Anomaly and Superconnection

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Work with Shigeki Sugimoto (YITP)

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Introduction to the anomaly

Motiation

We want to understand the QFTs.

- Known facts
  - Weak coupling theories are understood by the perturbation theory. (e.g. QED)
  - Strong coupling theories are very difficult. (e.g. QCD)
- Even though we know the Lagrangian of the theory in high energy(UV), it is difficult to understand it in low energy(IR).
  - Some (relevant) couplings become to larger values in the renormalization group flow (from UV to IR).
  - e.g. QCD, condensed matter theories, etc…
- The anomalies are useful tools to understand the IR theories!
  - ’t Hooft anomaly matching

  →IR effective theories (EFT) need to have the same anomalies of UV theories.

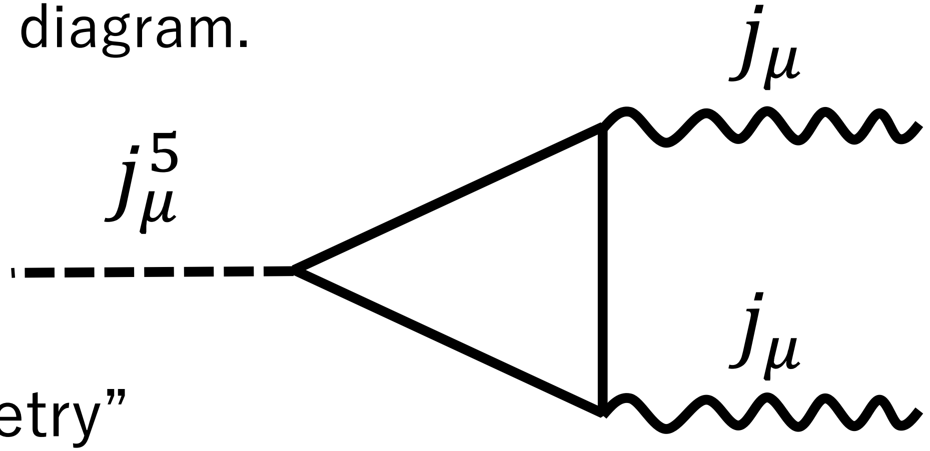
What is “anomaly”?

Anomaly (Quantum Anomaly)

An classical action has some symmetries, but sometimes these symmetries disappear in quantum theory.

$$\psi(x) \to e^{i\gamma_5 \alpha(x)} \psi(x),$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{i\gamma_5 \alpha(x)}$$

- e.g.) $U(1)_A$ anomaly
  - In QCD, there is a $U(1)_A$ symmetry in classical.
  - However, $U(1)_A$ is not a symmetry in quantum theory.
  - $U(1)_A$ current is not conserved through the triangle diagram.



’t Hooft anomaly

- ’t Hooft anomaly = “Anomaly of the global symmetry”
- Invariant from the RG flow.

’t Hooft anomaly matching (1)

In particular, the anomalies are important to the ground state theories!

- Let us take the IR limit.
    - There are two possibilities.

    (1)Gapped theory (The theory has a mass gap.)

    (2)Gapless theory (The theory has NO mass gap.)

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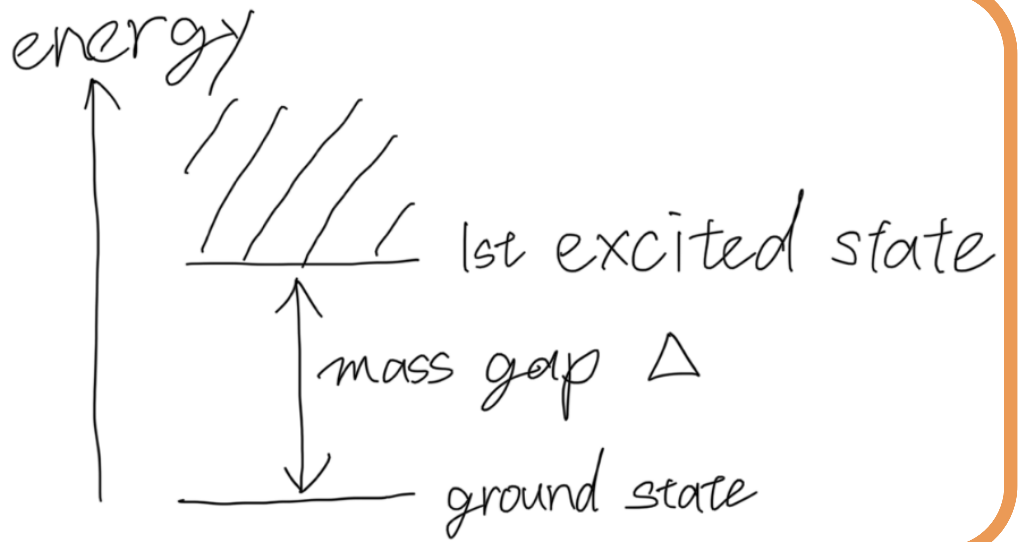
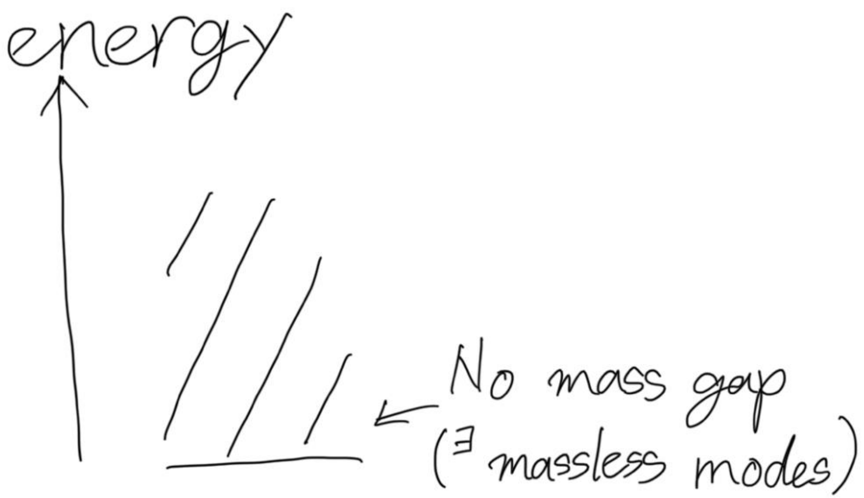
Gapped



