Anomaly and Superconnection

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What is “anomaly”? (1)

Anomaly (Quantum Anomaly)

An classical action have some symmetries, but sometimes these symmetries disappear in quantum theory.

e.g.) $\pi^0 \to 2\gamma$

- In massless QCD, there is a chiral symmetry $U(N_f)_L \times U(N_f)_R$.

N_f: # of flavors

$$S = \int d^4x \left\{ \bar{\psi} i \not{D} \psi - \frac{1}{2g^2} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu} \right] \right\}$$

$$= \int d^4x \left\{ \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) \psi - \frac{1}{2g^2} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu} \right] \right\}$$

What is “anomaly”? (2)

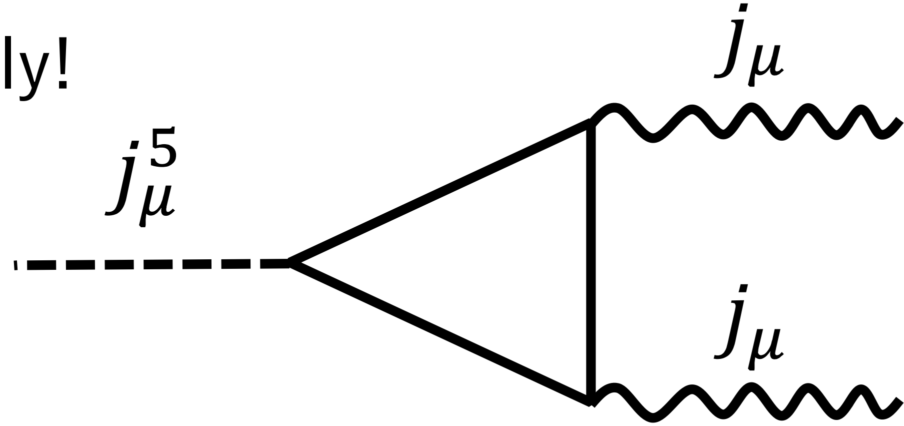
e.g.) $\pi^0 \to 2\gamma$

- In massless QCD, there is a chiral symmetry $U(N_f)_L \times U(N_f)_R$.
  - This chiral symmetry is broken if you add mass term.

$$S = \int d^4x \left\{ \bar{\psi} i \not{D} \psi - \frac{1}{2g^2} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu} \right] + m\bar{\psi}\psi \right\}$$

- If there is NO anomaly, π^0 never decay.
- However, π^0 decay into 2γ, because of an anomaly!

$U(N_f)_L \times U(N_f)_R \supset U(1)_A$ has an anomaly.



What is “anomaly”? (3)

e.g.) Gauge anomaly

- Let us consider a action include fermions and G gauge fields.
  - The theory has a gauged symmetry G, so that the action is invariant under G gauge transformation.
  - For example, consider $U(N_f)_L \times U(N_f)_R$ symmetry as G.

$$G = U(N_f)_L \times U(N_f)_R$$

$$S = \int d^4x \left\{ \bar{\psi} i \not{D} \psi - \frac{1}{2g_L^2} \mathrm{tr}\left[F^L_{\mu\nu} F^{L\mu\nu}\right] - \frac{1}{2g_R^2} \mathrm{tr}\left[F^R_{\mu\nu} F^{R\mu\nu}\right] \right\}$$

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$$S \to S' = S \qquad (U(N_f)_L \times U(N_f)_R)$$

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  - For example, consider $U(N_f)_L \times U(N_f)_R$ symmetry as G.
- The action is invariant under the G gauge transformation.
- How about the partition function Z?
  - If the gauge sym. does not have any anomaly, Z is also invariant. (e.g. Standard model)
  - If the gauge sym. has some anomalies, Z is not invariant!

    →This theory cannot be gauged!

$$G = U(N_f)_L \times U(N_f)_R$$

$$S = \int d^4x \left\{ \bar{\psi} i \not{D} \psi - \frac{1}{2g_L^2} \mathrm{tr}\left[F^L_{\mu\nu} F^{L\mu\nu}\right] - \frac{1}{2g_R^2} \mathrm{tr}\left[F^R_{\mu\nu} F^{R\mu\nu}\right] \right\}$$

$$S \to S' = S \qquad (U(N_f)_L \times U(N_f)_R)$$

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_L \mathcal{D}A_R \, \mathrm{e}^{-S}$$

$$Z \to Z' \quad (U(N_f)_L \times U(N_f)_R)$$

$$Z' = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_L \mathcal{D}A_R \, \mathrm{e}^{-S} \mathrm{e}^{i \int d^4x \alpha \mathcal{A}}$$

$$\neq Z$$

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$$Z \to Z' \quad (U(N_f)_L \times U(N_f)_R)$$

$$Z' = \underbrace{\boxed{\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_L \mathcal{D}A_R \, \mathrm{e}^{-S}}}_{= Z} \mathrm{e}^{i \int d^4x \alpha \mathcal{A}}$$

$$\neq Z$$

Theories what we want to think (1)

Let us consider 4dim action contains fermions.

$$S = \int d^4x \bar{\psi} i \not{D} \psi = \int d^4x \bar{\psi} i \gamma^\mu (\partial_\mu + A_\mu) \psi$$

- This action is massless, so it has a chiral symmetry $U(N_f)_L \times U(N_f)_R$.
- There also be a $U(1)_A$ anomaly.

