Hadron interactions in lattice QCD

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--- Effective Theories and Lattice ---
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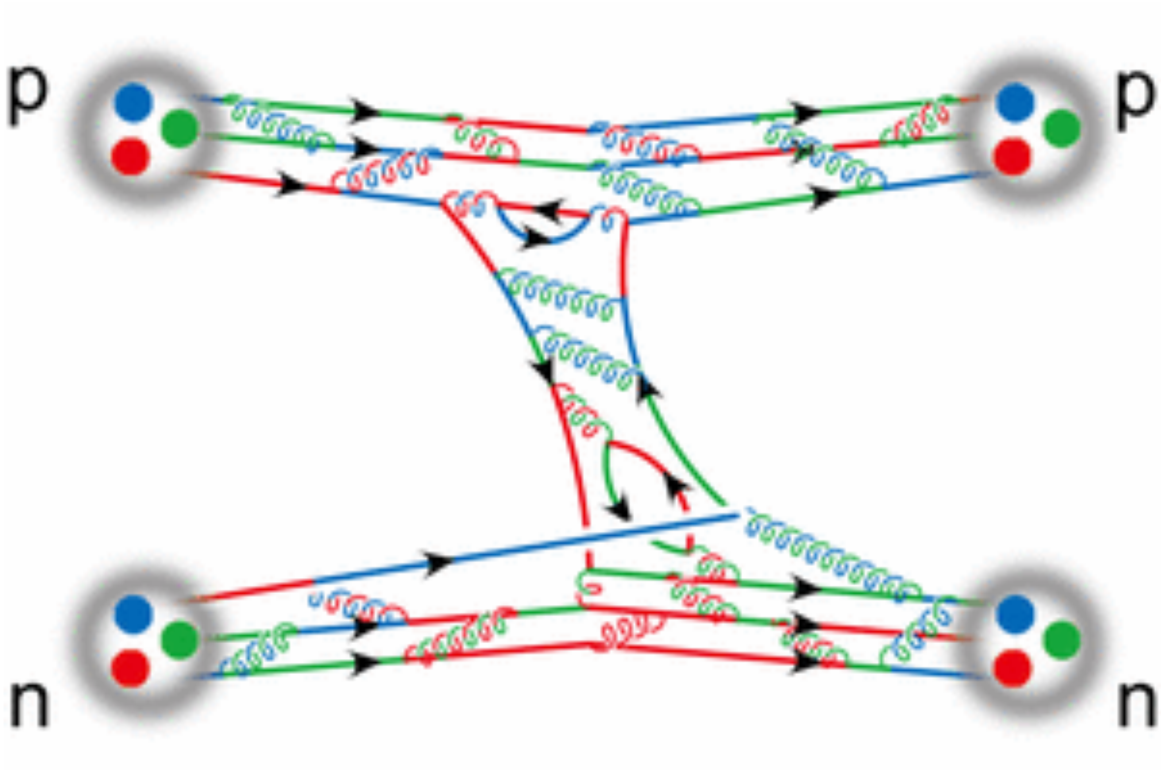
1. Introduction

