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Anomaly and Superconnection

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Motivation

QFTを理解したい

- 知っていること
  - 弱結合の理論(例:QED)は摂動論で理解できる
  - 強結合の理論(例:QCD)は良く分からない
- 高エネルギー(UV)の作用が分かっている理論でも、低エネルギー(IR)で強結合の場合、IRでどのような理論になるのか非自明
  - QCD
  - 物性理論

- アノマリー(量子異常)はIRの理論を理解するために使える!
  - ’t Hooft anomaly matching

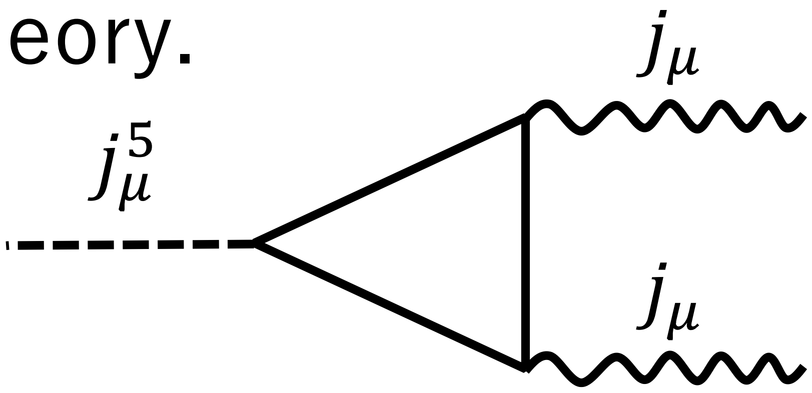
  →UVの理論の持つアノマリーをIRの理論(有効作用、EFT)は再現しないとならない

  →IRの理論の形を強く制限する

What is "anomaly"? (1)

Anomaly (Quantum Anomaly)

An classical action has some symmetries, but sometimes these symmetries disappear in quantum theory.



e.g.) $\pi^0 \to 2\gamma$

- In massless QCD, there is a chiral symmetry $SU(N_f)_L \times SU(N_f)_R$.

  N_f: # of flavors
- If there is NO anomaly, π^0 never decays.
- However, π^0 decays into 2γ, because of an anomaly!

Theories what we want to think (1)

Let us consider 4dim action contain fermions.

$$S = \int d^4x \bar{\psi} i \not{D} \psi = \int d^4x \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) \psi$$

- This action is massless, so it has a chiral symmetry $U(N_f)_L \times U(N_f)_R$.
- There also be a $U(1)_A$ anomaly.

- Add mass term
  - Mass term breaks the chiral symmetry.

$$S = \int d^4x \bar{\psi} \left(i \not{D} + m \right) \psi$$

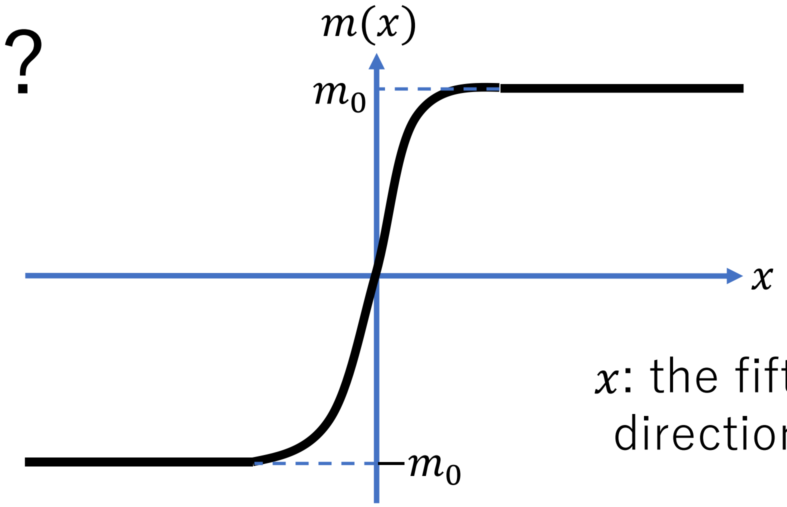
- Let the mass depend on the spacetime.
  - This mass is almost same as the Higgs field.
  - How change the symmetry and the anomaly?

$$S = \int d^4x \bar{\psi} \left(i \not{D} + m(x) \right) \psi$$

The spacetime dependent mass

What is “the spacetime dependent mass”?

- e.g.) Domain wall fermions
  - One way to realize chiral fermions on the lattice.
  - Consider 5dim spacetime, and realize 4dim fermions on $m(x) = 0$ subspace.



x: the fifth direction

- Chiral anomalies with Higgs fields
  - If Higgs fields change as bifundamental under the $U(N_f)_L \times U(N_f)_R$ chiral symmetry, the action is invariant for the symmetry.
  - It is known that chiral anomalies are not changed by adding Higgs fields.
  - See Fujikawa-san’s text book.

$$S = \int d^4x \bar{\psi} \left(i\not{D} + h(x) \right) \psi$$

The spacetime dependent mass

What is "the spacetime dependent mass"?

- e.g.) Dom
  - One wa … nions on the
  - Consid … d realize 4di
    fermio … ce.