- Add mass term
  - Mass term breaks the chiral symmetry.

$$S = \int d^4x \bar{\psi} \left(i \not{D} + m \right) \psi$$

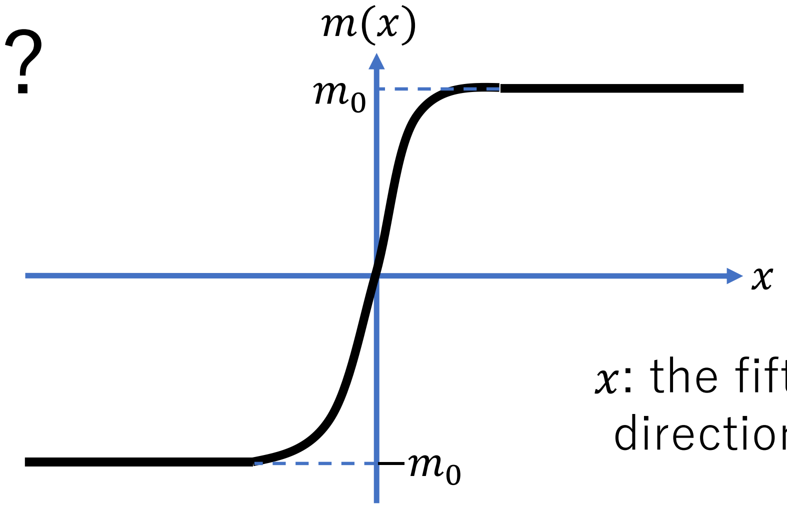
- Let the mass depend on the spacetime.
  - This mass is almost same as the Higgs field.
  - How change the symmetry and the anomaly?

$$S = \int d^4x \bar{\psi} \left(i \not{D} + m(x) \right) \psi$$

The spacetime dependent mass

What is “the spacetime dependent mass”?

- e.g.) Domain wall fermions
  - One way to realize chiral fermions on the lattice.
  - Consider 5dim spacetime, and realize 4dim fermions on $m(x) = 0$ subspace.



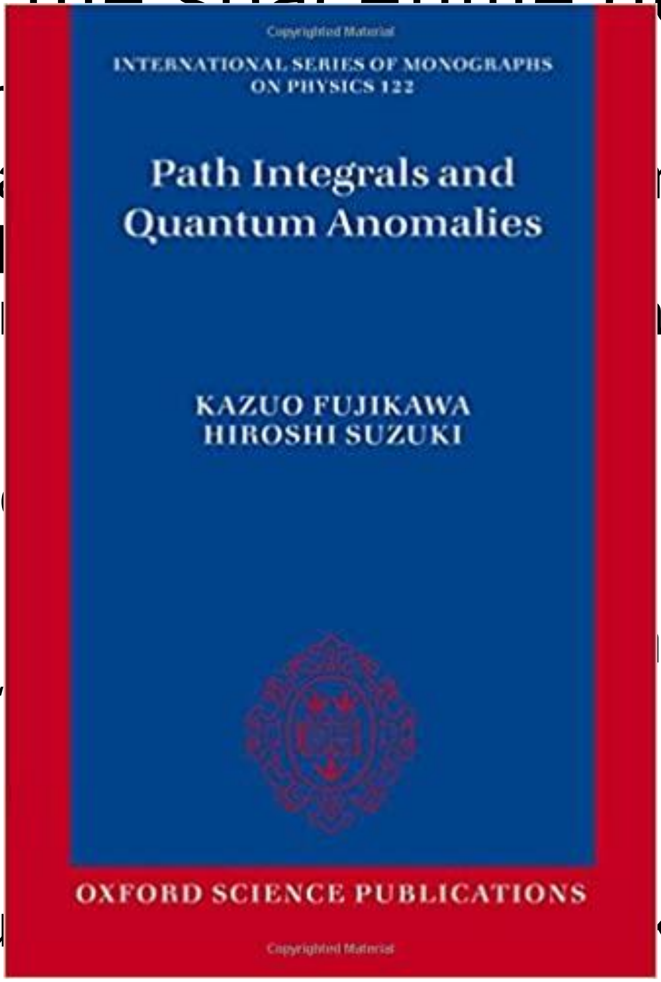
x: the fifth direction

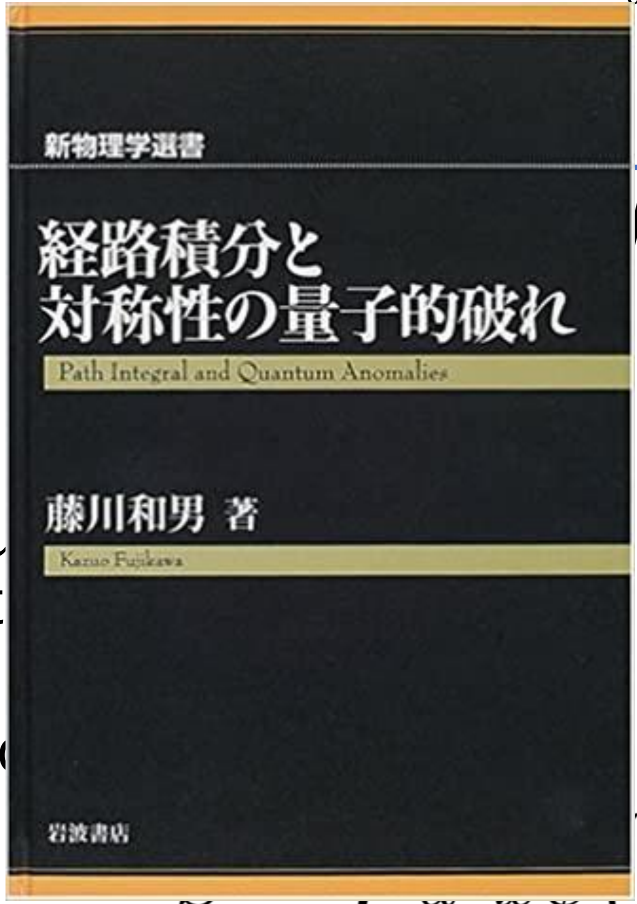
- Chiral anomalies on Higgs fields
  - If Higgs fields change as bifundamental under the $U(N_f)_L \times U(N_f)_R$ chiral symmetry, the action is invariant for the symmetry.
  - It is known that chiral anomalies are not changed by adding Higgs fields.
  - See Fujikawa-san’s text book.

