent bound on the parameter space than $\alpha_{1\mathrm{H}}$ does in the $(\mathcal{N}=2)$-SUSY (and hence one-loop) RG flow[8], as one can see from Figure 3.4. The $\alpha_{\mathrm{GUT}}$ coupling contributes to the beta function of $\alpha_{2\mathrm{H}}$ by less than 10 % of the one-loop contribution. The speed of the evolution of $\alpha_{2\mathrm{H}}$ is not modified by 10 %, since the $\mathrm{SU}(5)_{\mathrm{GUT}}$ effect is partially absorbed by shifts of $(\alpha_{2\mathrm{H}} - \alpha^{\lambda(')}_{2\mathrm{H}})$ and

[6]There is no IR-fixed relation in its strict meaning in the RG equations in the presence of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ interaction. The "IR-fixed relations" in Eq. (3.25) involve $\alpha^{\lambda(')}_{2\mathrm{H}}$ and $\alpha^{\lambda(')}_{1\mathrm{H}}$ in right-hand sides, and hence the "fixed relations" themselves change as the coupling constants flow. However, this change is small, since the right-hand sides of the Eqs. (3.25) are proportional to the $\alpha_{\mathrm{GUT}}$.

[7]Evolution of $\alpha^{\lambda(')}_{2\mathrm{H}}$ coupling constant is almost the same as that of $\alpha_{2\mathrm{H}}$ for a parameter on the $(\mathcal{N}=2)$-SUSY line. Effects of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge interaction are treated only as perturbation to the $(\mathcal{N}=2)$-SUSY RG flow.

[8]One can easily see from analytic calculation that this is true for any SUSY-breaking parameters.

$(\alpha_{1\mathrm{h}} - \alpha_{1\mathrm{H}}^{\lambda^{(\prime)}})$. Thus, the value of $\alpha_{2\mathrm{H}}(M_G)$ for the upper-bound value of $M_G$, which satisfies $(4\pi/(2\alpha_{2\mathrm{H}}))(M_G) \simeq 3.7$ irrespective of SUSY threshold corrections[9], is not changed by 10 %, either. As a result, the upper-bound value of $M_G$ is not modified by a factor more than $e^{(2\pi/4)(1/\alpha_{2\mathrm{H}})(M_G)(\pm 10\%)} \sim 10^{\pm 0.04}$, where $(1/\alpha_{\mathrm{L}} - 1/\alpha_{\mathrm{C}})(M_G) \simeq -4/(2\pi)\ln(M_G/M_{2-3})$ is used in addition to Eq. (3.16).

RG equations listed in the appendix A.1 are truncated at two-loop for gauge coupling constants and at one-loop for others. The analysis so far is based on those RG equations. Although higher-loop effects completely vanish in the limit of $\mathcal{N} = 2$ SUSY, they do not in the presence of $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge interaction. Then, the perturbative expansion does not converge when the coupling $\alpha_{2\mathrm{H}}$ exceeds the bound $(2\alpha_{2\mathrm{H}}/(4\pi) \lesssim 1)$. It is impossible to extract any definite statements on the RG evolution when the perturbative expansion is not valid. Scheme dependence of the RG coefficients is also a problem in higher loops. However, it follows that $(4\pi/(2\alpha_{2\mathrm{H}}))(M_G) \simeq 3.7$, for the upper-bound value of $M_G$, and hence the most part of the RG evolution is in the perturbative regime $((4\pi/(2\alpha_{2\mathrm{H}})) \gtrsim 1)$. Thus, we consider that the upper bound on the parameter $M_G$ obtained from the analysis based on RG equations in the appendix A.1 is fairly a good evaluation, unless non-perturbative effects happen to soften the evolution of the gauge coupling constant.

Finally, there are two remarks. First, the matching equations Eqs. (3.12)–(3.14) are those at one-loop level. Although we discussed the two-loop effects in the RG evolution, the threshold corrections at two-loop level is not discussed so far. At the two-loop level, mass parameters that were in the one-loop matching equations Eqs. (3.12)–(3.14) should be defined more accurately; renormalization point of the mass parameters were not specified in the analysis so far just because such a difference in the renormalization point leads only to the two-loop threshold corrections. Such extra threshold corrections almost vanish when the IR-fixed relations are satisfied also at UV ($M \sim M_{3V,3C}$). Thus, the approximate ($\mathcal{N} =$ 2)-SUSY relations are favourable even at renormalization point $M = M_{3V,3C}$ and possibly at more higher energy scale, because the perturbative analysis we performed in this section is completely reliable in that case. However, the required precision is not so high at such an energy scale compared with that in the IR ($M = M_G$) in Figure 3.4.

[9]The numerical value $(4\pi/(2\alpha_{2\mathrm{H}}))(M_G) = 3.7$ results from an equation

$$\frac{1}{\alpha_{2\mathrm{H}}}(M_G) = \frac{2}{2\pi}\ln\left(\frac{M_{3V}}{M_G}\right) \simeq \frac{2}{2\pi}\frac{1}{2}\ln\left(\frac{\alpha_{2\mathrm{H}}}{\alpha_{\mathrm{GUT}}}\right); \tag{3.26}$$

i.e., the $\alpha_{2\mathrm{H}}$ becomes infinity in the one-loop evolution when the renormalization point is at $M_{3V}$.

Secondly, the analysis so far is based on a renormalizable model, and we neglected non-renormalizable operators. They are not relevant to the RG flow (say, in the sense of Wilsonian RG equations) except at energy scale just below the cut-off scale $M_*$, and moreover, operators such as the fourth line of Eq. (2.22) affect the gauge coupling constants only at higher-loop level. However, a non-renormalizable operator

$$W = 2\,\mathrm{tr}\left(\left(\frac{1}{4g^2} + c\frac{\langle \bar{Q}Q \rangle}{M_*^2}\right)\mathcal{W}^\alpha \mathcal{W}_\alpha\right) \tag{3.27}$$

modifies the matching conditions of the gauge coupling constants even at tree level. Since $(v/M_*)^2 \simeq (v/M_{3V})^2 \simeq (1/(8\pi\alpha_{\mathrm{GUT}}))e^{-3.7} \simeq 0.024$ at the point in the parameter space indicated by an arrow in Figure 3.4, for example, the $\mathrm{SU}(5)_{\mathrm{GUT}}$-breaking contributions to the matching equations are of the order of

$$c \times 16\pi \times 0.023 \simeq 1.2 \times c. \tag{3.28}$$

There is no convincing argument, however, on what is the "natural value" of $c$, or on what is the "natural normalization" of the gauge kinetic term, since we do not know what is the "natural order of magnitude" of the gauge coupling constants.

## 3.3 Parameter Region of the $\mathrm{SU(5)}_{\mathrm{GUT}}\times\mathrm{U(3)}_{\mathrm{H}}$ Model

The same analysis as in section 3.2 can be performed for the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{3\mathrm{H}}$ model. Parameters $M_{8V}/M_{8C}$ and $\alpha_{3\mathrm{H}}$ in the matching equations Eqs. (3.4)–(3.6) play the same role as $M_{3V}/M_{3C}$ and $\alpha_{2\mathrm{H}}$, respectively. There is one extra parameter in the matching equations of this model, which is $(M_{H_c}M_{H_{\bar{c}}}/M_G^2)$. Thus, the parameter space is described by three free parameters $M_G$, $\log_{10}(M_{8V}/M_{8C})$ and $(1/2)\log_{10}(M_{H_c}M_{H_{\bar{c}}}/M_G^2)$. Three other parameters, namely, $\alpha_{\mathrm{GUT}}(M_G)$, $\alpha_{3\mathrm{H}}(M_G)$ and $\alpha_{1\mathrm{H}}(M_G)$, are expressed in terms of the three parameters. Perturbative analysis sets limits on the space of free parameters.

Figure 3.5 shows the result of the analysis; only a section of the parameter space at $(1/2)\log_{10}(M_{H_c}M_{H_{\bar{c}}}/M_G^2) = 0.3$ is shown. The parameter region is bounded from upper left at one-loop level in the figure because the $\mathrm{SU}(3)_{\mathrm{C}}$-**adj.** (whose masses are $M_{8V}$ and $M_{8C}$) becomes extremely heavy, and is bounded from right because the value $1/\alpha_{1\mathrm{H}}(M)$ at $M = M_G$ becomes too small that the coupling becomes large immediately in the RG evolution to UV. The region is bounded from below at around $M_{8V} \simeq M_{8C}$, because the coupling constant $\alpha_{3\mathrm{H}}^{\lambda}$ or $\alpha_h$ at $M = M_G$ is too large below the $(M_{8V}/M_{8C} \simeq 1)$-line that it becomes large

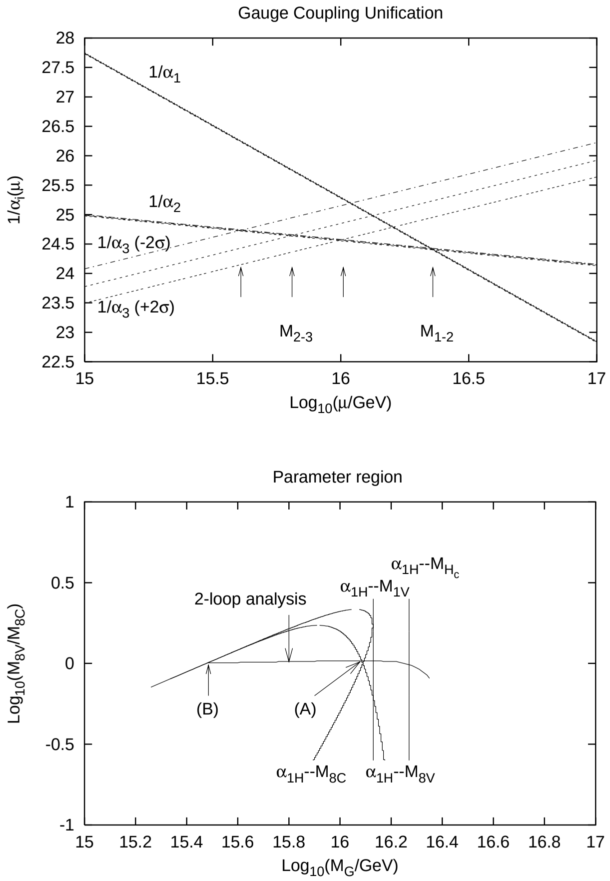


Figure 3.5: Parameter Region of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model.

immediately in the RG evolution. The range of $M_G$ is roughly the lower half of an interval between $M_{2-3}$ and $M_{1-2}$, as one can see from the figure, which is consistent with the naive expectation given just at the beginning of section 3.1. The parameter region is reduced to a line in this model after the two-loop effects are taken into account in the RG equations of gauge coupling constants. The line of allowed parameters is indicated by "2-loop analysis" in Figure 3.5.

Initial values of $h$ and $h'$ (values at $M = M_G$) are constrained only as a product $h \cdot h'$ in the matching equations Eqs. (3.4)–(3.6) with $M_*$ replaced with $M_G$. Since the RG equations are the same for $h$ and $h'$, when renormalization effects from the Yukawa couplings of quarks and leptons are neglected[10], we can set $h = h'$ during the RG evolution, and we define

[10]The Yukawa coupling constant of the top quark is extremely sensitive to the top Yukawa coupling

$\alpha_h \equiv (h = h')^2/(4\pi)$. This choice of the initial values makes the RG evolution the most stable. Initial values of $\alpha_{3\mathrm{H}}^{\lambda'}$ and $\alpha_{1\mathrm{H}}^{\lambda^{(\prime)}}$ are not determined in the matching equations, and hence we set

$$\alpha_{3\mathrm{H}}^{\lambda'} = \alpha_{3\mathrm{H}}^{\lambda}, \qquad \alpha_{1\mathrm{H}}^{\lambda} = \alpha_{1\mathrm{H}}^{\lambda'} = \alpha_{1\mathrm{H}}. \tag{3.29}$$

This choice also makes the RG evolution the most stable.

The line of allowed parameters is the ($\mathcal{N}=2$)-SUSY line. In the absence of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge interaction and cubic couplings of $h$ and $h'$ in the superpotential, there is a structure of the $\mathcal{N}=2$ SUSY in the GUT-symmetry-breaking sector; the chiral multiplet $X^{\alpha}{}_{\beta}$ can be regarded as an ($\mathcal{N}=2$)-SUSY partner of the $\mathrm{U}(3)_{\mathrm{H}}$ vector multiplet of the $\mathcal{N}=1$ SUSY, and the bi-fundamental representations $Q^{\alpha}{}_{k} + \bar{Q}^{k}{}_{\alpha}$ are regarded as hypermultiplets of the $\mathcal{N}$ $= 2$ SUSY. The $\mathcal{N}=2$ SUSY enhances in the sector when the ($\mathcal{N}=2$)-SUSY relations

$$g_{1\mathrm{H}} \simeq \lambda_{1\mathrm{H}}(\sim \lambda'_{1\mathrm{H}}), \qquad\qquad g_{3\mathrm{H}} \simeq \lambda_{3\mathrm{H}}(\sim \lambda'_{3\mathrm{H}}), \tag{3.30}$$

are satisfied, and in that case,

$$M_{8V} \simeq M_{8C}. \tag{3.31}$$

The ($\mathcal{N}=2$)-SUSY relation is an IR-fixed relation. Therefore, the required fine-tuning to the ($\mathcal{N}=2$)-SUSY line $(M_{8V}/M_{8C})(M = M_G) \simeq 1$ is just a result of the IR-fixed property of the system, and in particular, we can expect that there is parameter on the line on which the evolution of the coupling constants to UV are one-loop exact. In particular, the range of the parameter $M_G$ for a fixed value of $(1/2)\log_{10}(M_{H_c}M_{H_{\bar{c}}}/M_G^2)$ is given by the interval in Figure 3.5 indicated by two arrows at both end points. Lower bound and the upper bound of $M_G$ are determined as the minimum and the maximum value of the lower bound and upper bound, respectively, when the remaining parameter $(1/2)\log_{10}(M_{H_c}M_{H_{\bar{c}}}/M_G^2)$ is varied. Since it is evident from the figure that the lower bound of the range of $M_G$ leads to too fast proton decay that is already excluded by experiments, we focus only on the upper bound of the range in the following.

The effects of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge interaction and of the cubic coupling constants $h$ and $h'$ change the ($\mathcal{N}=2$)-SUSY one-loop RG evolution of $\alpha_{1\mathrm{H}}$ and $\alpha_{3\mathrm{H}}$. ($\mathcal{N}=2$)-SUSY-breaking effects are treated as perturbation. The evolution of $\alpha_{1\mathrm{H}}$ still puts stronger limits compared with $\alpha_{3\mathrm{H}}$, as shown below. The $\mathrm{SU}(5)_{\mathrm{GUT}}$ effect and $\alpha_h$ effect have opposite sign and cancel with each other. The $\alpha_h$-contribution is not too large compared with the $\alpha_{\mathrm{GUT}}$-contribution,

constant at the electroweak scale, and to the $\tan\beta$. Therefore, we did not include the top Yukawa loop to the renormalization equations.

since the parameter $(1/2)\ln(M_{H_c}M_{H_{\bar c}}/M_G^2) \simeq (1/2)\ln(\alpha_h/\alpha_{\rm GUT})$ is bounded from above; For a set of parameter that gives the largest value of $M_G$, which we are interested in, this parameter is at most 0.6. (this is because the coloured Higgs particles become the heaviest for the parameter larger than $\sim 0.6$, the upper boundary of the parameter region is determined by the line labeled by $\alpha_{\rm 1H} - M_{H_c}$ in the Figure, and this line shifts toward left as the value of the parameter becomes bigger.) The speed of evolution of $\alpha_{\rm 1H}$ coupling is changed at most by several %; this estimate comes from the ratio between the one-loop contribution and the $\mathrm{SU}(5)_{\rm GUT}$ contribution at two-loop. $\alpha_h$-effect is of the same order as the $\mathrm{SU}(5)_{\rm GUT}$-effect as long as the parameter $(1/2)\ln(M_{H_c}M_{H_{\bar c}}/M_G^2)$ is small compared with the unity, and its sign is opposite to that of the $\mathrm{SU}(5)_{\rm GUT}$ interaction. Then, the several % difference in the speed of the RG evolution leads to the change of the upper bound of $M_G$ by a factor of $10^{\pm(\text{a few})\times 0.01}$. The $\alpha_{\rm 3H}$ coupling also runs due to those effects, although it was invariant in the absence of the effects. However, the running of this coupling is also estimated from above by the running due purely to the $\alpha_{\rm GUT}$ or $\alpha_h$ effect. The resulting running of $\alpha_{\rm 3H}$ is, however, slower than that of $\alpha_{\rm 1H}$ unless $\alpha_h \sim \mathcal{O}(1)$, which is not the case as long as $(1/2)\log_{10}(M_{H_c}M_{H_{\bar c}}/M_G^2) \lesssim 0.6$. Thus, the change in the running of the $\alpha_{\rm 3H}$ does not change the upper bound of the parameter $M_G$.

Finally there is one remark on the validity of the perturbative expansion. For a set of parameter that gives the upper bound of the $M_G$, the value of $\alpha_{\rm 1H} \simeq \alpha_{\rm 1H}^{\lambda^{(\prime)}}$ satisfies

$$\frac{1}{\alpha_{\rm 1H}}(M_G) = \frac{6}{2\pi}\ln\left(\frac{M_{8V,8C}}{M_G}\right) \gtrsim \mathcal{O}(1). \tag{3.32}$$

Thus, $(4\pi/\alpha_{\rm 1H})(M_G) \gg 1$ and the perturbative expansion is valid for most part of the RG evolution of $\alpha_{\rm 1H}$. The value of $\alpha_{\rm 3H}(M_G)$ is roughly given by

$$\frac{1}{\alpha_{\rm 3H}}(M_G) = \frac{4}{2\pi}\ln\left(\frac{M_G}{M_{2-3}}\right) + \frac{1}{2\pi}\ln\left(\frac{M_{H_c}M_{H_{\bar c}}}{M_G^2}\right) = -\frac{2\epsilon_g}{\alpha_{\rm GUT}} + \frac{9}{7}\ln\left(\frac{M_{H_c}M_{H_{\bar c}}}{M_G^2}\right) - \frac{2}{7}\frac{1}{\alpha_{\rm 1H}}(M_G), \tag{3.33}$$

where

$$\frac{1}{\alpha_{\rm 1H}}(M_G) = \frac{-14}{2\pi}\ln\left(\frac{M_G}{M_{1-2}}\right) + \frac{1}{2\pi}\ln\left(\frac{M_{H_c}M_{H_{\bar c}}}{M_G^2}\right) \tag{3.34}$$

is used in the second equality and "unification threshold" $\epsilon_g$ is defined in terms of the MSSM gauge coupling constants as

$$\epsilon_g \equiv \frac{g_{\rm C}(M_{1-2}) - g_{\rm L}(M_{1-2})}{g_{\rm L}(M_{1-2})}. \tag{3.35}$$

Since $\epsilon_g \simeq -(0.01 - 0.03)$ for SUSY threshold correction from mSUGRA spectrum with $\alpha_s(M_Z) = 0.1172$ and is more negative for $\alpha_s(M_Z) = 0.1132$, the first term is no less than 0.5. The second and the third term are combined to be positive for the value of $(1/2)\log_{10}(M_{H_c}M_{H_{\bar{c}}}/M_G^2)$ that gives the upper bound of $M_G$, the value of $(4\pi)/(3\alpha_{3\mathrm{H}})(M_G)$ (and hence for arbitrary renormalization point in the one-loop approximation) is larger than unity, and the perturbative expansion in terms of $\alpha_{3\mathrm{H}}$ is also valid.