HAL QCD approach to Nuclear Force



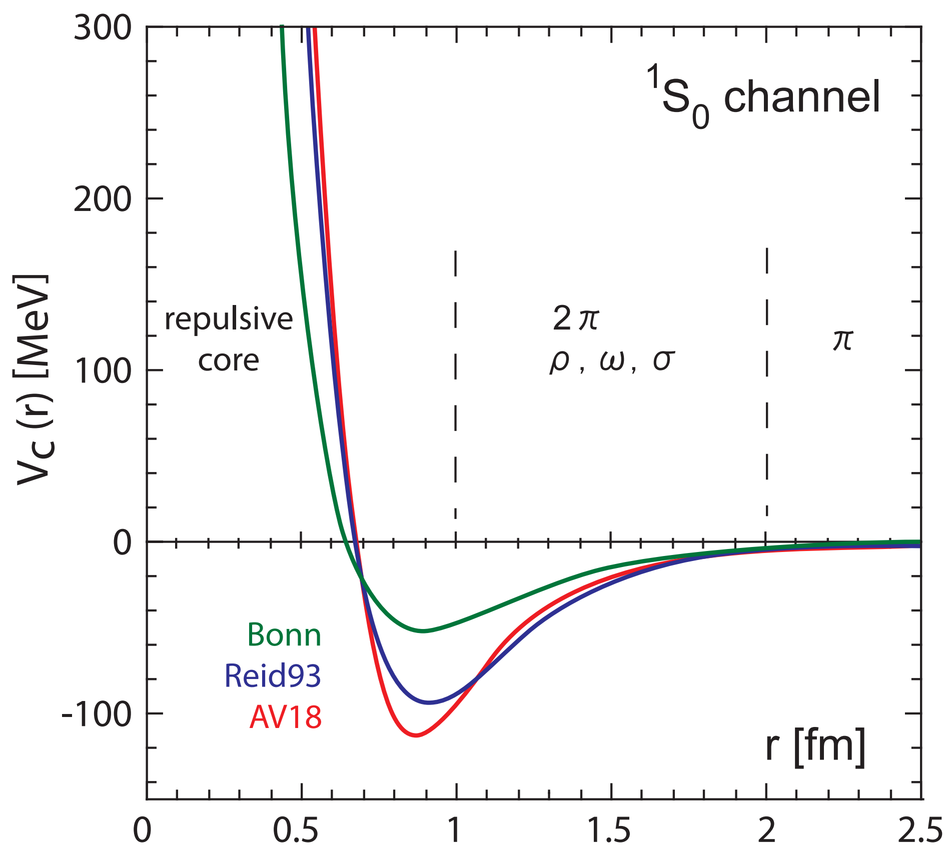
Potentials in QCD ?

"Potentials" themselves can NOT be directly measured.

scheme dependent, ambiguities in inelastic region cf. running coupling in QCD

potentials, but not unique

experimental data of scattering phase shifts



"Potentials" are still useful tools to extract observables such as scattering phase shift.

One may adopt a convenient definition of potentials as long as they reproduce correct physics of QCD.

HAL QCD strategy

Aoki, Hatsuda & Ishii, PTP123(2010)89.

Step 1

define (Equal-time) Nambu-Bethe-Salpeter (NBS) Wave function

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \langle 0 | N(\mathbf{x}+\mathbf{r},0) N(\mathbf{x},0) | NN, W_k \rangle$$

$$W_k = 2\sqrt{\mathbf{k}^2 + m_N^2}$$

energy

$N(x) = \varepsilon_{abc} q^a(x) q^b(x) q^c(x)$: local operator

Key Property 1

Lin et al., 2001; CP-PACS, 2004/2005

$$\varphi_{\mathbf{k}}(\mathbf{r}) \simeq \sum_{l,m} C_l \frac{\sin(kr - l\pi/2 + \delta_l(k))}{kr} Y_{ml}(\Omega_{\mathbf{r}})$$

$$r = |\mathbf{r}| \to \infty$$

$\delta_l(k)$ scattering phase shift (phase of the S-matrix by unitarity) in QCD.

Step 2

define non-local but energy-independent "potential" as

$$[\epsilon_k - H_0]\,\varphi_{\mathbf{k}}(\mathbf{x}) = \int d^3y\, \underline{U(\mathbf{x},\mathbf{y})}\varphi_{\mathbf{k}}(\mathbf{y})$$

$\mu = m_N/2$ reduced mass

$\epsilon_k = \dfrac{\mathbf{k}^2}{2\mu}$ $\quad H_0 = \dfrac{-\nabla^2}{2\mu}$

$$U_{\mathbf{k}}(\mathbf{x},\mathbf{y}) \to U(\mathbf{x},\mathbf{y}) \quad \Longleftrightarrow \quad V_{\mathbf{k}}(\mathbf{x})$$

general

Key Property 2

A non-local but energy-independent potential exists.