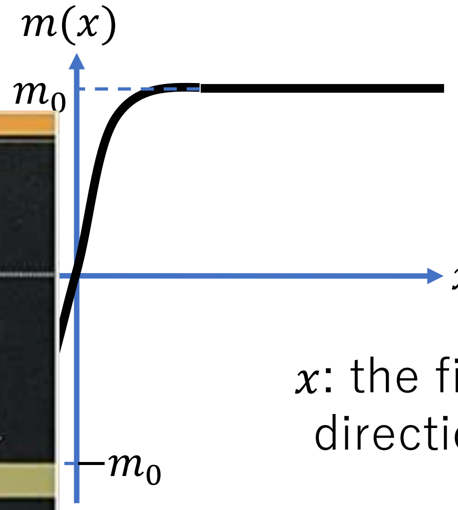
$$S = \int d^4x \bar{\psi} \left(i \not{D} + h(x) \right) \psi$$

The spacetime dependent mass

What is "the spacetime dependent mass"?







x: the fifth direction

- e.g.) Dom
  - One wa … nions on the
  - Consid … d realize 4di fermio … nce.
- Chiral an … ds
  - If Higg … ndamental u $U(N_f)_L$ … etry, the act invaria
  - It is kn … lies are not by add
  - See Fu … k.

$$i\not{D} + h(x)\Big)\psi$$

Theories what we want to think (2)

How about the spacetime dependent mass?

- The chiral anomaly is changed by the mass!!
  - Deference between Higgs and mass
    - Higgs field : bounded
    - Spacetime dependent mass : unbounded

$$S = \int d^d x \bar{\psi} \Big(i \not{D} + m(x) \Big) \psi$$

- If the mass diverge at some points, it contribute to the anomaly.
  - This contribution might be unknown.
  - We can find the anomaly in any dimension.

- The anomaly can be written by "superconnection."

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

Plan

1. Introduction (6)
    - What is anomaly?
    - Theories what we want to think

2. Fujikawa method (5)
    - How to calculate the anomaly
    - Calculation for massive case

3. Superconnection (3)
    - Definition of superconnection
    - Application for the anomaly

4. Application (10)
    - Kink
    - Vortex
    - With boundary
    - APS index theorem

5. String theory (4)
    - Relation to string theory
    - Tachyon condensation

(6. Detail of the derivation (7+1))

7. Conclusion

How to calculate anomalies ['79 Fujikawa]

Fujikawa method

- There are some ways to calculate anomalies.
- Today, we focus on Fujikawa method.
  - Consider path integral for fermions.
  - Anomaly = Jacobian comes from path integral measure
- We calculate $\log \mathcal{J}$ for anomalies in the last part of this talk.
- We focus on 4dim case at first.

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathrm{e}^{-S}$$

e.g.) $U(1)_V$ transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x),$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{-i\alpha(x)}$$

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \mathcal{J}\mathcal{D}\psi\mathcal{D}\bar{\psi}$$
$$= \mathrm{e}^{-i\int d^4x \alpha(x)\mathcal{A}(x)} \mathcal{D}\psi\mathcal{D}\bar{\psi}$$

anomaly

$$\log \mathcal{J} = -i \int d^4x \alpha(x) \mathcal{A}(x)$$

Chiral symmetry

Anomalous symmetries we calculate

- $U(N_f)_L \times U(N_f)_R$ chiral symmetry
  - For even dimension
  - Because chirality operators exist only even dimensions.
  - Fermions couple to $U(N_f)_L$ background gauge field A^L_μ and $U(N_f)_R$ background gauge field A^R_μ.
- $U(N_f)$ flavor symmetry
  - For odd dimension
  - No perturbative anomaly as usual.
  - With $U(N_f)$ background gauge field.

- We focus on **$U(1)$ parts** of these sym.
  - We calculate mixed anomaly between $U(1)_V$ and $SU(N_f)_L \times SU(N_f)_R \times U(1)_A$ for even dim, $U(1)$ and $SU(N_f)$ for odd dim.
  - Not $U(1)_A$ part, even for even dim.

$$S = \int d^dx \bar{\psi} i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A^R_\mu & 0 \\ 0 & A^L_\mu \end{pmatrix} \right\} \psi$$

For even dimension

$$S = \int d^dx \bar{\psi} i\gamma^\mu \left\{ \partial_\mu + A_\mu \right\} \psi$$

For odd dimension

The anomalies for massless cases

We focus on 4dim case.

- Mass less case
  - With $U(N_f)_L \times U(N_f)_R$ chiral sym.
  - $U(1)_V$ anomaly is written by the field strength.

$$S = \int d^4x \bar{\psi} i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} \psi$$

$$\begin{aligned} \log \mathcal{J} &= \frac{i}{32\pi^2} \int d^4x \alpha(x) \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\left[F_{\mu\nu}^R F_{\rho\sigma}^R - F_{\mu\nu}^L F_{\rho\sigma}^L \right] \\ &= \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F^R \wedge F^R - F^L \wedge F^L \right] \end{aligned}$$

- With a Higgs field
  - With $U(N_f)_L \times U(N_f)_R$ chiral sym.
  - The $U(1)_V$ anomaly is same for massless case.

$$S = \int d^4x \bar{\psi} \left(i\not{D} + h(x) \right) \psi$$

- How about the massive case?

For massive case

Let us consider spacetime dependent mass!