## 3.4 Conservative Upper Bound of the Proton Life-time

Analyses in sections 3.2 and 3.3 presented the way to extract the upper bound of the GUT-gauge-boson mass for both models. The upper bound, however, varies when the SUSY-breaking parameters are varied. In this section, the result of the upper bound of the proton life-time is presented for both models, as a function of the SUSY breaking parameters.

Let us consider the SUSY particle spectrum determined by the mSUGRA boundary condition. The spectrum is calculated by the *SOFTSUSY1.7* code [53], and the MSSM gauge coupling constants in the $\overline{\mathrm{DR}}$ scheme are obtained in an iterative procedure. We leave part of the parameters of the mSUGRA boundary condition fixed to be $\tan\beta = 10$, $A_0 = 0$ GeV and the sign of $\mu$ parameter is the standard one. This is because changes in these parameters did not change the result at all in section 3.1. Only the universal scalar mass $m_0$ and universal gaugino mass $m_{1/2}$ are varied.

Figure 3.6 is a contour plot of the upper bound of the proton life-time in the $\mathrm{SU}(5)_{\mathrm{GUT}}$ $\times\mathrm{U}(2)_{\mathrm{H}}$ model. QCD coupling $\alpha_s(M_Z) = 0.1212$ (coupling stronger than the centre value by $2\sigma$) is used, so that the upper bound becomes more conservative. Even with this value of the QCD coupling constant, the upper bound of the proton life-time is not longer than $4 \times 10^{33}$ yrs. Major difference from the estimation in section 3.1 comes from the fact the $\alpha_{2\mathrm{H}}$ and $\alpha_{1\mathrm{H}}$ coupling constants are required to be positive in this analysis. The $(m_0,m_{1/2})$-dependence in the figure can be understood intuitively, as follows. The upper-bound of the $M_G$ is such that

$$\frac{1}{\alpha_{2\mathrm{H}}}(M_G) = \frac{2}{2\pi}\ln\left(\frac{M_{3V}}{M_G}\right), \tag{3.36}$$

and hence $\frac{1}{\alpha_{2\mathrm{H}}}(M_G) = \frac{3.7}{2\pi}$ is satisfied, irrespective of the SUSY-breaking parameters. Thus, the upper bound of the $M_G$ is roughly obtained through

$$\left(\frac{1}{\alpha_{2\mathrm{H}}}(M_G) = \frac{3.7}{2\pi}\right) = \frac{-4}{2\pi}\ln\left(\frac{M_G}{M_{2-3}}\right), \tag{3.37}$$

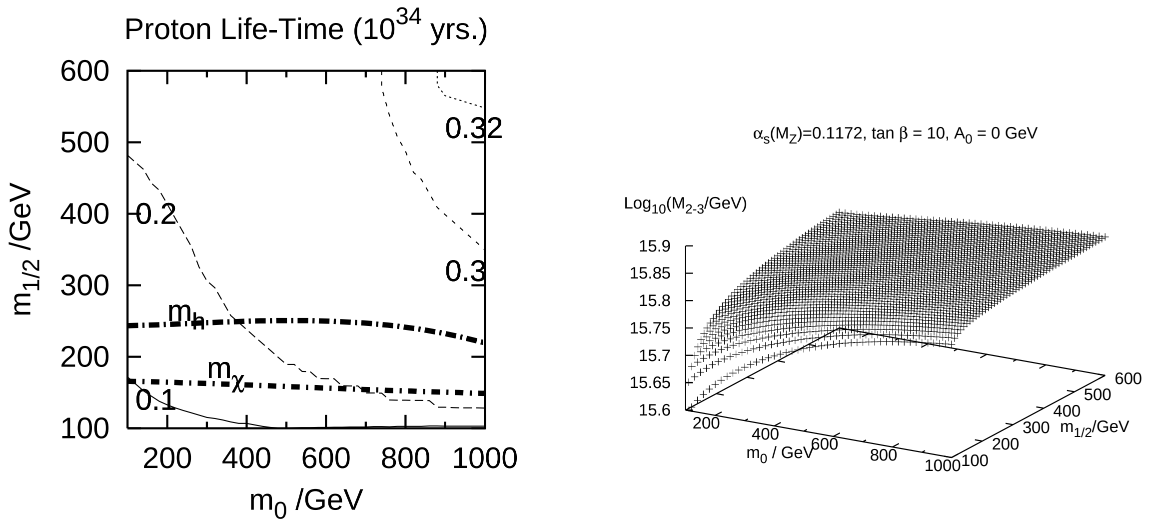


Figure 3.6: Contour plot of the upper bound of the proton life-time. LEP II bounds on the masses of the lightest Higgs and chargino are the same as those in Figure 3.2. The life-time varies within $(1.4 - 3.2) \times 10^{33}$ yrs. within the parameter space shown here. Panel in the right shows the behaviour of $M_{2-3}$ for reference.

and hence

$$M_G \lesssim e^{-\frac{3.7}{4}} M_{2-3}. \tag{3.38}$$

($m_0$,$m_{1/2}$)-dependence arises almost only through the variation of $M_{2-3}$. Behaviour in both panels in Figure 3.6 shows good agreement with each other.

It is now easy to see how much the prediction is changed when we adopt centre value of the QCD coupling constant $\alpha_s(M_Z) = 0.1172$. Since the choice of the QCD coupling directly changes the $M_{2-3}$, this severely affects the upper-bound of the $M_G$ in this model. $M_{2-3}$ is decreased by factor $e^{-\frac{2\pi}{4}\Delta\frac{1}{\alpha}}$, and the life-time is shortened by factor $e^{-2\pi 0.28} \simeq 0.2$. We confirmed that the upper bound of the life-time does not exceed $1 \times 10^{33}$ yrs., even when the SUSY-breaking parameters $m_0$ and $m_{1/2}$ are varied up to 2000 GeV if we adopt the centre value. Thus, it is unlikely from the analysis so far that the $\mathrm{SU(5)_{GUT}} \times \mathrm{U(2)_H}$ model is the model of the SUSY GUT. Dependence on the lattice form factor remains the same; just as in section 3.1. Extra particles $\mathbf{5}+\mathbf{5}^*$ at the electroweak scale enhances the unification coupling and the life-time becomes shorter just as in section 3.1, and moreover, threshold correction from these particles would make the $\mathrm{SU(3)_C}$ coupling weak. Thus, the upper bound of the life-time is further shortened.

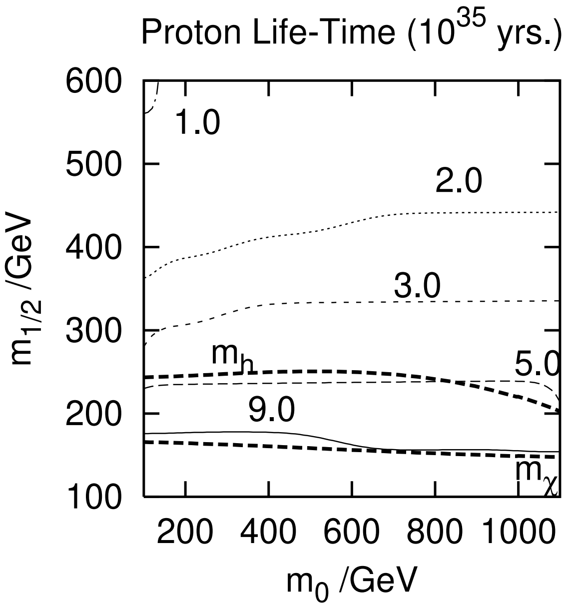


Figure 3.7: Contour plot of the upper bound of the proton life-time. LEP II bounds on the masses of the lightest Higgs and chargino are the same as those in Figure 3.2. The life-time varies within $(1-10)\times 10^{35}$ yrs. within the parameter space shown here.

Figure 3.7 is a contour plot of the upper bound of the proton life-time in the $\mathrm{SU}(5)_{\mathrm{GUT}}$ $\times\mathrm{U}(3)_{\mathrm{H}}$ model. QCD coupling $\alpha_s(M_Z) = 0.1132$ is used. The conservative upper bound of proton life-time is shorter than $5 \times 10^{35}$ yrs. for all the mSUGRA parameter space that is not excluded by the LEP II bound on the lightest Higgs boson mass. For the centre value of the QCD coupling, the upper bound does not change very much. Although the choice of the QCD coupling affects the lower bound of the life-time of the model, as one can easily see in Figure 3.5.

# Chapter 4

# Extension of Model I to Higher-Dimensional Space-Time

## 4.1 Motivation of Higher-Dimensional Extension

Let us begin by summarizing remarkable features that are common to the two models explained in section 2.2 (and discussed in chapter 3). First of all, the gauge groups of these "unification theories" have a product-group structure, and secondly, the $SU(5)_{GUT}$ gauge coupling constant is small while the rest of the gauge coupling constants are relatively large (Eqs. (2.26) and (2.33)). Thirdly, the GUT-symmetry-breaking sector has an approximate $\mathcal{N} = 2$ SUSY. The matter contents and interactions in the sector happen to support the approximate $\mathcal{N} = 2$ SUSY, and hence the evolution of the coupling constants has an IR-fixed relation, which is almost the same as the ($\mathcal{N} = 2$)-SUSY relation. Relatively large coupling constants of the extra gauge interactions in the sector are stabilized by this fixed relation even at the quantum level. Although the IR-fixed relation ($\simeq$ ($\mathcal{N} = 2$)-SUSY relation) does not have to be satisfied strictly at the cut-off scale of the models, large deviation from the relation would invalidate the approximate $SU(5)_{GUT}$ unification through the two-loop threshold corrections (see page 3.2.2). Therefore, it is favourable that the ($\mathcal{N} = 2$)-SUSY is not only respected by the matter contents and form of interactions but also satisfied (to some extent) by the value of coupling constants. Fourthly, the cut-off scale should lie below the Planck scale because of the asymptotically non-free running of the $U(1)_H$ gauge coupling constant. The lower cut-off scale is also required to explain the difference between the running Yukawa coupling constants of strange quark and muon at the GUT scale. Finally, these models have a symmetry that governs the models, yet this symmetry, which is an R symmetry, should be preserved in an accuracy better than the $10^{-14}$ level to keep the two Higgs doublets almost massless. A good hint to this issue is that mod-4 R symmetry can be gauged for the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model, and mod-$N$ R symmetry for arbitrary $N$ can for the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model.

It is shown in this section that the five features listed above are understood quite naturally when we embed the models in a SUSY brane world [68, 69, 70].

We consider the approximate $\mathcal{N}=2$ SUSY in the GUT-symmetry-breaking sector important. It is crucial in keeping the approximate SU(5) unification. It may be true that this approximate symmetry might be just an accident. Matter contents might happened to allow for an interpretation as ($\mathcal{N}=2$)-SUSY multiplets, their R charge assignment might happened to be such that the resulting interactions between them looks like those of ($\mathcal{N}=2$)-SUSY gauge theories, and the approximate ($\mathcal{N}=2$)-SUSY relation between the coupling constants might happened to be satisfied. However, we consider that this is not just an accident, but rather, that this approximate symmetry tells something about the nature.

It is quite reasonable to think of SUSY higher dimensions when one considers the approximate $\mathcal{N}=2$ SUSY in the GUT-symmetry-breaking sector as an indication of an extended SUSY at short distances. It is possible even in four-dimensional field theories to construct a model in which $\mathcal{N}=2$ SUSY is spontaneously broken down to $\mathcal{N}=1$ SUSY [71], but it would be quite difficult to construct a realistic model in which Yukawa couplings of quarks and leptons are generated and ($\mathcal{N}=2$)-SUSY partners of the MSSM particles acquire suitable masses to decouple from the renormalization group. In higher-dimensional field theories compactified to four dimensions, however, any extended SUSY can be easily broken down to the $\mathcal{N}=1$ SUSY through the compactified geometry; geometry breaks SUSY since metric is a super-partner of the gravitino, and the unwanted partners under the extended SUSY can decouple since there are infinite number of Kaluza–Klein particles to form mass terms with.

The GUT-symmetry-breaking sector, which is expected to preserve ($\mathcal{N}=2$)-SUSY approximately, should be localized in a compactified manifold. It should not observe the whole geometry, or otherwise only the $\mathcal{N}=1$ SUSY is left. Extended SUSY can be enhanced in the sector when it is affected by restricted part of geometry that preserves more SUSY.

On the other hand, the $\mathrm{SU}(5)_{\mathrm{GUT}}$ vector multiplet should propagate in the extra space dimensions. Indeed, gauge fields of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ should propagate around the localized sector, since the hypermultiplets in the bifundamental representation ($\bar{Q}^i{}_\alpha$,$Q^\alpha{}_i$) are charged under the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge group, while the $\mathrm{SU}(5)_{\mathrm{GUT}}$ vector multiplet should not be confined in the localized sector since we should not have chiral multiplets of the $\mathrm{SU}(5)_{\mathrm{GUT}}$-**adj.** representation (i.e. $\mathcal{N}=2$ SUSY partner) in the models explained in section 2.2.

When the compactified manifold has a moderately large volume in the units of the fundamental length $1/M_*$, the effective four-dimensional Planck scale $M_{\rm pl}$ is higher than the fundamental scale $M_*$. Einstein–Hilbert action in the (four+$\delta$)-dimensional space-time is given by

$$\mathcal{S} = M_*^{2+\delta} \int\int \sqrt{-g^{(4+\delta)}}\mathcal{R}\; d^4x d^\delta y, \tag{4.1}$$

where $y$ denotes the coordinates of the compactified manifold, and the integration over $dy$ leads to the action in four dimensions,

$$\mathcal{S}_4 = M_*^{2+\delta}(\text{volume}) \int \sqrt{-g}\mathcal{R}_4\; d^4x. \tag{4.2}$$

The coefficient in front of the integral must be the effective Planck scale $M_{\rm pl}$ in the four-dimensional space-time, and hence

$$M_{\rm pl}^2 \simeq M_*^2(M_*^\delta \times \text{volume}). \tag{4.3}$$

The cut-off scale $M_*$ lies below the effective Planck scale of the four-dimensional gravity since now we consider that $(M_*^\delta \times \text{volume}) > 1$ [72]. Therefore, the fourth feature is translated into the moderately large volume of the extra dimensions.

Now the gauge coupling constant of the $\mathrm{SU}(5)_{\rm GUT}$ Kaluza–Klein zero mode also becomes weak. The gauge coupling in the fundamental theory is defined as

$$\mathcal{S} = M_*^\delta \int d^4x \int d^\delta y \int d^2\theta \sqrt{-g^{(4+\delta)}}\frac{1}{16\pi\alpha_*}\mathcal{W}_\alpha\mathcal{W}^\alpha, \tag{4.4}$$

where $\alpha_*$ is the dimensionless gauge coupling constant of the fundamental theory. The integration over $y$'s yields a four-dimensional action as

$$\mathcal{S}_4 = M_*^\delta(\text{volume}) \int d^4x \int d^2\theta \sqrt{-g}\frac{1}{16\pi\alpha_*}\mathcal{W}_\alpha\mathcal{W}^\alpha. \tag{4.5}$$

Thus, the effective gauge coupling constant of the $\mathrm{SU}(5)_{\rm GUT}$ is given by

$$\frac{1}{\alpha_{\rm GUT}} = \frac{M_*^\delta \times \text{volume}}{\alpha_*}. \tag{4.6}$$

On the other hand, the gauge coupling of the $\mathrm{U}(2)_{\rm H}$ ($\mathrm{U}(3)_{\rm H}$) interaction will naturally be given by the dimensionless gauge coupling constant of the fundamental theory $\alpha_*$. Thus, the disparity in the gauge coupling constants (the second feature) follows naturally. It is interesting that $\mathrm{U}(2)_{\rm H}$ ($\mathrm{U}(3)_{\rm H}$) is expected to be localized while $\mathrm{SU}(5)_{\rm GUT}$ not because of a reason (i.e., $\mathcal{N}=2$ SUSY) completely independent of the disparity in the gauge coupling constants. We impose[1]

$$(M_*^\delta \times \text{volume}) \sim 10^2 \tag{4.7}$$

to maintain the approximate $\mathrm{SU}(5)_{\mathrm{GUT}}$ unification through Eqs. (2.26) and (2.33). Then, in turn, $M_* \simeq 10^{-1} M_{\mathrm{pl}}$ follows from Eq. (4.3), which is also a desirable value for the cut-off scale.

The above argument does not depend on the number of extra dimensions $\delta$. However, five-dimensional space-time ($\delta = 1$) is not favoured. This is because the Kaluza–Klein scale,

$$\frac{2\pi}{\text{length}} \sim 10^{-2} \times 2\pi M_* \sim 10^{-3} \times 2\pi M_{\mathrm{pl}} \sim 1.5 \times 10^{16}\mathrm{GeV}, \tag{4.8}$$

is comparable to the GUT scale, and hence the gauge coupling unification may be significantly affected.

Another benefit of higher dimensions is that R symmetries can be realized as discrete gauged symmetries below the compactification scale [73]. Since we consider a six-dimensional or more higher-dimensional space-time, discrete rotational symmetry is expected on the compactified geometry. It is, in general, recognized as an R symmetry below the compactification scale, because the SUSY generators transform as spinor representation under the rotation. The rotational symmetry is a gauge symmetry, since it is a subgroup of the extra-dimensional Lorentz group. The R symmetry is thus exact, unless broken spontaneously. The fifth feature finds its natural explanation when the mod-4 R symmetry is identified with a suitable rotational symmetry of the compactified manifold.

There are at least sixteen SUSY charges if one considers seven-dimensional or more higher-dimensional space-time. Even if six-dimensional space-time is considered, where the minimum number of SUSY charges is eight, sixteen SUSY charges are necessary at short-distance scale to realize the $\mathcal{N}=2$ SUSY on the localized sector, as shown below. Algebra of six-dimensional minimal SUSY is given by

$$\begin{aligned} \{\mathcal{Q}^{(6)}_{\mathbf{4}_-,B}, \mathcal{Q}^{(6)}_{\mathbf{4}_-,D}\} &= -i\epsilon_{BD}\left(\Gamma^{\mu}_{d=6} C^{T,-1}_{d=6}\right) P_\mu \\ &\quad +\tau^a_B{}^C \epsilon_{CD}\left(\Gamma^{\lambda\mu\nu}_{d=6} C^{T,-1}_{d=6}\right) C^a_{\lambda\mu\nu} + i\epsilon_{BD}\left(\Gamma^{\mu\nu\rho\sigma\tau}_{d=6} C^{T,-1}_{d=6}\right) C_{\mu\nu\rho\sigma\tau}, \end{aligned} \tag{4.9}$$

where the $C^a_{\lambda\mu\nu} (a = 1, 2, 3)$ in the second term and $C_{\mu\nu\rho\sigma\tau}$ in the third term in the right-hand side are central charges of objects that extend in three and five spatial dimensions. Reduction

[1]The volume for the gravity in Eq. (4.3) and the volume for the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge field in Eq. (4.6) are not necessarily the same, in general. However, we have no motivations to consider such a situation.

of this algebra into that of four-dimensional effective theory is given by

$$\{\mathcal{Q}^{(4)2}_{\alpha}, \bar{\mathcal{Q}}^{(4)}_{\dot{\beta},2}\} = i(i\sigma^i)_{\alpha\dot{\beta}}P_i - (i\sigma^1 i\bar{\sigma}^2 i\sigma^3)_{\alpha\dot{\beta}}(C^3_{123} - C_{12345}), \tag{4.10}$$

$$\{\mathcal{Q}^{(4)1}_{\alpha}, \bar{\mathcal{Q}}^{(4)}_{\dot{\beta},1}\} = i(i\sigma^i)_{\alpha\dot{\beta}}P_i + (i\sigma^1 i\bar{\sigma}^2 i\sigma^3)_{\alpha\dot{\beta}}(C^3_{123} - C_{12345}), \tag{4.11}$$

$$\{\mathcal{Q}^{(4)1}_{\alpha}, \mathcal{Q}^{(4)2}_{\beta}\} = \epsilon_{\alpha\beta}(-P_4 + iP_5), \tag{4.12}$$

where we dropped central charges that violate SO(3,1) Lorentz symmetry in the vacuum. Translational symmetries in extra dimensions $P_4$ and $P_5$ are not well-defined on the localized sector, and hence two SO(3,1)-irreducible SUSY generators $\mathcal{Q}^{(4)1}_{\alpha}$ and $\mathcal{Q}^{(4)2}_{\alpha}$ obtained through the dimensional reduction of an SO(5,1)-irreducible SUSY generator $\mathcal{Q}^{(6)}_{\mathbf{4}_-}$ are not well-defined simultaneously. One of the SO(3,1)-irreducible SUSY generator (i.e., $\mathcal{N}=1$ SUSY) is left unbroken if the localized sector satisfies a BPS condition

$$\left\langle P^0 \right\rangle = -\left\langle P_0 \right\rangle = \pm \left( \left\langle C^3_{123} \right\rangle - \left\langle C_{12345} \right\rangle \right), \tag{4.13}$$

i.e., tension localized there is related to the central charges. Otherwise, all SUSY charges would have been broken by the existence of the localized sector itself. Therefore, only half of the SUSY is left unbroken even if the BPS condition is satisfied, and hence sixteen SUSY charges are necessary in the short-distance physics in order for the $\mathcal{N}=2$ SUSY (eight SUSY charges) to be left unbroken on the localized sector. It is also necessary that the geometry around the localized sector preserves the $\mathcal{N}=2$ SUSY preserved by the localization.