Proof

$$U(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'}^{W_k, W_{k'} \le W_{\rm th}} [\epsilon_k - H_0]\,\varphi_{\mathbf{k}}(\mathbf{x})\eta^{-1}_{\mathbf{k},\mathbf{k}'}\varphi^\dagger_{\mathbf{k}'}(\mathbf{y})$$

inner product

$\eta^{-1}_{\mathbf{k},\mathbf{k}'}$: inverse of $\eta_{\mathbf{k},\mathbf{k}'} = (\varphi_{\mathbf{k}}, \varphi_{\mathbf{k}'})$

For $^\forall W_{\mathbf{p}} < W_{\rm th} = 2m_N + m_\pi$ (threshold energy)

$$\int d^3y\, U(\mathbf{x},\mathbf{y})\phi_{\mathbf{p}}(\mathbf{y}) = \sum_{\mathbf{k},\mathbf{k}'} [\epsilon_k - H_0]\,\varphi_{\mathbf{k}}(x)\eta^{-1}_{\mathbf{k},\mathbf{k}'}\eta_{\mathbf{k}',\mathbf{p}} = [\epsilon_p - H_0]\,\varphi_{\mathbf{p}}(x)$$

Step 3 expand the non-local potential in terms of derivative as

$$U(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\nabla)\delta^3(\mathbf{x}-\mathbf{y})$$

$$V(\mathbf{x},\nabla) = V_0(r) + V_\sigma(r)(\sigma_\mathbf{1}\cdot\sigma_\mathbf{2}) + V_T(r)S_{12} + V_{\rm LS}(r)\mathbf{L}\cdot\mathbf{S} + O(\nabla^2)$$

LO LO LO NLO NNLO

tensor operator $S_{12} = \frac{3}{r^2}(\sigma_\mathbf{1}\cdot\mathbf{x})(\sigma_\mathbf{2}\cdot\mathbf{x}) - (\sigma_\mathbf{1}\cdot\sigma_\mathbf{2})$

spins

Step 4 extract the local potential. At LO, for example, we simply have

$$V_{\rm LO}(\mathbf{x}) = \frac{[\epsilon_k - H_0]\varphi_\mathbf{k}(\mathbf{x})}{\varphi_\mathbf{k}(\mathbf{x})}$$

Step 5 solve the Schroedinger Eq. in the infinite volume with this potential.

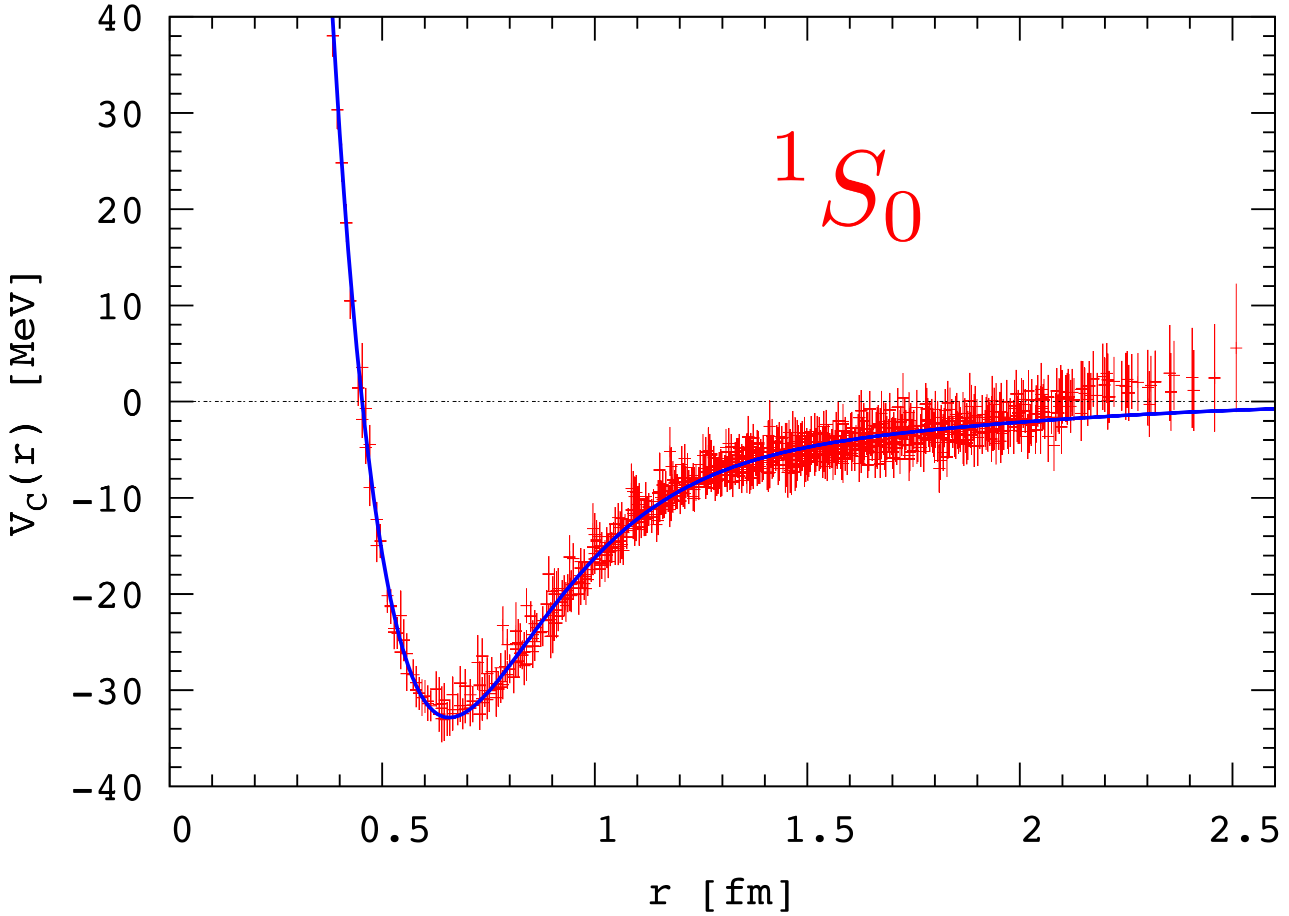
→ phase shifts and binding energy below inelastic threshold