Gapless



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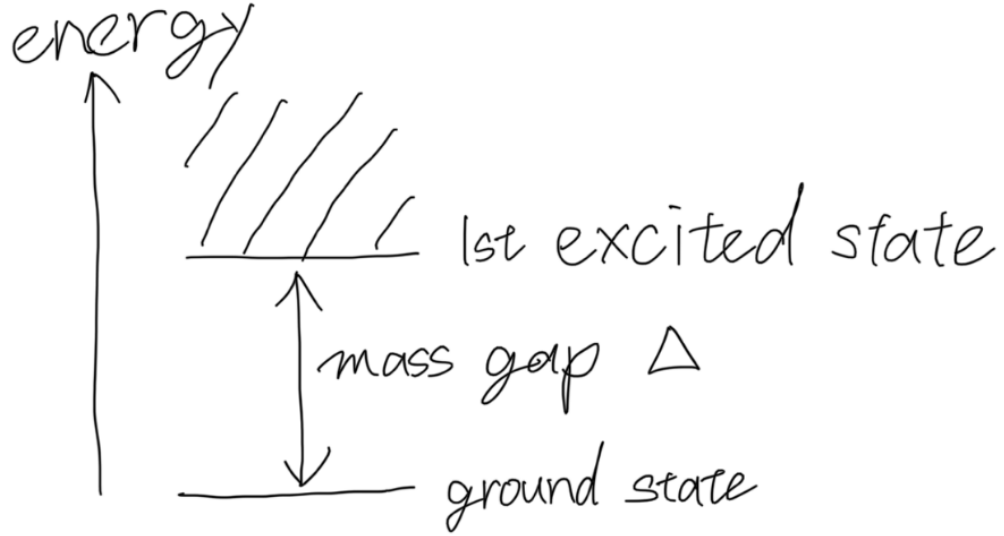
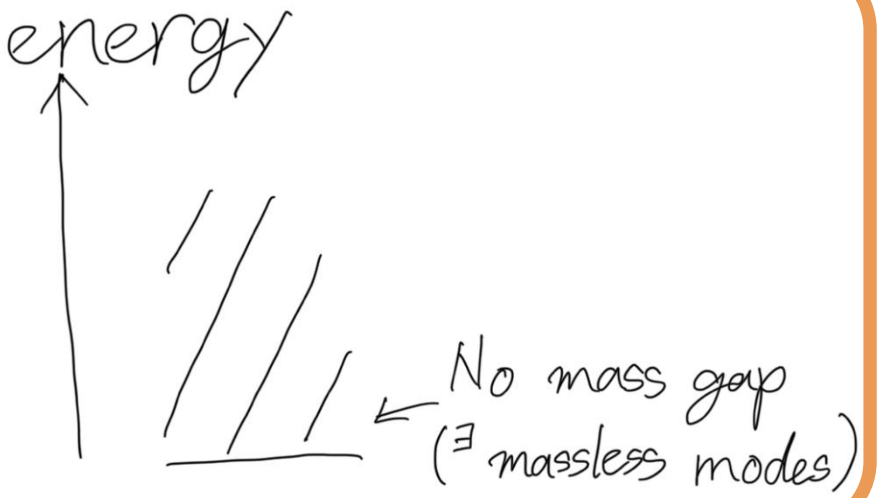
Gapped



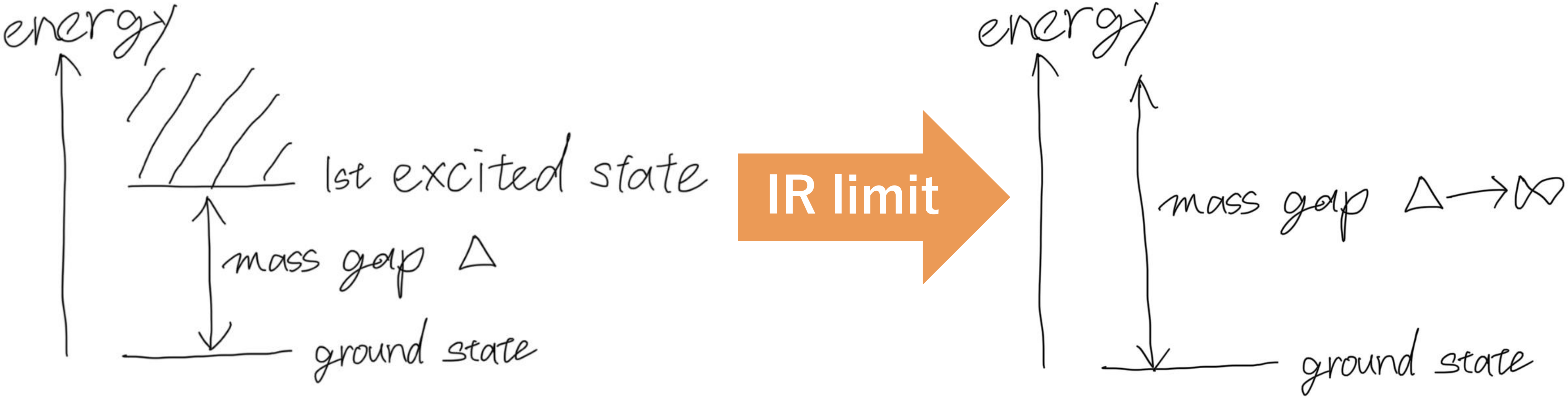
Gapless



’t Hooft anomaly matching (2)

(1)Gapped theory

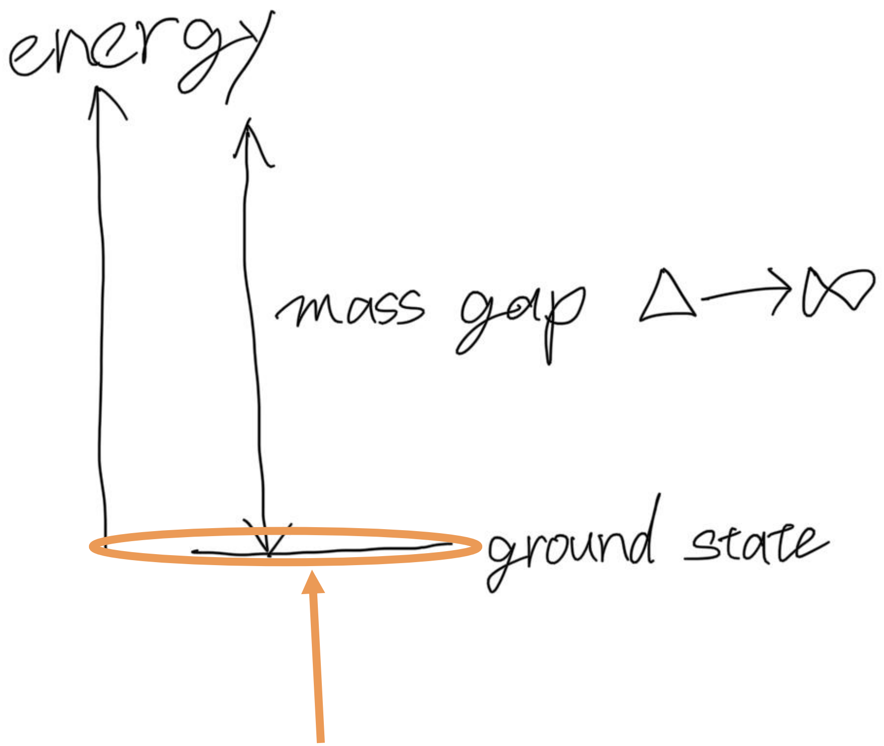
- In IR limit, there is no propagating modes. There is only the ground state.
- If UV theory have some ’t Hooft anomalies, IR theory should match these anomalies.
  - IR theory should have some degenerate vacua.
  - These vacua should have some topological labels. (to match anomalies)



’t Hooft anomaly matching (3)

(1)Gapped theory

- If UV theory have some ’t Hooft anomalies, its vacua should be topological order.
- Topological order
  - There are degenerate vacua and each vacua are distinguished by some topological labels.
  - These vacua can match the anomalies.
- IR theory should be some topological field theory(TQFT).
  - This TQFT is determined by the anomaly.
  - Degeneracy of vacua is also determined.