The whole geometry of the compactified manifold, on the other hand, is chosen so that it preserves only the $\mathcal{N}=1$ SUSY of the four-dimensional space-time. It is only the local geometry around the localized GUT-symmetry-breaking sector that is required to preserve the $\mathcal{N}=2$ SUSY.

As we have seen so far, an effective field theory with a localized sector and with a higher-dimensional SUSY is able to explain basic structures of the product-group unification model I. Both the extended SUSY and the localization of gauge fields are natural ingredients of higher-dimensional supergravities. It is usual the case that field theories localize on soliton solutions such as domain walls, strings, monopoles and instantons in field theories on four-dimensional space-time, and hence one might expect that gauge theories are localized on soliton solutions of higher-dimensional supergravity such as D-branes [74][2]. Once we adopt this picture, then the product-group structure of the "unified gauge group" (the first feature)

[2]Those solitonic solutions were formerly called "black $p$-branes" [74]. We make no distinction between "D-branes" in supergravities and "black $p$-branes".

is quite a natural consequence since each stack of D-branes provides each factor of the product group.

## 4.2 An Attempt to Realize the Model in String Theory — D3–D7 System in Type IIB Orientifolds

D-branes in string theories are characterized as Dirichlet boundary conditions of open strings, and hence gauge theories are predicted to be localized on them [75]. Thus, the M/string theory is a good candidate that provides a firm base to the speculative idea of brane-world effective theory in the previous section. We, therefore, consider that the M/string theory is not only one of promising candidates of quantum gravity, but is motivated also because it *does* have vacua which look like brane-world effective field theory in their low energy description.

Let us suppose that the GUT-symmetry-breaking sector is supported on D-branes. Then, the total charge of D$p$-branes in the vacuum should be zero, since the total volume of extra dimensions is finite and hence the $(p+2)$-form flux has nowhere to escape. Although anti-D$p$-branes have charges opposite to those of D$p$-branes, the SUSY charges they preserve are different from those of D$p$-branes. On the other hand, orientifold $p$-planes (O$p$-planes) in string theories [76] have opposite charges, yet preserve the same SUSY charges. Since the orientifold planes are associated with a symmetry that reverses the orientation of strings, they are formulated only in *string* theories[3]. Thus, this is another benefit of considering string theories behind the effective picture of brane-world in the previous section.

The $\mathcal{N}=2$ SUSY is preserved in a D3–D7 system of the Type IIB string theory [78]. Moreover, the D3-branes are localized in the D7-branes, i.e., there are four space dimensions within the D7-branes to which the D3-branes are not stretched. Thus, we first examine to what extent the phenomenological model of the GUT-symmetry breaking in a brane-world effective field theory can be obtained from the D3–D7 system. Possibilities other than the D3–D7 system are discussed in section 4.6.

To be more realistic, we consider that five D7-branes wrap on a holomorphic four-cycle of a compact Calabi–Yau three-fold, and $N(=2,3)$ D3-branes are on a point on the D7-branes. The $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge group are supposed to come form the D7-branes and the $\mathrm{U}(N)_{\mathrm{H}}$ $(N=2,3)$ gauge group from the D3-branes. In particular, the GUT-symmetry-breaking sector is on the D3-branes. The low energy physics preserves $\mathcal{N}=1$ SUSY. When

[3]The only exception at this moment is O6-planes. Configuration of metric is given in eleven-dimensional supergravity for what is believed to be O6-planes [77].

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| D3, O3 ($\mathrm{U}(N)_\mathrm{H}$) | o | o | o | o | × | × | × | × | × | × |
| D7, O7 ($\mathrm{SU}(5)_\mathrm{GUT}$) | o | o | o | o | o | o | o | o | × | × |

Table 4.1: Dimensions to which D3- and D7-branes are stretched. From $x_0$ to $x_3$ are the non-compact Minkowski four-dimensional space-time. Remaining six space dimensions form (an orbifold of) a six-dimensional torus $\mathbf{T}^6$. D7-branes wrap on a four-cycle of the $\mathbf{T}^6$.

the local geometry around the D3-branes preserves extended SUSY, or in other words, when the D3-branes are on a point of D7-branes where SUSY charges preserved by local geometry is enhanced, then the $\mathcal{N} = 2$ SUSY is (approximately) preserved in the GUT-symmetry-breaking sector on the D3-branes. We consider that the length scale of the Calabi–Yau three fold is less than inverse of the GUT scale, but moderately larger than the inverse of the Planck scale. Then the world volume of the D7-branes would also be moderately large, and the $\mathrm{SU}(5)_\mathrm{GUT}$ gauge interactions would have weak coupling constant relatively to those of the $\mathrm{U}(N)_\mathrm{H}$ gauge interactions.

It is not easy to treat generic curved Calabi–Yau manifolds. In the following three sections, we only treat orbifolds of six-dimensional torus $\mathbf{T}^6$ as the compact Calabi–Yau manifold[4], and make an attempt to find a suitable geometries and D-brane configurations. Although we could not find a global solution that contains quarks and leptons, in addition to the product-group unification model I, a local solution of the model was obtained. The $\mathrm{SU}(5)_\mathrm{GUT}\times\mathrm{U}(2)_\mathrm{H}$ model and the $\mathrm{SU}(5)_\mathrm{GUT}\times\mathrm{U}(3)_\mathrm{H}$ model on the D3–D7 system are described in section 4.3 and 4.4, respectively. Consistency conditions that are relevant to this local construction are discussed in section 4.5.

## 4.3 $\mathrm{SU}(5)_\mathrm{GUT}\times$ $\mathrm{U}(2)_\mathrm{H}$ Model

### 4.3.1 $\mathbf{T}^6/(\mathbf{Z}_{12}\,\langle\sigma\rangle\times\mathbf{Z}_2\,\langle R_{89}\Omega\rangle)$ Geometry

We consider that the D3-branes and D7-branes are stretched in various dimensions just as shown in Table 4.1. Then, the orbifold geometry we consider should contain orientifold planes in order to cancel the total D-brane charges in the compact extra-dimensional manifold. Thus, we consider orientifolds $\mathbf{T}^6/(\Gamma\times\mathbf{Z}_2\,\langle\Omega R_{89}\rangle)$, where $\Gamma$ is a discrete subgroup of SO(6) rotational symmetry of the six extra dimensions and $\Omega R_{89}$ is a generator of the orientifold

[4]We use the word "manifold" even if it has singularities. It is an abuse of terminology, though.

operation (simultaneous reversing of the orientation of strings and rotation of $(x_8 - x_9)$-plane by angle $\pi$) associated with the O7-planes parallel to the D7-branes shown in Table 4.1.

In order for the $\mathcal{N} = 1$ SUSY to be left unbroken below the Kaluza–Klein scale, the orbifold group $\Gamma$ should be contained in an SU(3) subgroup of the SO(6). A standard complex structure[5] is introduced to the $\mathbf{T}^6$; $z_1 \equiv x_4 + ix_5$, $z_2 \equiv x_6 + ix_7$ and $z_3 \equiv x_8 + ix_9$. The SU(3) subgroup is the special unitary rotation of the three complex planes. There are thirteen orbifold groups of $\mathbf{Z}_N$ type that preserve the $\mathcal{N} = 1$ SUSY [79]. We find the $\mathbf{Z}_{12}$ (1,−5,4)/12 is the best choice for our purpose; the reason is explained shortly.

The generator $\sigma$ of the present orbifold group, $\Gamma = \mathbf{Z}_{12} \langle \sigma \rangle$, acts on the three complex planes $\mathbf{C}^3$ as

$$\sigma : \mathbf{y} \equiv (z_b)|_{b=1,2,3} \in \mathbf{C}^3 \longmapsto \sigma \cdot \mathbf{y} \equiv (e^{2\pi i v_b} z_b)|_{b=1,2,3} \in \mathbf{C}^3 \tag{4.14}$$

with $(v_b)|_{b=1,2,3} = (1/12, -5/12, 4/12)$. Note that the order-two element of the orbifold group $\sigma^6$ induces reflection of the two complex planes $z_1$ and $z_2$, and hence $\sigma^6 \cdot \Omega R_{89} \in \Gamma \times \mathbf{Z}_2 \langle \Omega R_{89} \rangle$ guarantees that there are O3-planes in the manifold, which could cancel the charges of D3-branes. The six-dimensional torus $\mathbf{T}^6 \equiv \mathbf{C}^3/\Gamma_0$, where $\Gamma_0$ is a lattice of rank six in $\mathbf{C}^3$, should be chosen so that it has the $\mathbf{Z}_{12} \langle \sigma \rangle$ symmetry. Thus, the $\Gamma_0$, which determines the six-dimensional torus, is chosen as

$$\mathbf{e}_4 = (1, 1, 0)L_4, \qquad \mathbf{e}_5 = (\zeta, \zeta^{-5}, 0)L_4, \tag{4.15}$$

$$\mathbf{e}_6 = (\zeta^2, \zeta^2, 0)L_4, \qquad \mathbf{e}_7 = (\zeta^3, \zeta^{-3}, 0)L_4, \tag{4.16}$$

$$\mathbf{e}_8 = (0, 0, 1)L_2, \qquad \mathbf{e}_9 = (0, 0, \omega)L_2, \tag{4.17}$$

where $\zeta = e^{2\pi i/12}$, $\omega = e^{2\pi i/3}$ and $L_4, L_2$ are two independent length scales of the six-dimensional torus; $L_4$ corresponds to the size of the first four-dimensional torus and $L_2$ to that of the remaining two-dimensional torus.

The third complex plane of this orientifold looks like Figure 4.1. Loci of the D7-branes are specified by $z_3$ coordinates in the third complex plane. The $\mathbf{T}^6/(\mathbf{Z}_{12} \langle \sigma \rangle \times \mathbf{Z}_2 \langle \Omega R_{89} \rangle)$ orientifold has $\sigma$-fixed loci that is distant from the O7-planes (see Figure 4.1), and this is one of the reasons why we chose the $\mathbf{Z}_{12}$ as the orbifold group. We put the D7-branes on such fixed loci in section 4.3 and 4.4. Such fixed loci enables us to construct purely SU(5)-based (rather than SO(10)-based) unified theories.

The D7-branes are stretched in the $\mathbf{e}_{4,5,6,7}$ directions, on which the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge fields are expected to propagate. Therefore, we require that the $L_4$ be slightly larger than the

[5] The four cycle on which the D7-branes wrap is a holomorphic cycle in this complex structure.

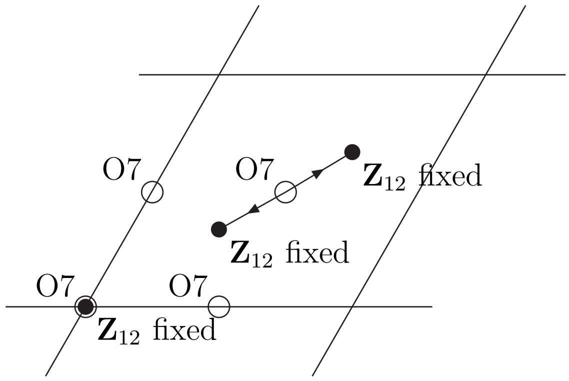


Figure 4.1: This figure shows a picture of the third complex plane (whose coordinate is $z_3$) of the $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle \times \mathbf{Z}_2\langle\Omega R_{89}\rangle)$ geometry. Circles are O7-plane positions and dots are $\mathbf{Z}_{12}\langle\sigma\rangle$ fixed loci. We put D7-branes for the $\mathrm{SU}(5)_{\mathrm{GUT}}$ on two dots without the circle. These two fixed loci are $\Omega R_{89}$-images of the other one and so are these D7-branes on them.

fundamental-scale inverse, because the volume of these extra four dimensions should be large enough to account for the disparity between gauge coupling constants of the $\mathrm{SU}(5)_{\mathrm{GUT}}$ and $\mathrm{U}(3)_{\mathrm{H}}$ or $\mathrm{U}(2)_{\mathrm{H}}$. Equations (4.6) and (4.7) imply that

$$\frac{1}{4}(M_* L_4)^4 \sim 10^2, \tag{4.18}$$

where $(1/12) \times 3(L_4)^4$ is the volume of extra dimensions where the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge field propagates. Here, the D7-branes that provide the $\mathrm{SU}(5)_{\mathrm{GUT}}$ are assumed to reside at a $\mathbf{Z}_{12}\langle\sigma\rangle$-fixed[6] locus. On the other hand, the $L_2$ is considered to be of the order of the inverse of the fundamental scale.

### 4.3.2 D-brane Configuration and Orbifold Projection

Matter contents below the Kaluza–Klein scale depend on the D-brane configuration (locations of D-branes on the orbifold geometry) and the orbifold projection conditions. Let us first describe how the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ model is obtained.

The gauge theory on $N$ coincident D7-branes (distant from any O7-planes) consists of a $\mathrm{U}(N)$ vector multiplet $(\Sigma^k{}_l)$ $(k, l = 1, ..., N)$ of the $\mathcal{N} = 1$ SUSY of eight-dimensional space-time. It is described in terms four-dimensional $\mathcal{N} = 1$ SUSY as a set of four chiral

[6]The orientifold projection $\mathbf{Z}_2\langle\Omega R_{89}\rangle$ does not reduce the volume on which $\mathrm{SU}(5)_{\mathrm{GUT}}$ propagates because it acts only in the directions transverse to the D7-branes.

superfields $(\Sigma_a)^k{}_l(x, \mathbf{y}', \theta)$ $(a = 0, 1, 2, 3)$, where $x$, $\theta$ and $\mathbf{y}'$ denote the four-dimensional spec-time coordinates, the Grassmann coordinates of the $\mathcal{N} = 1$ SUSY superspace and four space coordinates of extra four dimensions, respectively, and $\Sigma_0$ is the ordinary field strength tensor $\mathcal{W}_\alpha$. $M$ coincident D3-branes also have a U($M$) vector multiplet $(X^\alpha{}_\beta)$ $(\alpha, \beta = 1, ..., M)$ of the $\mathcal{N} = 4$ SUSY of four-dimensional space-time. It is described by four chiral superfields $(X_a)^\alpha{}_\beta(x, \theta)$ $(a = 0, 1, 2, 3)$ of the $\mathcal{N} = 1$ SUSY, where $X_0$ is the field strength tensor. There is a hypermultiplet $(\bar{Q}^k{}_\alpha, Q^\alpha{}_k)$ of the four-dimensional $\mathcal{N} = 2$ SUSY when the $M$ D3-branes are on the $N$ D7-branes. Here, the $\mathcal{N} = 1$ chiral multiplets $\bar{Q}^k{}_\alpha$ and $Q^\alpha{}_k$ are in the bi-fundamental representation under the U($N$)×U($M$) gauge group, transforming $(\mathbf{N}, \mathbf{M}^*)$ and $(\mathbf{N}^*, \mathbf{M})$, respectively. D7-branes are labelled by indices $k, l$, and D3-branes by $\alpha, \beta$.

We use twelve out of thirty-two D7-branes to provide the SU(5)$_{\rm GUT}$ vector multiplet. They should be put on (seven+one)-dimensional loci fixed under the $\mathbf{Z}_{12}\,\langle\sigma\rangle$ orbifold group, so that the vector multiplet with sixteen SUSY charges is projected out except for a four-dimensional $\mathcal{N} = 1$ SU(5)$_{\rm GUT}$ vector multiplet (without chiral multiplets in the SU(5)$_{\rm GUT}$-**adj.** representation). Six of them are put at a fixed locus $z_3 = \frac{\mathbf{e}_8 + 2\mathbf{e}_9}{3}$ and six remaining D7-branes are at the $\mathbf{Z}_2\,\langle \Omega R_{89}\rangle$-image of the fixed locus, i.e. at $z_3 = -\frac{\mathbf{e}_8 + 2\mathbf{e}_9}{3}$. The reason why we put six D7-branes at each fixed locus rather than five will become clear later in this subsection 4.3.2 (it is because we require that $\bar{Q}^6{}_\alpha$ and $Q^\alpha{}_6$ are obtained from the D3–D7 system).

The twenty D7-branes that have not been used are placed at the other fixed locus $z_a = \mathbf{0}$ or are floating in the bulk. Their existence is irrelevant to the dynamics of the SU(5)$_{\rm GUT}$-symmetry breaking, while they provide a room for constructing the SUSY-breaking sector, inflation sector and some other gauge theories we do not know yet.