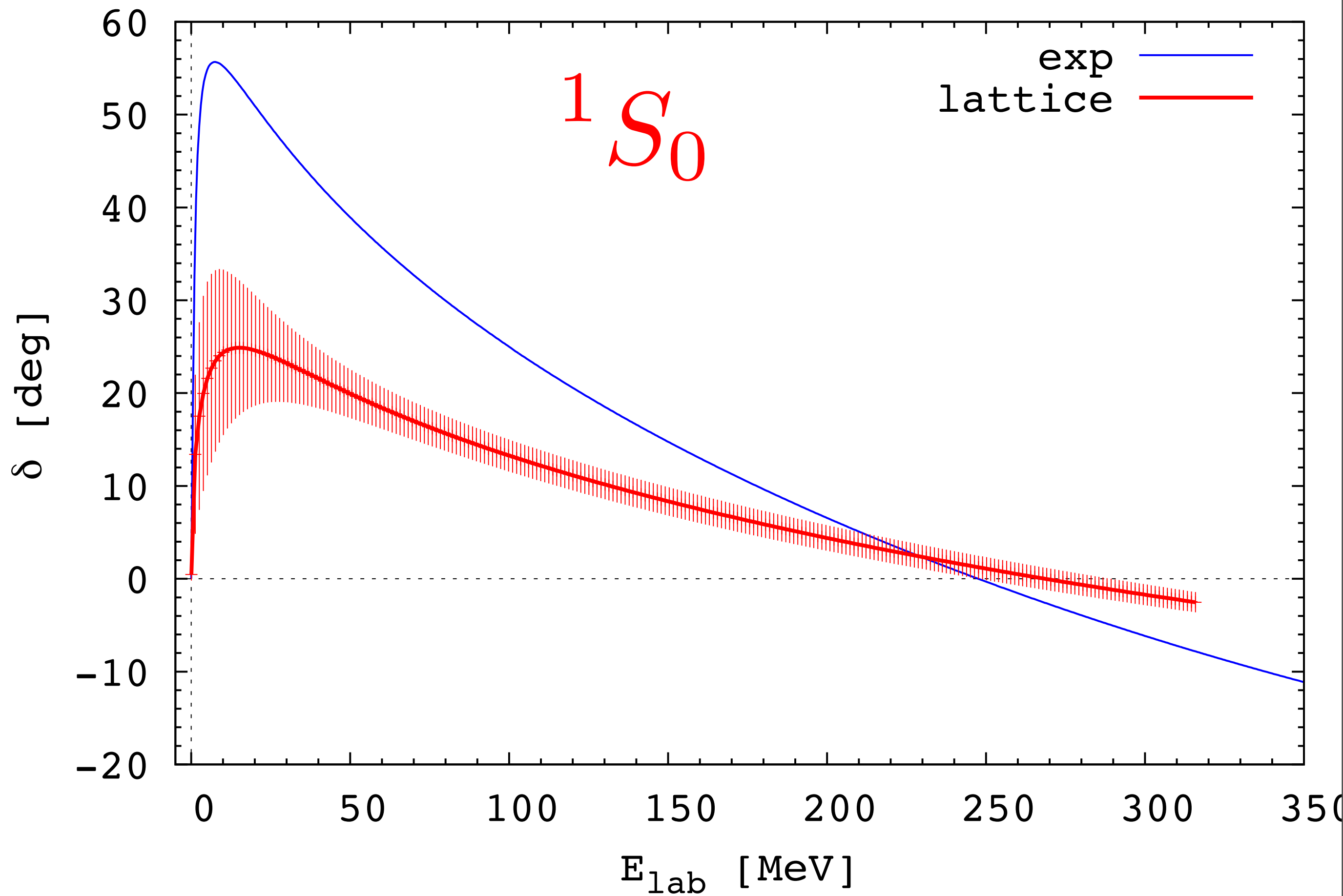
Example

2+1 flavor QCD a=0.09fm, L=2.9fm $m_\pi \simeq 700$ MeV

Ishii *et al.* (HALQCD), PLB712(2012) 437.

NN potential → phase shift





Qualitative features of NN potential are reproduced.

→ It has a reasonable shape.
The strength is weaker due to the heavier quark mass.

Need calculations at physical quark mass.

We can extend the HAL QCD method to inelastic and/or multi-particle scatterings.

Key Property 1

Asymptotic behaviors of NBS wave functions for more than 2 particles

Asymptotic behavior of NBS wave functions $\Longleftrightarrow$ "Phases" of S-matrix

S. Aoki, N. Ishii, T. Doi, Y. Ikeda, T. Inoue, PRD88 (2013) 014036.

Key Property 2

Existence of energy independent potentials above inelastic thresholds

Coupled channel

$$A + B \to C + D$$

S. Aoki, *et al.* , Proc. Jpn. Acad. Ser. B, 87 (2011) 509.

Particle production

$$A + B \to A + B + C$$

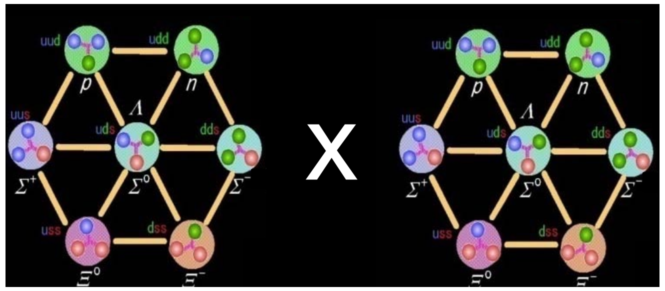
S. Aoki, B. Charron, T. Doi, T. Hatsuda, T. Inoue, N. Ishii, PRD87 (2013) 34512.

2. Applications: H-dibaryon

2-1. Baryon Potentials in the flavor SU(3) limit

$m_u = m_d = m_s$

1. First setup to predict YN, YY interactions not accessible in exp.
2. Origin of the repulsive core (universal or not)



$$8 \times 8 = \underbrace{27 + 8s + 1}_{\text{Symmetric}} + \underbrace{10^* + 10 + 8a}_{\text{Anti-symmetric}}$$