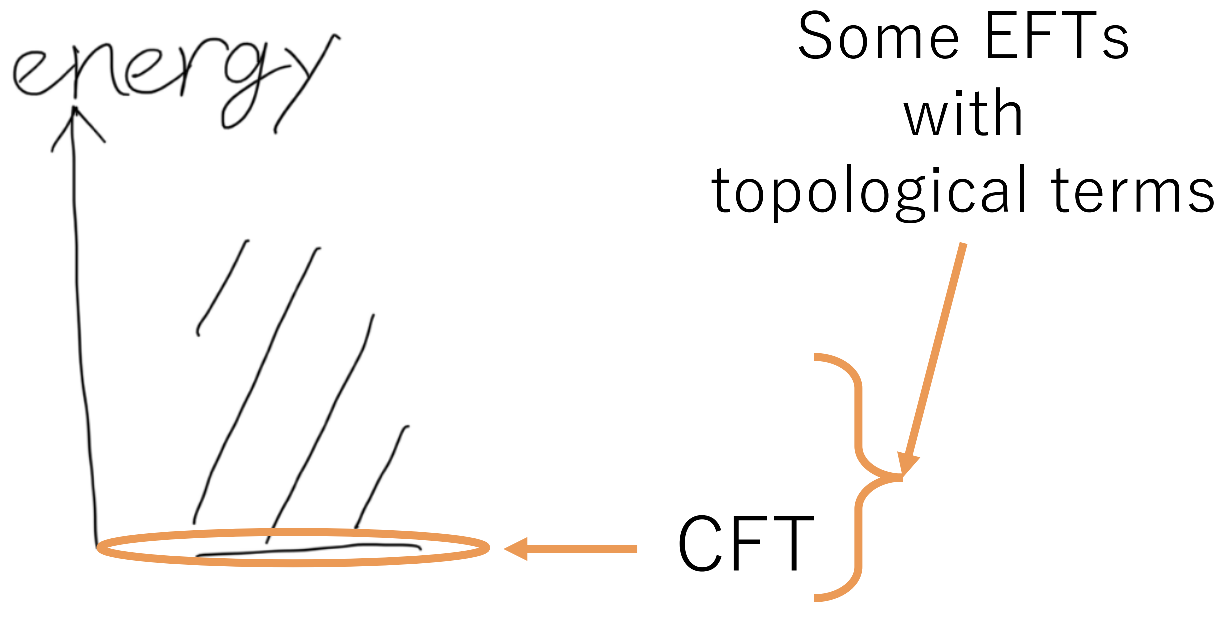


topological order,
TQFT

’t Hooft anomaly matching (4)

(2)Gapless theory

- In IR limit (on the IR fixed point), the theory becomes the conformal field theory(CFT).
- There are some massless propagating modes.
- If UV theory have some anomalies, IR theory (near the IR fixed point) need to have some topological terms. (to match anomalies)



→IR theory is almost determined by the UV anomalies!

e.g. massless QCD (1)

An example of the 't Hooft anomaly matching

- UV Lagrangian is known. ($SU(N_c)$ Yang-Mills + N_f massless quarks)

$$S = \int d^4x \left\{ -\frac{1}{2g^2} \mathrm{tr}\left[F^{\mu\nu} F_{\mu\nu}\right] + \bar{\psi} i \gamma^\mu \left(\partial_\mu + A_\mu\right) \psi \right\}$$

  - In classical theory, UV theory has $U(N_f)_L \times U(N_f)_R$ global symmetry.
  - In quantum, UV theory has $\frac{U(N_f)_L \times U(N_f)_R}{U(1)_A}$ global symmetry.
    - $U(1)_A$ is not symmetry in the quantum theory, because of the (ABJ-type) anomaly.
    - $U(1)_A$ transformation is :

$$\psi(x) \to e^{i\gamma_5 \alpha(x)} \psi(x),$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{i\gamma_5 \alpha(x)}$$

  - IR theory is highly non-trivial, because of the confinement.

e.g. massless QCD (2)

- IR theory should be gapless, because QCD has chiral SSB.
  - Chiral SSB : $\frac{U(N_f)_L \times U(N_f)_R}{U(1)_A} \to U(N_f)_V$
  - There should be massless NG bosons (pions) : massless propagating modes.
  - IR EFT is written by the pions : $\pi(x)$

$$U = \exp(i\pi(x)) \in SU(N_f) \sim \frac{U(N_f)_L \times U(N_f)_R}{U(1)_A \times U(N_f)_V}$$

- From the SSB pattern, the IR pion effective field theory is $SU(N_f)$ nonlinear sigma model.

$$S = \int d^4x \frac{f_\pi^2}{4} \mathrm{tr}[\partial_\mu U \partial^\mu U^\dagger]$$

$U = \exp\left(\frac{i\pi(x)}{f_\pi}\right) \in SU(N_f)$, f_π : the pion decay constant (∼QCD scale → Not CFT)

e.g. massless QCD (3)

’t Hooft anomaly matching

- UV anomaly is known → IR EFT need to have the same anomaly.
- From the ’t Hooft anomaly matching, the topological term is needed.
  - $SU(N_f)_1$ Wess-Zumino-Witten(WZW) term
  - Coupling between the pion and $U(N_f)_L \times U(N_f)_R$ background gauge fields.

→IR EFT : $SU(N_f)$ non-linear sigma model + WZW term

$$S = \int d^4x \frac{f_\pi^2}{4} \mathrm{tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] - \int k \frac{N_c}{240\pi^2} \mathrm{tr}\left[\left(U dU^\dagger\right)^5\right]$$

+(coupling to the background gauge fields)

e.g. massless QCD (4)

Why is this EFT important?

$\pi^0 \to 2\gamma$

- π^0 is unstable particle in the real world : decay into 2γ.
- $SU(N_f)$ non-linear sigma model cannot predict this decay.
- However, WZW term includes coupling between π^0 and γ.
  - SM gauge fields correspond to the background gauge fields.
  - $U(N_f)_L \times U(N_f)_R \supset SU(2)_L \times U(1)_Y$
- Because of the anomaly, π^0 becomes unstable!
  - It is hard to find pions in our daily lives!

The anomaly is very strong tool to understand QFTs.

Plan

1. Introduction to the anomaly (10)
    - What is anomaly?
    - 't Hooft anomaly matching
    - Massless QCD

2. Our work (Overview) (4)

3. Derivation (10)
    - How to calculate anomalies
    - The anomaly for massless case
    - The anomaly for massive case
    - Superconnection
    - The result

4. Application (7+5)
    - Kink, vortex
    - With boundary
    - Index theorem
        - Index for massive case
        - APS index theorem

5. String theory (5)
    - Tachyon condensation

6. Conclusion and future works (2)

2. Our work (Overview)

Relation between the our work and QCD

In our work, we focus on the chiral symmetry.

- We consider free fermion theory with N_f flavors.
- This theory has some global symmetries.
  - $U(N_f)_L \times U(N_f)_R$ symmetry for even dimension
  - $U(N_f)$ symmetry for odd dimension
- Basically, we consider the ’t Hooft anomaly for these theories.
  - Free fermions couple to the background gauge fields for each symmetries.

- What is the new point?

→ We focus on the massive theory with these symmetries.

Theories what we want to think (1)

Let us consider 4dim action contain fermions.

$$S = \int d^4x \bar{\psi} i \not{D} \psi = \int d^4x \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) \psi$$

- This action is massless, so it has a chiral symmetry $U(N_f)_L \times U(N_f)_R$.
- There also be a $U(1)_A$ anomaly.

- Add mass term
  - Mass term breaks the chiral symmetry.

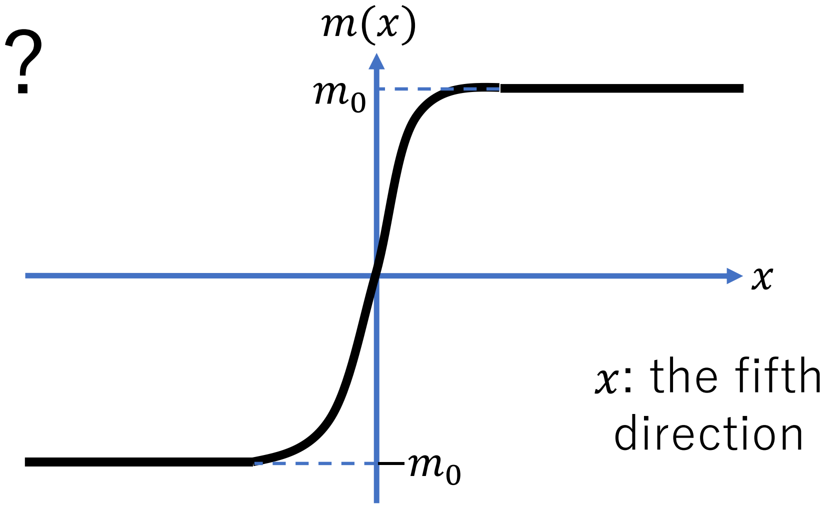
$$S = \int d^4x \bar{\psi} \left(i \not{D} + m \right) \psi$$

- Let the mass depend on the spacetime.
  - This mass is almost same as the Higgs field.
  - How change the symmetry and the anomaly?

$$S = \int d^4x \bar{\psi} \left(i \not{D} + m(x) \right) \psi$$

The spacetime dependent mass

What is "the spacetime dependent mass"?