- The action for general even dim has $U(N_f)_L \times U(N_f)_R$ symmetry.

$$S = \int d^4x \bar{\psi} \left[i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} + \begin{pmatrix} im(x) & 0 \\ 0 & im^\dagger(x) \end{pmatrix} \right] \psi$$

- For odd dim case, there is only $U(N_f)$ sym, we put $A_\mu = A_\mu^R = A_\mu^L$ and $m = m^\dagger$.
- We take $m(x)$ divergent.
  - I is some directions $m(x)$ change the values.

$$|m(x^I)| \to \infty \quad (|x^I| \to \infty)$$

- We calculated $U(1)_V$ anomaly for this action by Fujikawa method.
  - It is easy to get the anomaly for any dimension.
  - It is also easy to get the anomaly for $U(N_f)_L \times U(N_f)_R$, not only for $U(1)_V$.

The anomaly for massive case

The $U(1)_V$ anomaly is,

$\tilde{m} = m/\Lambda$

Λ is UV cut-off comes from heat kernel regularization.

$$
\begin{aligned}
\log \mathcal{J} = & \frac{i}{(2\pi)^2} \int d^4x \alpha(x) \mathrm{tr} \Bigg[\epsilon^{\mu\nu\rho\sigma} \Bigg\{ \frac{1}{8} \left(F^R_{\mu\nu} F^R_{\rho\sigma} - F^L_{\mu\nu} F^L_{\rho\sigma} \right) \\
& + \frac{1}{12} \Big(D_\mu \tilde{m}^\dagger D_\nu \tilde{m} F^R_{\rho\sigma} - D_\mu \tilde{m} D_\nu \tilde{m}^\dagger F^L_{\rho\sigma} + F^R_{\mu\nu} D_\rho \tilde{m}^\dagger D_\sigma \tilde{m} \\
& - F^L_{\mu\nu} D_\rho \tilde{m} D_\sigma \tilde{m}^\dagger - D_\mu \tilde{m} F^R_{\nu\rho} D_\sigma \tilde{m}^\dagger + D_\mu \tilde{m}^\dagger F^L_{\nu\rho} D_\sigma \tilde{m} \Big) \\
& + \frac{1}{24} \left(D_\mu \tilde{m}^\dagger D_\nu \tilde{m} D_\rho \tilde{m}^\dagger D_\sigma \tilde{m} - D_\mu \tilde{m} D_\nu \tilde{m}^\dagger D_\rho \tilde{m} D_\sigma \tilde{m}^\dagger \right) \Bigg\} \Bigg] \mathrm{e}^{-\tilde{m}^\dagger \tilde{m}}
\end{aligned}
$$

- This result seems very complicated…
- Can we write it more simple way?

3. Superconnection

Superconnection (1)

['85 Quillen]

- We define the superconnections for even and odd dimensions.
- This is made by Quillen, who is a mathematician, in 1985.

Even dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A_R & iT^\dagger \\ iT & A_L \end{pmatrix}$$

$A_R : U(N_f)_R$ gauge field (1-form)

$A_L : U(N_f)_L$ gauge field (1-form)

$T : U(N_f)_L \times U(N_f)_R$ bifundamental scalar field (0-form)

- Field strength

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$$

$$\equiv \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix}$$

- Supertrace

$$\mathrm{Str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

Superconnection (2)

['85 Quillen]

Odd dimension

- Superconnection

$$\mathcal{A} = \begin{pmatrix} A & iT \\ iT & A \end{pmatrix}$$

$A : U(N_f)$ gauge field (1-form)

$T : U(N_f)$ adjoint scalar field (0-form)

- Field strength

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A}^2 = \begin{pmatrix} F - T^2 & iDT \\ iDT & F - T^2 \end{pmatrix}$$

- Supertrace

$$\mathrm{Str}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i}\,\mathrm{tr}(b)$$

We apply superconnection to write the anomaly.

Rewrite the anomaly

- We can rewrite the $U(1)_V$ anomaly by superconnection.

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x) \mathrm{Str}\left[\mathrm{e}^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}}$$

$$\begin{aligned}\mathcal{F} =& d\mathcal{A} + \mathcal{A}^2 \\ \equiv& \begin{pmatrix} F^R - T^\dagger T & iDT^\dagger \\ iDT & F^L - TT^\dagger \end{pmatrix}\end{aligned}$$

$$\mathcal{F} \equiv \begin{pmatrix} F^R - \tilde{m}^\dagger \tilde{m} & i(D\tilde{m})^\dagger \\ iD\tilde{m} & F^L - \tilde{m}\tilde{m}^\dagger \end{pmatrix}$$

$$\mathrm{Str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

- For odd dimension case, put $A_\mu = A_\mu^R = A_\mu^L$ and $m = m^\dagger$. Then, we get $U(1)$ anomaly.
  - In odd dimension, the definition of Str is little different from even dim case.

$$\mathrm{Str}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \sqrt{2i}\mathrm{tr}(b)$$

- It is easy to check this for 4dim massless case.

4. Application

How can we apply the anomaly?

Mass means a wall for some cases!

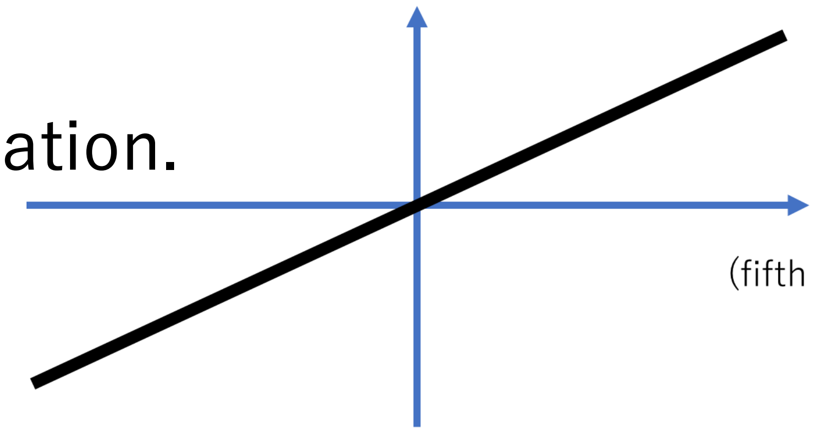
- If a fermion is massive enough, it does not have any propagating mode.
  - If the mass depends on spacetime, fermions are massless in some regions, but they can be massive in the others.
  - That means fermions localize in some areas!

