Universal behavior of the gluon saturation scale at high energy including full NLL BFKL effects

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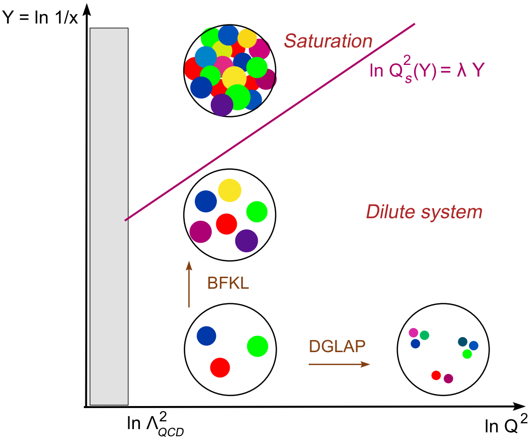
Outline

1. Introduction
   - Motivations
   - Fixed coupling case
2. Asymptotic behavior of the solutions of BK or similar equations, in the running coupling case
3. Saturation scale in the running coupling case
   - Calculation.
   - Discussion of NLL effects

G.B., arXiv:1008.0498
G.B., *in preparation.*

Kinematical regimes of DIS



Motivations

- Simple phenomenological model fitted on the data favors an evolution rate of $Q_s(Y)$

$$\lambda(Y) = \frac{1}{Q_s^2(Y)} \frac{\mathrm{d}}{\mathrm{d}Y} Q_s^2(Y)$$

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Can we understand why running coupling effects are so important? What is the potential impact of other NLL or higher order corrections not included in the simulations?

Generic gluon saturation equation

At leading logarithmic (LL) accuracy, the BK equation writes formally as

$$\partial_Y N(L,Y) \;=\; \bar{\alpha}\left\{\chi(-\partial_L)N(L,Y) - \text{Nonlinear damping}\right\}$$

with the BFKL eigenvalue $\chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)$,

- in position space for the dipole-target amplitude, with $L \sim -\log(r^2)$,
- in momentum space for the unintegrated gluon distribution, with $L \sim \log({k_\perp}^2)$.

Universality of traveling wave solutions

$$\partial_Y N(L,Y) \;=\; \bar{\alpha}\left\{\chi(-\partial_L)N(L,Y) - \text{Nonlinear damping}\right\}$$

Properties:

- Exponential growth and diffusion in the linear regime.
- Nonlinear damping.
- Family of uniformly translated wave-front solutions with a dispersion relation $v(\gamma) = \chi(\gamma)/\gamma$.

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Theorem:
For that type of equation, all solutions with steep enough initial conditions converge to the uniformly translated wave-front solution of minimal velocity $v_c = v(\gamma_c)$, in an universal way.
Bramson (1983)

Universality of traveling wave solutions

In QCD: color transparency ⇒ steep enough initial conditions ⇒ universality applies.

General method to calculate the universal relaxation of the solutions: Ebert, van Saarloos (2000)
Outcome: position of the wave-front / saturation scale:

$$
\begin{aligned}
L_s(Y) &\equiv \log\left(\frac{Q_s(Y)^2}{{Q_0}^2}\right) \\
&= v_c\,\bar{\alpha}Y - \frac{3}{2\gamma_c}\log(\bar{\alpha}Y) + \mathrm{Const.} \\
&\quad -\frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\chi''(\gamma_c)\,\bar{\alpha}Y}} + \mathcal{O}\left(\frac{1}{\bar{\alpha}Y}\right)
\end{aligned}
$$

From fixed to running coupling

$$\partial_Y N(L,Y) \;=\; \bar{\alpha}\left\{\chi(-\partial_L)N(L,Y) - \text{Nonlinear damping}\right\}$$

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Parent dipole/gluon prescriptions for the running coupling:

$$\bar{\alpha} \mapsto \frac{1}{bL}, \quad \text{with} \quad b = \frac{11N_c - 2N_f}{12N_c}$$

$$\text{and} \quad L = \log\left(\frac{{k_\perp}^2}{{\Lambda_{QCD}}^2}\right) \quad \text{or} \quad L = -\log\left(\frac{r^2{\Lambda_{QCD}}^2}{4}\right)$$

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& + \dots
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One can include the full NLL BFKL/BK corrections in that equation, and formally NNLL and higher order contributions.