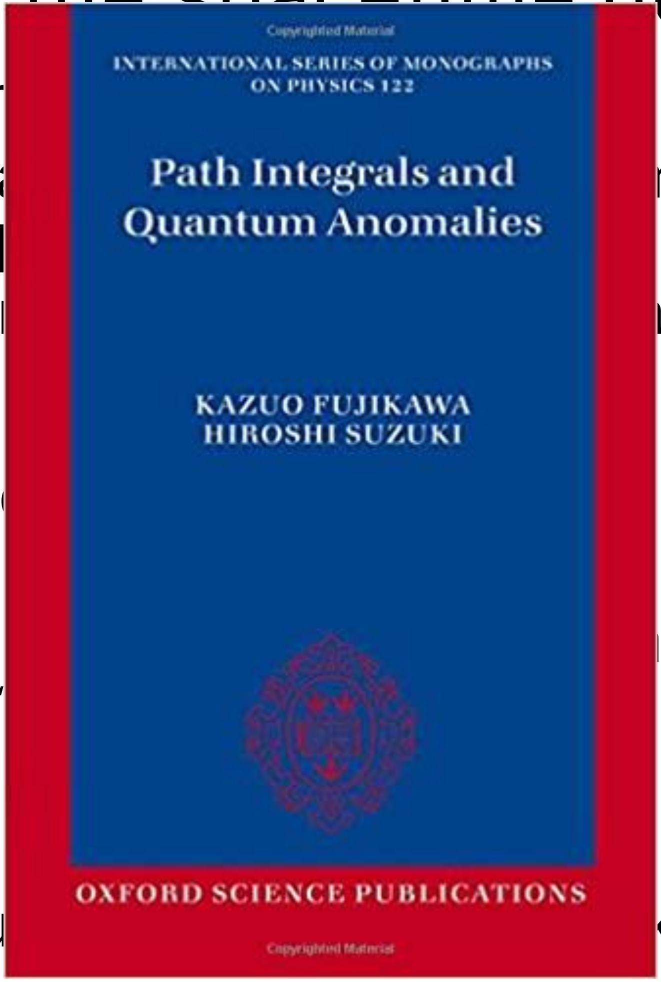
- e.g.) Domain wall fermions
  - One way to realize chiral fermions on the lattice.
  - Consider 5dim spacetime, and realize 4dim fermions on $m(x) = 0$ subspace.
- Chiral anomalies with Higgs fields
  - If Higgs fields change as bifundamental under the $U(N_f)_L \times U(N_f)_R$ chiral symmetry, the action is invariant for the symmetry.
  - It is known that chiral anomalies are not changed by adding Higgs fields.
  - See Fujikawa-san's text book.

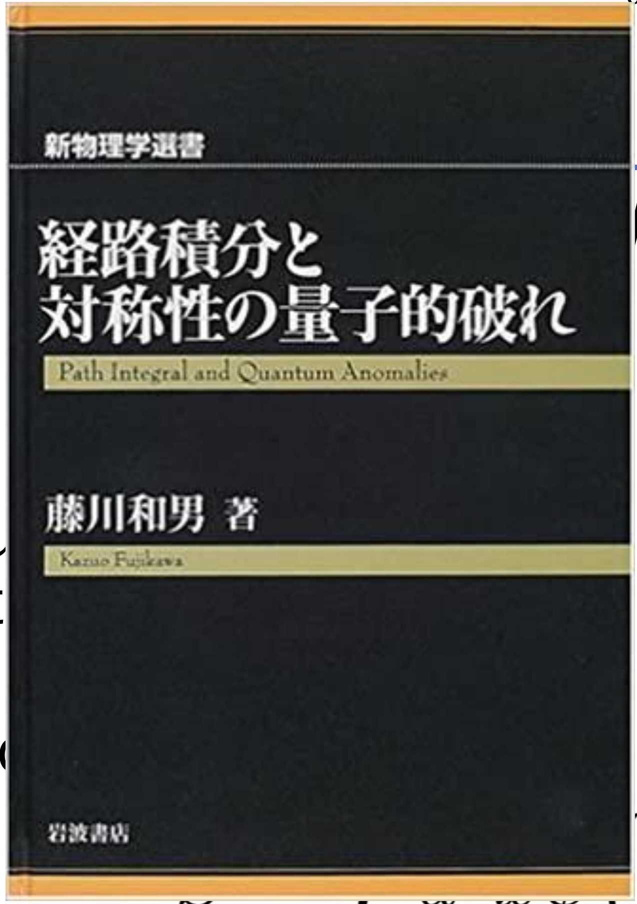
$$S = \int d^4x \bar{\psi}\Big(i\not{D} + h(x)\Big)\psi$$

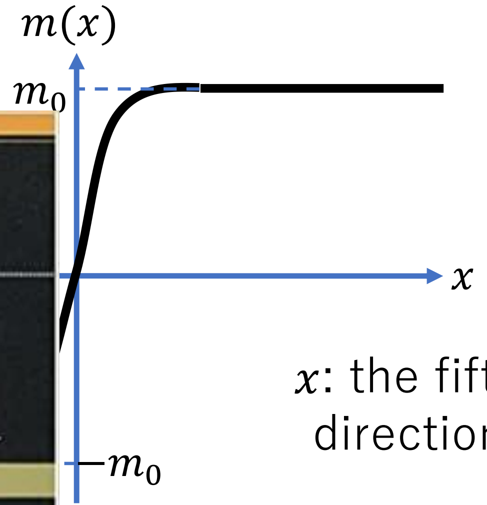
The spacetime dependent mass

What is “the spacetime dependent mass”?

- e.g.) Dom
  - One wa … nions on the
  - Consid … d realize 4di
    fermio … ce.
- Chiral an … elds
  - If Higg … ndamental u
    $U(N_f)_L$ … etry, the act
    invaria
  - It is kn … lies are not
    by add
  - See Fu … k.







x: the fifth direction

$$\int \ldots \left(i\not{D}+h(x)\right)\psi$$

Theories what we want to think (2)

How about the spacetime dependent mass?

- The chiral anomaly is changed by the mass!!
  - Deference between Higgs and mass
    - Higgs field : bounded
    - Spacetime dependent mass : unbounded

$$S = \int d^d x \bar{\psi} \left(i\not{D} + m(x) \right) \psi$$

- If the mass diverges at some points, it contributes to the anomaly.
  - This contribution might be unknown.
  - We can find the anomaly in any dimension.

- The anomaly can be written by "superconnection."

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

3. Derivation

How to calculate anomalies ['79 Fujikawa]

Fujikawa method

- There are several ways to calculate anomalies.
- Today, we focus on the Fujikawa method.
  - Consider path integral for fermions.
  - Anomaly = Jacobian comes from path integral measure
  - We only consider perturbative anomalies.
- We calculate $\log \mathcal{J}$ for anomalies in the last part of this talk.

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathrm{e}^{-S}$$

e.g.) local $U(1)_V$ transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x),$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{-i\alpha(x)}$$

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \to \mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \mathcal{J} \mathcal{D}\psi \mathcal{D}\bar{\psi}$$
$$= \mathrm{e}^{-i\int d^4x \alpha(x) \mathcal{A}(x)} \mathcal{D}\psi \mathcal{D}\bar{\psi}$$

anomaly

$$\log \mathcal{J} = -i \int d^4x \alpha(x) \mathcal{A}(x)$$

Chiral symmetry

Anomalous symmetries we calculate

- $U(N_f)_L \times U(N_f)_R$ chiral symmetry
  - For even dimension
  - Because chirality operators exist only even dimensions.
  - Weyl fermions couple to $U(N_f)_L$ background gauge field A_μ^L and $U(N_f)_R$ background gauge field A_μ^R.
- $U(N_f)$ flavor symmetry
  - For odd dimension
  - No perturbative anomaly as usual.
  - Dirac fermions couple to $U(N_f)$ background gauge field.