Fields on the D7-branes transform under three independent rotations $z_b \mapsto e^{i\alpha_b} z_b$ for $(b = 1, 2, 3)$ as

$$\Sigma_0(x, \mathbf{y}', \theta) \;\mapsto\; e^{i\tilde{\alpha}_0}\Sigma_0(x, \tilde{\mathbf{y}'}, e^{-i\tilde{\alpha}_0}\theta), \tag{4.19}$$

$$\Sigma_b(x, \mathbf{y}', \theta) \;\mapsto\; e^{i\alpha_b}\Sigma_b(x, \tilde{\mathbf{y}'}, e^{-i\tilde{\alpha}_0}\theta) \quad \text{for } b = 1, 2, 3 \tag{4.20}$$

where $\tilde{\alpha}_0 \equiv (\alpha_1 + \alpha_2 + \alpha_3)/2$, $\tilde{\alpha}_b \equiv \alpha_b - \tilde{\alpha}_0$ for $(b = 1, 2, 3)$, and $\tilde{\mathbf{y}'}$ is a point that is moved to the point $\mathbf{y}'$ under the rotation, i.e. $\tilde{\mathbf{y}'} = (e^{-i\alpha_1}z_1, e^{-i\alpha_2}z_2)$ and $\mathbf{y}' = (z_1, z_2)$. Let us now describe the orbifold projection we impose.

$$\Sigma_a(x, \mathbf{y}', \theta)^k{}_l = \widetilde{\Sigma_a}(x, \mathbf{y}', \theta)^k{}_l, \quad \text{where} \quad \sigma : \Sigma_a(x, \mathbf{y}', \theta)^k{}_l \mapsto \widetilde{\Sigma_a}(x, \mathbf{y}', \theta)^k{}_l. \tag{4.21}$$

Here, the $\sigma$-transformation of fields is primarily determined by the geometric rotation (4.14). However, we also have another degree of freedom in determining the $\sigma$-transformation of fields — the geometric rotation of fields can be accompanied by a non-trivial twist through a rigid gauge transformation by $\tilde{\gamma}_{\sigma;7} \in$U(6). Now, the $\sigma$-transformation is given as follows:

$$\sigma : \quad (\Sigma_0)^k{}_l(x,\mathbf{y}',\theta) \mapsto (\widetilde{\Sigma_0})^k{}_l(x,\mathbf{y}',\theta) \equiv \qquad (\tilde{\gamma}_{\sigma;7})^k{}_{k'}(\Sigma_0)^{k'}{}_{l'}(x,\sigma^{-1}\cdot\mathbf{y}',\theta)(\tilde{\gamma}^{-1}_{\sigma;7})^{l'}{}_l, \tag{4.22}$$
$$\sigma : \quad (\Sigma_b)^k{}_l(x,\mathbf{y}',\theta) \mapsto (\widetilde{\Sigma_b})^k{}_l(x,\mathbf{y}',\theta) \equiv e^{2\pi i v_b}(\tilde{\gamma}_{\sigma;7})^k{}_{k'}(\Sigma_b)^{k'}{}_{l'}(x,\sigma^{-1}\cdot\mathbf{y}',\theta)(\tilde{\gamma}^{-1}_{\sigma;7})^{l'}{}_l, \tag{4.23}$$

where $\tilde{\alpha}_a = 2\pi\tilde{v}_a$ are substituted into Eqs. (4.19)–(4.20) and we take the 6 by 6 unitary matrix[7] $\tilde{\gamma}_{\sigma;7}$ as

$$(\tilde{\gamma}_{\sigma;7})^k{}_l = \mathrm{diag}\left(\overbrace{e^{-\frac{1}{12}\pi i},\ldots,e^{-\frac{1}{12}\pi i}}^{5}, -e^{-\frac{1}{12}\pi i}\right). \tag{4.26}$$

The twist in the orbifold projection through the U(6) rigid transformation by $\tilde{\gamma}_{\sigma;7}$ breaks the U(6) symmetry down to U(5)$\times$U(1)$_6$.

Only the U(5)$\times$U(1)$_6$ vector multiplet of the four-dimensional $\mathcal{N}=1$ SUSY survive these orbifold projection in the low-energy spectrum. No Higgs multiplet appears after the projection in this model.

We identify the SU(5) part of the U(5) $\simeq$ SU(5)$\times$U(1)$_5$ gauge group with the SU(5)$_{\rm GUT}$ gauge group. The U(1)$_{5+6}$ symmetry decouples from this sector, and thus, the U(1)$_{5-6}$ symmetry is a candidate to be identified with the fiveness[8] gauge symmetry. There is no massless chiral multiplets arising from D7-branes. Although we do not need Higgs multiplets in the SU(5)$_{\rm GUT}\times$U(2)$_{\rm H}$ model, quarks and leptons (and right-handed neutrinos) are missing. It is not easy to accommodate all the three families of quarks and leptons along with the model of SU(5)$_{\rm GUT}$ breaking we discuss (see also discussion in sectiion 4.5 and 4.6).

Let us now derive from the D3–D7 bound state the matter contents of the SU(5)$_{\rm GUT}$-breaking sector. D3-branes are put on a fixed point that preserves $\mathcal{N}=2$ SUSY [69]. We refer such fixed points to the ($\mathcal{N}=2$)-SUSY fixed points. There would be the whole $\mathcal{N}=4$ SUSY vector multiiplet in the spectrum, including an unwanted SU(2)$_{\rm H}$-**adj.** hypermultiplet

[7]This 6 by 6 unitary matrix $\tilde{\gamma}_{\sigma;7}$ is related to 32 by 32 unitary matrices $\gamma_{\sigma;D7}$ found in references such as [80, 81, 82] through

$$\gamma_{\sigma;D7} = \tilde{\gamma}_{\sigma;7} \oplus \tilde{\gamma}^{-1}_{\sigma;7} \oplus (\text{20 by 20 matrix}). \tag{4.24}$$

The unitary matrix $\gamma_{\Omega R_{89};D7}$ associated to the projection condition of $\Omega R_{89}$ is expressed in this basis as

$$\gamma_{\Omega R_{89};D7} = \mathbf{1}_{6\times 6} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbf{1}_{10\times 10} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.25}$$

[8]The fiveness is equivalent to B$-$L in the standard model.

of $\mathcal{N}$ = 2 SUSY, if the D3-branes were not put at a fixed point; however, the multiplet structure of the $\mathcal{N}$ = 2 SUSY would be lost if they were put on a fixed point where the $\mathcal{N}$ = 2 SUSY is not preserved.

SUSY of local geometry at a given fixed point is determined by its isotropy group, a subgroup of the orbifold group that fixes the point. The isotropy group determines the local geometry around the fixed point, and hence the SUSY. Matter contents from D-branes located at that fixed point are also determined by imposing orbifold projection conditions associated to the isotropy group.

The ($\mathcal{N}$ = 2)-SUSY fixed points should have isotropy group contained in a particular SU(2) subgroup of the SU(4) $\simeq$ SO(6) rotation that acts as unitary rotation on the first two complex planes in order to preserve the $\mathcal{N}$ = 2 SUSY in the D3–D7 system. There are essentially two different candidates of the ($\mathcal{N}$ = 2)-SUSY fixed points on the D7-branes: two points at which the isotropy group is $\mathbf{Z}_2 \langle\sigma^6\rangle$ and a point at which the isotropy group is $\mathbf{Z}_4 \langle\sigma^3\rangle$. Note that $\sigma^3, \sigma^6$ and $\sigma^9$ belong to the above-mentioned SU(2) subgroup, since it does not rotate the third complex plane because $v_3 \times 3 \in \mathbf{Z}$.

We put the D3-branes at an ($\mathcal{N}$ = 2)-SUSY fixed point where the U(6) symmetry is enhanced on the D7-branes. Then, the $\mathcal{N} = 2$ hypermultiplet $(\bar{Q}^6{}_\alpha, Q^\alpha{}_6)$, which is necessary in the GUT-symmetry-breaking sector, is obtained in addition to the hypermultiplet $(\bar{Q}^i{}_\alpha, Q^\alpha{}_i)$ $(i = 1, ..., 5)$. Under the choice of $\tilde{\gamma}_{\sigma;7}$ in (4.26), one can see that the U(6) symmetry is not broken in $\sigma^{2k}$ projections $(k = 0, 1, 2, 3, 4, 5)$. Thus, both the $\mathcal{N}$ = 2 SUSY and the U(6) symmetry are preserved in the $\sigma^6$ projection. Therefore, we put two D3-branes on one of the fixed points where the isotropy group is $\mathbf{Z}_2 \langle\sigma^6\rangle$. There are two such fixed points on the $\mathbf{T}^6/(\mathbf{Z}_{12} \langle\sigma\rangle \times \mathbf{Z}_2 \langle\Omega R_{89}\rangle)$. Such a fixed point consists of twelve points in the covering space $\mathbf{T}^6$, which are identified under the $(\mathbf{Z}_{12} \langle\sigma\rangle \times \mathbf{Z}_2 \langle\Omega R_{89}\rangle)/\mathbf{Z}_2 \langle\sigma^6\rangle$. Thus, two D3-branes have to be introduced at each of these twelve points. Twenty-four D3-branes are necessary as a whole in the covering space $\mathbf{T}^6$. The rest of the D3-branes can be used for other sectors, which are not directly observed today.

We concentrate on a set of fields at one of these twelve images on which the orbifold projection associated with the isotropy group $\mathbf{Z}_2 \langle\sigma^6\rangle$ is imposed. Those fields consist of a U(2) vector multiplet $(X_0)^\alpha{}_\beta(x,\theta) \equiv (\mathcal{W}^{\mathrm{U}(2)})(x,\theta)$ of four-dimensional $\mathcal{N}$ = 1 SUSY and three chiral multiplets $(X_b)^\alpha{}_\beta(x,\theta)$ $(b = 1, 2, 3)$ in the U(2)-**adj.** representation, chiral superfields $Q^\alpha{}_k(x,\theta)$ in the $(\mathbf{6}^*,\mathbf{2})$ representation and $\bar{Q}^k{}_\alpha(x,\theta)$ in the $(\mathbf{6},\mathbf{2}^*)$ representation of the U(6)$\times$U(2) gauge group. They transform under the three independent rotational symmetries on the $\mathbf{C}^3 : z_b \mapsto e^{i\alpha_b} z_b$ for $b = 1, 2, 3$ as follows.

$$
\begin{aligned}
X_0(x,\theta) &\mapsto e^{i\tilde{\alpha}_0} X_0(x, e^{-i\tilde{\alpha}_0}\theta), &(4.27)\\
X_b(x,\theta) &\mapsto e^{i\alpha_b} X_b(x, e^{-i\tilde{\alpha}_0}\theta) \quad \text{for } b = 1,2,3. &(4.28)
\end{aligned}
$$

$$
\begin{aligned}
\bar{Q}(x,\theta) &\mapsto e^{i(\alpha_1+\alpha_2)/2} \bar{Q}(x, e^{-i\tilde{\alpha}_0}\theta), &(4.29)\\
Q(x,\theta) &\mapsto e^{i(\alpha_1+\alpha_2)/2} Q(x, e^{-i\tilde{\alpha}_0}\theta). &(4.30)
\end{aligned}
$$

Orbifold projection conditions

$$
X_a = \widetilde{X_a}, \quad Q = \widetilde{Q}, \quad \bar{Q} = \widetilde{\bar{Q}}, \tag{4.31}
$$

with $a = 0, 1, 2, 3$ are imposed, where the $\sigma^6$ transform $\widetilde{X_a}$, $\widetilde{Q}$ and $\widetilde{\bar{Q}}$ of these fields are given by

$$
\begin{aligned}
\sigma^6 : \quad & (X_0)^\alpha{}_\beta(x,\theta) \mapsto (\widetilde{X_0})^\alpha{}_\beta(x,\theta) \equiv (\tilde{\gamma}_{\sigma^6;3})^\alpha{}_{\alpha'} (X_0)^{\alpha'}{}_{\beta'}(x,\theta) (\tilde{\gamma}^{-1}_{\sigma^6;3})^{\beta'}{}_\beta, &(4.32)\\
\sigma^6 : \quad & (X_b)^\alpha{}_\beta(x,\theta) \mapsto (\widetilde{X_b})^\alpha{}_\beta(x,\theta) \equiv e^{2\pi i v_b 6} (\tilde{\gamma}_{\sigma^6;3})^\alpha{}_{\alpha'} (X_b)^{\alpha'}{}_{\beta'}(x,\theta) (\tilde{\gamma}^{-1}_{\sigma^6;3})^{\beta'}{}_\beta, &(4.33)
\end{aligned}
$$

$$
\begin{aligned}
\sigma^6 : \quad & Q^\alpha{}_k(x,\theta) \mapsto \widetilde{Q}^\alpha{}_k(x,\theta) \equiv e^{\pi i(v_1+v_2)6} (\tilde{\gamma}_{\sigma^6;3})^\alpha{}_{\alpha'} Q^{\alpha'}{}_{k'}(x,\theta) (\tilde{\gamma}^{-6}_{\sigma;7})^{k'}{}_k, &(4.34)\\
\sigma^6 : \quad & \bar{Q}^k{}_\alpha(x,\theta) \mapsto \widetilde{\bar{Q}}^k{}_\alpha(x,\theta) \equiv e^{\pi i(v_1+v_2)6} (\tilde{\gamma}^{6}_{\sigma;7})^k{}_{k'} \bar{Q}^{k'}{}_{\alpha'}(x,\theta) (\tilde{\gamma}^{-1}_{\sigma^6;3})^{\alpha'}{}_\alpha. &(4.35)
\end{aligned}
$$

We take the 2 by 2 unitary matrix for the rigid U(2) transformation[9] $\tilde{\gamma}_{\sigma^6;3}$ as

$$
\tilde{\gamma}_{\sigma^6;3} = \mathrm{diag}(e^{-\frac{1}{2}\pi i}, e^{-\frac{1}{2}\pi i}). \tag{4.37}
$$

The phase $e^{-\pi i/2}$ is chosen so that the $(\bar{Q}, Q)$ survive the orbifold projection conditions.

Massless modes that survive the orbifold projection are a U(2) vector multiplet $(X_0, X_3)$ of the four-dimensional $\mathcal{N} = 2$ SUSY and a hypermultiplet $(Q^\alpha{}_k, \bar{Q}^k{}_\alpha)$ $(k = 1, ..., 6)$. This is

[9] The 32 by 32 matrix $\gamma_{\sigma;D3}$ in Refs. [80, 81, 82] is expressed in terms of the $\tilde{\gamma}_{\sigma^6;3}$ as

$$
\gamma_{\sigma;D3} = \begin{pmatrix} & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{pmatrix} \otimes (\tilde{\gamma}_{\sigma^6;3})^{\frac{1}{6}} \oplus \begin{pmatrix} & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \end{pmatrix} \otimes (\tilde{\gamma}_{\sigma^6;3})^{-\frac{1}{6}} \oplus (8 \text{ by } 8 \text{ matrix}). \tag{4.36}
$$

.

exactly the matter content of the GUT-symmetry-breaking sector in the product-group unification model based on the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ gauge group. An unwanted $\mathcal{N}=2$ hypermultiplet in the U(2)-**adj.** representation ($X_1$,$X_2$) has been projected out. Neither $\mathrm{SU}(5)_{\mathrm{GUT}}$-**adj.** chiral multiplets for $\mathrm{SU}(5)_{\mathrm{GUT}}$ breaking nor -(**5**+**5**$^*$) for the Higgs fields are in the massless spectrum, which is also in good agreement with the model.

The most successful feature of this higher-dimensional construction is that the superpotential of $\mathcal{N}=2$ SUSY (the first and the second lines of (2.22)) is automatically obtained. The approximate $\mathcal{N}=2$ SUSY relation is naturally expected as a result of the extended SUSY in the UV physics.

Therefore, the global geometry $\mathbf{T}^6/(\mathbf{Z}_{12}\left\langle\sigma\right\rangle\times\mathbf{Z}_2\left\langle\Omega R_{89}\right\rangle)$ with a suitable choice of gauge bundles provides an explicit example in which the GUT-symmetry-breaking sector of the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model is perfectly obtained. It has a fixed point where the $\mathcal{N}=2$ SUSY and the U(6) symmetry on the D7-branes are enhanced On the other hand, quarks and leptons, $\mathrm{SU}(5)_{\mathrm{GUT}}$-**10** and -**5**$^*$, are missing in the spectrum obtained in our orbifold construction. There is no model using D-brane construction that has succeeded in obtaining all of (i) the three families of quarks and leptons, (ii) a gauge group of a unified theory and (iii) a sector to break that symmetry; this is not a difficulty limited to our construction. We considered that the GUT-symmetry-breaking sector with $\mathcal{N}=2$ SUSY is a strong indication of the structure of the higher-dimensional space, and therefore, we constructed the model so that this structure is manifestly realized. The orbifold geometry thus obtained provides a good description of the structure of the GUT-symmetry-breaking sector, but not of the quarks and leptons.

### 4.3.3 Discrete R Symmetry

Orbifold geometry preserves a discrete rotational symmetry, which is a subgroup of the SO(6) rotational symmetry of the six extra dimensions. This subsection is devoted to consequences of this symmetry.

Now that all the particles in the GUT-symmetry-breaking sector are obtained from D-branes, we know how those fields transform under the discrete rotational symmetry of the orbifold geometry. On the other hand, such a rotational symmetry is regarded as an internal symmetry at energies below the Kaluza–Klein scale. This symmetry, in general, rotates SUSY charges since these are in the spinor representation of the space rotational symmetry. Thus, it becomes an R symmetry below the Kaluza–Klein scale. Therefore, we can figure out how those fields transform under the discrete R symmetry.

The R symmetry obtained in this way is a gauged symmetry. Indeed the rotation is nothing but the combined action of a general coordinate transformation and a local Lorentz symmetry, both of which are gauged. Thus, the R symmetry is exact unless it is spontaneously broken. This is quite important because the R symmetry of the product-group unification model I should be preserved at the $10^{-14}$ level to keep the two Higgs doublets almost massless.

The R symmetry is expected to be spontaneously broken by vacuum condensation of the superpotential, which should be related to the spontaneous breaking of $\mathcal{N} = 1$ SUSY (mass of gravitino of the $\mathcal{N} = 1$ SUSY). The spontaneous SUSY breaking causes tadpoles of NS–NS fields, leading to instabilities of the geometry in some cases [83]. Deformations of geometry due to this instability (due to SUSY breaking) will lead to the spontaneous breaking of the discrete R symmetry.

The transformation properties of various fields under the rotation of extra dimensions have been already given in subsection 4.3.2. One can easily see that the R charge is properly assigned for all the particles in the GUT-symmetry-breaking sector when the R symmetry is identified[10] with the rotational symmetry of the third complex plane $z_3 \mapsto e^{i\alpha} z_3$.

The orbifold geometry breaks the SO(6) rotational symmetry down to its discrete subgroup. Since the third complex plane has angle-$\pi$ rotational symmetry (see Figure 4.1), the mod-4 R symmetry is relevant in the physics below the Kaluza–Klein scale. It is quite interesting that the mod-4 subgroup has completely independent characterization in section 2.2; anomaly cancellation based on low-energy matter contents implies that the mod-4 subgroup is the unique candidate that is a subgroup of a gauged symmetry.

It is discovered in [37] that the mod-4 R symmetry has a vanishing anomaly with the $\mathrm{SU}(2)_{\mathrm{H}}$ gauge group, and can be anomaly-free with $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge group, if there is an extra pair of $(\mathbf{5}, \mathbf{1})$ and $(\mathbf{5}^*, \mathbf{1})$ chiral multiplets of the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ gauge group. Therefore, the higher-dimensional construction suggests the existence of this $\mathrm{SU}(5)_{\mathrm{GUT}}$-charged vector-like pair at the TeV scale. The vector-like pair cannot have mass unless mod-4 R symmetry is broken. Notice, however, that the (mod-4 R)$[\mathrm{SU}(5)_{\mathrm{GUT}}]^2$ anomaly can be cancelled also by the generalized Green–Schwarz mechanism [84, 85].

The untwisted and twisted sectors of massless closed strings of the Type IIB string theory also provides Kaluza–Klein zero modes that survive the orbifold projection associated with $\mathbf{Z}_{12} \langle \sigma \rangle \times \mathbf{Z}_2 \langle \Omega R_{89} \rangle$. Some of their mass terms are forbidden by the R symmetry, and their

[10] The required R-charge assignment is properly obtained as long as $\alpha_1 = -\alpha_2$. We put $\alpha_1 = \alpha_2 = 0$ in the text just because of its simplicity. The unbroken subgroup discussed in the next paragraph is the mod 4 subgroup whenever $(\alpha_1 = -\alpha_2) \in \alpha_3 \mathbf{Z}$.

mass terms are allowed by inserting VEV's of the spontaneous breaking of the R symmetry. The masses will be of the order of the TeV scale. Since we do not specify the origin of the quarks and leptons, it is impossible to determine the R charges of those particles.