- Chiral an … elds
  - If Higg … ndamental u
    $U(N_f)_L$ … etry, the act
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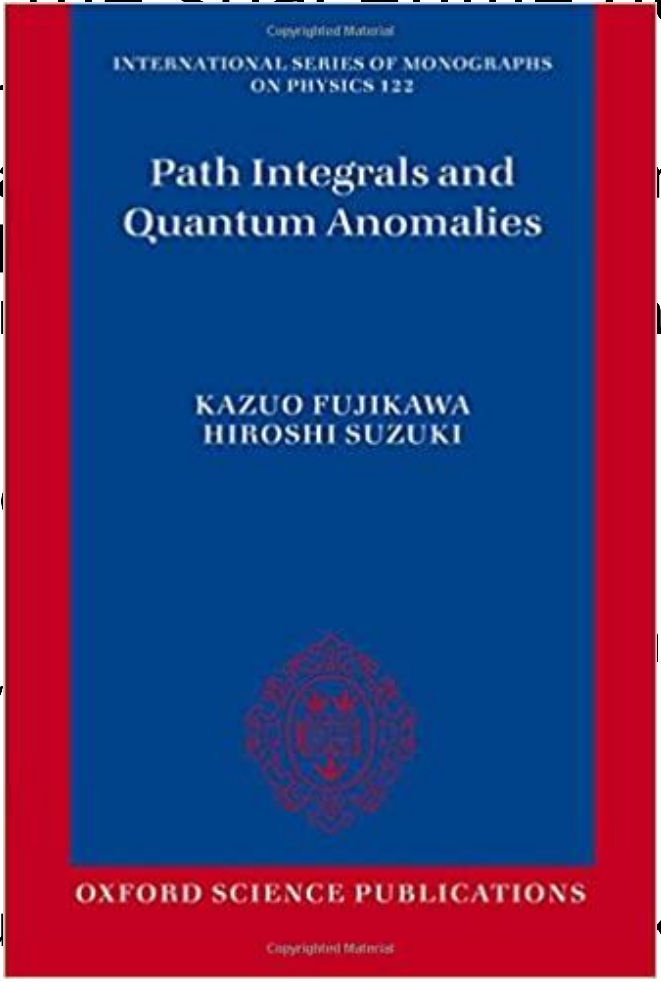
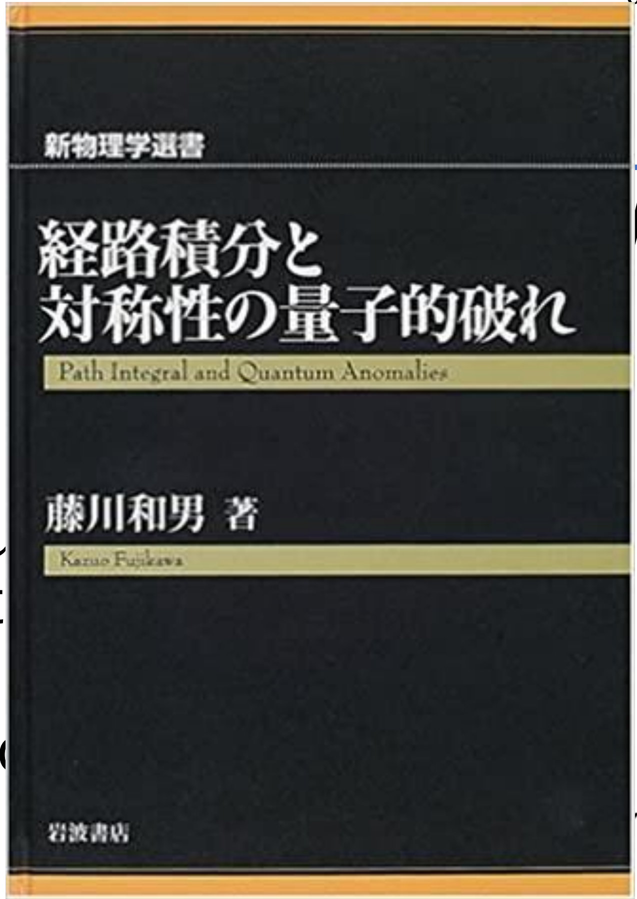
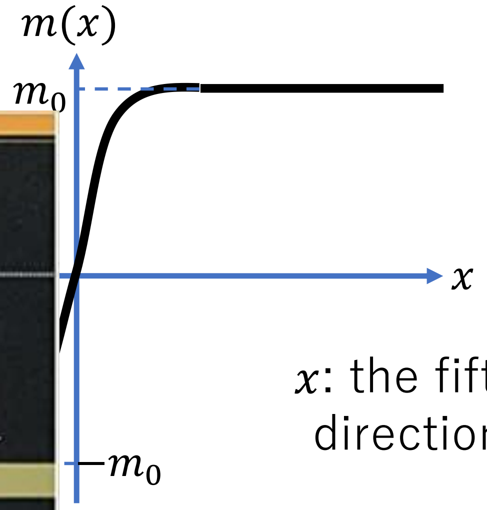
  - See Fu … k.







x: the fifth direction

$$\ldots \int \ldots \left(i\not{D} + h(x)\right)\psi$$

Theories what we want to think (2)

How about the spacetime dependent mass?

- The chiral anomaly is changed by the mass!!
    - Deference between Higgs and mass
        - Higgs field : bounded
        - Spacetime dependent mass : unbounded

$$S = \int d^d x \bar{\psi} \left(i \not{D} + m(x) \right) \psi$$

- If the mass diverges at some points, it contributes to the anomaly.
    - This contribution might be unknown.
    - We can find the anomaly in any dimension.

- The anomaly can be written by "superconnection."

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

Plan

1. Introduction (5)
    - What is anomaly?
    - Theories what we want to think

2. Derivation (10)
    - How to calculate anomalies
    - The anomaly for massless case
    - The anomaly for massive case
    - Superconnection
    - The result

3. Application (7)
    - Kink, vortex
    - With boundary

4. Index theorem (5)
    - Index for massive case
    - APS index theorem

5. String theory (5)
    - Tachyon condensation

6. Conclusion (1)

How to calculate anomalies

['79 Fujikawa]

Fujikawa method

- There are several ways to calculate anomalies.
- Today, we focus on the Fujikawa method.
  - Consider path integral for fermions.
  - Anomaly = Jacobian comes from path integral measure
  - We only consider perturbative anomalies.
- We calculate $\log \mathcal{J}$ for anomalies in the last part of this talk.

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathrm{e}^{-S}$$

e.g.) local $U(1)_V$ transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x),$$

$$\bar{\psi}(x) \to \bar{\psi}(x) e^{-i\alpha(x)}$$

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \to \mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \mathcal{J} \mathcal{D}\psi \mathcal{D}\bar{\psi}$$

$$= \mathrm{e}^{-i\int d^4x \alpha(x) \mathcal{A}(x)} \mathcal{D}\psi \mathcal{D}\bar{\psi}$$

anomaly

$$\log \mathcal{J} = -i \int d^4x \alpha(x) \mathcal{A}(x)$$

Chiral symmetry

Anomalous symmetries we calculate

- $U(N_f)_L \times U(N_f)_R$ chiral symmetry
  - For even dimension
  - Because chirality operators exist only even dimensions.
  - Weyl fermions couple to $U(N_f)_L$ background gauge field A_μ^L and $U(N_f)_R$ background gauge field A_μ^R.

- $U(N_f)$ flavor symmetry
  - For odd dimension
  - No perturbative anomaly as usual.
  - Dirac fermions couple to $U(N_f)$ background gauge field.