Balitsky's running coupling prescription

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$\rightarrow$ expanded version: same form as the all order equation

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but with

$$\chi_{NLL}(\gamma) = \frac{b}{2}\left[\chi(\gamma)^2 - \chi'(\gamma) - \frac{4}{\gamma}\chi(\gamma)\right]$$

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- Color transparency $\Rightarrow$ steepness of the relevant initial conditions.

The universality of the traveling wave solutions at fixed coupling is due on these properties and to the existence of a family of exact scaling solutions.
$\longrightarrow$ What about exact scaling solutions in the running coupling case?

Scaling properties and running coupling

$$\begin{aligned}\partial_Y N(L,Y) \;&=\; \frac{1}{bL}\left\{\chi(-\partial_L)N(L,Y) - \text{Nonlinear damping}\right\}\\&+\frac{1}{(bL)^2}\left\{\chi_{NLL}(-\partial_L)N(L,Y) - \text{Nonlinear damping}\right\}\\&+\ldots\end{aligned}$$

$\rightarrow$ discussion on the blackboard.

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- But infinitely many families of approximate scaling variables $s(L,Y)$ in the relevant regime $L, Y \to \infty$ with $s(L,Y)/L \to 0$.

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- No exact scaling solutions!
- But infinitely many families of approximate scaling variables $s(L,Y)$ in the relevant regime $L, Y \to \infty$ with $s(L,Y)/L \to 0$.
- Universal wave front formation independently of the chosen approximate scaling variable.

Scaling variables and running coupling

Examples of family of approximate scaling variables:

$$s(L, Y) = \frac{L}{2\delta}\left[1 - \left(\frac{2vY}{bL^2}\right)^{\delta}\right]$$

with a parameter $\delta > 0$.

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For generic δ: Critical solution with same parameters $v = v_c$ and $\gamma = \gamma_c$ as as at fixed coupling.

Scaling variables and running coupling

Approximate scaling variables for generic solutions with steep initial conditions:

$$s(L,Y) = \frac{L}{2\delta}\left[1-\left(\frac{2v_c Y}{bL^2}\right)^{\delta}\right]-\frac{3\xi_1}{4}(DL)^{1/3} + \eta + \mu(DL)^{-1/3} + \nu(DL)^{-2/3} + \mathcal{O}\left(L^{-1}\right)$$

$\rightarrow$ includes diffusive corrections.

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First one: already known. ξ_1: zero of Airy function. $D = \frac{\chi''(\gamma_c)}{2\,\chi(\gamma_c)}$.

Mueller, Triantafyllopoulos (2002); Munier, Peschanski (2004)

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Other coefficients η, μ and ν: new results calculated next.

Equation for the reduced critical front

Consider the pulled front universal solutions ($\gamma = \gamma_c$ and $v = v_c$), write them in the dilute regime as $N(L, Y) = e^{-\gamma_c s} f(s, L)$.

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$$0 = \frac{1}{D}\left[\frac{bL}{v_c}\left(\frac{\partial s}{\partial Y}\right) + \left(\frac{\partial s}{\partial L}\right)\right]\left[f(s,L) - \frac{1}{\gamma_c}\partial_s f(s,L)\right] - \frac{1}{\gamma_c D}\,\partial_L f(s,L)$$

$$+\sum_{n=2}^{\infty}(-1)^n\, K_n\left[\partial_s + \left(\frac{\partial s}{\partial L} - 1\right)(\partial_s - \gamma_c) + \partial_L\right]^n f(s,L)$$

$$+\frac{1}{DL}\sum_{n=0}^{\infty}(-1)^n\, N_n\left[\partial_s + \left(\frac{\partial s}{\partial L} - 1\right)(\partial_s - \gamma_c) + \partial_L\right]^n f(s,L)\,.$$