For odd dimension

$$S = \int d^d x \bar{\psi} i \gamma^\mu \left\{ \partial_\mu + A_\mu \right\} \psi$$

For even dimension

$$S = \int d^d x \bar{\psi} i \gamma^\mu \left\{ \partial_\mu + A_\mu^R P_+ + A_\mu^L P_- \right\} \psi = \int d^d x (\bar{\psi}_L, \bar{\psi}_R) i \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^{\mu\dagger} & 0 \end{pmatrix} \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

$$= \int d^d x \left\{ \bar{\psi}_R i \sigma^{\mu\dagger} (\partial_\mu + A_\mu^R) \psi_R + \bar{\psi}_L i \sigma^\mu (\partial_\mu + A_\mu^L) \psi_L \right\}$$

$$P_\pm = \frac{1 \pm \gamma_{d+1}}{2}$$

The anomaly for massless cases

e.g.) fermions in 4dim

- Mass less case
  - With $U(N_f)_L \times U(N_f)_R$ chiral sym.
  - $U(1)_V$ anomaly is written by the field strengths.
- With a Higgs field
  - With $U(N_f)_L \times U(N_f)_R$ chiral sym.
  - The $U(1)_V$ anomaly is same for massless case.
- How about the massive case?

$$S = \int d^4x \bar{\psi} i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^L & 0 \\ 0 & A_\mu^R \end{pmatrix} \right\} \psi$$

$$= \int d^4x \left\{ \bar{\psi}_R i\sigma^{\mu\dagger} (\partial_\mu + A_\mu^R) \psi_R + \bar{\psi}_L i\sigma^\mu (\partial_\mu + A_\mu^L) \psi_L \right\}$$

$$\log \mathcal{J} = \frac{i}{32\pi^2} \int d^4x \alpha(x) \epsilon^{\mu\nu\rho\sigma} \mathrm{tr} \left[F_{\mu\nu}^R F_{\rho\sigma}^R - F_{\mu\nu}^L F_{\rho\sigma}^L \right]$$

$$= \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr} \left[F^R \wedge F^R - F^L \wedge F^L \right]$$

$$S = \int d^4x \bar{\psi} \left(i\not{D} + h(x) \right) \psi$$

$$= \int d^4x \left\{ \bar{\psi}_R i\sigma^{\mu\dagger} D_\mu^R \psi_R + \bar{\psi}_L i\sigma^\mu D_\mu^L \psi_L + \bar{\psi}_L h \psi_R + \bar{\psi}_R h^\dagger \psi_L \right\}$$

The anomaly for massive case (1)

Let us consider spacetime dependent mass!

- The action for general even dim with $U(N_f)_L \times U(N_f)_R$ symmetry is,

$$S = \int d^d x \bar{\psi} \left[i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} + \begin{pmatrix} im(x) & 0 \\ 0 & im^\dagger(x) \end{pmatrix} \right] \psi$$

$$= \int d^d x \left\{ \bar{\psi}_R i\sigma^{\mu\dagger} D_\mu^R \psi_R + \bar{\psi}_L i\sigma^\mu D_\mu^L \psi_L + \bar{\psi}_L im(x) \psi_R + \bar{\psi}_R im^\dagger(x) \psi_L \right\}$$

$$\equiv \int d^d x \bar{\psi} \mathcal{D} \psi$$

- We assume this $\mathcal{D}$ is the Dirac op. for massive case.
  - $\mathcal{D}$ is non-Hermitian.
- For odd dim case, there is only $U(N_f)$ sym, we put $A_\mu = A_\mu^R = A_\mu^L$ and $m = m^\dagger$.

The anomaly for massive case (2)

Calculate the $U(1)_V$ anomaly for this action by Fujikawa method.

- We **take $m(x)$ divergent**.

$$|m(x^I)| \to \infty \quad (|x^I| \to \infty)$$

  - I denotes some directions which $m(x)$ changes its values.
  - If $m(x)$ does not diverge, the anomaly is same for the massless case.

- We use the heat kernel regularization.
  - We set a UV cut-off for the eigenvalues of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$.

$$\mathrm{e}^{-\frac{\mathcal{D}^\dagger\mathcal{D}}{\Lambda^2}}, \quad \mathrm{e}^{-\frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2}}$$

Λ is the UV cut-off.

- Generalizations
  - It is easy to get the anomaly for any dimension.
  - It is also easy to get the anomaly for $U(N_f)_L \times U(N_f)_R$, not only for $U(1)_V$.

The anomaly for massive case (3)

e.g.) In 4dim case, the $U(1)_V$ anomaly is, $\tilde{m} = m/\Lambda$

Λ is the UV cut-off comes from heat kernel regularization.

$$\log \mathcal{J} = \frac{i}{(2\pi)^2} \int d^4x \alpha(x) \mathrm{tr} \Bigg[\epsilon^{\mu\nu\rho\sigma} \Bigg\{ \frac{1}{8} \left(F^R_{\mu\nu} F^R_{\rho\sigma} - F^L_{\mu\nu} F^L_{\rho\sigma} \right)$$
$$+ \frac{1}{12} \Big(D_\mu \tilde{m}^\dagger D_\nu \tilde{m} F^R_{\rho\sigma} - D_\mu \tilde{m} D_\nu \tilde{m}^\dagger F^L_{\rho\sigma} + F^R_{\mu\nu} D_\rho \tilde{m}^\dagger D_\sigma \tilde{m}$$
$$- F^L_{\mu\nu} D_\rho \tilde{m} D_\sigma \tilde{m}^\dagger - D_\mu \tilde{m} F^R_{\nu\rho} D_\sigma \tilde{m}^\dagger + D_\mu \tilde{m}^\dagger F^L_{\nu\rho} D_\sigma \tilde{m} \Big)$$
$$+ \frac{1}{24} \left(D_\mu \tilde{m}^\dagger D_\nu \tilde{m} D_\rho \tilde{m}^\dagger D_\sigma \tilde{m} - D_\mu \tilde{m} D_\nu \tilde{m}^\dagger D_\rho \tilde{m} D_\sigma \tilde{m}^\dagger \right) \Bigg\} \Bigg] \mathrm{e}^{-\tilde{m}^\dagger \tilde{m}}$$

- This result seems very complicated…
- Can we rewrite it more simple way? → Superconnection!

Superconnection (1) ['85 Quillen]

- We define the superconnections for even and odd dimensions.
- This is made by Quillen, who is a mathematician, in 1985.

Even dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

$A_R : U(N_f)_R$ gauge field (1-form)

$A_L : U(N_f)_L$ gauge field (1-form)

$T : U(N_f)_L \times U(N_f)_R$ bifundamental scalar field (0-form)

- Field strength

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$$

$$\equiv \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix}$$

- Supertrace

$$\mathrm{Str} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

Superconnection (2)

['85 Quillen]

Odd dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A & iT \\ iT & A \end{pmatrix}$$

$A : U(N_f)$ gauge field (1-form)

$T : U(N_f)$ adjoint scalar field (0-form)

- Field strength

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = \begin{pmatrix} F - T^2 & iDT \\ iDT & F - T^2 \end{pmatrix}$$

- Supertrace

$$\mathrm{Str} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i}\,\mathrm{tr}(b)$$

We apply superconnection to write the anomaly.

The result (1)

- We can rewrite the $U(1)_V$ anomaly by superconnection.