  →We can make fermions localize by the mass!
- We can make some systems to decide mass configurations.
  - Kink, vortex and general codimension case
  - With boundary
- We also discuss about some index theorems.
  - APS index theorem
  - (Callias type index theorem)

Kink (1)

Mass kink for our set up

- For example, let's consider 5dim case.
- In this set up, "kink" means this mass configuration.



$$m(y) = uy \qquad\qquad y = x^5$$

  - This "mass" diverges at $y \to \pm\infty$.
  - 5dim fermions with $U(N_f)$ sym, and the mass depends on only y direction.
- The $U(1)$ anomaly is,

$$\log \mathcal{J} = \pm \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F \wedge F\right]$$

Corresponds to the sign of u.

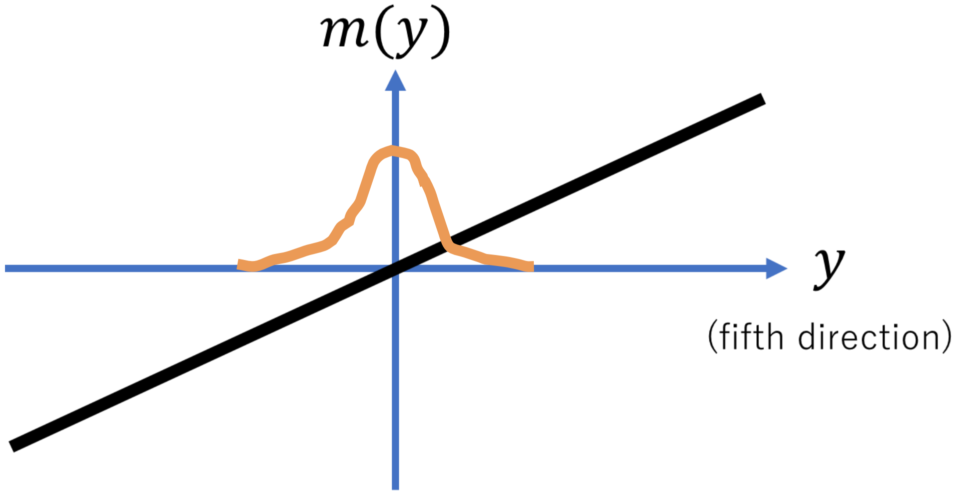
- Recall 4dim $U(1)_V$ anomaly,

$$\log \mathcal{J} = \frac{i}{8\pi^2} \int \alpha(x) \mathrm{tr}\left[F^R \wedge F^R - F^L \wedge F^L\right]$$

Kink (2)

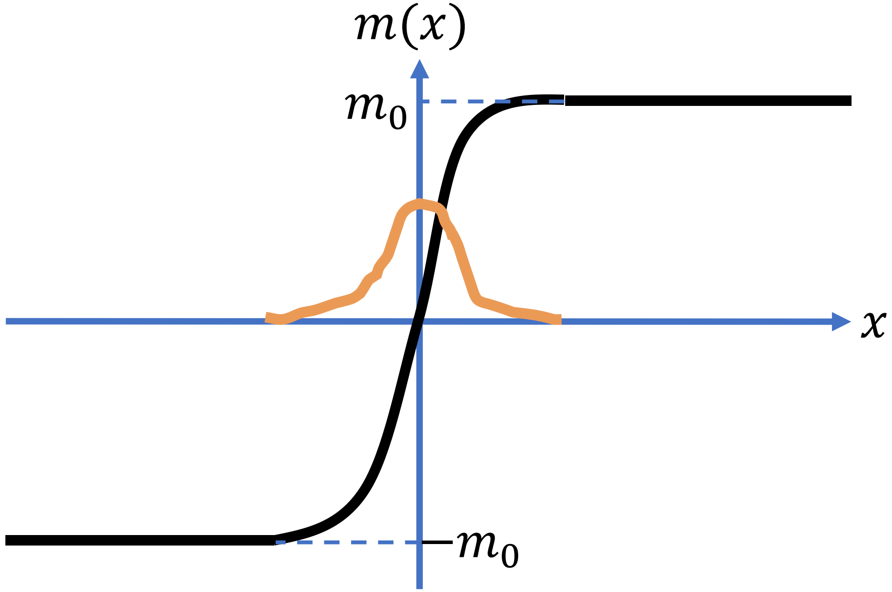
What is the meaning of the anomaly?

- 4dim Weyl fermions are localizing at $y = 0$.
    - When $u > 0$ corresponds to chirality $+$ (right-handed) fermion, and $u < 0$ corresponds to chirality $-$ (left-handed) fermion.



Domain wall fermion

- This Weyl fermions correspond to domain wall fermions.
    - But the regularization is different, so that I don't know the correspondence in detail.



Vortex

Next, we check codim-2 case.

$$m(z) = uz\mathbf{1}_{N\times N}$$
$$z = x^{\mu=2r+1} - ix^{\mu=2r+2}$$

- Vortex is 2dim topological object.
- Let us consider $2r+2$ dim.
  - $m(z)$ depends on 2 directions, and it is complex valued “mass”.
  - This mass diverges at $|z| \to \infty$.
- For simplicity, we put $A_L = A_R$ in $2r+2$dim.
- The $U(1)_V$ anomaly is,

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^r \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^F\right]\Big|_{2r-\mathrm{form}}$$

- This is $2r$dim $U(1)$ anomaly with $U(N_f)_R$ gauge field.
  - If you want to get chirality – (left-handed) result, use $m(\bar{z}) = u\bar{z}$, instead.