6 independent potentials in flavor-basis

$$V^{(27)}(r),\ V^{(8s)}(r),\ V^{(1)}(r) \quad \longleftarrow \quad {}^1S_0$$

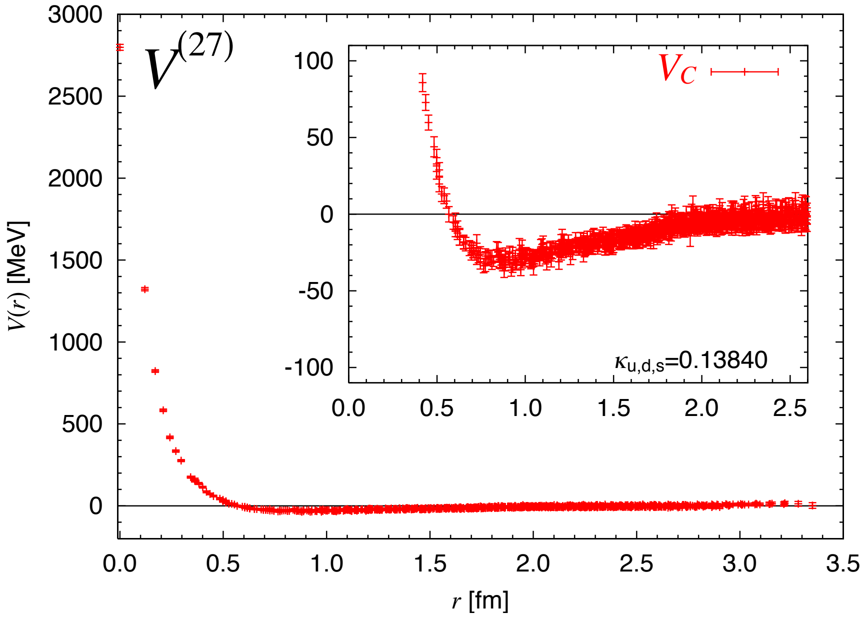
$$V^{(10^*)}(r),\ V^{(10)}(r),\ V^{(8a)}(r) \quad \longleftarrow \quad {}^3S_1$$

3-flavor QCD a=0.12 fm

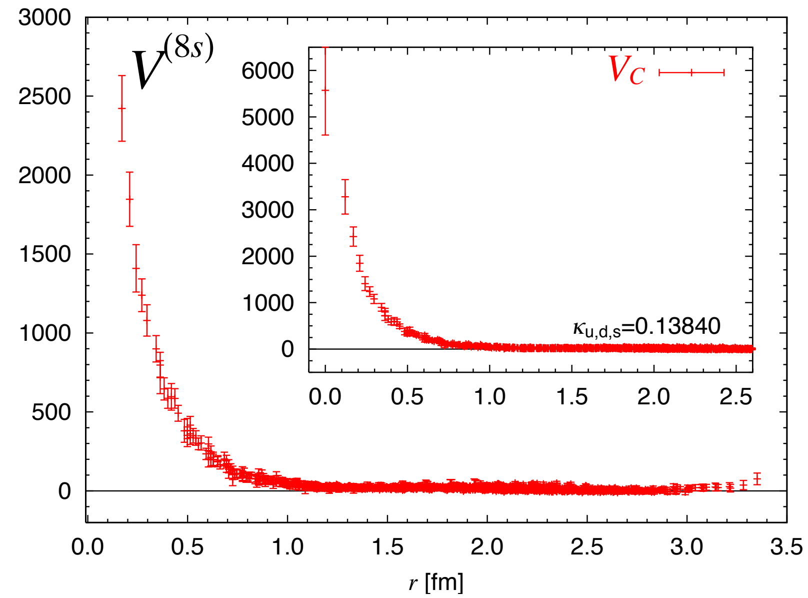
Inoue et al. (HAL QCD Coll.), PTP124(2010)591 L=2 fm

