

# Anomaly and Superconnection

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Work with Shigeki Sugimoto (YITP)



# Introduction to the anomaly

Introduction (9)

Overview (4)

Derivation (10)

Application (7)

String theory (5)

# Motiation

We want to understand the QFTs.

- Known facts
  - Weak coupling theories are understood by the perturbation theory. (e.g. QED)
  - Strong coupling theories are very difficult. (e.g. QCD)
- Even though we know the Lagrangian of the theory in high energy(UV), it is difficult to understand it in low energy(IR).
  - Some (relevant) couplings become to larger values in the renormalization group flow (from UV to IR).
  - e.g. QCD, condensed matter theories, etc...
- The anomalies are useful tools to understand the IR theories!
  - 't Hooft anomaly matching
  - IR effective theories (EFT) need to have the same anomalies of UV theories.

# What is “anomaly”?

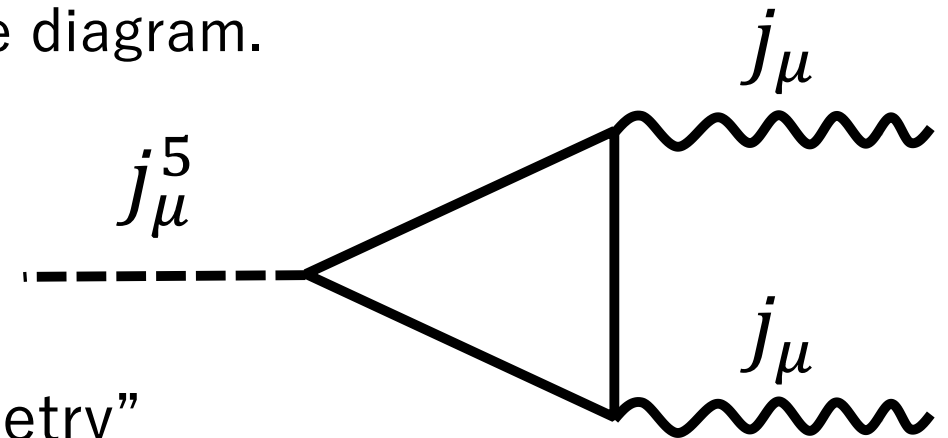
## Anomaly (Quantum Anomaly)

An classical action has some symmetries, but sometimes these symmetries disappear in quantum theory.

- e.g.)  $U(1)_A$  anomaly
  - In QCD, there is a  $U(1)_A$  symmetry in classical.
  - However,  $U(1)_A$  is not a symmetry in quantum theory.
  - $U(1)_A$  current is not conserved through the triangle diagram.

$$\psi(x) \rightarrow e^{i\gamma_5 \alpha(x)} \psi(x),$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{i\gamma_5 \alpha(x)}$$



## 't Hooft anomaly

- 't Hooft anomaly = “Anomaly of the global symmetry”
- Invariant from the RG flow.

# 't Hooft anomaly matching (1)

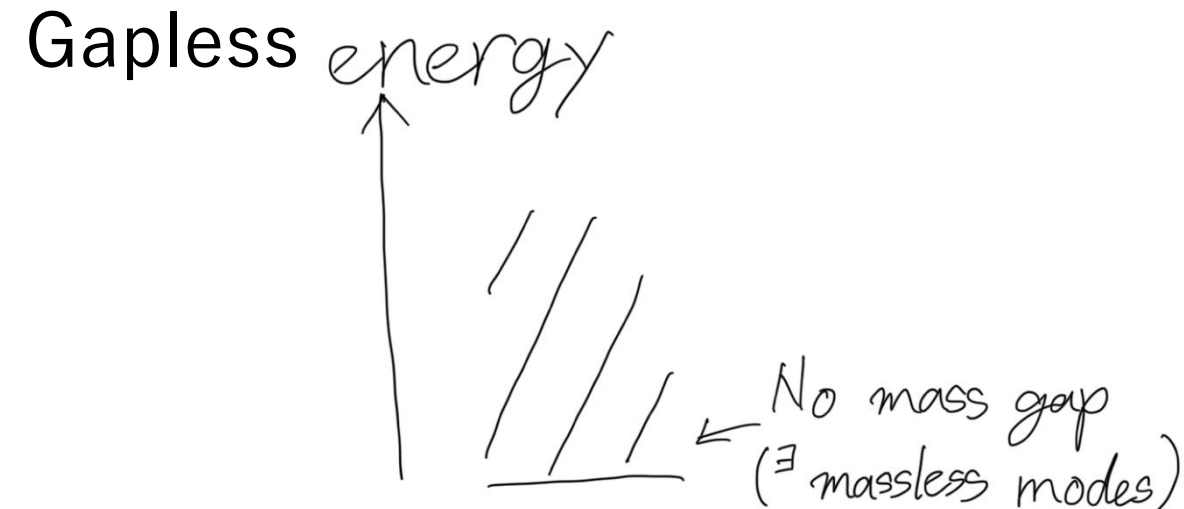
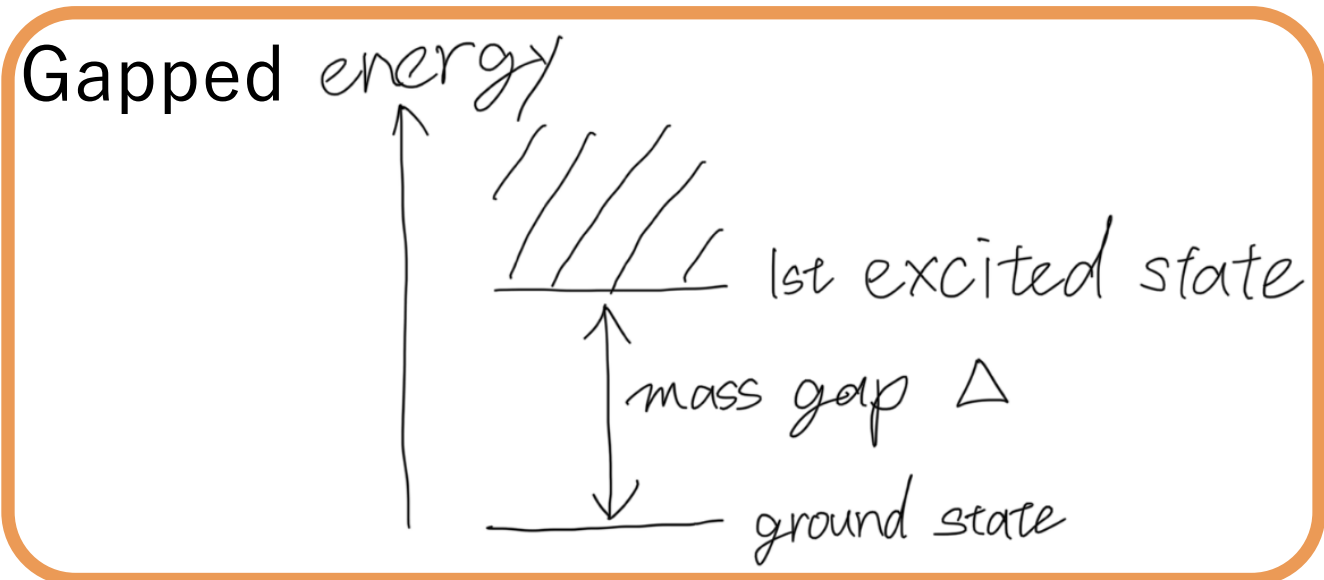
In particular, the anomalies are important to the ground state theories!

- Let us take the IR limit.
  - There are two possibilities.
    - (1) Gapped theory (The theory has a mass gap.)
    - (2) Gapless theory (The theory has NO mass gap.)