For odd dimension

$$S = \int d^d x \bar{\psi} i \gamma^\mu \left\{ \partial_\mu + A_\mu \right\} \psi$$

For even dimension

$$S = \int d^d x \bar{\psi} i \gamma^\mu \left\{ \partial_\mu + A_\mu^R P_+ + A_\mu^L P_- \right\} \psi = \int d^d x (\bar{\psi}_L, \bar{\psi}_R) i \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^{\mu\dagger} & 0 \end{pmatrix} \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

$$= \int d^d x \left\{ \bar{\psi}_R i \sigma^{\mu\dagger} (\partial_\mu + A_\mu^R) \psi_R + \bar{\psi}_L i \sigma^\mu (\partial_\mu + A_\mu^L) \psi_L \right\}$$

$$P_\pm = \frac{1 \pm \gamma_{d+1}}{2}$$

The anomaly for massless cases

e.g.) fermions in 4dim

- Mass less case
  - With $U(N_f)_L \times U(N_f)_R$ chiral sym.
  - $U(1)_V$ anomaly is written by the field strengths.
- With a Higgs field
  - With $U(N_f)_L \times U(N_f)_R$ chiral sym.
  - The $U(1)_V$ anomaly is same for massless case.
- How about the massive case?

$$S = \int d^4x \bar{\psi} i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A^L_\mu & 0 \\ 0 & A^R_\mu \end{pmatrix} \right\} \psi$$

$$= \int d^4x \left\{ \bar{\psi}_R i\sigma^{\mu\dagger}(\partial_\mu + A^R_\mu)\psi_R + \bar{\psi}_L i\sigma^\mu(\partial_\mu + A^L_\mu)\psi_L \right\}$$

$$\log \mathcal{J} = \frac{i}{32\pi^2} \int d^4x \alpha(x) \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\left[F^R_{\mu\nu} F^R_{\rho\sigma} - F^L_{\mu\nu} F^L_{\rho\sigma} \right]$$

$$= \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F^R \wedge F^R - F^L \wedge F^L \right]$$

$$S = \int d^4x \bar{\psi} \left(i\not{D} + h(x) \right) \psi$$

$$= \int d^4x \left\{ \bar{\psi}_R i\sigma^{\mu\dagger} D^R_\mu \psi_R + \bar{\psi}_L i\sigma^\mu D^L_\mu \psi_L + \bar{\psi}_L h \psi_R + \bar{\psi}_R h^\dagger \psi_L \right\}$$

The anomaly for massive case (1)

Let us consider spacetime dependent mass!

- The action for general even dim with $U(N_f)_L \times U(N_f)_R$ symmetry is,

$$S = \int d^d x \bar{\psi} \left[i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} + \begin{pmatrix} im(x) & 0 \\ 0 & im^\dagger(x) \end{pmatrix} \right] \psi$$

$$= \int d^d x \left\{ \bar{\psi}_R i\sigma^{\mu\dagger} D_\mu^R \psi_R + \bar{\psi}_L i\sigma^\mu D_\mu^L \psi_L + \bar{\psi}_L im(x)\psi_R + \bar{\psi}_R im^\dagger(x)\psi_L \right\}$$

$$\equiv \int d^d x \bar{\psi} \mathcal{D} \psi$$

- We assume this $\mathcal{D}$ is the Dirac op. for massive case.
  - $\mathcal{D}$ is non-Hermitian.
- For odd dim case, there is only $U(N_f)$ sym, we put $A_\mu = A_\mu^R = A_\mu^L$ and $m = m^\dagger$.

The anomaly for massive case (2)

Calculate the $U(1)_V$ anomaly for this action by Fujikawa method.

- We **take $m(x)$ divergent**.

$$|m(x^I)| \to \infty \quad (|x^I| \to \infty)$$

  - I denotes some directions which $m(x)$ changes its values.
  - If $m(x)$ does not diverge, the anomaly is same for the massless case.

- We use the heat kernel regularization.
  - We set a UV cut-off for the eigenvalues of $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$.

$$\mathrm{e}^{-\frac{\mathcal{D}^\dagger\mathcal{D}}{\Lambda^2}}, \quad \mathrm{e}^{-\frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2}}$$

Λ is the UV cut-off.

- Generalizations
  - It is easy to get the anomaly for any dimension.
  - It is also easy to get the anomaly for $U(N_f)_L \times U(N_f)_R$, not only for $U(1)_V$.

The anomaly for massive case (3)

e.g.) In 4dim case, the $U(1)_V$ anomaly is, $\tilde{m} = m/\Lambda$

Λ is the UV cut-off comes from heat kernel regularization.

$$\log \mathcal{J} = \frac{i}{(2\pi)^2}\int d^4x\alpha(x)\mathrm{tr}\Big[\epsilon^{\mu\nu\rho\sigma}\Big\{\frac{1}{8}\Big(F^R_{\mu\nu}F^R_{\rho\sigma} - F^L_{\mu\nu}F^L_{\rho\sigma}\Big)$$
$$+\frac{1}{12}\Big(D_\mu\tilde{m}^\dagger D_\nu\tilde{m}F^R_{\rho\sigma} - D_\mu\tilde{m}D_\nu\tilde{m}^\dagger F^L_{\rho\sigma} + F^R_{\mu\nu}D_\rho\tilde{m}^\dagger D_\sigma\tilde{m}$$
$$-F^L_{\mu\nu}D_\rho\tilde{m}D_\sigma\tilde{m}^\dagger - D_\mu\tilde{m}F^R_{\nu\rho}D_\sigma\tilde{m}^\dagger + D_\mu\tilde{m}^\dagger F^L_{\nu\rho}D_\sigma\tilde{m}\Big)$$
$$+\frac{1}{24}\Big(D_\mu\tilde{m}^\dagger D_\nu\tilde{m}D_\rho\tilde{m}^\dagger D_\sigma\tilde{m} - D_\mu\tilde{m}D_\nu\tilde{m}^\dagger D_\rho\tilde{m}D_\sigma\tilde{m}^\dagger\Big)\Big\}\Big]\mathrm{e}^{-\tilde{m}^\dagger\tilde{m}}$$

- This result seems very complicated…
- Can we rewrite it more simple way? → Superconnection!

Superconnection (1)

['85 Quillen]

- We define the superconnections for even and odd dimensions.
- This is made by Quillen, who is a mathematician, in 1985.

Even dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

$A_R : U(N_f)_R$ gauge field (1-form)

$A_L : U(N_f)_L$ gauge field (1-form)

$T : U(N_f)_L \times U(N_f)_R$ bifundamental scalar field (0-form)

- Field strength

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$$

$$\equiv \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix}$$

- Supertrace

$$\mathrm{Str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

Superconnection (2)

['85 Quillen]

Odd dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A & iT \\ iT & A \end{pmatrix}$$

$A : U(N_f)$ gauge field (1-form)

$T : U(N_f)$ adjoint scalar field (0-form)

- Field strength

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = \begin{pmatrix} F - T^2 & iDT \\ iDT & F - T^2 \end{pmatrix}$$

- Supertrace

$$\mathrm{Str}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i}\,\mathrm{tr}(b)$$

We apply superconnection to write the anomaly.