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}}$$

$$\mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

$$\tilde{m} = m/\Lambda$$

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2 \equiv \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix}$$

$$\mathrm{Str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

- For odd dimension case, put $A_\mu = A_\mu^R = A_\mu^L$ and $m = m^\dagger$. Then, we get $U(1)$ anomaly.
  - In odd dimension, the definition of Str is different from the even dim case.

$$\mathrm{Str}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i}\,\mathrm{tr}(b)$$

The result (2)

$$\log \mathcal{J} = -i \left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x) \mathrm{Str}\left[\mathrm{e}^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}}$$

$$\mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

- In this formula, $m(x)$ appears only as $\tilde{m}(x)$.

$$\tilde{m} = m/\Lambda$$

  - Λ is the UV cut-off comes from the heat kernel regularization.
  - If $m(x)$ is finite, the mass dependence disappears because we need to take $\Lambda \to \infty$.
  - The Λ dependence (or the regulator dependence) of the anomaly disappears after we integrate $\mathrm{Str}[\mathrm{e}^{\mathcal{F}}]$ over the spacetime.

- It is easy to check this anomaly is consistent with 4dim massless case.

$$\log \mathcal{J} = \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F^R \wedge F^R - F^L \wedge F^L\right]$$

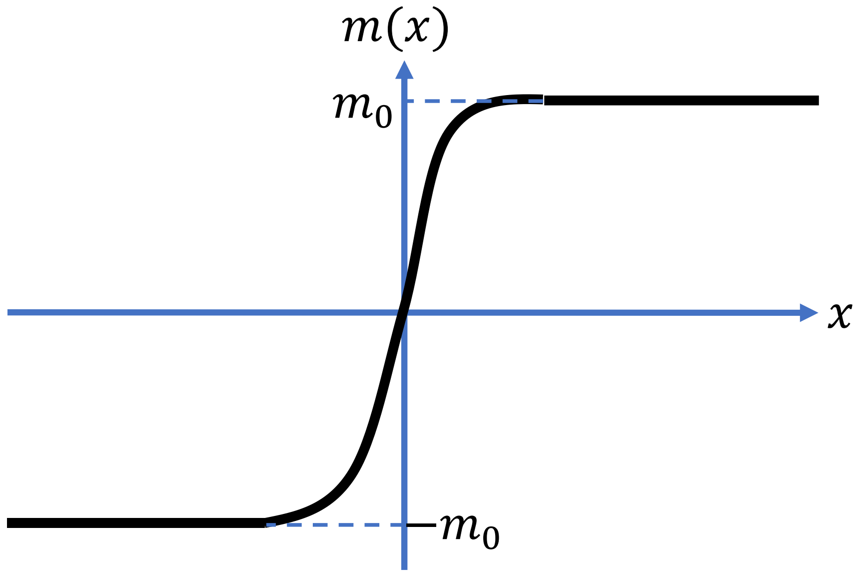
4. Application

How can we apply this anomaly?

Mass means a wall for some cases!

- e.g.) Domain wall
  - Can we make domain walls by this $m(x)$?

  →Yes!

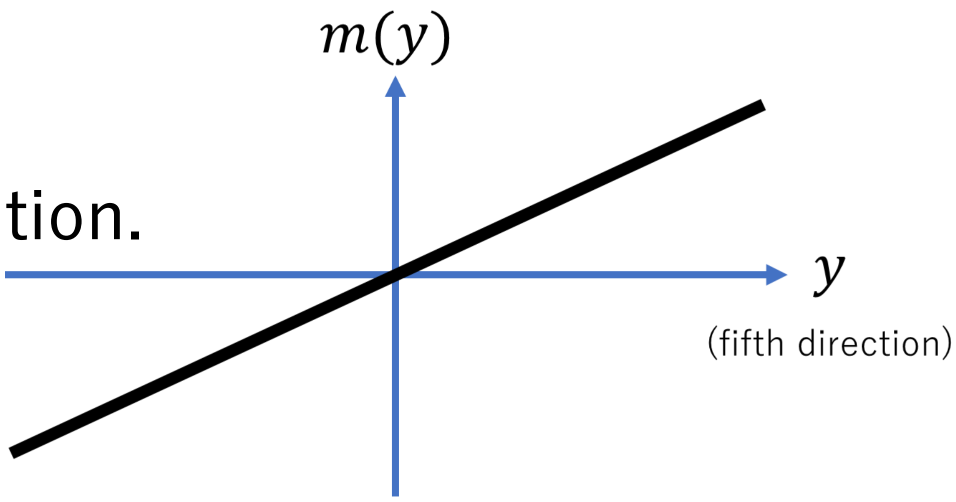


- We can make some systems with boundaries.
  - Kink, vortex and general codimension case
  - With boundary
- We also discuss about some index theorems.
  - APS index theorem
  - Callias type index theorem

Kink (1)

Mass kink for our set up

- For example, let's consider 5dim case.
- In our set up, "kink" means this mass configuration.



$$m(y) = uy \qquad y = x^5 \quad u \in \mathbb{R}$$

  - This "mass" diverges at $y \to \pm\infty$.
  - 5dim fermions with $U(N_f)$ sym, and the mass depends on only y direction.

- The $U(1)$ anomaly is,

$$\log \mathcal{J} = \pm \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F \wedge F\right]$$

Corresponds to the sign of u.

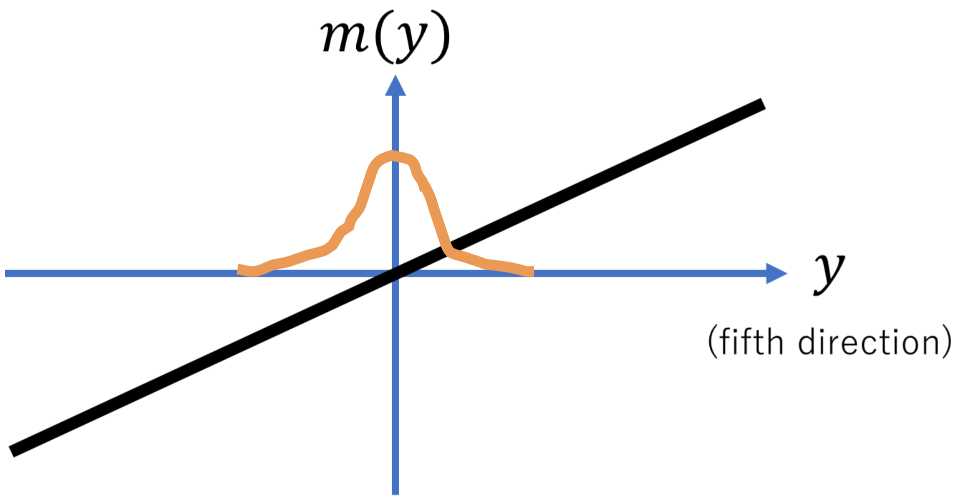
- Recall 4dim $U(1)_V$ anomaly,

$$\log \mathcal{J} = \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F^R \wedge F^R - F^L \wedge F^L\right]$$

Kink (2)

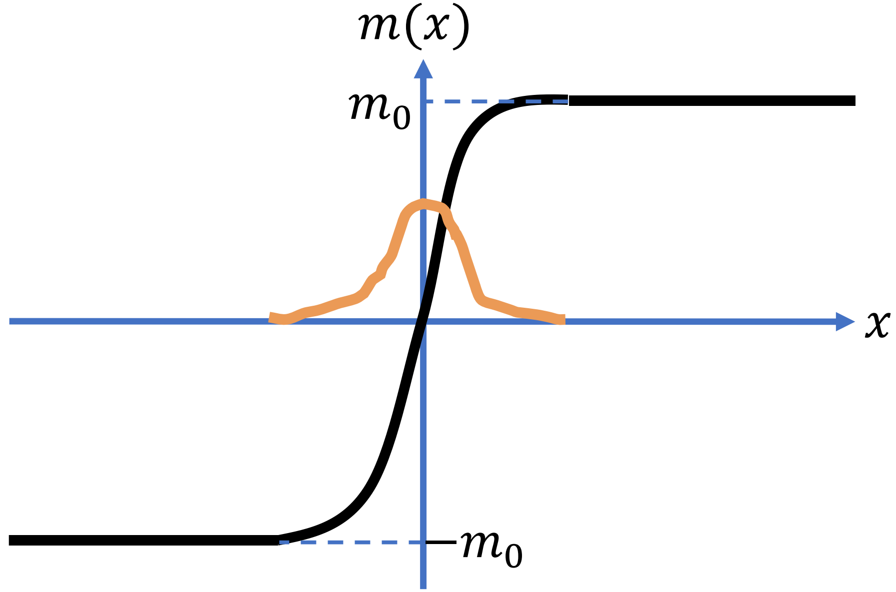
What is the meaning of the anomaly?