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Notations: $D = \dfrac{\chi''(\gamma_c)}{2\,\chi(\gamma_c)}$, $K_n = \dfrac{2\,\chi^{(n)}(\gamma_c)}{n!\,\chi''(\gamma_c)}$ and $N_n = \dfrac{\chi^{(n)}_{NLL}(\gamma_c)}{n!\,\chi(\gamma_c)\,b}$

Front interior expansion

$$N(L,Y) = e^{-\gamma_c s}\, f(s,L)$$

In the front interior regime $L \to \infty$ and $s = \mathcal{O}(1)$, structure of the solution:

$$f(s,L) = \mathcal{A}\left[s + \frac{1}{\gamma_c} + (DL)^{-2/3}\,\phi_2(s) + (DL)^{-1}\,\phi_3(s) + \mathcal{O}\left(L^{-4/3}\right) \right]$$

with polynomial functions $\phi_2(s)$, $\phi_3(s)$, ...

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Example:

$$\phi_2(s) \quad = \quad \frac{\xi_1}{6}\,s^3 + \frac{\xi_1}{2}\left[K_3 - (3\delta - 1)\gamma_c D\right] s^2 + \mathcal{E}$$

Leading edge expansion

Leading edge regime: $L \to \infty$, $s = \mathcal{O}(L^{1/3})$ and $z = \frac{s}{(DL)^{1/3}} + \xi_1 = \mathcal{O}(1)$:

$$f(s,L) = \mathcal{A}\left[(DL)^{1/3}\,\frac{\mathrm{Ai}(z)}{\mathrm{Ai}'(\xi_1)} + G_0(z) + (DL)^{-1/3}\,G_1(z) \right.$$
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The $G_n(z)$ functions verify a hierarchy of Airy equations with 2nd members.

- $G_n(\xi_1)$ and $G_n'(\xi_1)$ are fixed by matching with the front interior expansion.
- Additional requirement: regularity at $z \to \infty$. Forbids the appearance of $\mathrm{Bi}(z)$, and determine at each order one of the coefficients η, μ and ν in the scaling variable $s(L, Y)$.

Intermediate results

$$\eta = \frac{K_3}{2} - N_0$$

$$\mu = {\xi_1}^2 \left[\frac{9D\delta}{16} - \frac{D}{16} - \frac{1}{72\,\gamma_c^2} - \frac{K_3}{4\gamma_c} + \frac{3}{8}\,{K_3}^2 - \frac{3}{10}\,K_4 \right]$$

$$\nu = -3\mathcal{E} - \frac{(3\delta - 1)\xi_1 D}{2} \left[\frac{K_3}{2} - N_0 \right] + \xi_1 \left\{ \frac{5}{54\gamma_c^3} - \frac{D}{3\gamma_c} + K_3 \left[\frac{1}{6\gamma_c^2} + \frac{D}{4} - \frac{3K_3}{4\gamma_c} + \frac{11{K_3}^2}{4} \right] + K_4 \left[\frac{1}{\gamma_c} - 6K_3 \right] + 3K_5 + N_0 D - N_2 + \frac{1}{2\gamma_c^2} \, (1 + 3\gamma_c K_3) \left[\gamma_c N_1 - N_0 \right] \right\}$$

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Depend on arbitrary or undetermined parameters δ and $\mathcal{E}$!

Matching with the saturated regime

The expression obtained for $N(L, Y)$ gives a wave front with an unphysical maximum in the infrared of height

$$\mathcal{A}\left[\frac{1}{\gamma_c} + (DL)^{-2/3}\,\mathcal{E} + \mathcal{O}\left(L^{-1}\right)\right]$$

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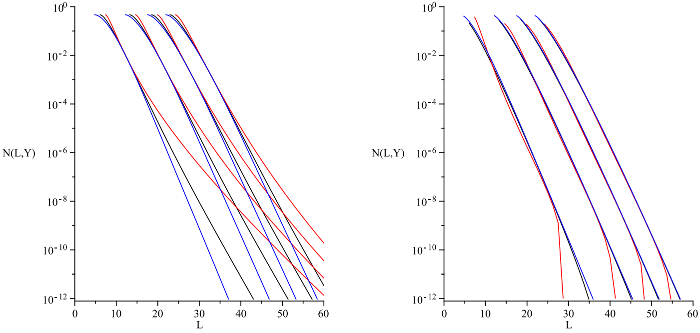
- The normalization $\mathcal{A}$ depends on the strength of the nonlinear damping but not on the initial conditions or on NLL terms.