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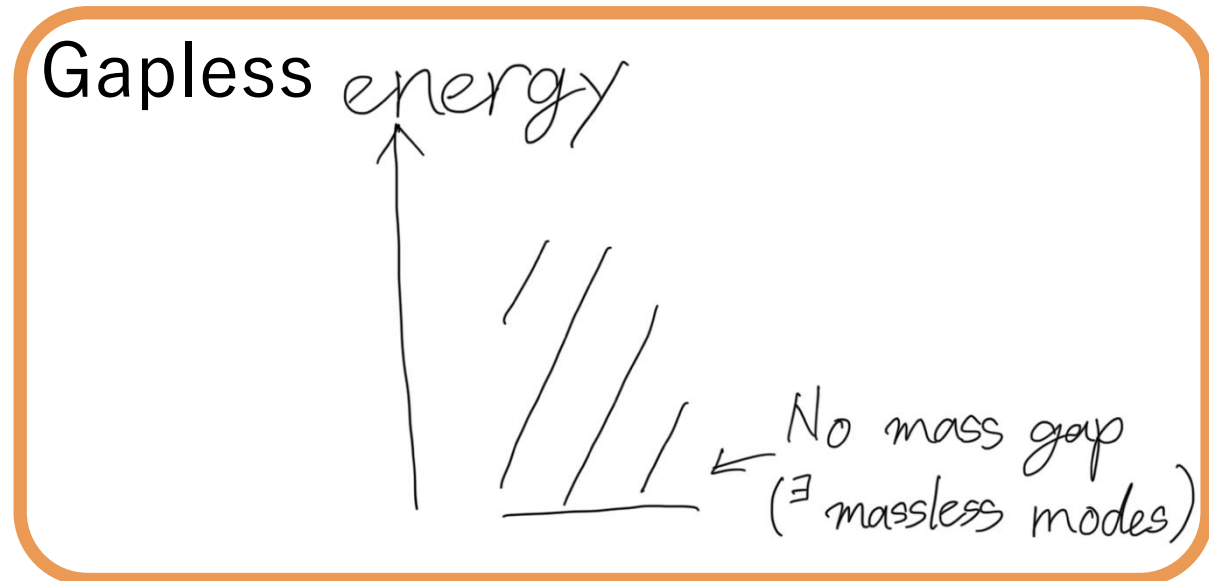
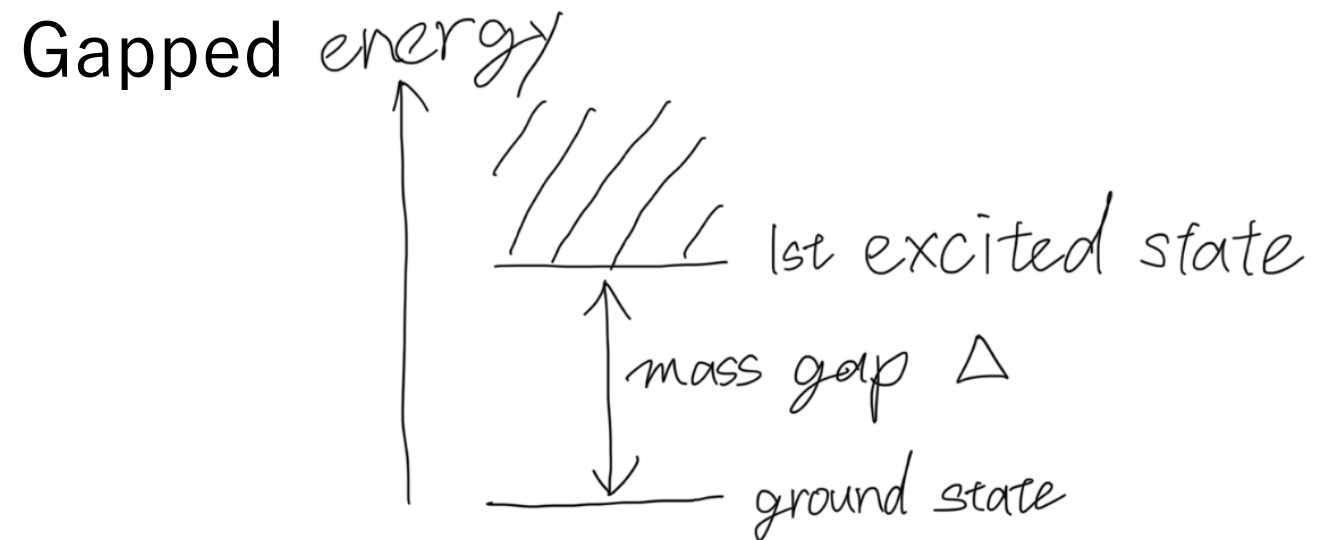
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    - (2) Gapless theory (The theory has NO mass gap.)



# 't Hooft anomaly matching (1)

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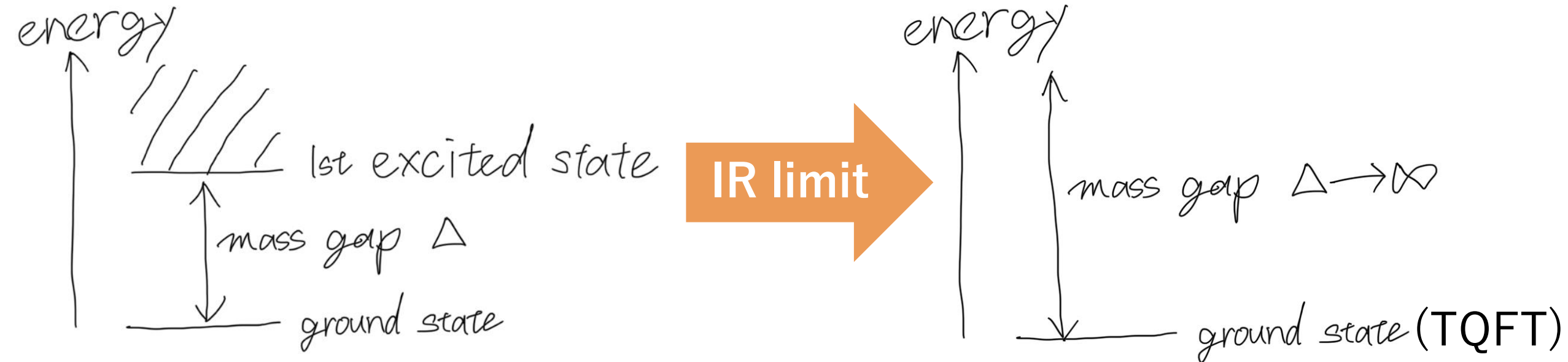
- Let us take the IR limit.
  - There are two possibilities.
    - (1) Gapped theory (The theory has a mass gap.)
    - (2) **Gapless theory** (The theory has NO mass gap.)



# 't Hooft anomaly matching (2)

## (1) Gapped theory

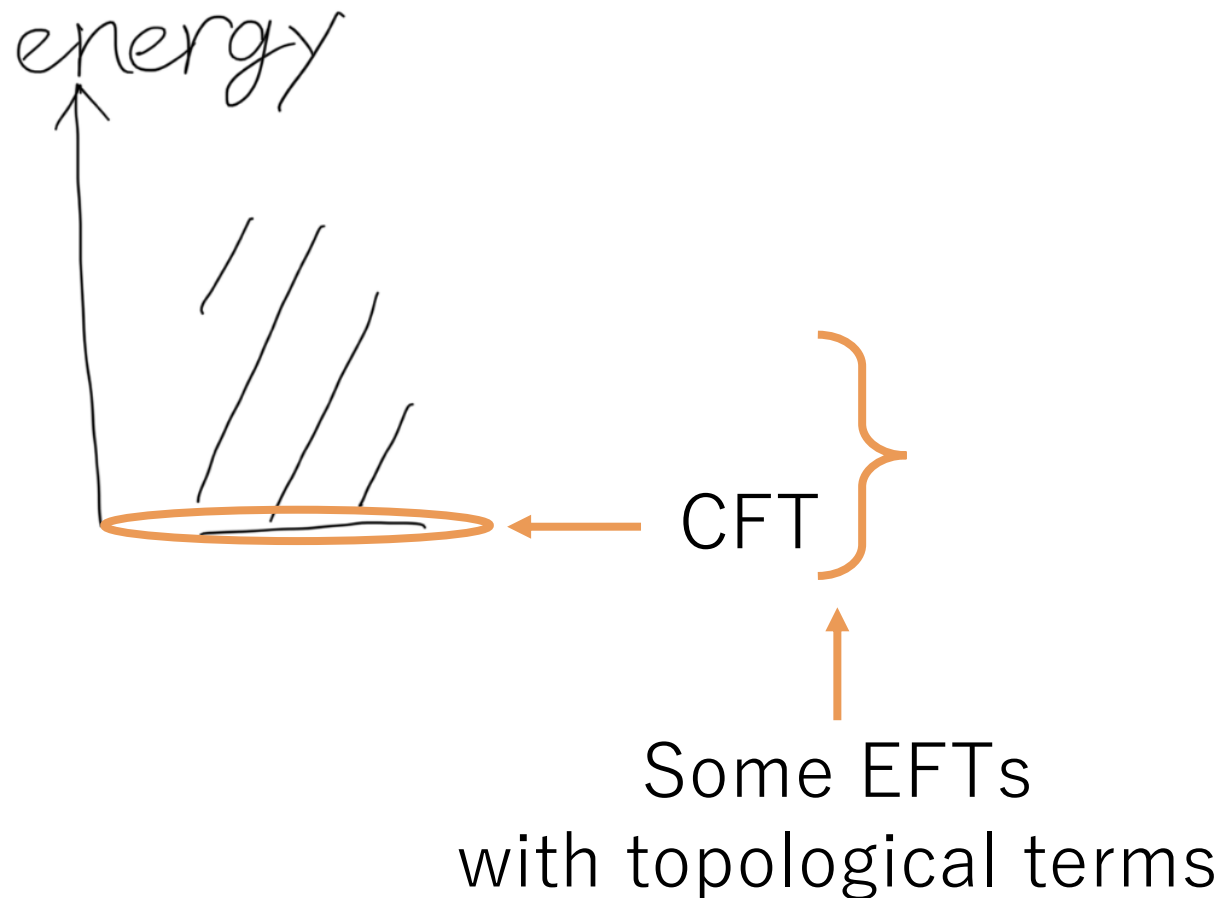
- In IR limit, there is no propagating modes. There is only the ground state.
- If UV theory have some 't Hooft anomalies, IR theory becomes the topological field theory(TQFT).