- 4dim Weyl fermions are localized at $y = 0$.
  - $u > 0$ corresponds to chirality $+$ (right-handed) fermion, and $u < 0$ corresponds to chirality $-$ (left-handed) fermion.



Domain wall fermion

- This Weyl fermions correspond to domain wall fermions.
  - But the regularization is different, so that I don't know the correspondence in detail.



Vortex

Next, we check codim-2 case.

$$m(z) = uz\mathbf{1}_{N\times N}$$
$$z = x^{\mu=2r+1} - ix^{\mu=2r+2}$$

- Vortex is 2dim topological object.
- Let us consider $2r + 2$ dim.
  - $m(z)$ depends on 2 directions, and it is complex valued “mass”.
  - This mass diverges at $|z| \to \infty$.
- For simplicity, we put $A_L = A_R$ in $2r + 2$dim.
- The $U(1)_V$ anomaly is,

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^r \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^F\right]\Big|_{2r-\mathrm{form}}$$

- This is $2r$dim $U(1)$ anomaly with $U(N_f)_R$ gauge field.
  - If you want to get chirality – (left-handed) result, use $m(\bar{z}) = u\bar{z}$, instead.

General defects

We can apply this formula to general codimension cases.

- When we think d dim system with n dim topological defects, we get $d-n$ dim $U(1)$ anomalies.
  - If $d-n$ is odd, we get nothing because odd dim mass less fermions are anomaly-free.
  - The mass configurations for general codimension is,

$$m(x) = u\sum_{I=1}^{n}\Gamma^I x^I$$

$$\gamma^I = \Gamma^I \qquad (n = odd)$$

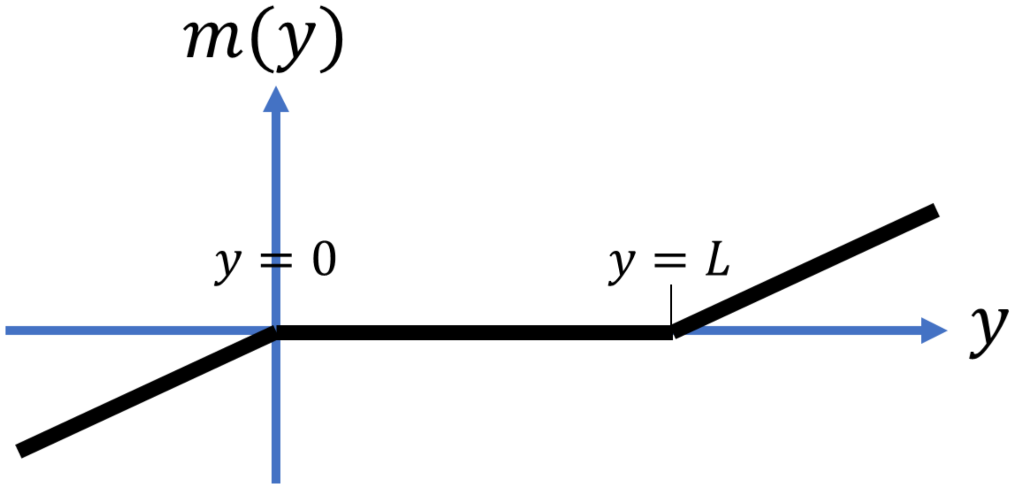
$$\gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix} (n = even)$$

- This results correspond to "tachyon condensation" in string theory.
  - We will discuss about it in section 5.

With boundary (1)

Let us make some boundaries.

- Fermions are massive = boundary



Odd dimension (2r dim)

- We realize localized fermions at $[0, L]$.
- The bulk is anomaly-free.
- The anomaly is,

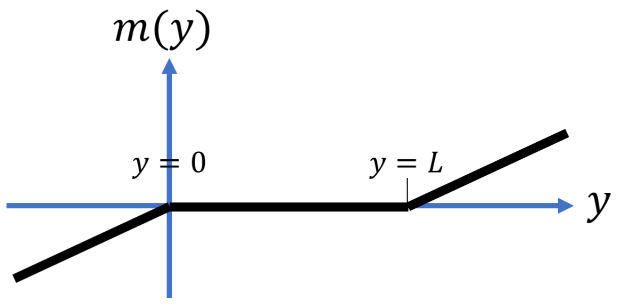
$$m(y) = \mu(y) 1_N = \begin{cases} (m_0 + u'(y-L)) 1_N & (L < y) \\ m_0 1_N & (0 \le y \le L) \\ (m_0 + uy) 1_N & (y < 0) \end{cases}$$

$$\log \mathcal{J} = i\kappa_- \int_{y=0} \alpha \left[\mathrm{ch}(F)\right]_{2r} + i\kappa_+ \int_{y=L} \alpha \left[\mathrm{ch}(F)\right]_{2r}$$

$$\kappa_- = \frac{1}{2}\mathrm{sgn}(u) \ , \quad \kappa_+ = \frac{1}{2}\mathrm{sgn}(u')$$

With boundary (2)

Even dimension



$$m(x) = \mu(y) g(x) = \begin{cases} u'(y-L)g(x) & (L < y) \\ 0 & (0 \le y \le L) \\ uyg(x) & (y < 0) \end{cases}$$

- The anomaly is,

$$\log \mathcal{J} = -i \int_{0<y<L} \alpha \left[\mathrm{ch}(F_+) - \mathrm{ch}(F_-) \right]_{2r} - i \int_{y=L} \alpha [\omega]_{2r-1} + i \int_{y=0} \alpha [\omega]_{2r-1}$$

  - ω is Chern-Simons form.
  - Anomaly from bulk + CS

5. String theory

String theory

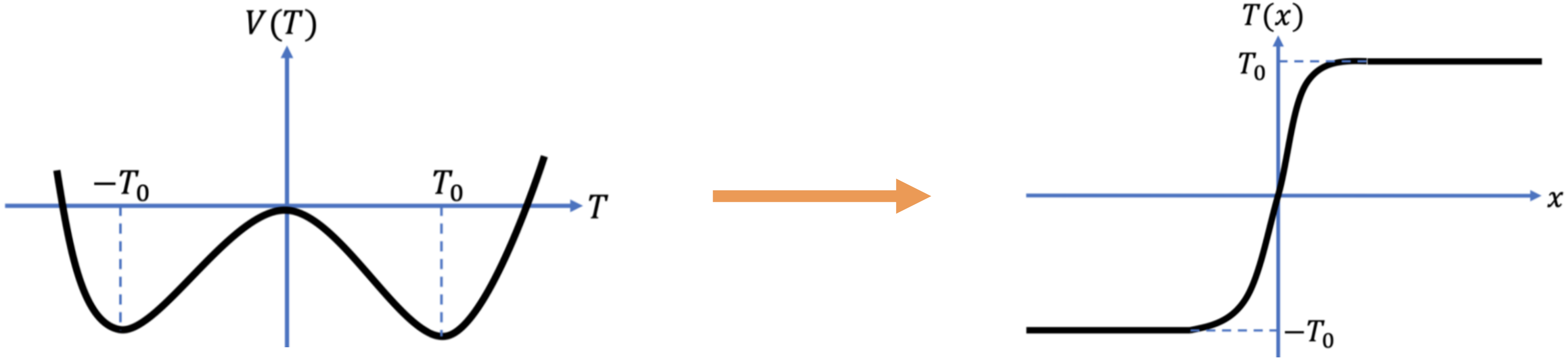
Let us check the relation between this anomaly and string theory.