Dependance of the solution on δ



Asymptotic solution of the BK equation with running coupling (no NLL kernel). $Y = 10, 30, 50, 70$.
Blue: $\delta = 1/4$. Black: $\delta = 1/2$. Red: $\delta = 1$.
Left: with $G_0(z) = G_1(z) = G_2(z) = \eta = \mu = \nu = 0$.
Right: with all the orders calculated.

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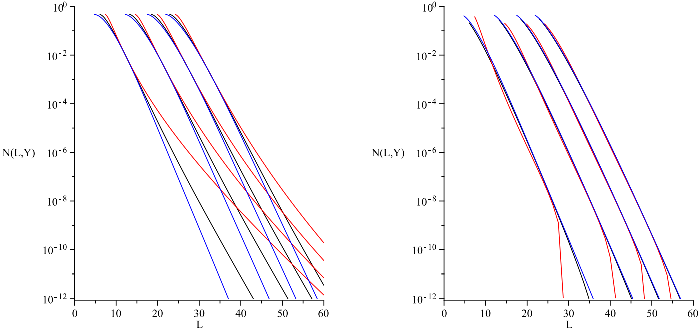

Dependance on δ weaker and weaker when increasing the accuracy of the calculations.

$\rightarrow$ Exact solution independent on the choice of scaling variable $s(L, Y)$.

Extraction of the saturation scale

The saturation scale $Q_s(Y)$ is defined by a level line:

$$N(L_s(Y), Y) = \kappa\,, \quad \text{with} \quad L_s(Y) \equiv \log \frac{Q_s^2(Y)}{\Lambda_{QCD}^2}$$

From the result

$$N(L, Y) = \mathcal{A}\, e^{-\gamma_c s} \left[s + \frac{1}{\gamma_c} + (DL)^{-2/3}\, \phi_2(s) + \mathcal{O}\left(L^{-1}\right) \right],$$

one gets $s(L_s(Y), Y) = \Sigma - \dfrac{1}{\gamma_c} + (D\, L_s(Y))^{-2/3}\, \dfrac{\phi_2(\Sigma - 1/\gamma_c)}{\gamma_c \Sigma - 1}\,,$

$$\text{where} \quad \Sigma \equiv -\frac{1}{\gamma_c}\, \mathrm{W}_{-1}\left(-\frac{\gamma_c\, \kappa}{\mathcal{A}\, e}\right)\,,$$

and $\mathrm{W}_{-1}(x)$ is the -1 branch of the Lambert function.

Universal part of the saturation scale

Finally, one obtains the expansion

$$
\begin{aligned}
L_s(Y) &= \log \frac{Q_s^2(Y)}{\Lambda_{QCD}^2} \\
&= \left(\frac{2v_c Y}{b}\right)^{1/2} + \frac{3\xi_1}{4}\left(D^2\,\frac{2v_c Y}{b}\right)^{1/6} + c_0 \\
&\quad + c_{-1/6}\left(D^2\,\frac{2v_c Y}{b}\right)^{-1/6} + c_{-1/3}\left(D^2\,\frac{2v_c Y}{b}\right)^{-1/3} \\
&\quad + \mathcal{O}\left(Y^{-1/2}\right),
\end{aligned}
$$

with 3 new universal coefficients c_0, $c_{-1/6}$ and $c_{-1/3}$.