General defects

We can apply this formula to general codimension cases.

- When we think d dim system with n dim topological defects, we get $d-n$ dim $U(1)$ anomalies.
  - If $d-n$ is odd, we get nothing because odd dim mass less fermions are anomaly-free.
  - The mass configurations for general codimension is,

$$m(x) = u\sum_{I=1}^{n}\Gamma^I x^I$$

$$\gamma^I = \Gamma^I \qquad (n = odd)$$

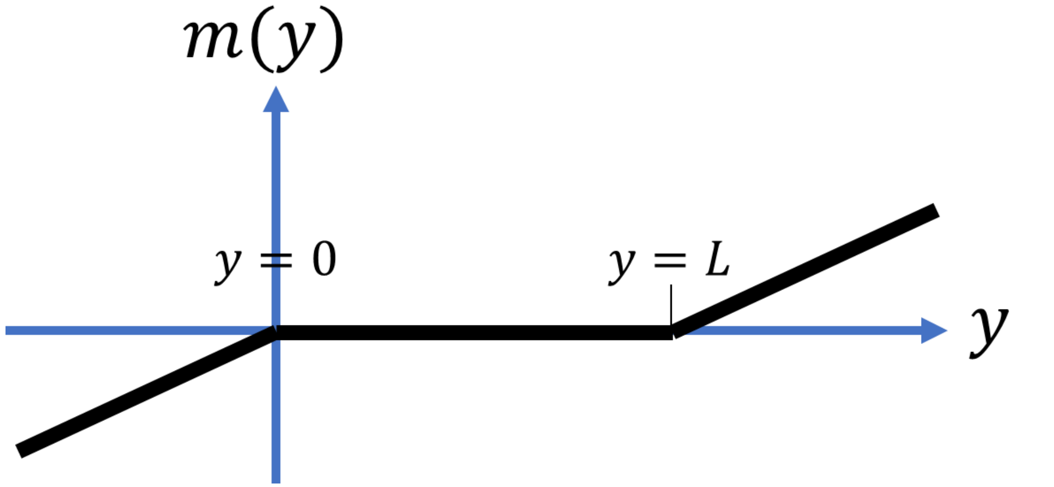
$$\gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix} (n = even)$$

- This results correspond to "tachyon condensation" in string theory.
  - We will discuss it in the next section.

With boundary (1)

Next, we will make boundary.

- Fermions are massive = boundary



Odd dimension

- We realize localized fermions at $[0,L]$.
- The bulk is anomaly-free.
- The anomaly is,

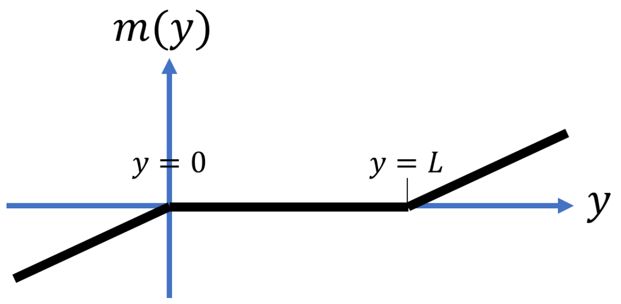
$$m(y) = \mu(y) 1_N = \begin{cases} (m_0 + u'(y-L))1_N & (L < y) \\ m_0 1_N & (0 \le y \le L) \\ (m_0 + uy)1_N & (y < 0) \end{cases}$$

$$\log \mathcal{J} = i\kappa_- \int_{y=0} \alpha \left[\mathrm{ch}(F)\right]_{2r} + i\kappa_+ \int_{y=L} \alpha \left[\mathrm{ch}(F)\right]_{2r}$$

$$\kappa_- = \frac{1}{2}\mathrm{sgn}(u) \ , \quad \kappa_+ = \frac{1}{2}\mathrm{sgn}(u')$$

With boundary (2)

Even dimension



$$m(x)=\mu(y)g(x)=\begin{cases} u'(y-L)g(x) & (L<y) \\ 0 & (0\leq y\leq L) \\ uyg(x) & (y<0) \end{cases}$$

- The anomaly is,

$$\log\mathcal{J}=-i\int_{0<y<L}\alpha\left[\mathrm{ch}(F_+)-\mathrm{ch}(F_-)\right]_{2r}-i\int_{y=L}\alpha[\omega]_{2r-1}+i\int_{y=0}\alpha[\omega]_{2r-1}$$

  - ω is Chern-Simons form.
  - Anomaly from bulk + CS

Index theorem (1)

- We will discuss index theorems for the massive Dirac operator $\mathcal{D}$.

$$S = \int d^d x \bar{\psi}(x) \mathcal{D} \psi(x)$$

$$\begin{aligned}\mathcal{D} =& i\gamma^\mu \left\{ \partial_\mu + \begin{pmatrix} A_\mu^R & 0 \\ 0 & A_\mu^L \end{pmatrix} \right\} + \begin{pmatrix} im(x) & 0 \\ 0 & im^\dagger(x) \end{pmatrix} \\ =& \begin{pmatrix} im(x) & i\sigma^\mu(\partial_\mu + A_\mu^L) \\ i\sigma^{\mu\dagger}(\partial_\mu + A_\mu^R) & im^\dagger(x) \end{pmatrix} \\ \equiv& i \begin{pmatrix} m(x) & \not{D}_L \\ \not{D}_R & m^\dagger(x) \end{pmatrix}\end{aligned}$$

Chern character

- Before discuss about index theorems, we define Chern character for $\mathcal{F}$.
- The Chern character for massive case is,