## 4.4 SU(5)$_{\mathrm{GUT}}\times$U(3)$_{\mathrm{H}}$ Model

### 4.4.1 D-brane Configuration and Orbifold Projection

Let us now describe how the GUT-symmetry-breaking sector of the SU(5)$_{\mathrm{GUT}}\times$U(3)$_{\mathrm{H}}$ model is derived. We adopt the same geometry, $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle\times\mathbf{Z}_2\langle\Omega R_{89}\rangle)$, as in the previous section. The difference between the construction of the two models is the D-brane configuration, and also the choice of unitary matrices $\tilde{\gamma}_{\sigma^k;7}$ and $\tilde{\gamma}_{\sigma^k;3}$ that appear in the orbifold projection conditions.

We put seven D7-branes at the two $\mathbf{Z}_{12}\langle\sigma\rangle$-fixed loci $z_3=\frac{\mathbf{e}_8+2\mathbf{e}_9}{3}$ and $z_3=-\frac{\mathbf{e}_8+2\mathbf{e}_9}{3}$. The $\mathbf{Z}_2\langle\Omega R_{89}\rangle$ only identifies the fields on both fixed loci, and the identified fields are subject only to the orbifold projection conditions of the $\mathbf{Z}_{12}\langle\sigma\rangle$. Projection conditions are written in the same way as in Eqs. (4.22), (4.23) and (4.21). The only difference is that the vector multiplet $(\Sigma)^k{}_l$ $(k,l=1,...,7)$ is of the U(7) gauge group rather than of the U(6). The 7 by 7 unitary matrix $\tilde{\gamma}_{\sigma;7}$ is now chosen as

$$(\tilde{\gamma}_{\sigma;7})^k{}_l=\mathrm{diag}\left(\overbrace{e^{-\frac{1}{12}\pi i},...,e^{-\frac{1}{12}\pi i}}^{5},e^{-\frac{9}{12}\pi i},e^{-\frac{11}{12}\pi i}\right), \tag{4.38}$$

instead of (4.26). The U(7) symmetry on the D7-branes is broken down to SU(5)$\times$U(1)$_5\times$ U(1)$_6\times$U(1)$_7$.

The Kaluza–Klein zero modes that survive the orbifold projection in Eq. (4.21) are as follows: ($\mathcal{N}=1$)-SUSY vector multiplets of U(5)$\times$U(1)$_6\times$U(1)$_7$, where the SU(5) subgroup of the U(5)$\simeq$ SU(5)$\times$U(1)$_5$ is identified with the SU(5)$_{\mathrm{GUT}}$ gauge group, and chiral multiplets, $(\Sigma_3)^6{}_i$ , $(\Sigma_2)^i{}_7$ and $(\Sigma_1)^7{}_6$, which transform $(\mathbf{5}^*)^{(-1,1,0)}$, $(\mathbf{5})^{(1,0,-1)}$ and $(\mathbf{1})^{(0,-1,1)}$ under the gauge group SU(5)$_{\mathrm{GUT}}\times$U(1)$_5\times$U(1)$_6\times$U(1)$_7$. The index "$i$" now runs from 1 to 5, not to 7. We identify $(\Sigma_3)^6{}_i$ with a Higgs multiplet $\bar{H}_i(\mathbf{5}^*)$ at the end of this section. Then, it would be natrual to identify $(\Sigma_2)^i{}_7$ with the other Higgs multiplet $H^i(\mathbf{5})$.

Three D3-branes are put on a fixed point on the D7-branes where the isotropy group[11] is $\mathbf{Z}_4\langle\sigma^3\rangle$. There is only one such candidate in the $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle\times\mathbf{Z}_2\langle\Omega R_{89}\rangle)$ geometry.

[11]The SU(5)$_{\mathrm{GUT}}\times$U(3)$_{\mathrm{H}}$ model cannot be constructed by putting the D3-branes at a fixed point where

These three D3-branes provide the $U(3)_H$ gauge group of the GUT-symmetry-breaking sector. Eighteen D3-branes (six images of three D3-branes) are necessary as a whole within the covering space $\mathbf{T}^6$, yet fourteen remaining D3-branes can be used for other sectors.

Fields that come from D3–D3 open strings or D3–D7 open strings are under the orbifold projection only by the isotropy group $\mathbf{Z}_4 \langle \sigma^3 \rangle$. The $\mathcal{N} = 2$ SUSY is preserved in the $\sigma^3$ projection. The U(6) symmetry is preserved in the $\sigma^3$ projection, while the U(7) symmetry not, as one can see from the Chan–Paton matrix (4.38).

Fields on the D3-branes transform under the $\mathbf{Z}_4 \langle \sigma^3 \rangle$ as

$$\sigma^3 : \quad (X_0)^\alpha{}_\beta \mapsto (\widetilde{X_0})^\alpha{}_\beta \equiv \quad (\tilde{\gamma}_{\sigma^3;3})^\alpha{}_{\alpha'} (X_0)^{\alpha'}{}_{\beta'} (\tilde{\gamma}^{-1}_{\sigma^3;3})^{\beta'}{}_\beta, \tag{4.39}$$

$$\sigma^3 : \quad (X_b)^\alpha{}_\beta \mapsto (\widetilde{X_b})^\alpha{}_\beta \equiv e^{2\pi i \tilde{v}_b 3} (\tilde{\gamma}_{\sigma^3;3})^\alpha{}_{\alpha'} (X_b)^{\alpha'}{}_{\beta'} (\tilde{\gamma}^{-1}_{\sigma^3;3})^{\beta'}{}_\beta, \tag{4.40}$$

$$\sigma^3 : \quad Q^\alpha{}_k \mapsto \widetilde{Q}^\alpha{}_k \equiv e^{\pi i (v_1+v_2)3} (\tilde{\gamma}_{\sigma^3;3})^\alpha{}_{\alpha'} Q^{\alpha'}{}_{k'} (\tilde{\gamma}^{-3}_{\sigma;7})^{k'}{}_k, \tag{4.41}$$

$$\sigma^3 : \quad \bar{Q}^k{}_\alpha \mapsto \widetilde{\bar{Q}}^k{}_\alpha \equiv e^{\pi i (v_1+v_2)3} (\tilde{\gamma}^3_{\sigma;7})^k{}_{k'} \bar{Q}^{k'}{}_{\alpha'} (\tilde{\gamma}^{-1}_{\sigma^3;3})^{\alpha'}{}_\alpha, \tag{4.42}$$

where the $(X_a)$'s form a U(3) vector multiplet of the four-dimensional $\mathcal{N} = 4$ SUSY and the $(\bar{Q}, Q)$ is a hypermultiplet of the four-dimensional $\mathcal{N} = 2$ SUSY in the $(\mathbf{7},\mathbf{3}^*)$ representation of the U(7)×U(3) gauge group. The 3 by 3 unitary matrix $\tilde{\gamma}_{\sigma^3;3}$ is chosen as

$$\tilde{\gamma}_{\sigma^3;3} = \mathrm{diag}(e^{\frac{3}{4}\pi i}, e^{\frac{3}{4}\pi i}, e^{\frac{3}{4}\pi i}), \tag{4.43}$$

so that the hypermultiplets $(\bar{Q}^k{}_\alpha, Q^\alpha{}_k)$ $(k = 1, ..., 6;\ \alpha = 1, 2, 3)$ survive the following orbifold projection conditions. The orbifold projection imposes Eq. (4.31). Projected out by these conditions are the $\mathcal{N} = 2$ hypermultiplet $(X_1, X_2)$ in the U(2)-**adj.** representation and $(\bar{Q}^7{}_\alpha, Q^\alpha{}_7)$. What is left is exactly the matter contents of the GUT-symmetry-breaking sector. It is very encouraging that the full multiplets for the GUT-symmetry-breaking sector are obtained with the $\mathcal{N} = 2$ SUSY structure.

We have obtained the $SU(5)_{GUT} \times U(1)_5 \times U(1)_6 \times U(1)_7$ vector multiplet and three chiral multiplets $S \equiv (\Sigma_1)^7{}_6$, $\bar{H}_i(\mathbf{5}^*) \equiv (\Sigma_3)^6{}_i$ and $H^i(\mathbf{5}) \equiv (\Sigma_2)^i{}_7$ from the D7-branes. The GUT-symmetry-breaking sector is exactly obtained on the D3-branes. Interactions determined by extended SUSY provide tree-level interactions of these fields. Some of them are written in

the isotropy group is $\mathbf{Z}_2 \langle \sigma^6 \rangle$. This is because thirty-six D3-branes (twelve images of three D3-branes) are necessary within the covering space $\mathbf{T}^6$ in this case. Thirty-six D3-branes are too many to cancel their 3-brane charges by negative charges of O3-planes.

the superpotential as:

$$W = \sqrt{2}g_{\rm H}\bar{Q}^{i}{}_{\alpha}(X_3)^{\alpha}{}_{\beta}Q^{\alpha}{}_{i} + \sqrt{2}g_{\rm H}\bar{Q}^{6}{}_{\alpha}(X_3)^{\alpha}{}_{\beta}Q^{\alpha}{}_{6} \tag{4.44}$$

$$+\sqrt{2}g_{\rm GUT}Q^{\alpha}{}_{6}(\Sigma_3)^{6}{}_{i}\bar{Q}^{i}{}_{\alpha} + \sqrt{2}g_{\rm GUT}(\Sigma_1)^{7}{}_{6}(\Sigma_3)^{6}{}_{i}(\Sigma_2)^{i}{}_{7}. \tag{4.45}$$

The first line is the $\mathcal{N}=2$ SUSY interaction in (2.29), whose natural explanation is one of the main purposes of our higher-dimensional construction. We identify the $(\Sigma_3)^6{}_i$ as one of the Higgs multiplets $\bar{H}_i(\mathbf{5}^*)$, because the first term in the second line gives the first term of the fourth line of the superpotential (2.29). Once the $(\Sigma_2)^i{}_7$ is identified with the other Higgs multiplet $H^i(\mathbf{5})$, then the last term implies that there exists a tri-linear term

$$W = \sqrt{2}g_{\rm GUT}S\bar{H}_iH^i. \tag{4.46}$$

### 4.4.2 Discrete R Symmetry

The matter contents obtained from D-branes are the whole GUT-symmetry-breaking sector and three chiral multiplets $(\Sigma_1)^7{}_6$, $(\Sigma_2)^i{}_7$ and $(\Sigma_3)^6{}_i$. We regard the rotational symmetry $z_b \mapsto e^{i\alpha_b}z_b$ of $\mathbf{C}^3$ with $\alpha_1=-\alpha_2=\alpha_3$ as the origin of the discrete R symmetry. Note that the mod-4 subgroup is preserved by the geometry, since it is generated by the rotation of three complex planes by an angle $\pi$. The Giudice–Masiero mechanism for the $\mu$-term works when the R symmetry at low energies is the mod-4 subgroup in the $\mathrm{SU}(5)_{\rm GUT}\times\mathrm{U}(3)_{\rm H}$ model. This is also quite remarkable. The zero modes have the following R charges under this rotation: 2 for $(X_3)^{\alpha}{}_{\beta}$, 0 for $\bar{Q}^k{}_{\alpha}$ and $Q^{\alpha}{}_k$, 2 for $(\Sigma_1)^7{}_6$, $-2$ for $(\Sigma_2)^i{}_7$ and 2 for $(\Sigma_3)^6{}_i$.

We consider that the mod-4 R symmetry is a suitable linear combination of the rotational symmetry and anomaly-free $\mathrm{U}(1)_{6+7}$ symmetry[12,13], so that the $(\Sigma_3)^6{}_i=\bar{H}_i(\mathbf{5}^*)$ has R charge 0. Then, it follows that $(\Sigma_2)^i{}_7=H^i(\mathbf{5})$ also has R charge 0. The R charges (mod 4) of all the zero modes are now obtained exactly as in Table 2.3, including those of the $\bar{Q}^6{}_{\alpha}$ and $Q^{\alpha}{}_6$. We also note here that the $\mathrm{SU}(5)_{\rm GUT}$-singlet $(\Sigma_1)^7{}_6$ has R charge 2.

The R symmetry given in the previous paragraph allows a superpotential

$$W = \lambda S^3 + m^2 S. \tag{4.47}$$

[12]The $\mathrm{U}(1)_{6-7}$ gauge symmetry is anomalous. The anomaly is cancelled through the generalized Green–Schwarz mechanism, and hence the $\mathrm{U}(1)_{6-7}$ symmetry is spontaneously broken.

[13]The mod-4 R symmetry can be a linear combination of the $\mathrm{U}(1)_{\rm H}$ and the $\mathrm{U}(1)_5$ symmetry in addition to the geometric rotation and the $\mathrm{U}(1)_{6+7}$. We do not exclude this possibility. The choice made in the text is just to simplify the description.

However, the order of magnitude of $m$ does not allow any expectation since it highly depends on UV the cut-off. Higher-dimensional construction so far on an orbifold geometry suggests interactions (4.46) and (4.47), which look like those of the NMSSM.

## 4.5 Consistency Conditions and U(1) Symmetries

### 4.5.1 Relevant Ramond–Ramond tadpole cancellation conditions

The main focus in preceding sections was in obtaining the correct spectrum. The consistency of the models was not discussed there. Ramond–Ramond tadpole cancellation condition should be satisfied in string vacua [80]. One has to specify the model completely in order to examine whether the condition is satisfied or not. One might be able to find a model that is consistent as a string vacuum, through determining the configuration of the remaining D-branes and orbifold projection conditions on them. However, there is almost no prospect that quarks and leptons are obtained from D-branes as long as we restrict ourselves in the models in preceding sections based on torus orientifolds[14]. Thus, the model is not consistent phenomenologically even before the consistency in the string theory is examined.

What can one learn from the construction in preceding sections? It is clear that they do not have meanings as global geometries, but the local geometry around the GUT-symmetry-breaking sector may be of importance. If the local piece is/can be found to be consistent as a string vacuum, then the local construction of the GUT models can be embedded in more generic Calabi–Yau manifolds in a suitable way so that the quarks and leptons are also obtained. Therefore, we only discuss the consistency condition that is relevant to the construction of the GUT-symmetry-breaking sector.

Ramond–Ramond tadpole cancellation condition of the Type IIB $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle\times\mathbf{Z}_2\langle\Omega R_{89}\rangle)$ orientifold is given in the following [82, 86]: the tadpole of the untwisted Ramond–Ramond field is cancelled when there are thirty-two D7-branes and thirty-two D3-branes in $\mathbf{T}^6$, i.e., $\mathrm{tr}(\gamma_{\sigma^0;7}) = 32$ and $\mathrm{tr}(\gamma_{\sigma^0;3}) = 32$ [87, 80]. Here, the Chan–Paton matrix $\gamma_{\sigma^k;7}$ denotes a 32 by 32 unitary matrix $(\gamma_{\sigma;D7})^k$ obtained from (4.24), and $\gamma_{\sigma^k;3}$ is $(\gamma_{\sigma;D3})^k$ obtained from (4.36). The Chan–Paton matrices are replaced by corresponding ones when the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$

[14] We constructed the models of the GUT-symmetry breaking in such a way that the D3–D7 system is separated from O7-planes. This is because we did not want extra massless particles to come into the spectrum. On the other hand, $\mathrm{SU}(5)_{\mathrm{GUT}}$-**10** representation of quarks and leptons is obtained when D-branes and orientifold planes coexist at least on a point. Thus, both ideas cannot be compatible with each other within the Type IIB torus orientifolds; D7-branes and O7-branes are parallel to one another, and hence they cannot be separated at one point and they co-exist at another point.

model is considered. For the Ramond–Ramond fields in the $\sigma^k$-twisted sectors with $k = 3, 6, 9$, tadpoles are cancelled when

$$\mathrm{tr}(\gamma_{\sigma^k;7}) + 4\,\mathrm{tr}_{\,\mathbf{y}'\in\mathrm{Eq.}(B.5)}(\gamma_{\sigma^k;3}) = 0 \quad \text{for} \quad k = 6, \tag{4.48}$$

$$\mathrm{tr}(\gamma_{\sigma^k;7}) + 2\,\mathrm{tr}_{\,\mathbf{y}'\in\mathrm{Eq.}(B.6)}(\gamma_{\sigma^k;3}) = 0 \quad \text{for} \quad k = 3, 9. \tag{4.49}$$

Here, Eq. (4.48) and Eq. (4.49) should be satisfied for each $\sigma^k$-fixed singularity[15] (i.e., for each $\mathbf{y}'$ in Eq. (B.5) and Eq. (B.6)). All the D7-branes in $\mathbf{T}^6$ contribute to these conditions, whereas only D3-branes that are on one of $\sigma^k$-fixed singularities ($k = 3, 6, 9$) contribute to the corresponding conditions. D7-branes and D3-branes contribute to these conditions no matter what their $z_3$ coordinates are. This is because the $\sigma^k$-twisted sector ($k = 3, 6, 9$) propagates in the third complex plane, and hence it couples to all these D-branes. For the tadpole cancellation of the other twisted sectors ($\sigma^k$-twisted sectors with $k \neq 0, 3, 6, 9,$),

$$\mathrm{tr}(\tilde{\gamma}_{\sigma^k;7} \oplus \tilde{\gamma}^{-1}_{\sigma^k;7}) - \quad \mathrm{tr}_{\,\mathbf{y}'\in\mathrm{Eq.}(B.8)}(\tilde{\gamma}_{\sigma^k;3} \oplus \tilde{\gamma}^{-1}_{\sigma^k;3}) = 0 \quad \text{for} \quad k = 1, 2, 5, 7, 10, 11, \tag{4.50}$$

$$\mathrm{tr}(\tilde{\gamma}_{\sigma^k;7} \oplus \tilde{\gamma}^{-1}_{\sigma^k;7}) + 3\,\mathrm{tr}_{\,\mathbf{y}'\in\mathrm{Eq.}(B.7)}(\tilde{\gamma}_{\sigma^k;3} \oplus \tilde{\gamma}^{-1}_{\sigma^k;3}) = 0 \quad \text{for} \quad k = 4, 8 \tag{4.51}$$

are required. The above conditions should be satisfied for each twisted sector, i.e., for each $\sigma^k$-fixed points. D7-branes at $z_3 = \pm(\mathbf{e}_8 + 2\mathbf{e}_9)/3$ and D3-branes at the corresponding singularities with $z_3 = \pm(\mathbf{e}_8 + 2\mathbf{e}_9)/3$ contribute to the above conditions. Although there are extra conditions for the $\sigma^k$-twisted sector ($k \neq 0, 3, 6, 9$) at fixed points with $z_3 = 0$, we do not list them here, since they are totally irrelevant to the construction in previous sections.

Now it would be clear that the tadpole cancellation conditions for $\sigma^k$-twisted sectors with $k = 3, 6, 9$ are relevant to the construction of the GUT-symmetry-breaking sector, and others are irrelevant; the sector is *on* these singularities, while other singularities do not play crucial roles for the sector. Since D7- and D3-branes that are *not* on $z_3 = \pm(\mathbf{e}_8 + 2\mathbf{e}_9)/3$ also contribute to those conditions, they can be satisfied when one chooses the Chan–Paton matrix for those D-branes. Thus, the D-brane construction of the GUT-symmetry-breaking sector in preceding sections can be consistent as a string vacuum.