Inoue et al. (HAL QCD Coll.), NPA881(2012)28 L=2-4 fm

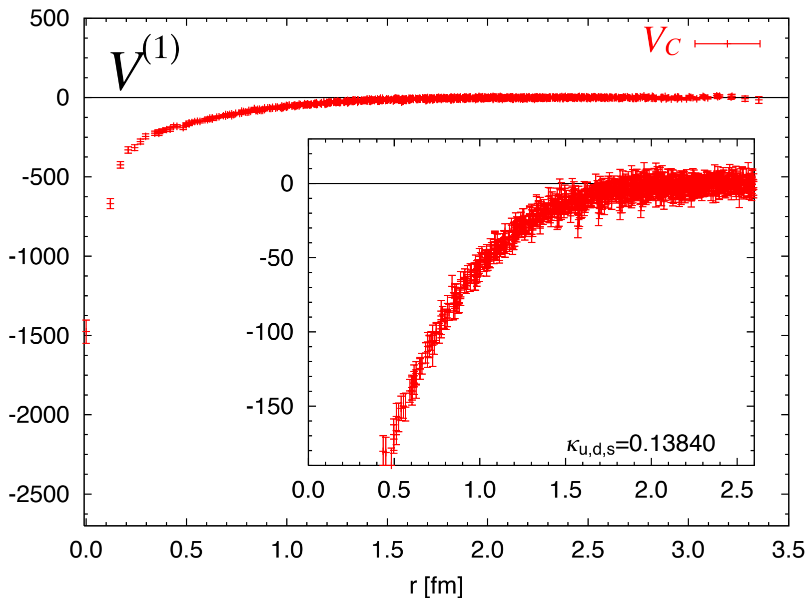
$L \simeq 4$ fm, $\quad m_\pi \simeq 470$ MeV



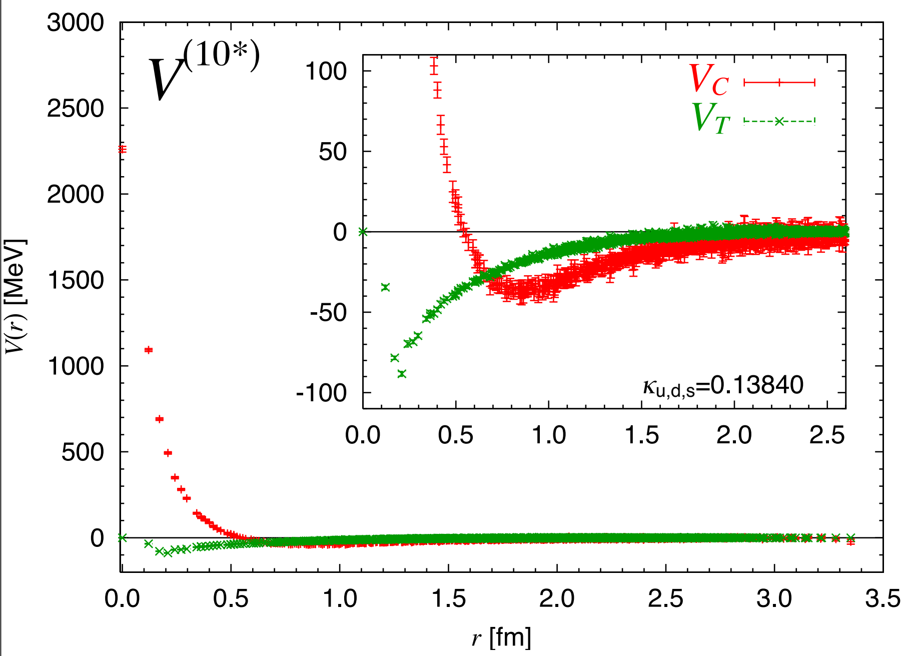
same as NN



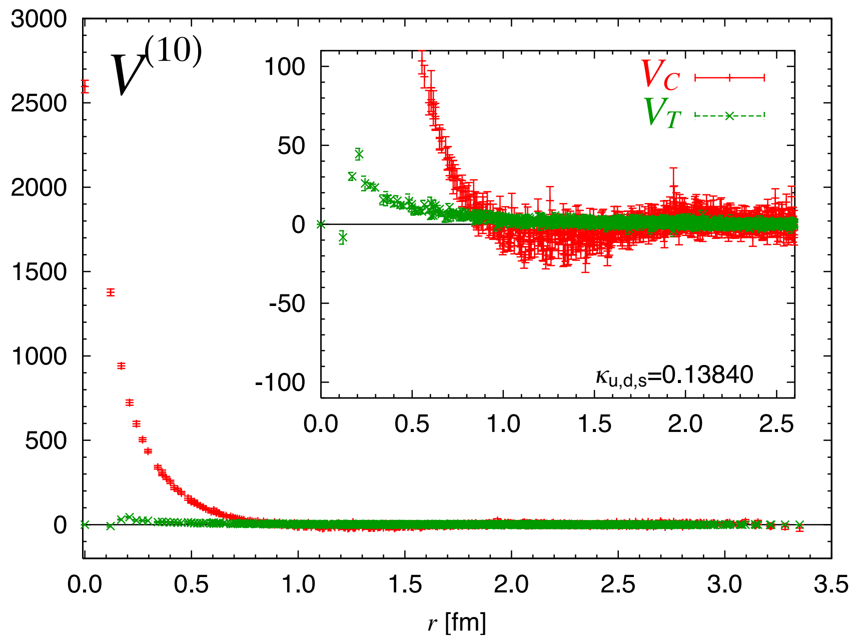
8s: strong repulsive core. repulsion only.