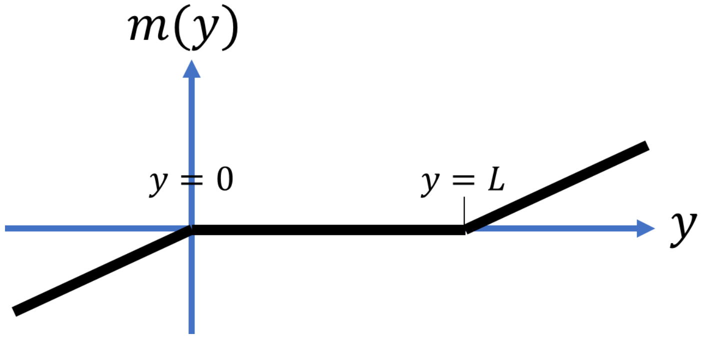
$$\mathrm{ch}(\mathcal{F}) = \sum_{k \leq 0} \left(\frac{i}{2\pi} \right)^{\frac{k}{2}} \mathrm{Str}\left[\mathrm{e}^{\mathcal{F}} \right] \Big|_{k-\mathrm{form}}$$

Index theorem (2)

- We can write the $U(1)$ anomaly by the Chern character.

$$\log \mathcal{J} = -i\left(\frac{i}{2\pi}\right)^{\frac{d}{2}} \int \alpha(x)\mathrm{Str}\left[\mathrm{e}^{\mathcal{F}}\right]\Big|_{d-\mathrm{form}} = -i\int \alpha(x)\mathrm{ch}(\mathcal{F})$$

- The index for the massive Dirac operator is, $\mathrm{Ind}(\mathcal{D}) = \int \mathrm{ch}(\mathcal{F})$
- If $m(x) = 0$, this index becomes Atiyah-Singer(AS) index.
- Let us consider $2r$ dimensional system with boundary.
    - The index will be Atiyah-Patodi-Singer(APS) index.
    - Let's check the index!



Index theorem (3)

APS index theorem

- The index is,

$$\mathrm{Ind}(\mathcal{D}) = \int_{0<y<L} \mathrm{ch}(\mathcal{F})|_{2r} - \frac{1}{2}\left[\eta(i\not{D}_R^{2r-1}) - \eta(i\not{D}_L^{2r-1})\right]_{y=0}^{y=L}$$

  - This is the APS index theorem for the massless Dirac operators.

- To apply this form, we get well-known relation between eta invariant and Chern-Simons form.

$$\int \omega_{2r-1} = -\frac{1}{2}\left(\eta(i\not{D}_R^{2r-1}) - \eta(i\not{D}_L^{2r-1})\right) \qquad (\mathrm{mod}\ \mathbb{Z})$$

5. String theory

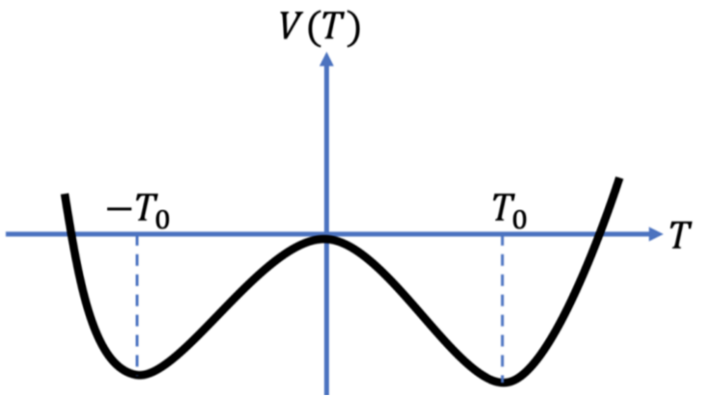
String theory

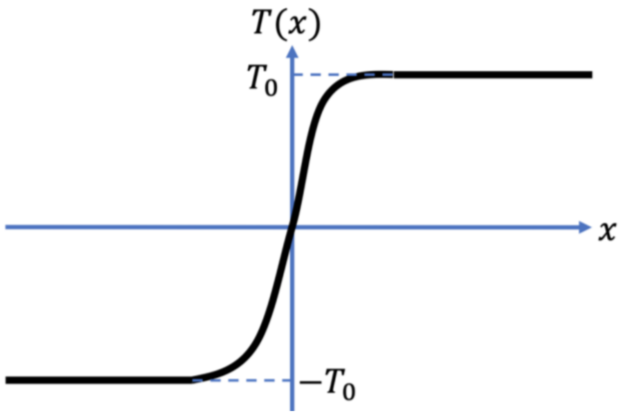
Let us see the relation between the anomaly and string theory.

- Type IIA or IIB string theory
  - 10dim theory
- In string theory, we can think D_p-branes.
  - $p+1$ dim subspace in 10dim.
  - Open strings have their ends on D-branes.
  - Excitation modes of these open strings → Fields on D-branes
  - Open strings on D-branes → QFT in $p+1$ dim
- In some cases, the excitation modes of the strings have tachyon modes.
  - Lowest excitation modes are $m^2 < 0$. (Tachyon)
  - Non-BPS states have tachyons.
  - This tachyonic modes are unstable. → Tachyon condensation

Tachyon condensation (1)

- Tachyonic modes are unstable, so the tachyons have non-zero VEV.
  - Non-trivial configuration of tachyon is also realizable.