# 't Hooft anomaly matching (3)

## (2) Gapless theory

- In IR limit (on the IR fixed point), the theory becomes the conformal field theory (CFT).
- There are some massless propagating modes.
- If UV theory have some anomalies, IR theory (near the IR fixed point) need to have some topological terms.



→ **IR theory is almost determined by the UV anomalies!**

# e.g. massless QCD (1)

An example of the 't Hooft anomaly matching

- UV Lagrangian is known. ( $SU(N_c)$  Yang-Mills +  $N_f$  massless quarks)

$$S = \int d^4x \left\{ -\frac{1}{2g^2} \text{tr}[F^{\mu\nu} F_{\mu\nu}] + \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) \psi \right\}$$

- In classical theory, UV theory has  $U(N_f)_L \times U(N_f)_R$  global symmetry.
- In quantum, UV theory has  $\frac{U(N_f)_L \times U(N_f)_R}{U(1)_A}$  global symmetry.
  - $U(1)_A$  is not symmetry in the quantum theory, because of the (ABJ-type) anomaly.
  - $U(1)_A$  transformation is :
$$\begin{aligned} \psi(x) &\rightarrow e^{i\gamma_5 \alpha(x)} \psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x) e^{i\gamma_5 \alpha(x)} \end{aligned}$$
- IR theory is highly non-trivial, because of the confinement.

# e.g. massless QCD (2)

- IR theory should be gapless, because QCD has chiral SSB.

- Chiral SSB :  $\frac{U(N_f)_L \times U(N_f)_R}{U(1)_A} \rightarrow U(N_f)_V$

- There should be massless NG bosons (pions) : massless propagating modes.

- IR EFT is written by the pions :  $\pi(x)$

$$U = \exp(i\pi(x)) \in SU(N_f) \sim \frac{U(N_f)_L \times U(N_f)_R}{U(1)_A \times U(N_f)_V}$$

- From the SSB pattern, the IR pion effective field theory is  $SU(N_f)$  nonlinear sigma model.

$$S = \int d^4x \frac{f_\pi^2}{4} \text{tr}[\partial_\mu U \partial^\mu U^\dagger]$$

$$U = \exp\left(\frac{i\pi(x)}{f_\pi}\right) \in SU(N_f), \quad f_\pi : \text{the pion decay constant } (\sim \text{QCD scale} \rightarrow \text{Not CFT})$$

# e.g. massless QCD (3)

## 't Hooft anomaly matching

- UV anomaly is known  $\rightarrow$  IR EFT need to have the same anomaly.
- From the 't Hooft anomaly matching, the topological term is needed.
  - $SU(N_f)_1$  Wess-Zumino-Witten(WZW) term
  - Coupling between the pion and  $U(N_f)_L \times U(N_f)_R$  background gauge fields.

$\rightarrow$  IR EFT :  $SU(N_f)$  non-linear sigma model + WZW term

$$S = \int d^4x \frac{f_\pi^2}{4} \text{tr} \left[ \partial_\mu U \partial^\mu U^\dagger \right] - \int k \frac{N_c}{240\pi^2} \text{tr} \left[ (U dU^\dagger)^5 \right]$$

+ (coupling to the background gauge fields)

# e.g. massless QCD (4)

Why is this EFT important?

$$\pi^0 \rightarrow 2\gamma$$

- $\pi^0$  is unstable particle in the real world : decay into  $2\gamma$ .
- $SU(N_f)$  non-linear sigma model cannot predict this decay.
- However, WZW term includes coupling between  $\pi^0$  and  $\gamma$ .
  - SM gauge fields correspond to the background gauge fields.
  - $U(N_f)_L \times U(N_f)_R \supset SU(2)_L \times U(1)_Y$
- Because of the anomaly,  $\pi^0$  becomes unstable!
  - It is hard to find pions in our daily lives!

The anomaly is very strong tool to understand QFTs.

# Plan

## 1. Introduction to the anomaly (9)

- What is anomaly?
- 't Hooft anomaly matching
- Massless QCD

## 2. Our work (Overview)(4)

## 3. Derivation (10)

- How to calculate anomalies
- The anomaly for massless case
- The anomaly for massive case
- Superconnection
- The result

## 4. Application (7+5)

- Kink, vortex
- With boundary
- Index theorem
  - Index for massive case
  - APS index theorem

## 5. String theory (5)

- Tachyon condensation

## 6. Conclusion (1)

## 2. Our work (Overview)

# Relation between the our work and QCD

In our work, we focus on the chiral symmetry.

- We consider free fermion theory with  $N_f$  flavors.
- This theory has some global symmetries.
  - $U(N_f)_L \times U(N_f)_R$  symmetry for even dimension
  - $U(N_f)$  symmetry for odd dimension
- Basically, we consider the 't Hooft anomaly for these theories.
  - Free fermions couple to the background gauge fields for each symmetries.
- What is the new point?  
→ We focus on the **massive theory** with these symmetries.

# Theories what we want to think (1)

Let us consider 4dim action contain fermions.

$$S = \int d^4x \bar{\psi} i \not{D} \psi = \int d^4x \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) \psi$$

- This action is massless, so it has a chiral symmetry  $U(N_f)_L \times U(N_f)_R$ .
- There also be a  $U(1)_A$  anomaly.

- Add mass term

- Mass term breaks the chiral symmetry.

$$S = \int d^4x \bar{\psi} \left( i \not{D} + \textcircled{m} \right) \psi$$

- Let the mass depend on the spacetime.

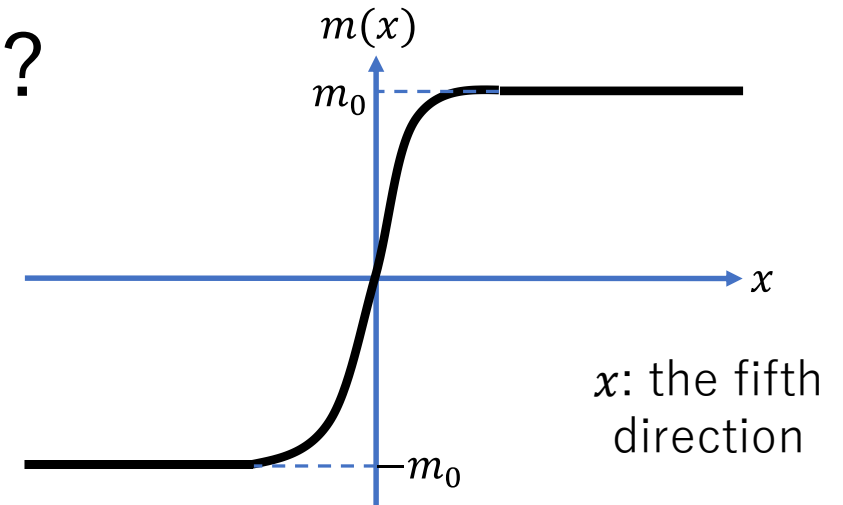
- This mass is almost same as the Higgs field.
- How change the symmetry and the anomaly?

$$S = \int d^4x \bar{\psi} \left( i \not{D} + \textcircled{m(x)} \right) \psi$$

# The spacetime dependent mass

What is “the spacetime dependent mass”?