- Consider type IIA or IIB string theory with D-branes.
  - Open strings have their ends on D-branes.
  - Excitation modes of these open strings → Fields on D-branes
  - Open strings on D_p-branes → QFT in $p+1$ dim

- In some cases, excitation modes of the strings have tachyon modes.
  - Lowest excitation modes are $m^2 < 0$. (Tachyon mode)
  - Non-BPS states have tachyons.
  - This tachyonic modes are unstable. → Tachyon condensation
  - See Sen's review [hep-th/9904207].

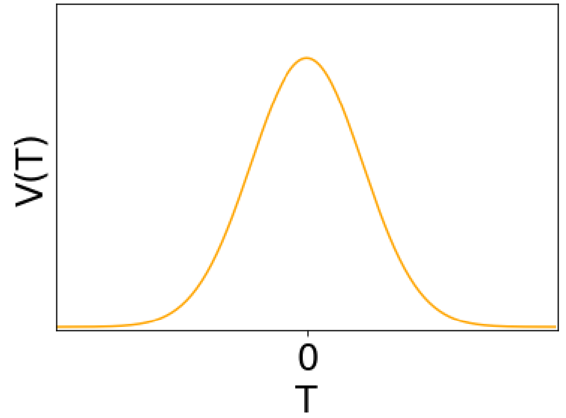
Tachyon condensation (1)

- Tachyonic modes are unstable, so the tachyons have non-zero VEV.
  - Non-trivial configuration of tachyon is also realizable.



e.g.) D-brane and anti D-brane ($\bar{D}$-brane) system

- Non-BPS state
- Tachyonic modes appear in $D - \bar{D}$ string.
  - The shape of tachyon potential is known. $V(T) = \mathrm{e}^{-T^\dagger T}$
  - If tachyon configuration is trivial, the D-branes disappear.



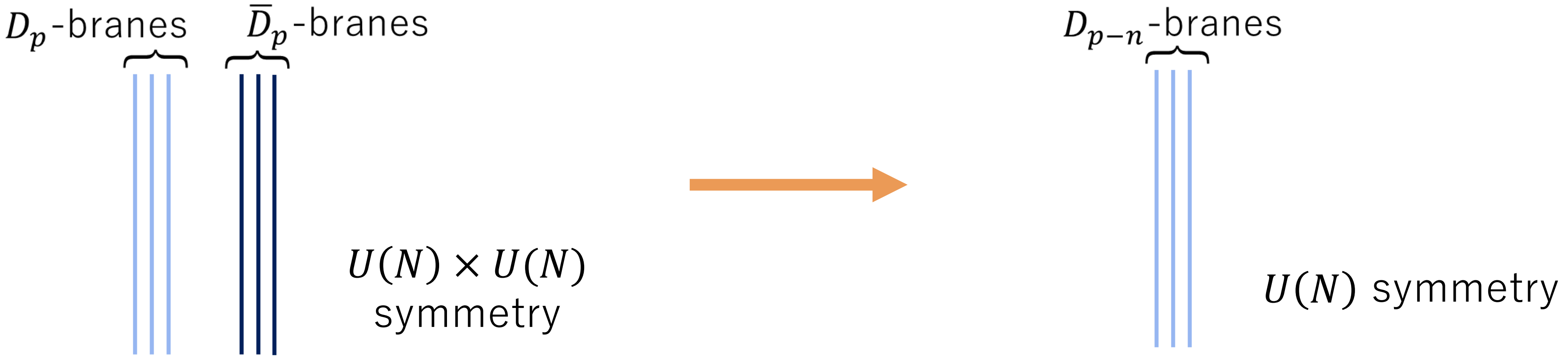
Tachyon condensation (2)

Kink on tachyon in $D_p - \overline{D}_p$ system

- Tachyonic kinks for this system is,

$$T(x) = u \sum_{I=1}^{n} \Gamma^I x^I \qquad \gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix}$$

- We get D_{p-n}-branes from this tachyon.
  - If D_{p-n}-branes are non-BPS, tachyons still exist on the D-branes.
  - In this case, tachyon condensation occur again.



Tachyon condensation (3)

- The superconnection is used in the context of tachyon condensation.
  - This structure comes from RR-coupling of D-branes.

cf.) ['98 Witten] [hep-th/9810188]
['99 Kennedy-Wilkins] [hep-th/9905195]
['01 Kuraus-Larsen] [hep-th/0012198]
['01 Takayanagi-Terashima-Uesugi] [hep-th/0012210]

$$S = T_{D9} \int C \wedge \mathrm{Str}\ e^{2\pi\alpha' i\mathcal{F}}$$

- The tachyon configuration is given by ['98 Witten].
  - In this paper, relation between tachyon condensation and K-theory is discussed.
  - This tachyon configuration comes from ['64 Atiyah-Bott-Shapiro]

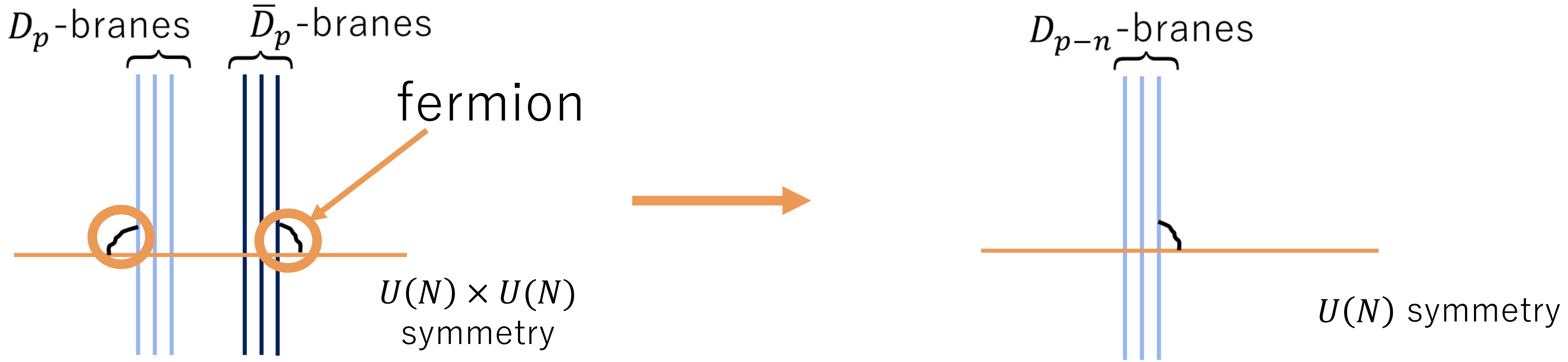
$$T(x) = u \sum_{I=1}^{n} \Gamma^I x^I$$

Relation between the anomaly and string

- This tachyon configuration is same for the mass defect in section 3!

$$T(x) = u\sum_{I=1}^{n}\Gamma^I x^I \qquad\qquad m(x) = u\sum_{I=1}^{n}\Gamma^I x^I$$

- This anomaly can be understood from string theory.
  - Fermions are found where D-branes intersect.
  - This is similar to flavor symmetry on holographic QCD model. (Sakai-Sugimoto model)



Conclusion and future works (1)

Conclusion

- We discussed about perturbative anomaly with spacetime dependent mass.
  - If value of the mass diverge, non-trivial contribution of the mass appears.
- The anomaly can be written by superconnection.

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x) \mathrm{Str}\left[\mathrm{e}^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}} \qquad \mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

- There are some applications.
  - Kink, vortex, ...
  - With boundary
  - Index theorem

Conclusion and future works (2)

Future (or on-going) works

- Anomaly mathching for massive fermion
  - This new anomaly should be matched for all theories.
  - Wess-Zumino-Witten term is important for this case.
- Get Wess-Zumino-Witten term for massive fermion
  - On-going work with Sugimoto-san.
  - Some applications??

- Is there any application to condensed matter physics?
  - Application to higher Berry curvature : Choi and Ohmori [hep-th/2205.02188]
    - In this set up, the structure of mass parameter space is important.
  - Any more applications??