The result (1)

- We can rewrite the $U(1)_V$ anomaly by superconnection.

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}}$$

$$\mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

$$\tilde{m} = m/\Lambda$$

$$\begin{aligned} \mathcal{F} =& d\mathcal{A} + \mathcal{A}^2 \\ \equiv& \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix} \end{aligned}$$

$$\mathrm{Str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

- For odd dimension case, put $A_\mu = A_\mu^R = A_\mu^L$ and $m = m^\dagger$. Then, we get $U(1)$ anomaly.
  - In odd dimension, the definition of Str is different from the even dim case.

$$\mathrm{Str}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i}\,\mathrm{tr}(b)$$

The result (2)

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}} \qquad \mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger\tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

- In this formula, $m(x)$ appears only as $\tilde{m}(x)$. $\qquad \tilde{m} = m/\Lambda$
  - Λ is the UV cut-off comes from the heat kernel regularization.
  - If $m(x)$ is finite, the mass dependence disappears because we need to take $\Lambda \to \infty$.
  - The Λ dependence (or the regulator dependence) of the anomaly disappears after we integrate $\mathrm{Str}[\mathrm{e}^{\mathcal{F}}]$ over the spacetime.

- It is easy to check this anomaly is consistent with 4dim massless case.

$$\log \mathcal{J} = \frac{i}{8\pi^2}\int \alpha(x)\mathrm{tr}\left[F^R\wedge F^R - F^L\wedge F^L\right]$$

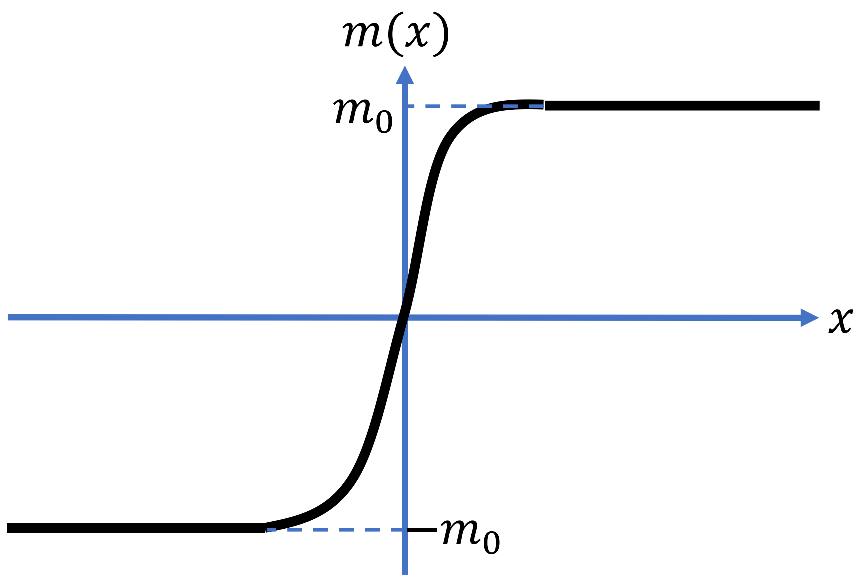
3. Application

How can we apply this anomaly?

Mass means a wall for some cases!

- e.g.) Domain wall
  - Can we make domain walls by this $m(x)$?

  →Yes!

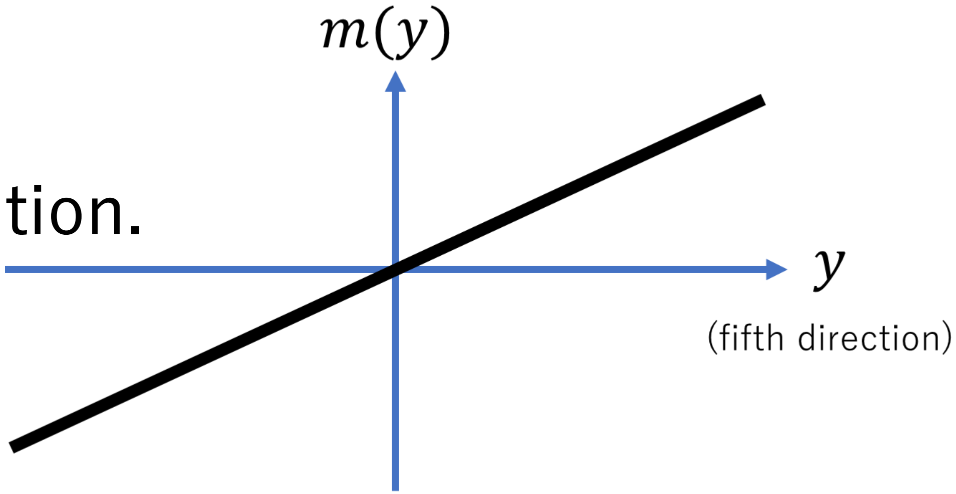


- We can make some systems with boundaries.
  - Kink, vortex and general codimension case
  - With boundary
- We also discuss about some index theorems.
  - APS index theorem
  - Callias type index theorem

Kink (1)

Mass kink for our set up

- For example, let's consider 5dim case.
- In our set up, "kink" means this mass configuration.



$$m(y) = uy \qquad y = x^5 \quad u \in \mathbb{R}$$

  - This "mass" diverges at $y \to \pm\infty$.
  - 5dim fermions with $U(N_f)$ sym, and the mass depends on only y direction.
- The $U(1)$ anomaly is,

$$\log \mathcal{J} = \pm \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F \wedge F\right]$$

Corresponds to the sign of u.