- e.g.) Domain wall fermions
  - One way to realize chiral fermions on the lattice.
  - Consider 5dim spacetime, and realize 4dim fermions on  $m(x) = 0$  subspace.
- Chiral anomalies with Higgs fields
  - If Higgs fields change as bifundamental under the  $U(N_f)_L \times U(N_f)_R$  chiral symmetry, the action is invariant for the symmetry.
  - It is known that chiral anomalies are not changed by adding Higgs fields.
  - See Fujikawa-san's text book.



$$S = \int d^4x \bar{\psi} \left( i \not{D} + h(x) \right) \psi$$

# The spacetime dependent mass

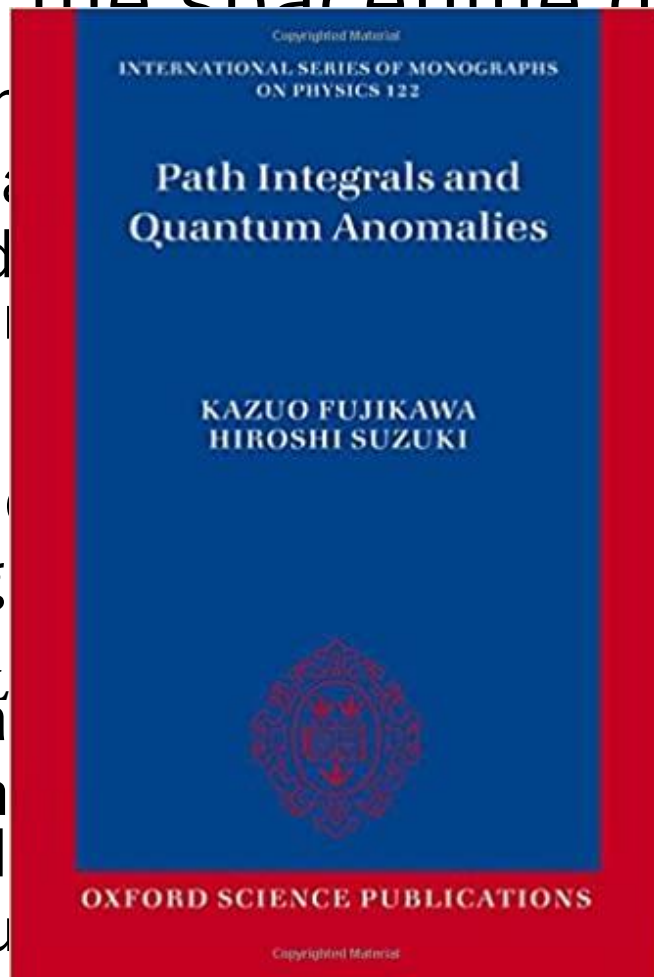
What is “the spacetime dependent mass”?

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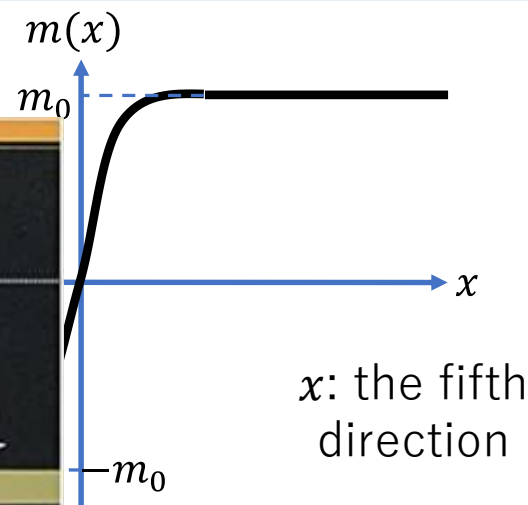
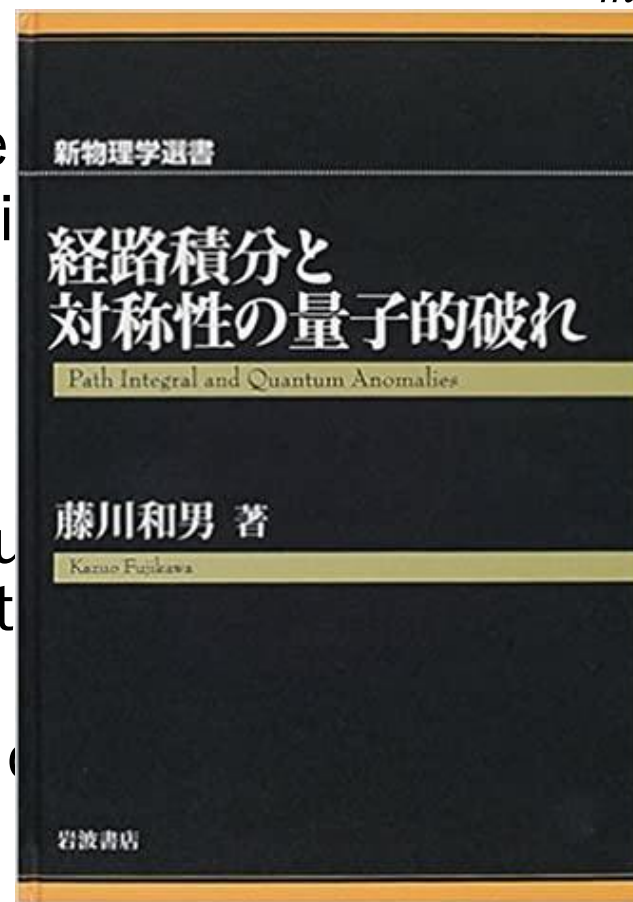


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$$\int \dots \left( i \not{D} + h(x) \right) \psi$$

# Theories what we want to think (2)

How about the spacetime dependent mass?

- The chiral anomaly is changed by the mass!!

- Difference between Higgs and mass
  - Higgs field : bounded
  - Spacetime dependent mass : unbounded

$$S = \int d^d x \bar{\psi} \left( i \not{D} + m(x) \right) \psi$$

- If the mass diverges at some points, it contributes to the anomaly.
  - This contribution might be unknown.
  - We can find the anomaly in any dimension.

- The anomaly can be written by “superconnection.”  $\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$

# 3. Derivation

# How to calculate anomalies

[ '79 Fujikawa ]

## Fujikawa method

- There are several ways to calculate anomalies.
- Today, we focus on the Fujikawa method.
  - Consider path integral for fermions.
  - Anomaly = Jacobian comes from path integral measure
  - We only consider perturbative anomalies.
- We calculate  $\log \mathcal{J}$  for anomalies in the last part of this talk.