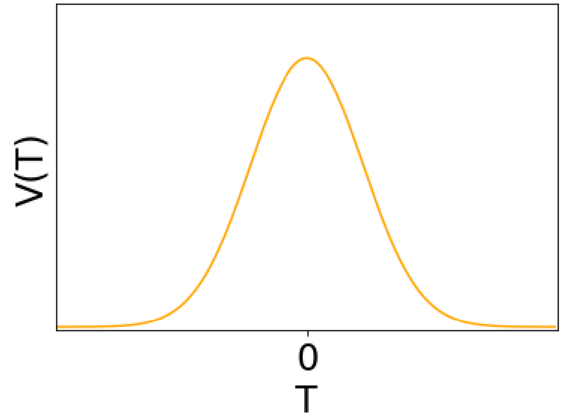




e.g.) D-brane and anti D-brane ($\overline{D}$-brane) system

- Non-BPS state
- Tachyonic modes appear in $D-\overline{D}$ string.
  - The tachyon potential is known.
  - If tachyon configuration is trivial, the D-branes disappear.

$$V(T) = \mathrm{e}^{-T^\dagger T}$$



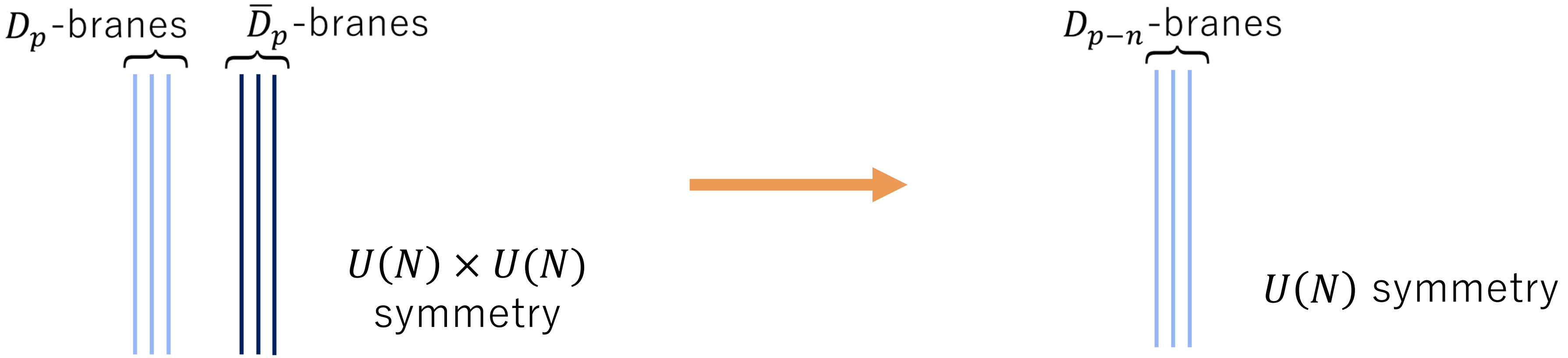
Tachyon condensation (2)

Kink on tachyon in $D_p - \overline{D}_p$ system

- Tachyonic kinks for this system is,

$$T(x) = u\sum_{I=1}^{n}\Gamma^I x^I \qquad \gamma^I = \begin{pmatrix} 0 & \Gamma^I \\ \Gamma^{I\dagger} & 0 \end{pmatrix}$$

- We get D_{p-n}-branes from this tachyon.
  - If D_{p-n}-branes are non-BPS, tachyons still exist on the D-branes.
  - In this case, tachyon condensation occur again.

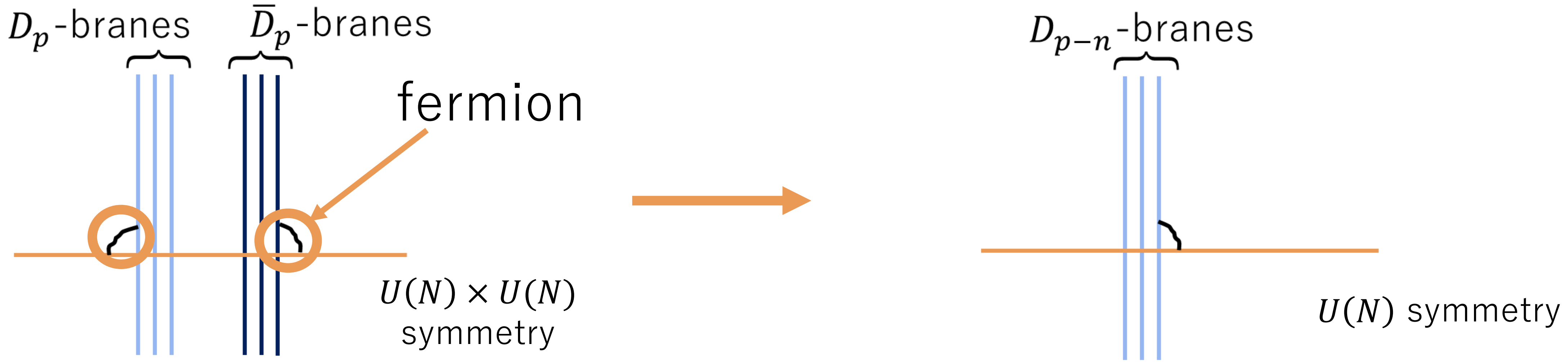


Relation between the anomaly and string

- This tachyon configuration is same for the mass defect in section 4!

$$T(x) = u\sum_{I=1}^{n}\Gamma^I x^I \qquad\qquad m(x) = u\sum_{I=1}^{n}\Gamma^I x^I$$

- The anomalies can be understood from string theory.
  - Fermions are found where D-branes intersect.
  - This is similar to flavor symmetry on holographic QCD model.
    (Sakai-Sugimoto model)



Conclusion

- We discussed about perturbative anomaly with spacetime dependent mass.
  - $U(N_f)_L \times U(N_f)_R$ chiral symmetry for even dimension
  - $U(N_f)$ flavor symmetry for odd dimension
  - We focused on $U(1)$ anomalies for these systems.
- The anomaly can be written by superconnection.
  - This formula comes from string theory, in particular tachyon condensation.
- There are some applications.
  - Kink, vortex, …
  - With boundary
  - Index theorem