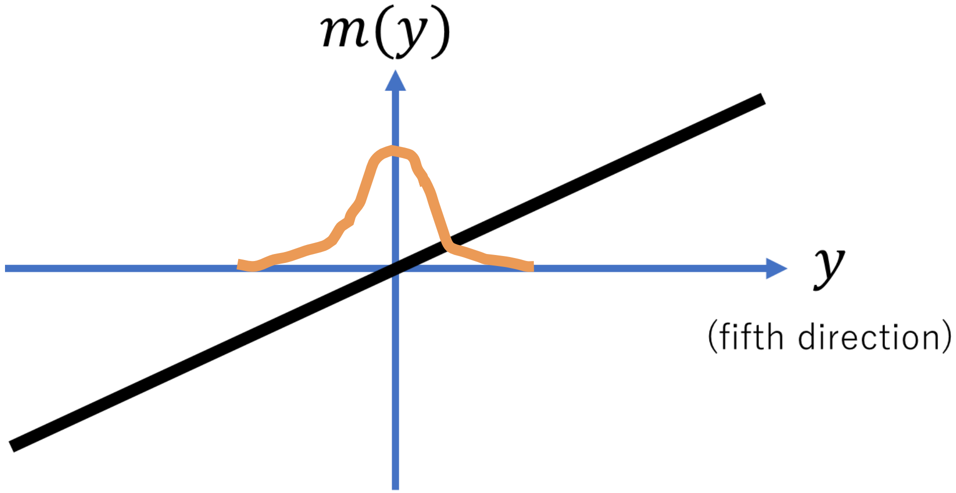
- Recall 4dim $U(1)_V$ anomaly,

$$\log \mathcal{J} = \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F^R \wedge F^R - F^L \wedge F^L\right]$$

Kink (2)

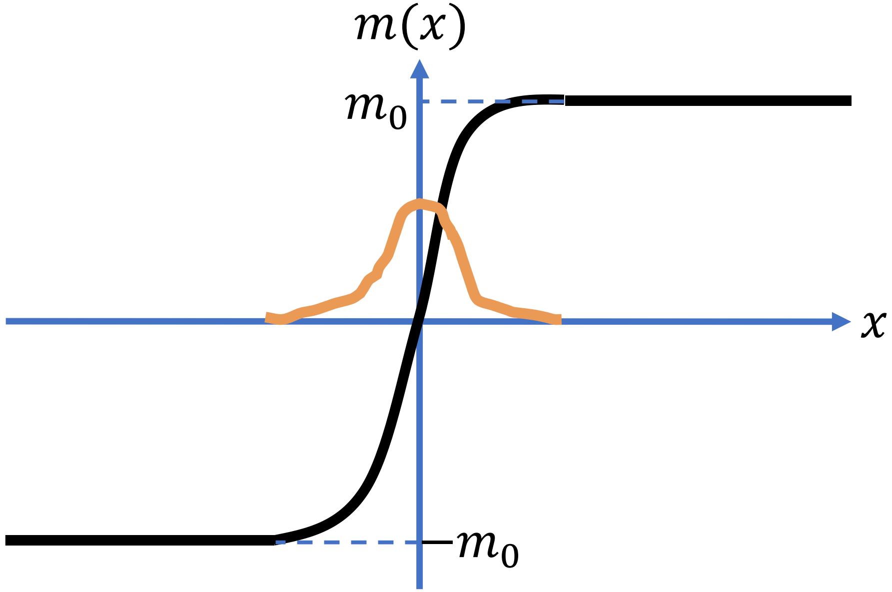
What is the meaning of the anomaly?

- 4dim Weyl fermions are localized at $y = 0$.
  - $u > 0$ corresponds to chirality $+$ (right-handed) fermion, and $u < 0$ corresponds to chirality $-$ (left-handed) fermion.



Domain wall fermion

- This Weyl fermions correspond to domain wall fermions.
  - But the regularization is different, so that I don't know the correspondence in detail.



Vortex

Next, we check codim-2 case.

$$m(z) = uz\mathbf{1}_{N\times N}$$
$$z = x^{\mu=2r+1} - ix^{\mu=2r+2}$$

- Vortex is 2dim topological object.
- Let us consider $2r+2$ dim.
  - $m(z)$ depends on 2 directions, and it is complex valued “mass”.
  - This mass diverges at $|z| \to \infty$.
- For simplicity, we put $A_L = A_R$ in $2r+2$dim.
- The $U(1)_V$ anomaly is,

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^r \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^F\right]\Big|_{2r-\mathrm{form}}$$

- This is $2r$dim $U(1)$ anomaly with $U(N_f)_R$ gauge field.
  - If you want to get chirality – (left-handed) result, use $m(\bar{z}) = u\bar{z}$, instead.

General defects

We can apply this formula to general codimension cases.

- When we think d dim system with n dim topological defects, we get $d-n$ dim $U(1)$ anomalies.
  - If $d-n$ is odd, we get nothing because odd dim mass less fermions are anomaly-free.
  - The mass configurations for general codimension is,

$$m(x) = u\sum_{I=1}^{n}\Gamma^I x^I$$

$$\gamma^I = \Gamma^I \qquad (n = odd)$$

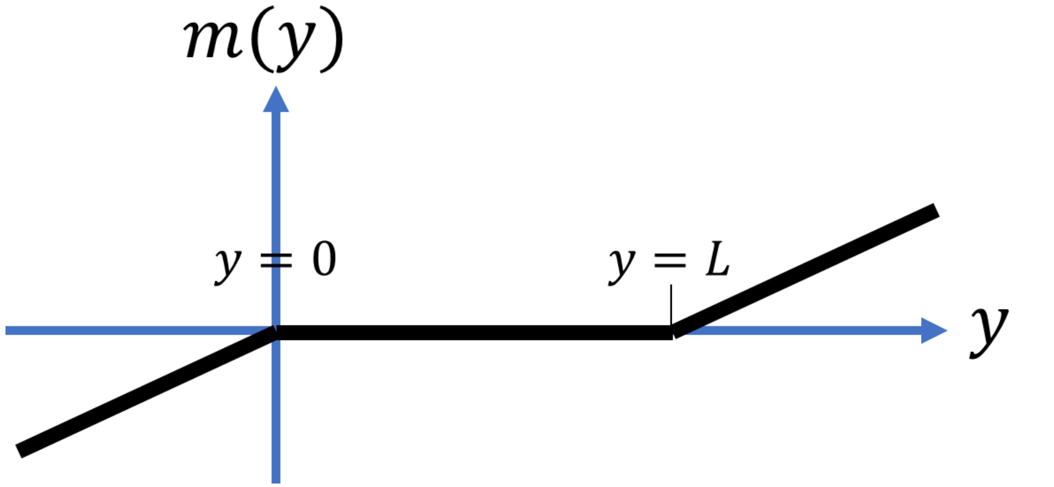
$$\gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix} (n = even)$$

- This results correspond to “tachyon condensation” in string theory.
  - We will discuss about it in section 5.

With boundary (1)

Let us make some boundaries.