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S}$$

e.g.) local  $U(1)_V$  transformation

$$\begin{aligned} \psi(x) &\rightarrow e^{i\alpha(x)} \psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x) e^{-i\alpha(x)} \end{aligned}$$

$$\begin{aligned} \mathcal{D}\psi \mathcal{D}\bar{\psi} &\rightarrow \mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \mathcal{J} \mathcal{D}\psi \mathcal{D}\bar{\psi} \\ &= e^{-i \int d^4x \alpha(x) \mathcal{A}(x)} \mathcal{D}\psi \mathcal{D}\bar{\psi} \end{aligned}$$

anomaly

$$\log \mathcal{J} = -i \int d^4x \alpha(x) \mathcal{A}(x)$$

# Chiral symmetry

## Anomalous symmetries we calculate

- $U(N_f)_L \times U(N_f)_R$  chiral symmetry
  - For **even** dimension
  - Because chirality operators exist only even dimensions.
  - Weyl fermions couple to  $U(N_f)_L$  background gauge field  $A_\mu^L$  and  $U(N_f)_R$  background gauge field  $A_\mu^R$ .

$$S = \int d^d x \bar{\psi} i \gamma^\mu \left\{ \partial_\mu + A_\mu^R P_+ + A_\mu^L P_- \right\} \psi = \int d^d x (\bar{\psi}_L, \bar{\psi}_R) i \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^{\mu\dagger} & 0 \end{pmatrix} \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

$$= \int d^d x \left\{ \bar{\psi}_R i \sigma^{\mu\dagger} (\partial_\mu + A_\mu^R) \psi_R + \bar{\psi}_L i \sigma^\mu (\partial_\mu + A_\mu^L) \psi_L \right\}$$

For **odd** dimension

- $U(N_f)$  flavor symmetry
  - For **odd** dimension
  - No perturbative anomaly as usual.
  - Dirac fermions couple to  $U(N_f)$  background gauge field.

$$S = \int d^d x \bar{\psi} i \gamma^\mu \left\{ \partial_\mu + A_\mu \right\} \psi$$

For **even** dimension

$$P_\pm = \frac{1 \pm \gamma_{d+1}}{2}$$

# The anomaly for massless cases

e.g.) fermions in 4dim

- Mass less case
  - With  $U(N_f)_L \times U(N_f)_R$  chiral sym.
  - $U(1)_V$  anomaly is written by the field strengths.
- With a Higgs field
  - With  $U(N_f)_L \times U(N_f)_R$  chiral sym.
  - The  $U(1)_V$  anomaly is same for massless case.
- How about the massive case?

$$\begin{aligned} S &= \int d^4x \bar{\psi} i \gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^L & 0 \\ 0 & A_\mu^R \end{pmatrix} \right\} \psi \\ &= \int d^4x \left\{ \bar{\psi}_R i \sigma^{\mu\dagger} (\partial_\mu + A_\mu^R) \psi_R + \bar{\psi}_L i \sigma^\mu (\partial_\mu + A_\mu^L) \psi_L \right\} \end{aligned}$$

$$\begin{aligned} \log \mathcal{J} &= \frac{i}{32\pi^2} \int d^4x \alpha(x) \epsilon^{\mu\nu\rho\sigma} \text{tr} [F_{\mu\nu}^R F_{\rho\sigma}^R - F_{\mu\nu}^L F_{\rho\sigma}^L] \\ &= \frac{i}{8\pi^2} \int \alpha(x) \text{tr} [F^R \wedge F^R - F^L \wedge F^L] \end{aligned}$$

$$\begin{aligned} S &= \int d^4x \bar{\psi} (i \not{D} + h(x)) \psi \\ &= \int d^4x \left\{ \bar{\psi}_R i \sigma^{\mu\dagger} D_\mu^R \psi_R + \bar{\psi}_L i \sigma^\mu D_\mu^L \psi_L + \bar{\psi}_L h \psi_R + \bar{\psi}_R h^\dagger \psi_L \right\} \end{aligned}$$

# The anomaly for massive case (1)

Let us consider spacetime dependent mass!

- The action for general even dim with  $U(N_f)_L \times U(N_f)_R$  symmetry is,

$$\begin{aligned} S &= \int d^d x \bar{\psi} \left[ i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} + \begin{pmatrix} im(x) & 0 \\ 0 & im^\dagger(x) \end{pmatrix} \right] \psi \\ &= \int d^d x \left\{ \bar{\psi}_R i\sigma^{\mu\dagger} D_\mu^R \psi_R + \bar{\psi}_L i\sigma^\mu D_\mu^L \psi_L + \bar{\psi}_L im(x) \psi_R + \bar{\psi}_R im^\dagger(x) \psi_L \right\} \\ &\equiv \int d^d x \bar{\psi} \mathcal{D} \psi \end{aligned}$$

- We assume this  $\mathcal{D}$  is the Dirac op. for massive case.
  - $\mathcal{D}$  is non-Hermitian.
- For odd dim case, there is only  $U(N_f)$  sym, we put  $A_\mu = A_\mu^R = A_\mu^L$  and  $m = m^\dagger$ .

# The anomaly for massive case (2)

Calculate the  $U(1)_V$  anomaly for this action by Fujikawa method.

- We **take  $m(x)$  divergent**.

$$|m(x^I)| \rightarrow \infty \quad (|x^I| \rightarrow \infty)$$

- $I$  denotes some directions which  $m(x)$  changes its values.
- If  $m(x)$  does not diverge, the anomaly is same for the massless case.

- We use the heat kernel regularization.

- We set a UV cut-off for the eigenvalues of  $\mathcal{D}^\dagger \mathcal{D}$  and  $\mathcal{D} \mathcal{D}^\dagger$ .

$$e^{-\frac{\mathcal{D}^\dagger \mathcal{D}}{\Lambda^2}}, \quad e^{-\frac{\mathcal{D} \mathcal{D}^\dagger}{\Lambda^2}}$$

$\Lambda$  is the UV cut-off.

- Generalizations

- It is easy to get the anomaly for any dimension.
- It is also easy to get the anomaly for  $U(N_f)_L \times U(N_f)_R$ , not only for  $U(1)_V$ .

# The anomaly for massive case (3)

e.g.) In 4dim case, the  $U(1)_V$  anomaly is,  $\tilde{m} = m/\Lambda$   $\Lambda$  is the UV cut-off comes from heat kernel regularization.

$$\begin{aligned} \log \mathcal{J} = & \frac{i}{(2\pi)^2} \int d^4x \alpha(x) \text{tr} \left[ \epsilon^{\mu\nu\rho\sigma} \left\{ \frac{1}{8} \left( F_{\mu\nu}^R F_{\rho\sigma}^R - F_{\mu\nu}^L F_{\rho\sigma}^L \right) \right. \right. \\ & + \frac{1}{12} \left( D_\mu \tilde{m}^\dagger D_\nu \tilde{m} F_{\rho\sigma}^R - D_\mu \tilde{m} D_\nu \tilde{m}^\dagger F_{\rho\sigma}^L + F_{\mu\nu}^R D_\rho \tilde{m}^\dagger D_\sigma \tilde{m} \right. \\ & \left. \left. - F_{\mu\nu}^L D_\rho \tilde{m} D_\sigma \tilde{m}^\dagger - D_\mu \tilde{m} F_{\nu\rho}^R D_\sigma \tilde{m}^\dagger + D_\mu \tilde{m}^\dagger F_{\nu\rho}^L D_\sigma \tilde{m} \right) \right. \\ & \left. \left. + \frac{1}{24} \left( D_\mu \tilde{m}^\dagger D_\nu \tilde{m} D_\rho \tilde{m}^\dagger D_\sigma \tilde{m} - D_\mu \tilde{m} D_\nu \tilde{m}^\dagger D_\rho \tilde{m} D_\sigma \tilde{m}^\dagger \right) \right\} \right] e^{-\tilde{m}^\dagger \tilde{m}} \end{aligned}$$

- This result seems very complicated...
- Can we rewrite it more simple way?  $\rightarrow$  **Superconnection!**

# Superconnection (1)

[’85 Quillen]

- We define the superconnections for even and odd dimensions.
- This is made by Quillen, who is a mathematician, in 1985.

## Even dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

$A_R : U(N_f)_R$  gauge field (1-form)

$A_L : U(N_f)_L$  gauge field (1-form)

$T : U(N_f)_L \times U(N_f)_R$  bifundamental scalar field (0-form)

- Field strength

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$$

$$\equiv \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix}$$

- Supertrace

$$\text{Str} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{tr}(a) - \text{tr}(d)$$

# Superconnection (2)

[’85 Quillen]

Odd dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A & iT \\ iT & A \end{pmatrix} \quad \begin{array}{l} A : U(N_f) \text{ gauge field (1-form)} \\ T : U(N_f) \text{ adjoint scalar field (0-form)} \end{array}$$

- Field strength  $\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2$

$$= \begin{pmatrix} F - T^2 & iDT \\ iDT & F - T^2 \end{pmatrix}$$

- Supertrace

$$\text{Str} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i} \text{tr}(b)$$

We apply superconnection to write the anomaly.