Universal part of the saturation scale

$$c_0 = \Sigma - \frac{1}{\gamma_c} - \frac{K_3}{2} + N_0$$

$$c_{-1/6} = {\xi_1}^2 \left[\frac{1}{72\,\gamma_c^2} - \frac{D}{32} + \frac{K_3}{4\gamma_c} - \frac{3}{8}{K_3}^2 + \frac{3}{10}K_4 \right]$$

$$\begin{aligned} c_{-1/3} \; = \; & \xi_1 \Bigg\{ \frac{\Sigma^2}{6\gamma_c} + \left[\frac{K_3}{2\gamma_c} - \frac{1}{3\gamma_c^2} \right] \Sigma + \frac{2}{27\gamma_c^3} + \frac{D}{3\gamma_c} \\ & + K_3 \left[-\frac{2}{3\gamma_c^2} + \frac{3K_3}{4\gamma_c} - \frac{11{K_3}^2}{4} \right] + K_4 \left[-\frac{1}{\gamma_c} + 6K_3 \right] - 3K_5 \\ & + \frac{1}{2\gamma_c^2} \left[1 + 3\gamma_c K_3 \right] \left[N_0 - \gamma_c N_1 \right] + N_2 - D N_0 \Bigg\} \end{aligned}$$

Remarks

- Initial condition effects on $\log Q_s(Y)$ delayed until the order $Y^{-1/2}$ as expected.
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  Rummukainen, Weigert (2004); Albacete et al. (2005)
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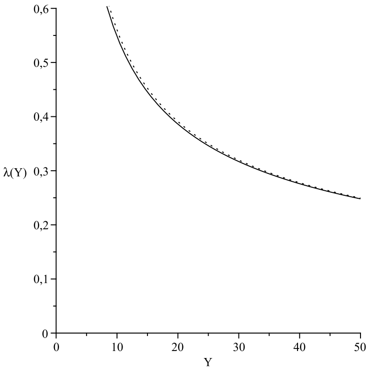
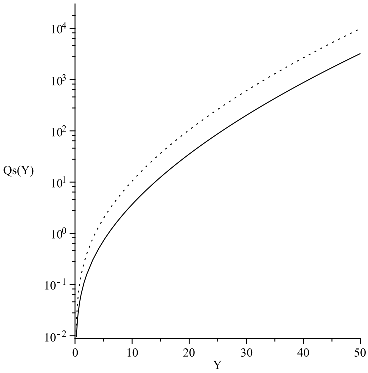
- Renormalization scheme independant result for $Q_s(Y)$.

$Q_s(Y)$: Parent dipole *vs.* Balitsky's prescription





Solid line: with Balitsky's prescription.
Dotted line: with the parent dipole prescription.

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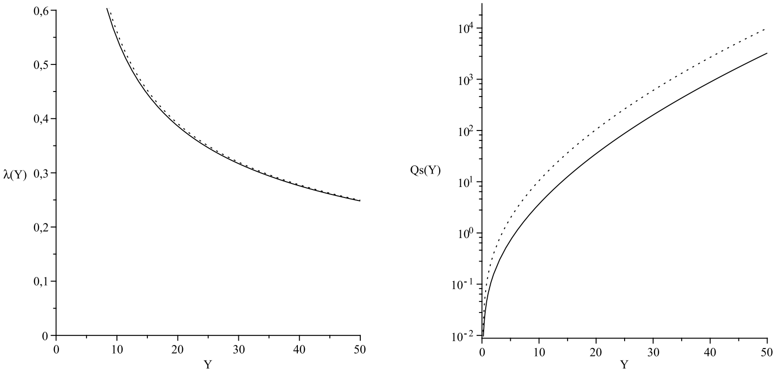
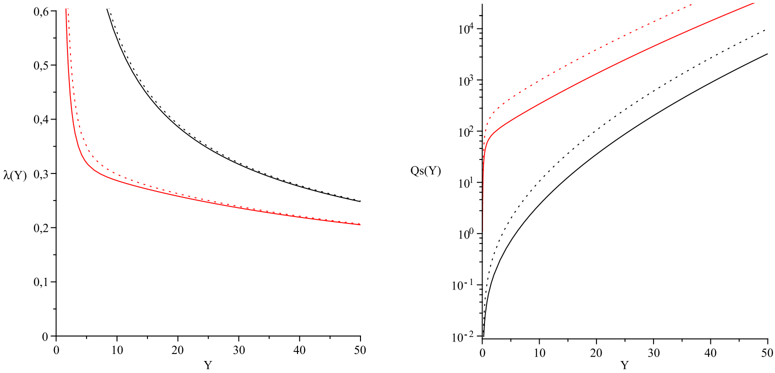


⇒ Running coupling prescription: important for the normalization of $Q_s(Y)$ but not for its evolution.