### 4.5.2 Anomaly cancellation, U(1) symmetries and related issues

Gauge theories on orbifold geometry generically have anomalies localized on orbifold singularities. They have to be cancelled at each point. The appendix B.2 summarizes the method

[15] The coefficients 4 and 2 in Eq. (4.48) and Eq. (4.49) can be interpreted as square roots of the number of the corresponding $\sigma^k$-fixed singularities 16 and 4, respectively. The coefficient 3 in Eq. (4.51) is a square root of the number of $\sigma^4$-fixed points on the D7-branes.

for calculating the anomaly distribution. It can be shown through explicit calculation [70] that all the triangle anomalies vanish at all the fixed points in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model of section 4.3, and all the triangle anomalies except for those involving $\mathrm{U}(1)_{6-7}$ symmetry are cancelled at all the fixed points in the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model in section 4.4. Box anomalies that arise on $\sigma^k$ ($k = 3, 6, 9$) cannot be calculated without specifying the Chan–Paton matrices for D7-branes and for D3-branes that are not on the $z_3 = \pm(\mathbf{e}_8 + 2\mathbf{e}_9)/3$ loci. (These models are not phenomenologically satisfactory, though, since quarks and leptons are absent.)

On the other hand, we do not have to check that the mod-4 R symmetry has vanishing anomaly at each orbifold singularity; it is only the total sum of the anomalies over the geometry that has to be cancelled. This is because the angle-$\pi$ rotation, in which we are interested, is a rigid rotation of the whole orbifold. We are not interested in a space rotation with the angle varying point by point.

It is known that there are close relations between the anomaly cancellation at orbifold singularities and the Ramond–Ramond tadpole cancellations for twisted sectors. The anomaly distribution can be attributed to components, each of which corresponds to an element of the orbifold group, as explained briefly in the appendix B.2. Each component is localized on the orbifold singularities of the corresponding element. Requiring the anomaly cancellation at all the orbifold singularities is generically equivalent to requiring each component to vanish. The Ramond–Ramond tadpole cancellation condition also has the same properties; the twisted sectors are localized on corresponding orbifold singularities, and the condition requires that certain quantities should vanish for each element of the orbifold group.

Indeed, it is shown that the Ramond–Ramond tadpole cancellation guarantees that non-Abelian irreducible anomalies on ten-dimensional space-time, on six-dimensional and on four-dimensional orbifold singularities vanish in [87], [88] and [89], respectively. The results on irreducible-box-anomaly cancellation in models compactified only to six-dimensions can be applicable to models compactified down to four dimensions; this is obvious from the group structure of both cancellation conditions and also from the fact that both conditions are local in some sense.

The reducible part of the box anomalies are cancelled by the generalized Green–Schwarz mechanism [84, 85] carried by Ramond–Ramond fields in the $\sigma^k$-twisted sector ($k = 3, 6, 9$) [90, 88]. A number of U(1) gauge symmetries become massive through bilinear coupling with the twisted sector fields [90, 81]. One the other hand, the $\mathrm{U}(1)_{\mathrm{H}}$ symmetry should not acquire mass term before the spontaneous breakdown due to the superpotential (2.22) or (2.29). Under the condition that this is true, there are two consequences. One is that the

$\mathrm{U}(1)_\mathrm{H}$ should be a linear combination of the U(1) part of $N(=2,3)$ D3-branes and another U(1) symmetry, which would arises from other D3-branes in the same $\mathbf{Z}_2$ $\langle\sigma^6\rangle$-fixed or $\mathbf{Z}_4$ $\langle\sigma^3\rangle$-fixed singularity in the $\mathrm{SU}(5)_\mathrm{GUT}\times\mathrm{U}(N)_\mathrm{H}$ model ($N=2,3$), respectively. Thus, the $\mathrm{U}(N)_\mathrm{H}$ gauge group is rather $\mathrm{SU}(N)_\mathrm{H}\times\mathrm{U}(1)_\mathrm{H}$, and in particular, the $\mathrm{U}(N)_\mathrm{H}$ relations such as Eq. (3.3) are no longer expected. The other consequence is that the auxiliary fields of the ($\mathcal{N}=2$)-SUSY vector multiplet of such $\mathrm{U}(1)_\mathrm{H}$ do not have coupling with the NS–NS fields in the relevant twisted sector [81]. Thus, the Fayet–Iliopoulos parameter in the third line of (2.22) and (2.29) are not obtained from the VEV of the NS–NS fields in the $\mathcal{N}=2$ SUSY limit.

## 4.6 Discussion

The GUT-symmetry-breaking sector was properly obtained from D-branes in the Type IIB torus orientifolds, and we could also find the discrete rotational symmetry that properly reproduces the R symmetry of the phenomenological models. On the other hand, quarks and leptons were not obtained in the models in section 4.3 and 4.4, and hence they are not satisfactory phenomenologically. Possible ways to construct a satisfactory model is briefly discussed in this section.

Two important ingredients of the models in section 4.3 and 4.4 are (i) the $\mathcal{N}=2$ SUSY of the D3–D7 system, and (ii) the U(6) symmetry. enhanced at the GUT-symmetry-breaking sector. The simplest extension is to embed the D3–D7 system in generic Calabi–Yau three-folds. D7-branes are wrapped on a holomorphic four-cycle so that $\mathcal{N}=1$ SUSY is preserved [91]. The local geometry of the D3-branes should be chosen the same as those in section 4.3 and 4.4, i.e., $\mathbf{C}^2/\mathbf{Z}_n\times\mathbf{C}$. The gauge bundle on the D7-branes is chosen so that the U(6) symmetry is enhanced on the D3-branes. The absence of ($\mathcal{N}=1$)-SUSY chiral multiplet in the $\mathrm{SU}(5)_\mathrm{GUT}$-**adj.** representation is translated in terms of topological nature of the D7-branes world volume [92]. (The D3–D7 system can be replaced by D5–D9 system, when the world volume of the D5-branes should be a torus.) Two consequences discussed at the end of subsection 4.5.2, which have some relevance in phenomenology, still hold in this case. The discrete R symmetry is realized just as described in section 4.3 and 4.4, at least locally around the GUT-symmetry-breaking sector. But there is no guarantee that the R symmetry can be extended to the whole geometry of the Calabi–Yau manifolds.

The other extension is an intersecting D$p$–D$p$ system with $p=5,6,7$ [93] embedded in a Calabi–Yau three fold instead of the D3–D7 system. D$p$-branes with $p=5,7$ should be wrapped on holomorphic cycles, and D6-branes should be on a special Lagrangian [91]. The

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4, x_5$ | $x_6, x_7$ | $x_8, x_9$ |
|---|---|---|---|---|---|---|---|
| D5 | o | o | o | o | $x_5 = 0$ | $x_7 = 0$ | $\times, \times$ |
| D5 | o | o | o | o | $x_4 \sin\theta + x_5 \cos\theta = 0$ | $-x_4 \sin\theta + x_5 \cos\theta = 0$ | $\times, \times$ |
| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4, x_5$ | $x_6, x_7$ | $x_8, x_9$ |
| D6 | o | o | o | o | $x_5 = 0$ | $x_7 = 0$ | o,$\times$ |
| D6 | o | o | o | o | $x_4 \sin\theta + x_5 \cos\theta = 0$ | $-x_4 \sin\theta + x_5 \cos\theta = 0$ | o,$\times$ |
| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4, x_5$ | $x_6, x_7$ | $x_8, x_9$ |
| D7 | o | o | o | o | $x_5 = 0$ | $x_7 = 0$ | o,o |
| D7 | o | o | o | o | $x_4 \sin\theta + x_5 \cos\theta = 0$ | $-x_4 \sin\theta + x_5 \cos\theta = 0$ | o,o |

Table 4.2: Intersecting D$p$–D$p$ systems that preserve $\mathcal{N} = 2$ SUSY. The angle $\theta$ of intersection is arbitrary. "o" denotes that the D-branes are stretched in that direction and "$\times$" denotes that the D-branes are not stretched in that direction.

intersection of these D-branes should look locally like those in Table 4.2 so that the $\mathcal{N} = 2$ SUSY is preserved there. The U(6) symmetry has to be enhanced at the intersection on the D-branes for the $\mathrm{SU}(5)_{\mathrm{GUT}}$ gauge group, which is the same as in the D3–D7 system However, the numbers of *both* ($\mathcal{N} = 1$)-SUSY chiral multiplets in the $\mathrm{SU}(5)_{\mathrm{GUT}}$-**adj.** and in the $\mathrm{U}(N)_{\mathrm{H}}$-**adj.** representations are translated in terms of topological numbers of the both D-branes world volume [92]. One also has to require that the volume of two $(p-3)$-cycles on which each stack of D$p$-branes wrap are fairly different; one for the $\mathrm{U}(N)_{\mathrm{H}}$ gauge group has volume of the order one, while that for the $\mathrm{SU}(5)_{\mathrm{GUT}}$ should have moderately large volume. Two consequences discussed at the end of subsection 4.5.2 no longer hold in this case. This is because the mechanism to eliminate unwanted matter multiplets in the $\mathrm{U}(N)_{\mathrm{H}}$-**adj.** representations are completely different between the two cases. The argument on the discrete R symmetry is no longer valid, either.

The Higgs particles $\bar{H}_i(\mathbf{5}^*)$ and $H^i(\mathbf{5})$ were obtained in the $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(3)_{\mathrm{H}}$ model in section 4.4, where the construction started with the U(7) gauge symmetry and an $\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(3)_{\mathrm{H}}$-singlet chiral multiplet is obtained simultaneously. This singlet may have interactions that look like those of the NMSSM, provided the mod-4 R symmetry controls the model. Although the model with the U(7) symmetry at short distance may not be the only way to obtain the Higgs multiplets, it is fairy good since an order one coupling for the missing-partner mass terms are expected because of non-vanishing wave functions of the Higgs multiplets at the GUT-symmetry-breaking sector.

# Chapter 5

# Conclusions and Outlook

We started our discussion on SUSY GUT's in this thesis with a question why the two Higgs doublets are so light. It seems that a parameter smaller than $10^{-14}$ is necessary. It is a conventional wisdom that there must be some symmetry when there is such an exceedingly small parameter. It is true that considering a symmetry might just be a translation of a problem into another terminology, yet it *is* of importance to translate into a problem that might be easy to understand. Moreover, there is strong and independent support from experiments for the symmetry that forbids the mass term of the Higgs doublets. Experimental bound from Super-Kamiokande implies that the proton decay through dimension-five operators are suppressed at least by one order of magnitude compared with the amplitude naturally expected. Actually the symmetry mentioned above also forbids these operators. (Here, unification of quarks and leptons is assumed implicitly.) Thus, we consider that SUSY GUT's with this symmetry is quite promising, and hence, only such models were discussed in this thesis.

We did not discuss the following two types of SUSY-GUT models. First, the $\mu$ parameter, the coefficient of the bilinear of two Higgs doublets, should be of the order of the electroweak scale $\sim 10^{2-3}$ GeV for a phenomenological reason. Then, it is natural to consider that this $\mu$ parameter has a good reason to be of the order of the SUSY-breaking scale. Therefore, we did not discuss seriously the models of SUSY GUT's where the $\mu$ parameter is given by $M_{\rm pl} \times (\text{GUT scale}/M_{\rm pl})^5$ or so. There, the $\mu$ parameter only happened to be of the order of the electroweak scale.

Secondly, there have been proposed other models of SUSY GUT's in which the above symmetry is partially broken, i.e., when a continuous U(1) symmetry is broken exclusively by vacuum expectation values of fields that are charged positive, it happens that the Higgs bilinear is not allowed by the symmetry, while the dimension-five operators are allowed with a suppression factor (GUT scale/$M_{\rm pl}$)$^{\rm a\ few}$. Since the Higgs bilinear should be suppressed by $10^{-14}$, while the latter only by $10^{-1}$ or so, such an idea is still viable. Now that the Super-Kamiokande experiment (22.5 kt fiducial volume) is already sensitive to the dimension-five operators that is $10^{-1}$-suppressed from the natural order of magnitude, future project with fiducial volume 500 kt would be sensitive to operators that are further suppressed by 0.2. It is not surprising that the dimension-five proton decay is observed in the future in this type of models, though further phenomenological study is necessary before definite conclusion is drawn on the detectability. This type of models *proposed so far* also predict fast proton decay through dimension-six operators; the predicted life-time is around the current experimental limit from Super-Kamiokande. However, it is not clear whether this is an inevitable consequence of this type of models.

Now let us go back to the main stream of this thesis. The models discussed in this thesis have a symmetry that suppresses the mass term of the two Higgs doublets and also the dimension-five proton-decay operators. Thus, the dimension-five proton decay is not expected in the models at all.

The existence of the unbroken symmetry puts stringent constraint on the possible structure of the mass matrix of Higgs multiplets. We arrived at three possibilities. One is to go to higher-dimensional field theories, where infinite number of Kaluza–Klein particles allow a structure of mass matrix that is impossible with a finite number of four-dimensional particles. The second possibility is to introduce extra gauge group(s) into the "unified gauge group" that is(are) different from SU(5). This makes it possible to solve the doublet–triplet splitting problem with minimal number of four-dimensional particles. The final possibility is to introduce an extra SU(5) gauge group and assume that one of Higgs doublets is charged under the different SU(5) rather than the original SU(5). This assumption changes structure of the mass matrix considerably, and the doublet–triplet splitting problem is solved within four-dimensional field theories. Two classes of models were explained in detail in section 2.2 and 2.3 for the last two possibilities. It is interesting that both classes of models in four-dimensional field theories require product gauge group.

We referred to the models based on the second possibility as "product-group unification model I." There are two models. One of them is based on an $\mathrm{SU}(5)_{\rm GUT}\times \mathrm{U}(2)_{\rm H}$ gauge group, and the other on an $\mathrm{SU}(5)_{\rm GUT}\times \mathrm{U}(3)_{\rm H}$ gauge group. Both models have an unbroken R symmetry. There is a discrete subgroup of this R symmetry that can be a subgroup of a gauged symmetry. The anomaly is cancelled only by introducing certain amount of extra particles around the electroweak scale. It is interesting because such particles might be discovered at energy-frontier experiments in the near future. It was also shown based only on the low energy matter contents that the non-R symmetry could never be a subgroup of a gauged symmetry within the second possibility. Thus, it is also quite interesting that the symmetry compatible with the models happened to be R symmetries; those symmetries are exact when they really originate from gauged symmetries.

We referred to the models based on the final possibility as "product-group unification model II." The model explained is based on $SU(5)_1 \times SU(5)_2$ gauge group. The anomaly cancellation of these two factors predicts extra particles (possibly at low energy). This model has $\mathbf{Z}_N$ symmetry, which can also be a subgroup of a gauged symmetry, if it is a non-R symmetry. It might be difficult to construct a fully realistic model in which the $\mu$ parameter and supersymmetric masses of the extra particles are obtained with a suitable order of magnitude in this type of model. The particle spectrum around the GUT scale suggests that the proton decay through dimension-six operators tends to be slow. The life-time varies as the parameters of the model are varied. It is not bounded from above, but it would be bounded from below. The lower bound is not expected to be extremely low, but detailed calculation is further necessary to derive a definite prediction.

We analyzed the proton-decay amplitude of the product-group unification model I in chapter 3. Rough estimate of the life-time is $\tau(p \to e^+\pi^0) \sim (6 \times 10^{33} - 5 \times 10^{34})$ yrs., within a typical parameter space of the mSUGRA SUSY-breaking parameters. This estimate is beyond the current experimental bound from Super-Kamiokande ($\tau(p \to e^+\pi^0) \simeq 4.4 \times 10^{33}$ yrs., 90% C.L.), but is within the $3\sigma$-discovery limit ($\tau(p \to e^+\pi^0) \lesssim 3 \times 10^{34}$ yrs.) of next generation water Cerenkov detectors (based on ten-years running with the 500 kt fiducial volume) for most part of the surviving parameter region.

Uncertainties in this estimate come both within and outside the GUT models. The uncertainties outside the models arise first from the experimental value of the QCD coupling constant. $1\sigma$-deviation changes the life-time by $\times(0.4 - 2.)$. They also arise from the systematic error in the hadron form factor calculated by JLQCD collaboration, which might affect the life-time by almost the same amount as above. The life-time is shortened by a factor $\times 0.6$ when there is an extra pair of SU(5)-($\mathbf{5}+\mathbf{5}^*$), since the unified coupling constant becomes larger in the presence of extra matter particles.

There are a couple of sources of uncertainties within the GUT models. Typical estimate of the uncertainties for both the $SU(5)_{GUT} \times U(2)_H$ model and the $SU(5)_{GUT} \times U(3)_H$ model are $\times(0.2 - 5.)$ and $\times(0.3 - 3.)$, respectively, or more larger. Thus, the uncertainties in the estimate are dominated by those within the GUT models.

We further derived the upper bound of the proton life-time by adopting another way of analysis. It is a crucial information in distinguishing these models from others. Only three parameters are fixed by the experimental values of the MSSM gauge coupling constants, and the rest of the parameters are left free. Perturbation analysis puts limits on the space of free parameters. We required that all the coupling constants in the models should stay finite at least below the heaviest particle of the model. All the parameter region was exploited for both the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model and the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model. The upper bound of the parameter $M_G$, the GUT-gauge-boson mass, is obtained in this way.

We concluded that the proton life-time is upper-bounded by $\tau(p \rightarrow e^+\pi^0) \lesssim (1.-4.)\times 10^{33}$ yrs. for the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(2)_{\mathrm{H}}$ model. The QCD coupling constant $\alpha_s^{\overline{\mathrm{MS}},(5)}(M_Z) = 0.1212$ is used, which is a value stronger than the centre value by $2\sigma$. SUSY-breaking-parameter affects the upper bound of the life-time almost only through the energy scale $M_{2-3}$ where the coupling constants of the $\mathrm{SU}(3)_{\mathrm{C}}$ and the $\mathrm{SU}(2)_{\mathrm{L}}$ become the same. Dependence on the experimental value of the QCD coupling constant, the lattice form factor remain the same. Extra particles $\mathbf{5}+\mathbf{5}^*$ at the electroweak scale enhances the unification coupling and the life-time becomes shorter by factor $\times 0.6$, and moreover, threshold correction from these particles would make the $\mathrm{SU}(3)_{\mathrm{C}}$ coupling weak. Thus, the upper bound of the life-time is further shortened.

The proton life-time is bounded from above by $\tau(p \rightarrow e^+\pi^0) \lesssim (1.-5.)\times 10^{35}$ yrs. for the $\mathrm{SU}(5)_{\mathrm{GUT}}\times\mathrm{U}(3)_{\mathrm{H}}$ model. The QCD coupling constant $\alpha_s^{\overline{\mathrm{MS}},(5)}(M_Z) = 0.1132$ is used, which is a value weaker than the centre value by $2\sigma$. SUSY-breaking-parameter dependence seems to arise almost only through the energy scale $M_{1-2}$ where the gauge coupling constants of the $\mathrm{SU}(2)_{\mathrm{L}}$ and (SU(5)-normalized) $\mathrm{U}(1)_{\mathrm{Y}}$ become the same. The upper bound does not depend on the experimental value of the QCD coupling, which is also understood because the upper bound is rather related to $M_{1-2}$ rather than to $M_{2-3}$ in this model. Dependence on the lattice form factor remains the same.

In the course of this analysis, we found the the approximate $\mathcal{N}=2$ SUSY in the GUT-symmetry-breaking sector of both models plays an important role in preserving the approximate SU(5) unification of the MSSM gauge coupling constants. Particle contents of the GUT-symmetry-breaking sector of both models can be regarded as multiplets of $\mathcal{N}=2$ SUSY, and the superpotential that is directly relevant to the GUT-symmetry breaking is just the form expected in the ($\mathcal{N}=2$)-SUSY gauge theories. As one of consequences, the renormalization group evolution of the coupling constants of the models have IR-fixed relation, which is almost the same as the ($\mathcal{N}=2$)-SUSY relation (slightly perturbed by interactions that do not respect the $\mathcal{N} = 2$ SUSY). Thus, coupling constants of the model flow almost into the ($\mathcal{N} = 2$)-SUSY relation as the renormalization point goes from the cut-off scale down to the GUT scale. As the second consequence, a large threshold correction from the $\mathrm{SU}(2)_{\mathrm{H}}$ or $\mathrm{SU}(3)_{\mathrm{H}}$ vector multiplet is cancelled by their ($\mathcal{N} = 2$)-SUSY partner, and hence these models have a mechanism that cancels the threshold corrections within themselves.