# The result (1)

- We can rewrite the  $U(1)_V$  anomaly by superconnection.

$$\log \mathcal{J} = -i \left( \frac{i}{2\pi} \right)^{\frac{d}{2}} \int \alpha(x) \text{Str} [e^{\mathcal{F}}] \Big|_{d\text{-form}} \quad \left| \begin{array}{l} \tilde{m} = m/\Lambda \\ \mathcal{F} = d\mathcal{A} + \mathcal{A}^2 \\ \equiv \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix} \\ \text{Str} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{tr}(a) - \text{tr}(d) \end{array} \right.$$

$$\mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

- For odd dimension case, put  $A_\mu = A_\mu^R = A_\mu^L$  and  $m = m^\dagger$ . Then, we get  $U(1)$  anomaly.
  - In odd dimension, the definition of Str is different from the even dim case.

$$\text{Str} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i} \text{tr}(b)$$

# The result (2)

$$\log \mathcal{J} = -i \left( \frac{i}{2\pi} \right)^{\frac{d}{2}} \int \alpha(x) \text{Str} [e^{\mathcal{F}}] \Big|_{d\text{-form}} \quad \mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

- In this formula,  $m(x)$  appears only as  $\tilde{m}(x)$ .  
 $\tilde{m} = m/\Lambda$ 
  - $\Lambda$  is the UV cut-off comes from the heat kernel regularization.
  - If  $m(x)$  is finite, the mass dependence disappears because we need to take  $\Lambda \rightarrow \infty$ .
  - The  $\Lambda$  dependence (or the regulator dependence) of the anomaly disappears after we integrate  $\text{Str}[e^{\mathcal{F}}]$  over the spacetime.
- It is easy to check this anomaly is consistent with 4dim massless case.

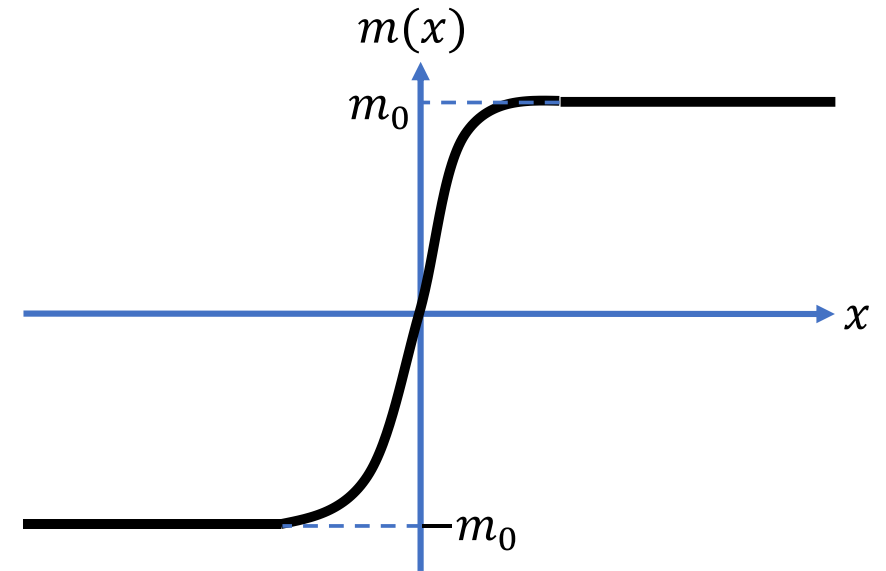
$$\log \mathcal{J} = \frac{i}{8\pi^2} \int \alpha(x) \text{tr} [F^R \wedge F^R - F^L \wedge F^L]$$

# 3. Application

# How can we apply this anomaly?

Mass means a wall for some cases!

- e.g.) Domain wall
  - Can we make domain walls by this  $m(x)$ ?  
→ Yes!
- We can make some systems with boundaries.
  - Kink, vortex and general codimension case
  - With boundary
- We also discuss about some index theorems.
  - APS index theorem
  - Callias type index theorem



# Kink (1)

## Mass kink for our set up

- For example, let's consider 5dim case.
- In our set up, “kink” means this mass configuration.

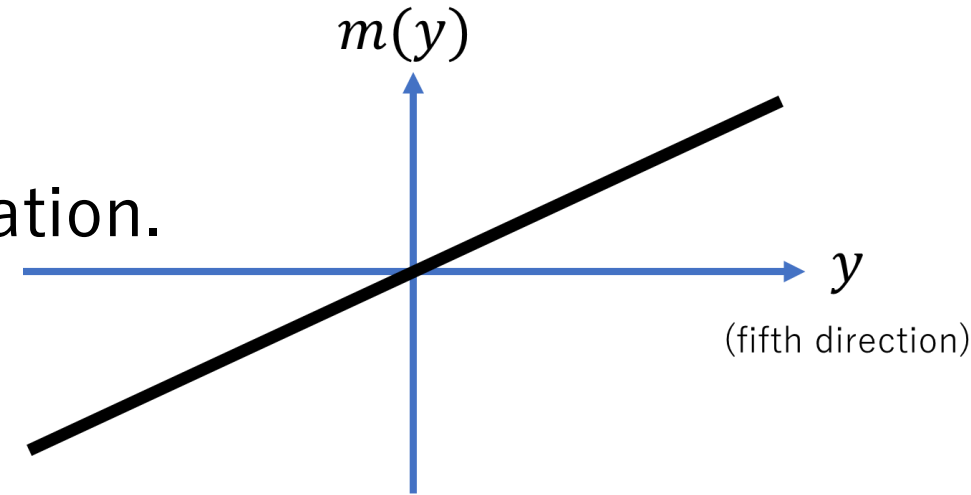
$$m(y) = uy \quad y = x^5 \quad u \in \mathbb{R}$$

- This “mass” diverges at  $y \rightarrow \pm\infty$ .
- 5dim fermions with  $U(N_f)$  sym, and the mass depends on only  $y$  direction.
- The  $U(1)$  anomaly is,

$$\log \mathcal{J} = \pm \frac{i}{8\pi^2} \int \alpha(x) \text{tr} [F \wedge F]$$

- Recall 4dim  $U(1)_V$  anomaly, Corresponds to the sign of  $u$ .

$$\log \mathcal{J} = \frac{i}{8\pi^2} \int \alpha(x) \text{tr} [F^R \wedge F^R - F^L \wedge F^L]$$



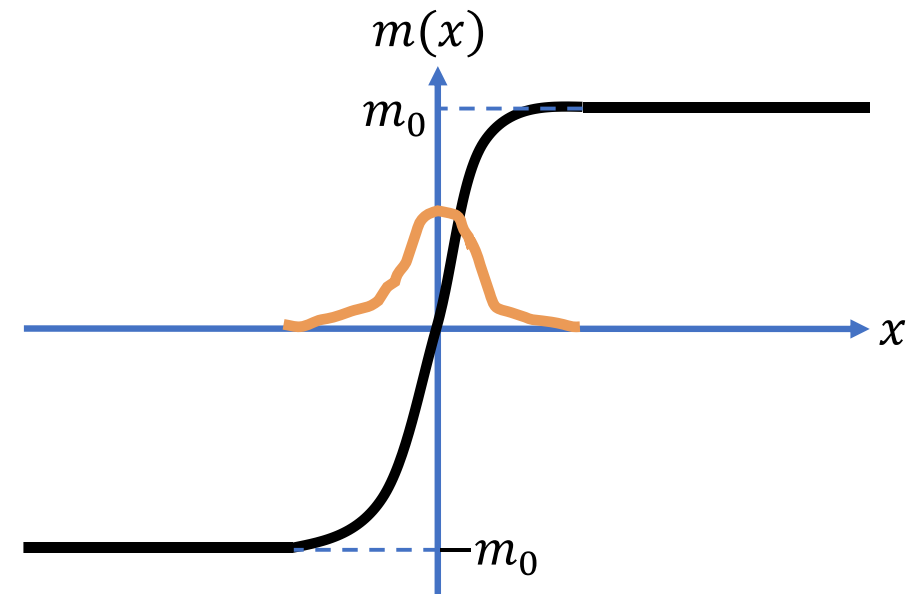
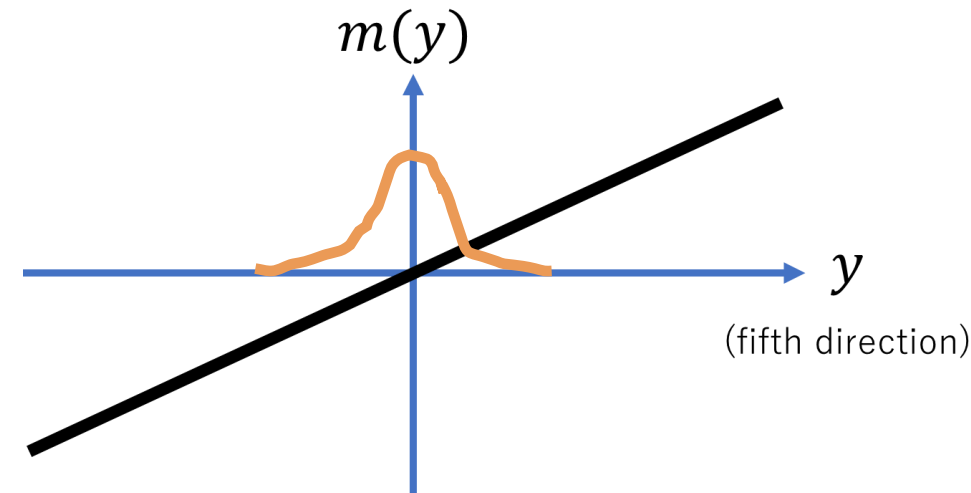
# Kink (2)