- Fermions are massive = boundary



Odd dimension (2r dim)

- We realize localized fermions at $[0,L]$.
- The bulk is anomaly-free.
- The anomaly is,

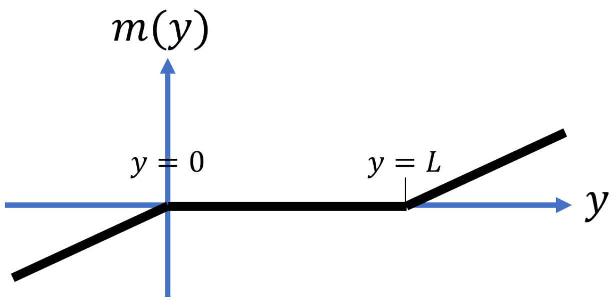
$$m(y) = \mu(y) 1_N = \begin{cases} (m_0 + u'(y-L))1_N & (L < y) \\ m_0 1_N & (0 \leq y \leq L) \\ (m_0 + uy)1_N & (y < 0) \end{cases}$$

$$\log \mathcal{J} = i\kappa_- \int_{y=0} \alpha\, [\mathrm{ch}(F)]_{2r} + i\kappa_+ \int_{y=L} \alpha\, [\mathrm{ch}(F)]_{2r}$$

$$\kappa_- = \frac{1}{2}\mathrm{sgn}(u)\ , \quad \kappa_+ = \frac{1}{2}\mathrm{sgn}(u')$$

With boundary (2)

Even dimension



$$m(x) = \mu(y)g(x) = \begin{cases} u'(y-L)g(x) & (L < y) \\ 0 & (0 \le y \le L) \\ uyg(x) & (y < 0) \end{cases}$$

- The anomaly is,

$$\log \mathcal{J} = -i\int_{0<y<L} \alpha\left[\mathrm{ch}(F_+) - \mathrm{ch}(F_-)\right]_{2r} - i\int_{y=L} \alpha[\omega]_{2r-1} + i\int_{y=0} \alpha[\omega]_{2r-1}$$

  - ω is Chern-Simons form.
  - Anomaly from bulk + CS

4. Index theorem

Index for massive Dirac op. (1)

- We will discuss index theorems for the massive Dirac operator $\mathcal{D}$.
  - We just consider flat spacetime.

$$\mathcal{D} = i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} + \begin{pmatrix} im(x) & 0 \\ 0 & im^\dagger(x) \end{pmatrix} \qquad S = \int d^d x \bar{\psi}(x) \mathcal{D} \psi(x)$$

$$\mathrm{Ind}(\mathcal{D}) = \dim\ \ker(\mathcal{D}) - \dim\ \ker(\mathcal{D}^\dagger)$$

Chern character

- We need to define Chern character for the superconnection.
- The Chern character for massive case is,

$$\mathrm{ch}(\mathcal{F}) = \sum_{k\geq 0} \left(\frac{i}{2\pi} \right)^{\frac{k}{2}} \mathrm{Str}\left[\mathrm{e}^{\mathcal{F}} \right] \Big|_{k-\mathrm{form}} \qquad \mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

Index for massive Dirac op. (2)

- We can write the $U(1)$ anomaly by the Chern character.

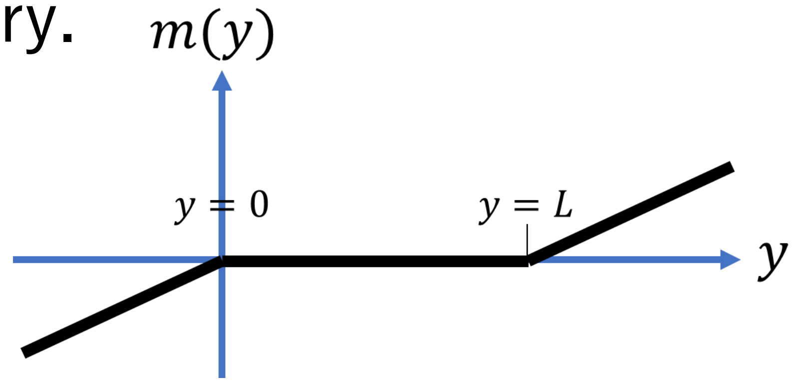
$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}} = -i\int \alpha(x)\mathrm{ch}(\mathcal{F})$$

- The index for the massive Dirac operator is,

$$\mathrm{Ind}(\mathcal{D}) = \int \mathrm{ch}(\mathcal{F})$$

  - For closed manifolds, this is Atiyah-Singer index theorem.

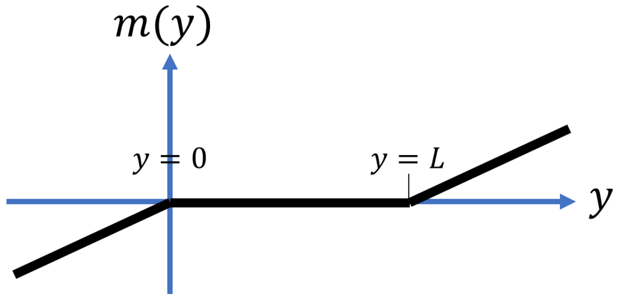
- Let us consider $2r$ dimensional system with boundary.
  - We can make the boundary with the mass.
  - The index will be Atiyah-Patodi-Singer(APS) index.
  - Let's check the index!



APS index theorem (1) ['75 Atiyah-Patodi-Singer]

APS index theorem

- APS index is the index for open manifold with boundaries.
  - APS index = bulk index + η-invariant on the boundaries
  - In APS paper, they introduce APS boundary condition, which is non-local boundary condition for fermions.
- It is known that APS index is realized with local boundary condition.
  - ['17 Fukaya-Onogi-Yamaguchi] [hep-th/1710.03379]
  - If you use domain walls for boundaries, you can use local boundary condition for fermions.



- Let us consider both boundary conditions in our set up.
  - We can realize boundaries by mass; this is very similar to DW set up.