Although the IR-fixed relation ($\simeq$ ($\mathcal{N} = 2$)-SUSY relation) does not have to be satisfied strictly at the cut-off scale of the models, large deviation from the relation would invalidate the approximate $\mathrm{SU}(5)_{\mathrm{GUT}}$ unification through the two-loop threshold corrections (see page 46). Therefore, it is favourable that the $\mathcal{N} = 2$ SUSY is not only respected by the matter contents and by form of interactions but is also satisfied (to some extent) by the values of coupling constants.

The product-group unification model I constructed in four-dimensional space-time has a number of interesting features. Models use product group as a "unified gauge group" with relatively strong gauge coupling constants for the extra gauge groups. The $\mathcal{N} = 2$ SUSY is necessary to maintain the strong coupling, and the structures of the $\mathrm{SU}(5)_{\mathrm{GUT}}$-breaking sectors of these models accommodate the $\mathcal{N} = 2$ SUSY. The cut-off scale of the models should lie somewhat lower than the Planck scale. Finally, the symmetry principle of these models, namely, the mod-4 R symmetry, can be a discrete subgroup of a gauged symmetry, and this fact sheds some light on the required $10^{-14}$ precision necessary in keeping Higgs-doublets' mass at the TeV scale.

All these features, some of them are conventionally considered unfavourable and some of them are mysterious, can be naturally explained when these models are embedded into an extra-dimensional space with extended SUSY, where the $\mathrm{SU}(5)_{\mathrm{GUT}}$-breaking sector is expected to be localized on a point in the extra dimensions, i.e., the brane-world picture fits very well.

It is better to go to string theories in firmly formulating the speculative idea of the brane-world. D3–D7 system in Type IIB string theory will be the most natural candidate that accomodates the brane-world picture suggested from phenomenology. We put the D3–D7 system in $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle \times \mathbf{Z}_2\langle\Omega R_{89}\rangle)$ orientifold, and obtained spectrum of the GUT-symmetry-breaking sector completely. The discrete rotational symmetry of the extra-dimensional geometry was identified with the discrete R symmetry in the models. Charge assignment of the symmetry was reproduced for particles obtained from D-branes. However, quarks and leptons were not obtained from the D-branes. This is too difficult a task to obtain all the particles as long as we restrict ourselves in orbifolds of a torus. Two possible way to extend the construction in the Type IIB orientifolds were briefly mentioned.

It is also an important subject to understand very well higher-dimensional field theory so that one can derive predictions on the proton decay amplitude in GUT models based on higher-dimensional space-time. We listed three possibilities of SUSY GUT models that have a suitable symmetry in chapter 2. The first possibility uses Hosotani mechanism, and hence essentially higher-dimensional calculation is crucial in the proton decay amplitude. The second possibility we mainly pursued in this thesis also suggests existence of higher dimensions. The analysis in chapter 3 is only the result for models on four-dimensional space-time. The prediction might be changed in the models on higher-dimensional space-time. Thus, it is interesting and is important to consider how much the proton-decay amplitude changes between the four-dimensional models and higher-dimensional models. This is still an open question. The last possibility also has natural higher-dimensional extension [30]. The proton decay in this model is discussed recently [94].

# Acknowledgments

First of all, the author thanks his collaborators for allowing to use the results of the collaborations as materials of his Ph.D thesis. Chapter 3 is based on Refs. [46] and [47] that were done in collaboration with Masaaki Fujii and Masahiro Ibe, respectively. Chapter 4 is on Refs. [68] with Yosuke Imamura and Tsutomu Yanagida, and on [69, 70] with Tsutomu Yanagida. He thanks to them also because he learned many things from them during the collaborations. Discussion with Mirjam Cvetic (Penn. U.) helped him to improve section 4.6. He also thanks the theory division of CERN for hospitality, where earlier part of this thesis was written.

Akishi Kato (Dept. Math, Univ. Tokyo) held a seminar for students on the Seiberg–Witten theory, when the author was a master-course student. This seminar led him to have an interest in the $\mathcal{N} = 2$ SUSY gauge theories, and this interest eventually played an important role in this thesis. Discussion with Yosuke Imamura and lectures by Yutaka Matsuo gave him elementary and useful knowledge in string theories. It helped him very much.

The author learned a lot during his collaboration with Masahiro Kawasaki, Koshiro Suzuki and Ken-Iti Izawa, and is grateful to them, although results of the collaborations with them are not included in this thesis. He also thanks Arthur Hebecker (DESY), Yasunori Nomura (FNAL) and former and current members of the High-Energy-Physics Theory Group at University of Tokyo, especially Koichi Hamaguchi, Masashi Hamanaka, Kent Ichikawa, Kazuhiro Sakai and Hiromitsu Takayanagi for fruitful discussion. Yasuaki Hikida and Koshiro Suzuki helped him very much with their knowledge in computer system and programming.

Finally, the author expresses his cordial gratitude to his advisor Prof. Tsutomu Yanagida for educating, advising and encouraging him for five years. He owes his advisor for most part of what he has in physics.

# Appendix A

# Appendix to Chapter 3

## A.1 Renormalization Group Equations

In this section, renormalization group of coupling constants of the models are summarized.

$\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(2)_{\mathrm{H}}$ **model**

$$
\begin{aligned}
\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha_{2\mathrm{H}}}(\mu)\right) &= \frac{-2}{2\pi} \qquad\qquad (\mathrm{one-loop}) \\
&- \frac{2}{2\pi}\frac{(6\alpha_{2\mathrm{H}} - 5\alpha_{2\mathrm{H}}^{\lambda} - \alpha_{2\mathrm{H}}^{\lambda'})}{2\pi} \\
&- \frac{1}{2\pi}\frac{6(3\alpha_{2\mathrm{H}} + \alpha_{1\mathrm{H}}) - 5(3\alpha_{2\mathrm{H}}^{\lambda} + \alpha_{1\mathrm{H}}^{\lambda}) - (3\alpha_{2\mathrm{H}}^{\lambda'} + \alpha_{1\mathrm{H}}^{\lambda'})}{4\pi} \\
&- \frac{5}{2\pi}\left(\frac{\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}}\right).
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha_{2\mathrm{H}}^{\lambda}}(\mu)\right) &= \frac{-2}{2\pi}\left(\frac{\alpha_{2\mathrm{H}}}{\alpha_{2\mathrm{H}}^{\lambda}}\right) + \frac{1}{\alpha_{2\mathrm{H}}^{\lambda}}\frac{(6\alpha_{2\mathrm{H}} - 5\alpha_{2\mathrm{H}}^{\lambda} - \alpha_{2\mathrm{H}}^{\lambda'})}{2\pi} \\
&+ \frac{2}{\alpha_{2\mathrm{H}}^{\lambda}}\frac{(3\alpha_{2\mathrm{H}} + \alpha_{1\mathrm{H}}) - (3\alpha_{2\mathrm{H}}^{\lambda} + \alpha_{1\mathrm{H}}^{\lambda})}{4\pi} \\
&+ \frac{2}{\alpha_{2\mathrm{H}}^{\lambda}}\left(\frac{\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}}\right).
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha_{2\mathrm{H}}^{\lambda'}}(\mu)\right) &= \frac{-2}{2\pi}\left(\frac{\alpha_{2\mathrm{H}}}{\alpha_{2\mathrm{H}}^{\lambda'}}\right) + \frac{1}{\alpha_{2\mathrm{H}}^{\lambda'}}\frac{(6\alpha_{2\mathrm{H}} - 5\alpha_{2\mathrm{H}}^{\lambda} - \alpha_{2\mathrm{H}}^{\lambda'})}{2\pi} \\
&+ \frac{2}{\alpha_{2\mathrm{H}}^{\lambda'}}\frac{(3\alpha_{2\mathrm{H}} + \alpha_{1\mathrm{H}}) - (3\alpha_{2\mathrm{H}}^{\lambda'} + \alpha_{1\mathrm{H}}^{\lambda'})}{4\pi}.
\end{aligned}
\tag{A.3}
$$

$$\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha_{1\mathrm{H}}}(\mu)\right) = \frac{-6}{2\pi} \qquad (\mathrm{one-loop}) \tag{A.4}$$
$$- \frac{1}{2\pi}\frac{6(3\alpha_{2\mathrm{H}}+\alpha_{1\mathrm{H}}) - 5(3\alpha^{\lambda}_{2\mathrm{H}}+\alpha^{\lambda}_{1\mathrm{H}}) - (3\alpha^{\lambda'}_{2\mathrm{H}}+\alpha^{\lambda'}_{1\mathrm{H}})}{4\pi}$$
$$-\frac{5}{2\pi}\left(\frac{\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}}\right).$$

$$\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha^{\lambda}_{1\mathrm{H}}}(\mu)\right) = \frac{-6}{2\pi}\left(\frac{\alpha_{1\mathrm{H}}}{\alpha^{\lambda}_{1\mathrm{H}}}\right) + \frac{1}{\alpha^{\lambda}_{1\mathrm{H}}}\frac{(6\alpha_{1\mathrm{H}} - 5\alpha^{\lambda}_{1\mathrm{H}} - \alpha^{\lambda'}_{1\mathrm{H}})}{2\pi} \tag{A.5}$$
$$+ \frac{2}{\alpha^{\lambda}_{1\mathrm{H}}}\frac{(3\alpha_{2\mathrm{H}}+\alpha_{1\mathrm{H}}) - (3\alpha^{\lambda}_{2\mathrm{H}}+\alpha^{\lambda}_{1\mathrm{H}})}{4\pi}$$
$$+\frac{2}{\alpha^{\lambda}_{1\mathrm{H}}}\left(\frac{\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}}\right).$$

$$\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha^{\lambda'}_{1\mathrm{H}}}(\mu)\right) = \frac{-6}{2\pi}\left(\frac{\alpha_{1\mathrm{H}}}{\alpha^{\lambda'}_{1\mathrm{H}}}\right) + \frac{1}{\alpha^{\lambda'}_{1\mathrm{H}}}\frac{(6\alpha_{1\mathrm{H}} - 5\alpha^{\lambda}_{1\mathrm{H}} - \alpha^{\lambda'}_{1\mathrm{H}})}{2\pi} \tag{A.6}$$
$$+ \frac{2}{\alpha^{\lambda'}_{1\mathrm{H}}}\frac{(3\alpha_{2\mathrm{H}}+\alpha_{1\mathrm{H}}) - (3\alpha^{\lambda'}_{2\mathrm{H}}+\alpha^{\lambda'}_{1\mathrm{H}})}{4\pi}.$$

$\mathrm{SU}(5)_{\mathrm{GUT}} \times \mathrm{U}(3)_{\mathrm{H}}$ **model**

$$\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha_{3\mathrm{H}}}(\mu)\right) = 0 \qquad (\mathrm{one-loop}) \tag{A.7}$$
$$- \frac{3}{2\pi}\frac{(6\alpha_{3\mathrm{H}} - 5\alpha^{\lambda}_{3\mathrm{H}} - \alpha^{\lambda'}_{3\mathrm{H}})}{2\pi}$$
$$- \frac{1}{2\pi}\frac{6(8\alpha_{3\mathrm{H}}+\alpha_{1\mathrm{H}}) - 5(8\alpha^{\lambda}_{3\mathrm{H}}+\alpha^{\lambda}_{1\mathrm{H}}) - (8\alpha^{\lambda'}_{3\mathrm{H}}+\alpha^{\lambda'}_{1\mathrm{H}})}{6\pi}$$
$$+\left(-\frac{5}{2\pi}\frac{\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}} + \frac{10}{2\pi}\frac{1}{2\pi}\alpha_h\right).$$

$$\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha^{\lambda}_{3\mathrm{H}}}(\mu)\right) = 0 + \frac{1}{\alpha^{\lambda}_{3\mathrm{H}}}\frac{(6\alpha_{3\mathrm{H}} - 5\alpha^{\lambda}_{3\mathrm{H}} - \alpha^{\lambda'}_{3\mathrm{H}})}{2\pi} \tag{A.8}$$
$$+ \frac{2}{\alpha^{\lambda}_{3\mathrm{H}}}\frac{(8\alpha_{3\mathrm{H}}+\alpha_{1\mathrm{H}}) - (8\alpha^{\lambda}_{3\mathrm{H}}+\alpha^{\lambda}_{1\mathrm{H}})}{6\pi}$$
$$+\frac{2}{\alpha^{\lambda}_{3\mathrm{H}}}\left(\frac{\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}} - \frac{1}{2\pi}\alpha_h\right).$$

$$\frac{\partial}{\partial \ln \mu}\left(\frac{1}{\alpha^{\lambda'}_{3\mathrm{H}}}(\mu)\right) = 0 + \frac{1}{\alpha^{\lambda'}_{3\mathrm{H}}}\frac{(6\alpha_{3\mathrm{H}} - 5\alpha^{\lambda}_{3\mathrm{H}} - \alpha^{\lambda'}_{3\mathrm{H}})}{2\pi} \tag{A.9}$$

$$
\begin{aligned}
& + \quad \frac{2}{\alpha_{3\mathrm{H}}^{\lambda'}}\frac{(8\alpha_{3\mathrm{H}}+\alpha_{1\mathrm{H}})-(8\alpha_{3\mathrm{H}}^{\lambda'}+\alpha_{1\mathrm{H}}^{\lambda'})}{6\pi} \\
& \quad +\frac{2}{\alpha_{3\mathrm{H}}^{\lambda'}}\left( \qquad\qquad\qquad -\frac{5}{2\pi}\alpha_h\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \ln\mu}\left(\frac{1}{\alpha_{1\mathrm{H}}}(\mu)\right) &= \frac{-6}{2\pi} \qquad\qquad (\mathrm{one-loop}) \\
&- \frac{1}{2\pi}\frac{6(8\alpha_{3\mathrm{H}}+\alpha_{1\mathrm{H}})-5(8\alpha_{3\mathrm{H}}^{\lambda}+\alpha_{1\mathrm{H}}^{\lambda})-(8\alpha_{3\mathrm{H}}^{\lambda'}+\alpha_{1\mathrm{H}}^{\lambda'})}{6\pi} \\
&\quad +\left(-\frac{5}{2\pi}\frac{\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}}+\frac{10}{2\pi}\frac{1}{2\pi}\alpha_h\right).
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \ln\mu}\left(\frac{1}{\alpha_{1\mathrm{H}}^{\lambda}}(\mu)\right) &= \frac{-6}{2\pi}\left(\frac{\alpha_{1\mathrm{H}}}{\alpha_{1\mathrm{H}}^{\lambda}}\right)+\frac{1}{\alpha_{1\mathrm{H}}^{\lambda}}\frac{(6\alpha_{1\mathrm{H}}-5\alpha_{1\mathrm{H}}^{\lambda}-\alpha_{1\mathrm{H}}^{\lambda'})}{2\pi} \\
&+ \frac{2}{\alpha_{1\mathrm{H}}^{\lambda}}\frac{(8\alpha_{3\mathrm{H}}+\alpha_{1\mathrm{H}})-(8\alpha_{3\mathrm{H}}^{\lambda}+\alpha_{1\mathrm{H}}^{\lambda})}{6\pi} \\
&\quad +\frac{2}{\alpha_{1\mathrm{H}}^{\lambda}}\left(\frac{\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}}-\frac{1}{2\pi}\alpha_h\right).
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \ln\mu}\left(\frac{1}{\alpha_{1\mathrm{H}}^{\lambda'}}(\mu)\right) &= \frac{-6}{2\pi}\left(\frac{\alpha_{1\mathrm{H}}}{\alpha_{1\mathrm{H}}^{\lambda'}}\right)+\frac{1}{\alpha_{1\mathrm{H}}^{\lambda'}}\frac{(6\alpha_{1\mathrm{H}}-5\alpha_{1\mathrm{H}}^{\lambda}-\alpha_{1\mathrm{H}}^{\lambda'})}{2\pi} \\
&+ \frac{2}{\alpha_{1\mathrm{H}}^{\lambda'}}\frac{(8\alpha_{3\mathrm{H}}+\alpha_{1\mathrm{H}})-(8\alpha_{3\mathrm{H}}^{\lambda'}+\alpha_{1\mathrm{H}}^{\lambda'})}{6\pi} \\
&\quad +\frac{2}{\alpha_{1\mathrm{H}}^{\lambda'}}\left( \qquad\qquad\qquad -\frac{5}{2\pi}\alpha_h\right).
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \ln\mu}\left(\frac{1}{\alpha_h}(\mu)\right) &= \frac{1}{\alpha_h}\frac{2(8\alpha_{3\mathrm{H}}+\alpha_{1\mathrm{H}})-(8\alpha_{3\mathrm{H}}^{\lambda}+\alpha_{1\mathrm{H}}^{\lambda})-(8\alpha_{3\mathrm{H}}^{\lambda'}+\alpha_{1\mathrm{H}}^{\lambda'})}{6\pi} \\
&\quad +\left(\frac{1}{\alpha_h}\right)\left(\frac{2\times\frac{24}{10}}{\pi}\alpha_{\mathrm{GUT}}-\frac{(1+5+3)}{2\pi}\alpha_h\right).
\end{aligned} \tag{A.13}
$$

# Appendix B

# Appendix to Chapter 4

## B.1 $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle \times \mathbf{Z}_2\langle R_{89}\Omega\rangle)$ Geometry

In this section, geometry of the $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle \times \mathbf{Z}_2\langle\Omega R_{89}\rangle)$ orientifold is explained. $\mathbf{T}^6$ denotes a six-dimensional torus $(\mathbf{C}^3 = \{(z_1, z_2, z_3)|z_1, z_2, z_3 \in \mathbf{C}\})/\Gamma_0$, in which the lattice $\Gamma_0$ is spanned by six base vectors $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_6$, $\mathbf{e}_7$, $\mathbf{e}_8$, and $\mathbf{e}_9$. In other words, two points $\mathbf{y}, \tilde{\mathbf{y}} \in \mathbf{C}^3$ are identified with each other if and only if

$$\tilde{\mathbf{y}} = \mathbf{y} + n_m \mathbf{e}_m \qquad (n_m \in \mathbf{Z}, \quad m = 4, \cdots, 9). \tag{B.1}$$

The base vectors $\mathbf{e}_{4,5,6,7,8,9}$ are chosen as in Eq. (4.17). $\mathbf{e}_{4,5,6,7}$ span the first two complex planes $\mathbf{C}^2 = \{(z_1, z_2)|z_1, z_2 \in \mathbf{C}\}$, and $\mathbf{e}_{8,9}$ for the last complex plane. The $\mathbf{T}^6$ obtained in this way posesses $\mathbf{Z}_{12}\langle\sigma\rangle$ symmetry generated by an SU(3) rotation

$$\sigma : \mathbf{y} \equiv (z_b)|_{b=1,2,3} \in \mathbf{C}^3 \longmapsto \sigma \cdot \mathbf{y} \equiv (e^{2\pi i v_b} z_b)|_{b=1,2,3} \in \mathbf{C}^3 \tag{B.2}$$

with $(v_b)|_{b=1,2,3} = (1/12, -5/12, 4/12)$.

There are a number of singularities on the orientifold $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle \times \mathbf{Z}_2\langle\Omega R_{89}\rangle)$. There are two types of singularities. One is the orientifold planes and the other is orbifold singularities.