What is the meaning of the anomaly?

- 4dim Weyl fermions are localized at  $y = 0$ .
  - $u > 0$  corresponds to chirality + (right-handed) fermion, and  $u < 0$  corresponds to chirality - (left-handed) fermion.

## Domain wall fermion

- These Weyl fermions correspond to domain wall fermions.
  - But the regularization is different, so that I don't know the correspondence in detail.



# Vortex

Next, we check codim-2 case.

- Vortex is 2dim topological object.

- Let us consider  $2r + 2$  dim.

$$m(z) = uz \mathbf{1}_{N \times N}$$

$$z = x^{\mu=2r+1} - ix^{\mu=2r+2}$$

- $m(z)$  depends on 2 directions, and it is complex valued “mass”.

- This mass diverges at  $|z| \rightarrow \infty$ .

- For simplicity, we put  $A_L = A_R$  in  $2r + 2$ dim.

- The  $U(1)_V$  anomaly is,

$$\log \mathcal{J} = -i \left( \frac{i}{2\pi} \right)^r \int \alpha(x) \text{Str} [e^F] \Big|_{2r\text{-form}}$$

- This is  $2r$ dim  $U(1)$  anomaly with  $U(N_f)_R$  gauge field.

- If you want to get chirality – (left-handed) result, use  $m(\bar{z}) = u\bar{z}$ , instead.

# General defects

We can apply this formula to general codimension cases.

- When we think  $d$  dim system with  $n$  dim topological defects, we get  $d - n$  dim  $U(1)$  anomalies.
  - If  $d - n$  is odd, we get nothing because odd dim mass less fermions are anomaly-free.
  - The mass configurations for general codimension is,

$$m(x) = u \sum_{I=1}^n \Gamma^I x^I$$
$$\gamma^I = \Gamma^I \quad (n = \text{odd})$$
$$\gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix} \quad (n = \text{even})$$

- This results correspond to “tachyon condensation” in string theory.
  - We will discuss about it in section 5.

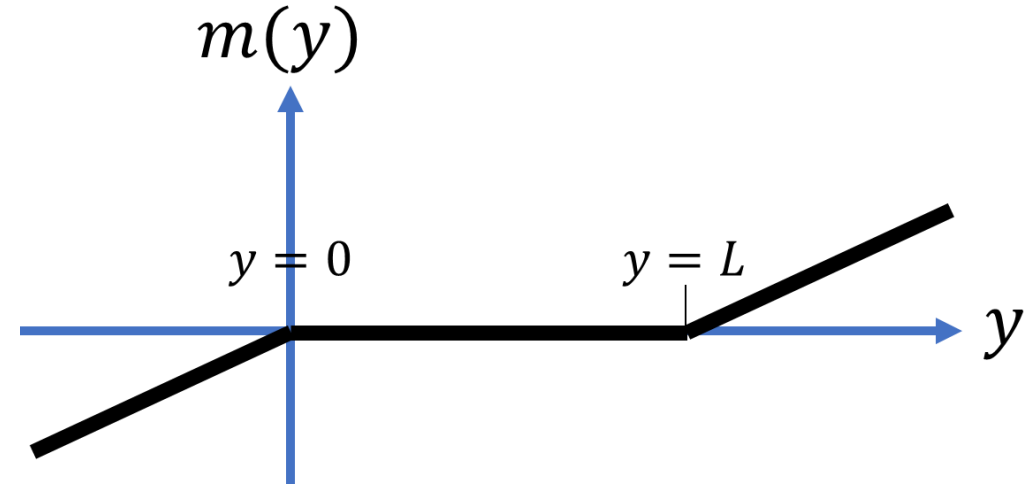
# With boundary (1)

Let us make some boundaries.

- Fermions are massive = boundary

Odd dimension ( $2r$  dim)

- We realize localized fermions at  $[0, L]$ .
- The bulk is anomaly-free.
- The anomaly is,



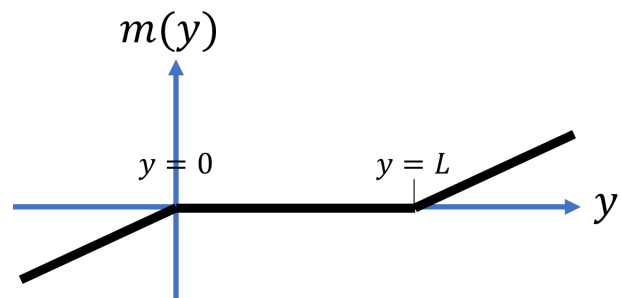
$$m(y) = \mu(y)1_N = \begin{cases} (m_0 + u'(y - L))1_N & (L < y) \\ m_0 1_N & (0 \leq y \leq L) \\ (m_0 + uy)1_N & (y < 0) \end{cases}$$

$$\log \mathcal{J} = i\kappa_- \int_{y=0} \alpha [\text{ch}(F)]_{2r} + i\kappa_+ \int_{y=L} \alpha [\text{ch}(F)]_{2r}$$

$$\kappa_- = \frac{1}{2} \text{sgn}(u) , \quad \kappa_+ = \frac{1}{2} \text{sgn}(u')$$

# With boundary (2)

Even dimension



$$m(x) = \mu(y)g(x) = \begin{cases} u'(y-L)g(x) & (L < y) \\ 0 & (0 \leq y \leq L) \\ uyg(x) & (y < 0) \end{cases}$$

- The anomaly is,

$$\log \mathcal{J} = -i \int_{0 < y < L} \alpha [\text{ch}(F_+) - \text{ch}(F_-)]_{2r} - i \int_{y=L} \alpha[\omega]_{2r-1} + i \int_{y=0} \alpha[\omega]_{2r-1}$$

- $\omega$  is Chern-Simons form.
- Anomaly from bulk + CS

# 5. String theory

# String theory

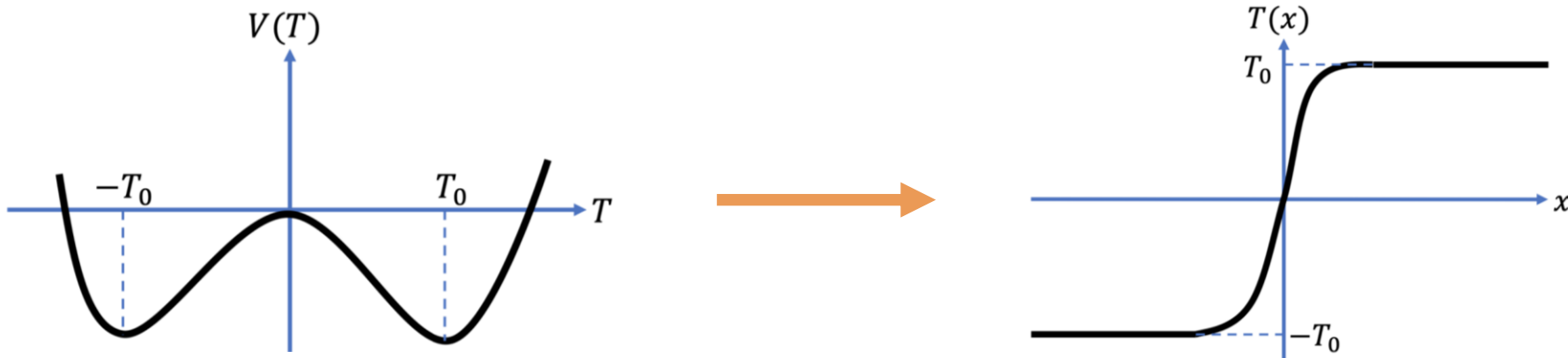
Let us check the relation between this anomaly and string theory.