APS index theorem (2)

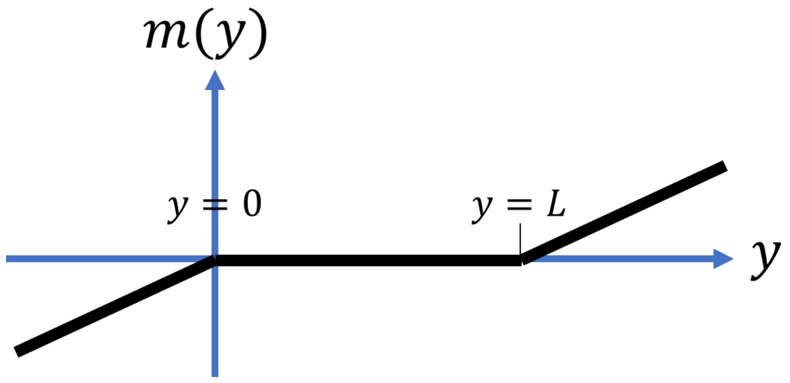
APS index in our set up

- The index is,

$$\mathrm{Ind}(\mathcal{D}) = \lim_{\Lambda\to\infty}\int_{y_-<y<y_+}[\mathrm{ch}(\mathcal{F})]_{2r} + \frac{1}{2}[\eta(H_y)]_{y=y_-}^{y=y_+}$$

  - This is the APS index theorem for the massive Dirac operators in 2r dim with boundaries at $y = y_\pm$.
  - We just consider $m(x)$ depends on only one direction y.

- Let us consider APS index for 2r dim system in previous section.
  - If you take $y_+ = L$ and $y_- = 0$, then you will get APS index with APS boundary condition.
  - But in this case, mass does not work because the Dirac op. is massless in $0 < y < L$.

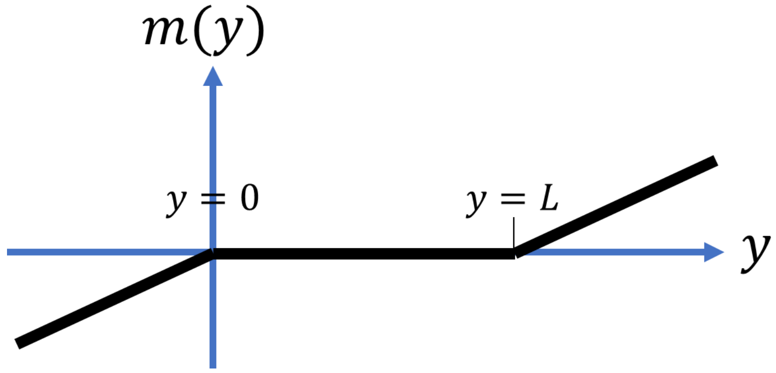


$$\mathrm{Ind}(\mathcal{D}|_{[0,L]}) = \int_{0<y<L}[\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r} - \frac{1}{2}\left[\eta(i\not{D}_+^{(2r-1)}) - \eta(i\not{D}_-^{(2r-1)})\right]_{y=0}^{y=L}$$

APS index theorem (3)

APS index with mass

- Let us put $y_+ = \infty$ and $y_- = -\infty$.
  - Boundaries come from the mass.
  - APS index is,



$$\mathrm{Ind}(\mathcal{D}) = \int_{0<y<L} [\mathrm{ch}(F_+) - \mathrm{ch}(F_-)]_{2r} + \int_{y=L} [\omega]_{2r-1} - \int_{y=0} [\omega]_{2r-1}$$

  - This is APS index with "physicist-friendly" boundary condition.
    - cf. ['17 Fukaya-Onogi-Yamaguchi]
    - In our set up, massive $\simeq$ domain wall.

- To apply this form, we get a relation between eta invariant and Chern-Simons form ω.

$$\int [\omega]_{2r-1} = \frac{1}{2}\left(\eta(i\not{D}_-^{(2r-1)}) - \eta(i\not{D}_+^{(2r-1)})\right) \pmod{\mathbb{Z}}$$

5. String theory

String theory

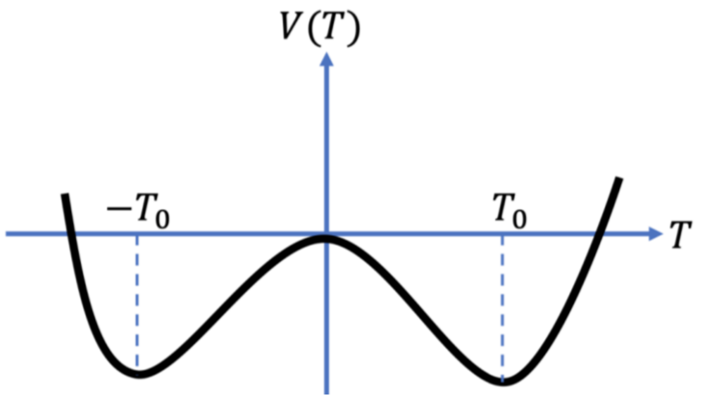
Let us check the relation between this anomaly and string theory.

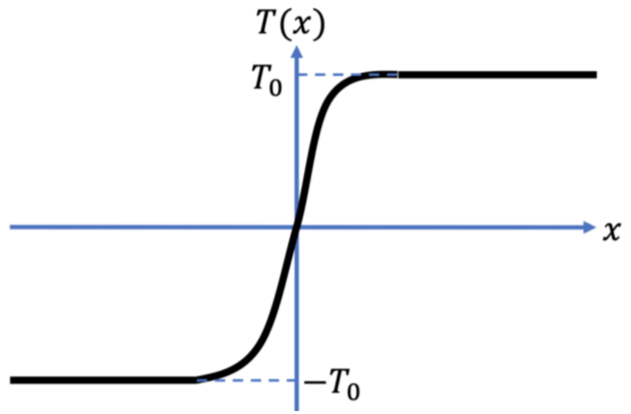
- Consider type IIA or IIB string theory with D-branes.
  - Open strings have their ends on D-branes.
  - Excitation modes of these open strings → Fields on D-branes
  - Open strings on D_p-branes → QFT in $p+1$ dim

- In some cases, excitation modes of the strings have tachyon modes.
  - Lowest excitation modes are $m^2 < 0$. (Tachyon mode)
  - Non-BPS states have tachyons.
  - This tachyonic modes are unstable. → Tachyon condensation
  - See Sen's review [hep-th/9904207].

Tachyon condensation (1)

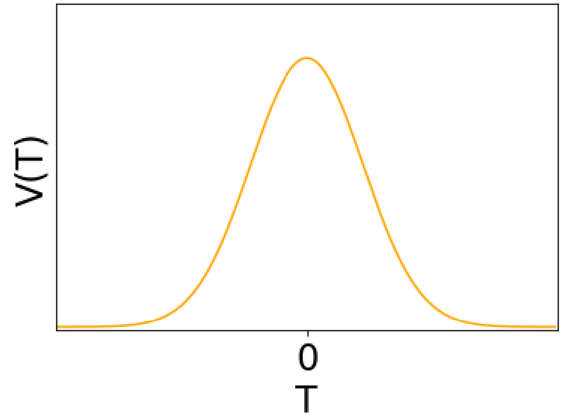
- Tachyonic modes are unstable, so the tachyons have non-zero VEV.
  - Non-trivial configuration of tachyon is also realizable.





e.g.) D-brane and anti D-brane ($\bar{D}$-brane) system

- Non-BPS state
- Tachyonic modes appear in $D-\bar{D}$ string.
  - The shape of tachyon potential is known. $V(T)=\mathrm{e}^{-T^\dagger T}$
  - If tachyon configuration is trivial, the D-branes disappear.