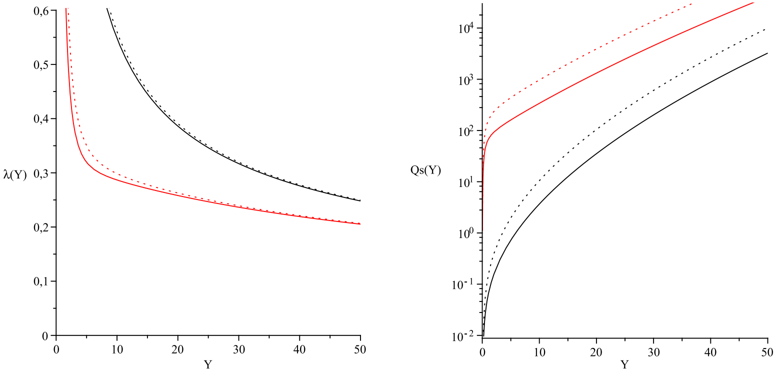
$Q_s(Y)$: Parent dipole *vs.* Balitsky's prescription



Black: universal terms only.

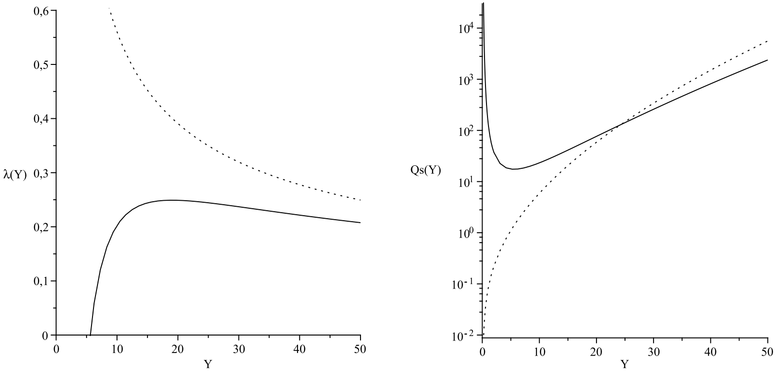
Red: with a shift $Y \mapsto Y + 20$ in the leading term only.

$Q_s(Y)$: Parent dipole *vs.* Balitsky's prescription



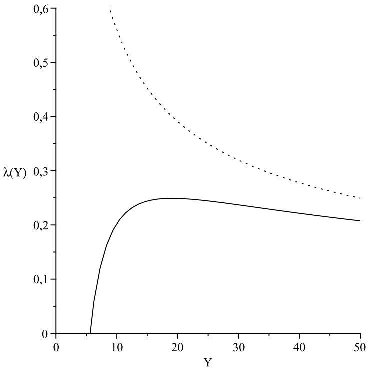
$\Rightarrow$ Very large initial conditions effects required to reduce $\lambda(Y)$ to 0.2 - 0.3 at moderate Y. But leads to a too large $Q_s(Y)$.

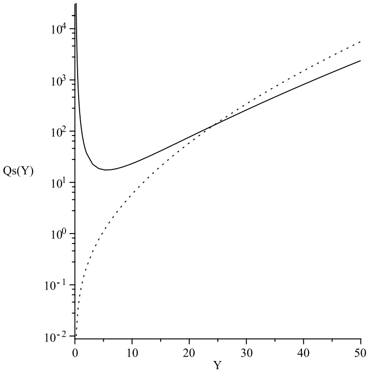
$Q_s(Y)$ with the full NLL BFKL kernel



Solid line: with the full NLL BFKL kernel.
Dotted line: with non NLL contribution besides running coupling with parent gluon $k_\perp$.