- Consider type IIA or IIB string theory with D-branes.
  - Open strings have their ends on D-branes.
  - Excitation modes of these open strings  $\rightarrow$  Fields on D-branes
  - Open strings on  $D_p$ -branes  $\rightarrow$  QFT in  $p + 1$  dim
- In some cases, excitation modes of the strings have tachyon modes.
  - Lowest excitation modes are  $m^2 < 0$ . (Tachyon mode)
  - Non-BPS states have tachyons.
  - These tachyonic modes are unstable.  $\rightarrow$  **Tachyon condensation**
  - See Sen's review [hep-th/9904207].



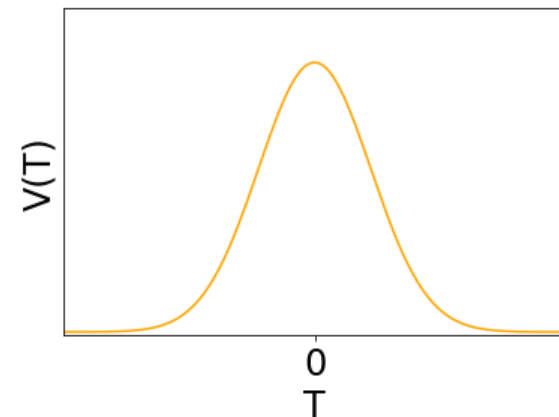
# Tachyon condensation (1)

- Tachyonic modes are unstable, so the tachyons have non-zero VEV.
  - Non-trivial configuration of tachyon is also realizable.



e.g.)  $D$ -brane and anti  $D$ -brane ( $\bar{D}$ -brane) system

- Non-BPS state
- Tachyonic modes appear in  $D - \bar{D}$  string.
  - The shape of tachyon potential is known.  $V(T) = e^{-T^\dagger T}$
  - If tachyon configuration is trivial, the  $D$ -branes disappear.



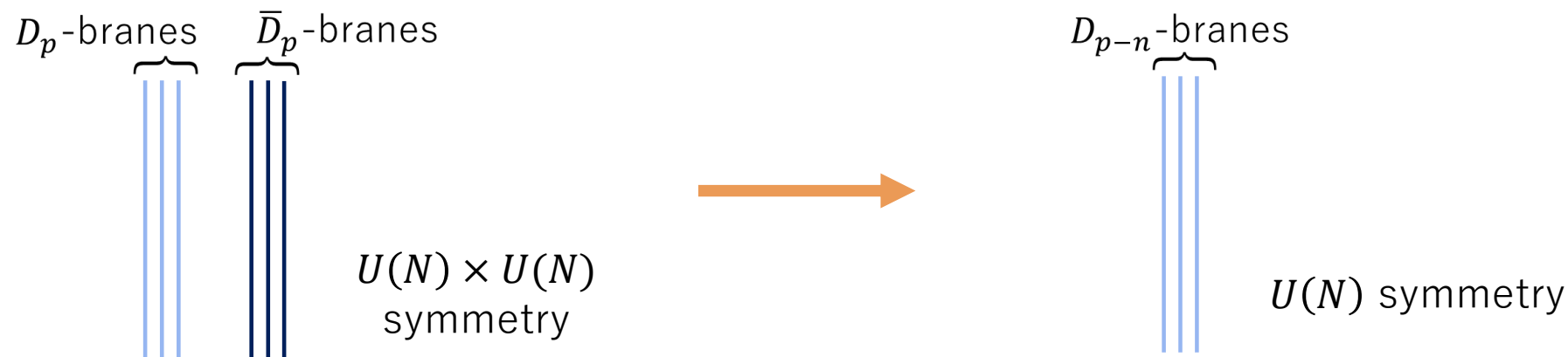
# Tachyon condensation (2)

Kink on tachyon in  $D_p - \bar{D}_p$  system

- Tachyonic kinks for this system is,

$$T(x) = u \sum_{I=1}^n \Gamma^I x^I \quad \gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix}$$

- We get  $D_{p-n}$ -branes from this tachyon.
  - If  $D_{p-n}$ -branes are non-BPS, tachyons still exist on the  $D$ -branes.
  - In this case, tachyon condensation occur again.



# Tachyon condensation (3)

- The superconnection is used in the context of tachyon condensation.
  - This structure comes from RR-coupling of D-branes.

cf.) ['98 Witten] [hep-th/9810188]

['99 Kennedy-Wilkins] [hep-th/9905195]

['01 Kuraus-Larsen] [hep-th/0012198]

['01 Takayanagi-Terashima-Uesugi] [hep-th/0012210]

$$S = T_{D9} \int C \wedge \text{Str } e^{2\pi\alpha' i\mathcal{F}}$$

- The tachyon configuration is given by ['98 Witten].
  - In this paper, relation between tachyon condensation and **K-theory** is discussed.
  - This tachyon configuration comes from ['64 Atiyah-Bott-Shapiro]

$$T(x) = u \sum_{I=1}^n \Gamma^I x^I$$

# Relation between the anomaly and string

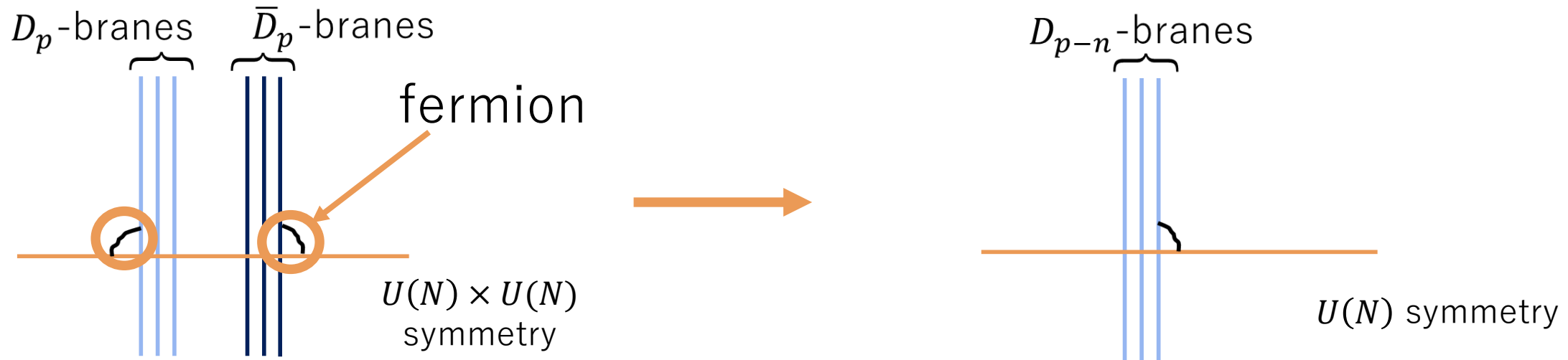
- This tachyon configuration is same for the mass defect in section 3!

$$T(x) = u \sum_{I=1}^n \Gamma^I x^I$$

$$m(x) = u \sum_{I=1}^n \Gamma^I x^I$$

- This anomaly can be understood from string theory.
  - Fermions are found where  $D$ -branes intersect.
  - This is similar to flavor symmetry on holographic QCD model.

(Sakai-Sugimoto model)



# Conclusion

- We discussed about perturbative anomaly with **spacetime dependent mass**.
  - If value of the mass **diverge**, non-trivial contribution of the mass appears.
- The anomaly can be written by **superconnection**.

$$\log \mathcal{J} = -i \left( \frac{i}{2\pi} \right)^{\frac{d}{2}} \int \alpha(x) \text{Str} [e^{\mathcal{F}}] \Big|_{d\text{-form}} \quad \mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

- There are some applications.
  - Kink, vortex, ...
  - With boundary
  - Index theorem