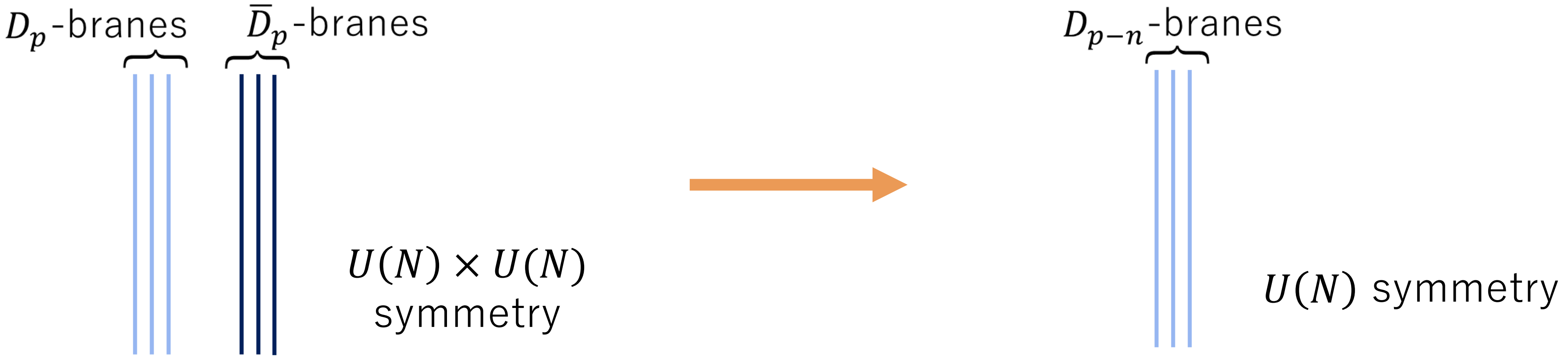
Tachyon condensation (2)

Kink on tachyon in $D_p - \overline{D}_p$ system

- Tachyonic kinks for this system is,

$$T(x) = u \sum_{I=1}^{n} \Gamma^I x^I \qquad \gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix}$$

- We get D_{p-n}-branes from this tachyon.
    - If D_{p-n}-branes are non-BPS, tachyons still exist on the D-branes.
    - In this case, tachyon condensation occur again.



Tachyon condensation (3)

- The superconnection is used in the context of tachyon condensation.
  - This structure comes from RR-coupling of D-branes.

cf.) ['98 Witten] [hep-th/9810188]
['99 Kennedy-Wilkins] [hep-th/9905195]
['01 Kuraus-Larsen] [hep-th/0012198]
['01 Takayanagi-Terashima-Uesugi] [hep-th/0012210]

$$S = T_{D9} \int C \wedge \mathrm{Str}\; e^{2\pi\alpha' i \mathcal{F}}$$

- The tachyon configuration is given by ['98 Witten].
  - In this paper, relation between tachyon condensation and K-theory is discussed.
  - This tachyon configuration comes from ['64 Atiyah-Bott-Shapiro]

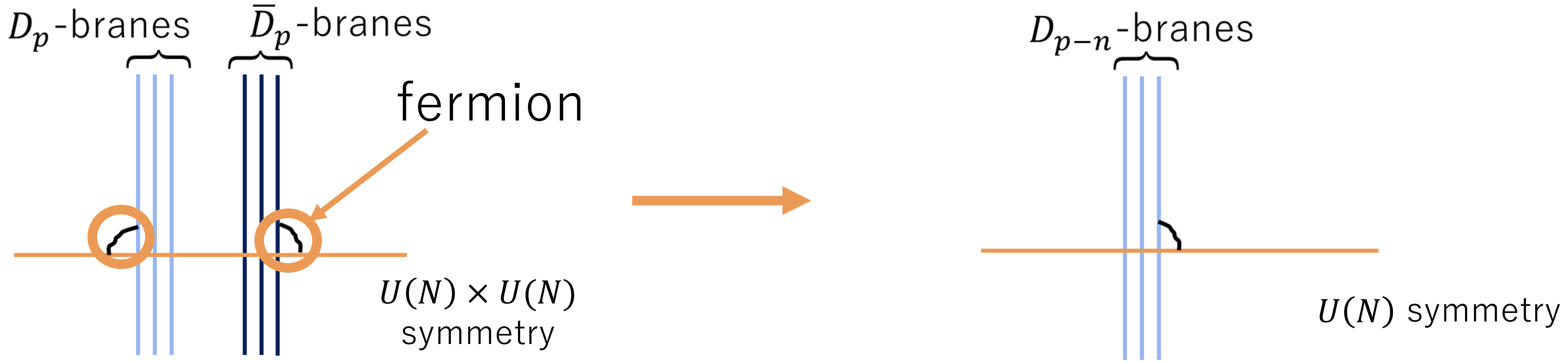
$$T(x) = u \sum_{I=1}^{n} \Gamma^I x^I$$

Relation between the anomaly and string

- This tachyon configuration is same for the mass defect in section 3!

$$T(x) = u \sum_{I=1}^{n} \Gamma^I x^I \qquad\qquad m(x) = u \sum_{I=1}^{n} \Gamma^I x^I$$

- This anomaly can be understood from string theory.
  - Fermions are found where D-branes intersect.
  - This is similar to flavor symmetry on holographic QCD model. (Sakai-Sugimoto model)



Conclusion

- We discussed about perturbative anomaly with spacetime dependent mass.
  - If value of the mass diverge, non-trivial contribution of the mass appears.

- The anomaly can be written by superconnection.

$$\log \mathcal{J} = -i \left(\frac{i}{2\pi} \right)^{\frac{d}{2}} \int \alpha(x) \mathrm{Str} \left[\mathrm{e}^{\mathcal{F}} \right] \Big|_{d-\mathrm{form}} \qquad \mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

- There are some applications.
  - Kink, vortex, …
  - With boundary
  - Index theorem