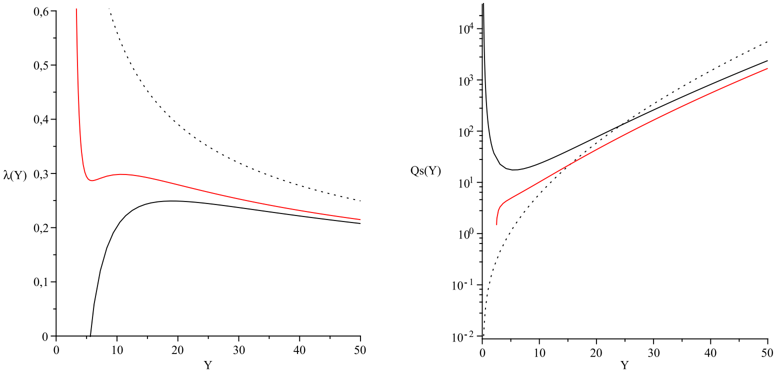
$Q_s(Y)$ with the full NLL BFKL kernel





$\Rightarrow$ Full NLL effects stabilize $\lambda(Y)$ in the phenomenologically preferred range 0.2 - 0.3.

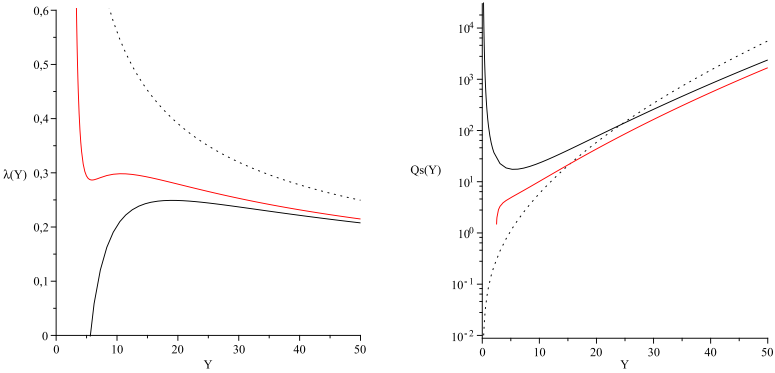
$Q_s(Y)$ with the full NLL BFKL kernel



Black: universal terms only.

Red: with a shift $Y \mapsto Y - 2.5$ in the leading term only.

$Q_s(Y)$ with the full NLL BFKL kernel



⇒ Minor initial conditions effects sufficient to bring both $\lambda(Y)$ and $Q_s(Y)$ in agreement with phenomenological results.

Conclusion

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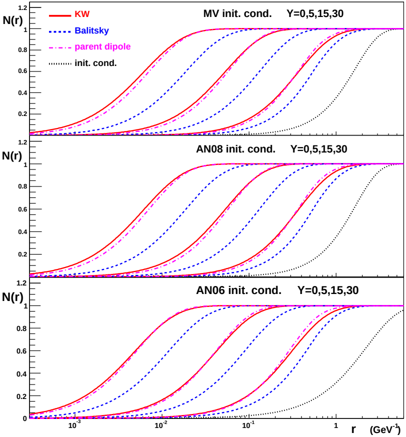
Conclusion

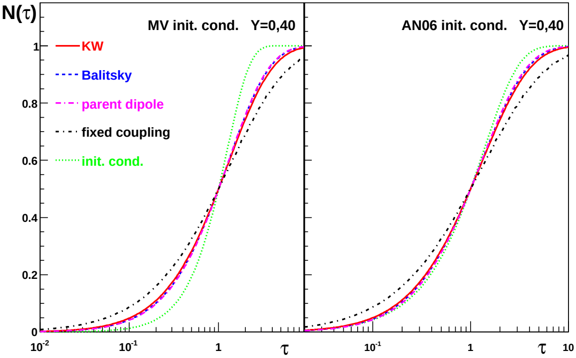
- Running coupling effects must be taken into account for the high energy evolution with the gluon saturation (different universality class).
- All universal terms in $\log Q_s^2(Y)$ at large Y with running coupling have been calculated, including two NLL sensitive ones.
- Absence of exact scaling solutions in the running coupling case does not spoil the universality of the generic solutions.
- Complete NLL effects lead naturally to the observed values and behavior of $Q_s(Y)$.
  $\Rightarrow$ Full NLL equations required for the next generation of gluon saturation phenomenology.

Qualitative agreement with numerics.



Numerical results: Albacete, Kovchegov (2007)

Universal and non universal solutions.



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