Orientifold planes are loci of points fixed under any order-2 elements of $\mathbf{Z}_{12}\langle\sigma\rangle \times \mathbf{Z}_2\langle\Omega R_{89}\rangle$ that involve reversing the orientation of strings $\Omega$. To be more explicit, they are loci of fixed points of $\Omega R_{89}$ and $\sigma^6 \cdot \Omega R_{89}$. There are four loci of $(\Omega R_{89})$-fixed points (which we call O7-planes). Their location are specified by their $z_3$ coordinates:

$$z_3 = \frac{1}{2} n_{m''} \mathbf{e}_{m''}, \qquad (\text{mod } n_{m''}\mathbf{e}_{m''}|_{m''=8,9} \quad \text{where } n_{m''} \in \mathbf{Z}). \tag{B.3}$$

They are described in Figure 4.1. Points fixed under the $(\sigma^6 \cdot \Omega R_{89})$ are O3-planes. Their location are specified by their coordinates in the $\mathbf{T}^6$:

$$\mathbf{y}' = (z_1, z_2) = \frac{1}{2} n_{m'} \mathbf{e}_{m'}|_{m'=4,5,6,7} \quad \text{and} \quad z_3 = \frac{1}{2} n_{m''} \mathbf{e}_{m''}|_{m''=8,9} \tag{B.4}$$

mod $n_m \mathbf{e}_m|_{m=4,\ldots,9}$, where $n_m \in \mathbf{Z}$. There are sixty-four O3-planes in the $\mathbf{T}^6$.

There are two types among orbifold singularities. Loci of points fixed under $\sigma^{3,6,9}$ extend in the third complex plane, while orbifold singularities assosicted with $\sigma^{1,2,4,5,7,8,10,11}$ are only points in the $\mathbf{T}^6/(\mathbf{Z}_{12}\langle\sigma\rangle \times \mathbf{Z}_2\langle\Omega R_{89}\rangle)$ orientifold.

The location of the $\sigma^{3,6,9}$-fixed singularities are specified by coordinates in the first two complex planes, $\mathbf{y}' = (z_1, z_2) \in \mathbf{C}^2$:

$$\sigma^6\text{-fixed} \quad \mathbf{y}' = n_{m'} \frac{\mathbf{e}_{m'}}{2}|_{m'=4,\cdots,7}, \qquad ({}^\forall z_3), \tag{B.5}$$

$$\sigma^3\text{-fixed} \quad \mathbf{y}' = n_4 \frac{\mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6}{2} + n_5 \frac{\mathbf{e}_4 + \mathbf{e}_7}{2} + n_6 \mathbf{e}_6 + n_7 \mathbf{e}_7, \qquad ({}^\forall z_3), \tag{B.6}$$

where $n_{m'}|_{m'=4,5,6,7}$ are integers. There are sixteen $\sigma^6$-fixed singularities within the covering space $\mathbf{T}^6$, four of which are $\sigma^3$-fixed singularities. These sixteen singularities are classified into four $\mathbf{Z}_{12}\langle\sigma\rangle$-orbits. One singularity at $\mathbf{y}' = \mathbf{0}$ stays fixed as a locus under the $\mathbf{Z}_{12}\langle\sigma\rangle$ transformation. Three other $\sigma^3$-fixed singularities form an orbit under the action of $\mathbf{Z}_{12}\langle\sigma\rangle/\mathbf{Z}_4\langle\sigma^3\rangle$. Remaining twelve $\sigma^6$-fixed singularities form two orbits under the action of $\mathbf{Z}_{12}\langle\sigma\rangle/\mathbf{Z}_2\langle\sigma^6\rangle$.

Other singularities are points in the $\mathbf{T}^6$ whose coordinates $\mathbf{y} = (\mathbf{y}', z_3) \in \mathbf{C}^3$ are

$$\sigma^4\text{-fixed} \qquad \mathbf{y}' = n_4 \frac{\mathbf{e}_4 + \mathbf{e}_6}{3} + n_5 \frac{\mathbf{e}_5 + \mathbf{e}_7}{3} + n_6 \mathbf{e}_6 + n_7 \mathbf{e}_7, \tag{B.7}$$

$$\sigma^{2,1}\text{-fixed} \qquad \mathbf{y}' = n_{m'} \mathbf{e}_{m'}|_{m'=4,\ldots,7} \tag{B.8}$$

in the first two complex planes, where $n_{m'}|_{m'=4,\ldots,7}$ are integers, and

$$\sigma^{4,2,1}\text{-fixed} \quad z_3 = \mathbf{0}, \pm \frac{\mathbf{e}_8 + 2\mathbf{e}_9}{3} \qquad (\text{mod } \mathbf{e}_{8,9} \quad \text{where } n_{8,9} \in \mathbf{Z}) \tag{B.9}$$

in the last complex plane. Note that all fixed points of $\sigma^2$ are fixed also under the $\sigma^1$ in this geometry.

## B.2 Anomaly Distribution

In gauge theories on orbifolds such as $\mathbf{T}^6/\Gamma$, anomalies generically arise at orbifold singularities. Thus, it should be examined whether the anomaly is cancelled at each orbifold singularity. We present formula for the distribution of anomalies on orbifold. We also see that the anomaly distribution can be attributed to pieces, each of which cooresponds to an element of the orbifold group $\Gamma$.

We adopt D5–D9 system in this subsection, which is T-dual to the D3–D7 system discussed in section 4.2, 4.3, 4.4 and 4.5. We adopt this description only because the description becomes simpler; Kaluza–Klein modes is more familiar in field theories than winding modes. The $z_3$-coordinates of the D3-branes and D7-branes are now implemented by Wilson lines on the third complex plane carried by gauge fields of D5- and D9-branes.

The Fujikawa method [96] is useful when one calculates the anomaly distribution on orbifolds [95]. Anomalies are understood as anomalous variations of the functional measure [96]. The measure of a Weyl fermion in four-dimensional space-time yields an anomalous variation of the action under a chiral transformation:

$$\delta S = \mathrm{Tr}\left(\delta^4(\tilde{x} - x)\gamma_5\right), \tag{B.10}$$

where the trace is taken over space-time coordinates $\tilde{x} = x$, spinor indices and gauge indices. One can think of anomaly of a more general symmetry transformation by replacing $\gamma_5$ to the generator $t^a$ of the symmetry transformation. It is straightforward to extend the formalism for field theories on higher-dimensional space-time compactified over orbifold geometry. The extension of Eq. (eq:chi-anomaly) is given [95] by

$$\delta S = \mathrm{Tr}_{co,sp,\omega}\left(\left(\frac{1}{\#\Gamma}\sum_{g\in\Gamma}\delta^6(\tilde{\mathbf{y}} - g\cdot\mathbf{y})\delta^4(\tilde{x} - x)\rho_{sp}(g)Ad(\gamma_g)\right)t^a\right), \tag{B.11}$$

where $t^a$ is a suitable representation of the generator of the transformation of which we consider the anomaly; the space-time delta function ($\delta^4(\tilde{x}-x)$) in four dimensions is replaced by its orbifold analogue; the delta function on the orbifold geometry is given by

$$\frac{1}{\#\Gamma}\sum_{g\in\Gamma}\delta^6(\tilde{\mathbf{y}} - g\cdot\mathbf{y})\delta^4(\tilde{x} - x)\rho_{sp}(g)Ad(\gamma_g). \tag{B.12}$$

Here, we include contributions from closed strings and D9–D9 open strings. Those from D5–D9 open strings and D5–D5 open strings are not yet included. Coordinates of six extra dimensions are denoted by $\mathbf{y}, \tilde{\mathbf{y}} \in \mathbf{T}^6 \equiv \mathbf{C}^3/\Gamma_0$. $\mathbf{y}$ and $g\cdot\mathbf{y}$ ($g \in \Gamma$) are the same point on the orbifold $\mathbf{T}^6/\Gamma$. Fields on these two points are identified up to two internal transformations, $\rho_{sp}(g)$, which denotes the effect of local Lorentz rotation under $g \in \Gamma(\subset \mathrm{SU}(3))$, and $Ad(\gamma_g)$, which denotes the adjoint action of the Chan–Paton matrix $\gamma_g$ on the D9–D9 open strings.

When $g$ rotates three complex planes separately, i.e., $g : z_b \mapsto e^{2\pi i v_b} z_b$ for $b = 1, 2, 3$ with $\sum_b v_b = 0$, then the $\rho_{sp}(g)$ is given by

$$\rho_{sp}(g) \equiv \prod_{b=1}^{3} \left( \cos(\pi v_b) + \sin(\pi v_b) \Gamma^{(2b+2)(2b+3)} \right). \tag{B.13}$$

The trace is taken over the space-time coordinates (i.e. summation over $x$ and $\mathbf{y}$ with $\tilde{x} = x$ and $\tilde{\mathbf{y}} = \mathbf{y}$ imposed), over indices of representations of local Lorentz symmetry (for a Weyl fermion of D9–D9 gauginos, index runs from 1 to 16.) and over weights $\omega$ (roots for D9–D9 open strings) of D9–D9 gauge symmetry.

The complex conjugate of chiral representations of SO(9,1) are still isomorphic to themselves. The D9–D9 open strings are in the **adj.** representation, which is a vector-like representation. Thus, the contributions to anomaly from CPT conjugates, which give the Hermitian conjugate of (B.11), are the same as the (B.11) in ten-dimensional space-time.

The trace over the coordinate in the $(x, \mathbf{y})$-base can be rewritten in the momentum-space base as

$$\begin{aligned}
\delta S &= \int \left(\frac{dp}{2\pi}\right)^6 \frac{1}{\mathrm{vol}(\mathbf{T}^6)} \sum_{\mathbf{p} \in \Lambda_0} \int d^4\tilde{x} d^6\tilde{\mathbf{y}} \int d^4x d^6\mathbf{y} \delta^4(\tilde{x} - x) && \text{(B.14)} \\
&\quad \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \mathrm{tr}_{\,sp,\omega} \left( e^{-i(p\cdot\tilde{x} + \mathbf{p}\cdot\tilde{\mathbf{y}})} \delta^6(\tilde{\mathbf{y}} - g \cdot \mathbf{y}) \rho_{sp}(g) Ad(\gamma_g) e^{i(p\cdot x + \mathbf{p}\cdot\mathbf{y})} t^a \right), \\
&= \int d^4x d^6\mathbf{y} \int \left(\frac{dp}{2\pi}\right)^4 \frac{1}{\mathrm{vol}(\mathbf{T}^6)} \sum_{\mathbf{p} \in \Lambda_0} \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \mathrm{tr}_{\,sp,\omega} \left( e^{-i\mathbf{p}\cdot(g\cdot\mathbf{y} - \mathbf{y})} \rho_{sp}(g) Ad(\gamma_g) t^a \right), && \text{(B.15)}
\end{aligned}$$

where now the trace runs only over the spinor and gauge indices. The $\Lambda_0$ is the dual lattice of the $\Gamma_0$. Kaluza–Klein momenta run over this dual lattice. In the presence of the Wilson line, the lattice of Kaluza–Klein momenta is shifted by $\mathbf{p}_\omega \equiv \langle \omega, \text{Wilson line} \rangle$ for each root $\omega$, although we avoid expressing this effect in the above equation in order to keep the formula simple.

This anomalous variation of the functional measure is regularized in a gauge-invariant way by inserting $e^{\not{D} \cdot \not{D}/2M^2}$, where $M$ is an energy scale of the regulator and $\not{D}$ the Dirac operator. We adopt, here, the most symmetric regularization. Then,

$$\begin{aligned}
\delta S &= \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \int d^4x d^6\mathbf{y} \; \frac{1}{\mathrm{vol}(\mathbf{T}^6)} \sum_{\mathbf{p} \in \Lambda_0} e^{-i(g^{-1}\cdot\mathbf{p} - \mathbf{p})\cdot\mathbf{y}} && \text{(B.16)} \\
&\quad \lim_{M \to \infty} i \left(\frac{M^2}{2\pi}\right)^2 \mathrm{tr}_{\,sp,\omega} \left( e^{-\frac{\mathbf{p}\cdot\mathbf{p} + \frac{i}{2} F_{AB}\Gamma^{AB}}{2M^2}} e^{-i(g^{-1}\cdot\mathbf{p}_\omega - \mathbf{p}_\omega)\cdot\mathbf{y}} \rho_{sp}(g) Ad(\gamma_g) t^a \right).
\end{aligned}$$

The space-time indices $A$ and $B$ run from 0 to 9. The anomalies are decomposed into parts, each of which corresponds to each element $g$ of the orbifold group $\Gamma$. Each component has its own distribution function determined by $g$. The distribution function

$$\frac{1}{\mathrm{vol}(\mathbf{T}^6)} \sum_{\mathbf{p} \in \Lambda_0} e^{-i(g^{-1}\cdot\mathbf{p}-\mathbf{p})\cdot\mathbf{y}} \tag{B.17}$$

is supported on $g$-fixed points, and is equally weighted at each $g$-fixed point. The weight is modified by phase factor $e^{-i(g^{-1}\cdot\mathbf{p}_\omega-\mathbf{p}_\omega)\cdot\mathbf{y}}$ at each fixed point $\mathbf{y}$ in the presence of the Wilson line. The anomaly of $g$-component for $g \in \Gamma$ is given by

$$\frac{1}{\#\Gamma} \times \lim_{M\to\infty} i \left(\frac{M^2}{2\pi}\right)^2 e^{-\frac{\mathbf{p}\cdot\mathbf{p}}{2M^2}} \mathrm{tr}_{\,sp,\omega} \left(e^{-i\frac{\Gamma\cdot F}{4M^2}} \rho_{sp}(g) Ad(\gamma_g) t^a\right). \tag{B.18}$$

For $g = \mathbf{id} \in \Gamma$, the $g$-component anomaly is distributed homogeneously in ten-dimensional space-time, and (B.18) is nothing but local anomalies of ten-dimensional field theories. Thus, the anomaly is cancelled because of the well-known anomaly cancellation in Type I theory.

For $g \in \Gamma$ that has only fixed *points* in $\mathbf{T}^6$, (i.e., when all the fixed points are isolated from one another), the distribution function is supported on $g$-fixed points and the triangle anomalies are obtained from (B.18) in the following way. The $\rho_{sp}(g)$ in the above expression contains a term $\prod_{b=1}^3 \sin(\pi v_b)\Gamma^{456789}$. Thus, the trace over spinor indices becomes non-vanishing when the regulator $e^{-i(\Gamma\cdot F)/(4M^2)}$ provides $-(\Gamma\cdot F)^2/(2(4M^2)^2)$. The decoupling limit of the regulator $M\to\infty$ leaves convergent and (generally) non-vanishing quantities. The $g$-component of triangle anomalies from D9–D9 gauge fermions is given by

$$\begin{aligned} & \frac{1}{\#\Gamma} i \left(\frac{M^2}{2\pi}\right)^2 \left(\prod_{b=1}^3 \sin(\pi v_b)\right) \mathrm{tr}_{\,sp,\omega} \left(Ad(\gamma_g) t^a \frac{-1}{2(4M^2)} (\Gamma^{MN}F_{MN})^2 \Gamma^{456789}\right) \qquad \text{(B.19)} \\ = \; & \frac{\mathrm{tr}_{\,\omega}(t^a\{t^b,t^c\})|_{I=\mathbf{fund.}}}{32\pi^2} (F^b\cdot\tilde{F}^c)|_{4D} \frac{\mathrm{tr}_{\,sp}\left(\Gamma^{0123456789}_{16\text{ by }16}\right)}{16} \times \\ & \qquad \frac{4}{\#\Gamma}\left(\prod_{b=1}^3 \sin(\pi v_b)\right) \sum_I (-iAd(\gamma_g)_I A_I) e^{-i(g^{-1}\cdot\mathbf{p}_I-\mathbf{p}_I)\cdot\mathbf{y}}. \end{aligned}$$

Here, the summation over gauge indecies (roots of **adj.** representation D9–D9 gauge group) is grouped into summation of weights within an irreducible representation $I$ of unbroken gauge groups and then the summation over irreducible representations are taken. Anomaly coefficient $A_I$ of an irreducible representation is defined by

$$\mathrm{tr}(\{t_I^b,t_I^c\}t_I^a) = A_I\,\mathrm{tr}(\{t_I^b,t_I^c\}t_I^a)|_{I=\mathbf{fund.}}, \tag{B.20}$$

$Ad(\gamma_g)_I$ is a phase of $Ad(\gamma_g)$ on an irreducible representation $I$, and $\mathbf{p}_I$ is the shift of the dual lattice $\mathbf{p}_\omega$, which is common in an irreducible representation $I$.

For $g \in \Gamma$ under which one of three complex planes is not rotated, the above triangle anomaly vanishes because one of $\sin(\pi v_b)$'s is zero. This is because the orbifold projection preserves the $\mathcal{N} = 2$ SUSY around the $g$-fixed singularities, gauge theories around them are vector-like, and hence the triangle anomalies is cancelled. However, the $g$-fixed singularities extend in the unrotated complex plane, and hence the box anomalies can be localized. Indeed, the $g$-component anomaly has distribution function supported on the six-dimensioal singularities in this case.

The $g$-component anomaly (B.18) yields box anomalies in this case. Indeed, the $\rho_{sp}(g)$ does not contain the $\Gamma^{456789}$ term, since $\sin(\pi v_3)) = 0$ ($v_3 = 0$ is assumed here), but rather another term

$$\prod_{b=1}^{2}(\sin(\pi v_b))\cos(\pi v_3)\Gamma^{4567}. \tag{B.21}$$

The trace over the spinor indices becomes non-trivial when the regulator provides $i(\Gamma \cdot F)^3/(3!(4M^2)^3)$. One can see that the decoupling limit of the regulator $M \to \infty$ leaves convergent and (generally) non-vanishing results $\propto \mathrm{tr}_{sp}((F \wedge F \wedge F)|_{6D} t^a)$.

One does not have to consider the pentagonal anomalies in eight-dimensional space-time, since there is no eight-dimensional orbifold singularities.

Anomaly arising from D5–D9 open strings and D5–D5 open strings have not yet included so far. First, D5–D9 open strings contribute to anomaly of the D5–D5 gauge symmetry as well as to that of the gravity and the D9–D9 gauge symmetry. For D5-branes fixed by $g \in \Gamma$, D5–D9 open strings contribute to $g \in \Gamma$ components of the triangle anomalies and box anomalies. It also has to be noted that the gravitational and D9–D9 box anomaly is shifted from one 5-brane to another through the anomaly inflow mechanism [97, 98, 99, 100].

Second, the "local" anomalies from D5–D5 open strings are formulated in almost the same way as those from D9–D9 open strings. The only difference is that the D5–D5 spectrum contains the winding modes instead of the Kaluza–Klein modes. Anomaly from them are also classified into $g \in \Gamma$ components, yet their distrbution functions cannot be determined for the hexagonal ($g = \mathbf{id}.$) and some of box anomalies. This is because the winding modes play the central role there. Although these anomalies cannot be interpreted as anomalies on specific points of space-time, such as orbifold singularities, they become the ordinary local anomalies after T-dual transformation.

In conclusion, anomalies on orbifold geometry are attributed to components corresponding to elements $g$ of the orbifold group $\Gamma$. Each component of anomalies for $g \in \Gamma$ is localized on the $g$-fixed singularities, as long as the anomaly is interpreted as local anomaly on space-time. Others are also interpreted as local anomalies localized on the $g$-fixed singularities after T-duality transformation. The amount of anomaly of each component for $g \in \Gamma$ is easily calculated in terms of Chan–Paton matrix $\gamma_g$ for D9-branes and D5-branes.

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