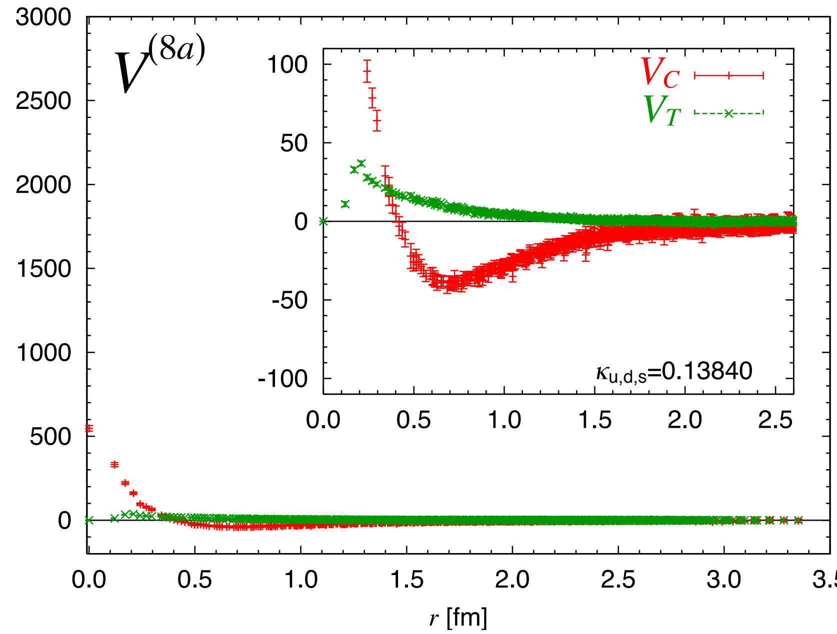
1: attractive instead of repulsive core ! attraction only . H-dibaryon.



same as NN



10: strong repulsive core. weak attraction.



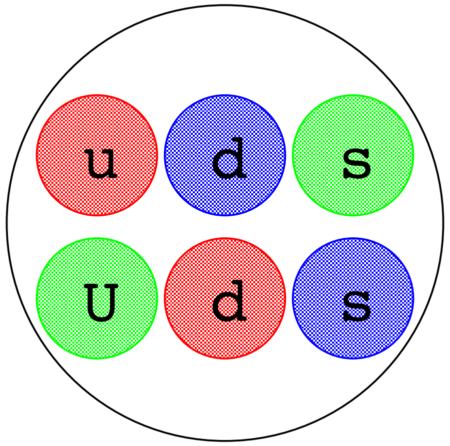
8a: weak repulsive core. strong attraction.

Flavor dependences of BB interactions become manifest in SU(3) limit !

H-dibaryon:

a possible six quark state(uuddss)

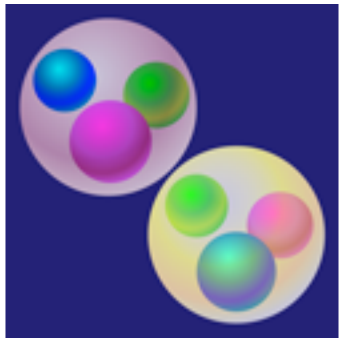
predicted by the model but not observed yet.



http://physics.aps.org/synopsis-for/10.1103/PhysRevLett.106.162001

Binding baryons on the lattice

April 26, 2011



2-2. H-dibaryon in the flavor SU(3) limit

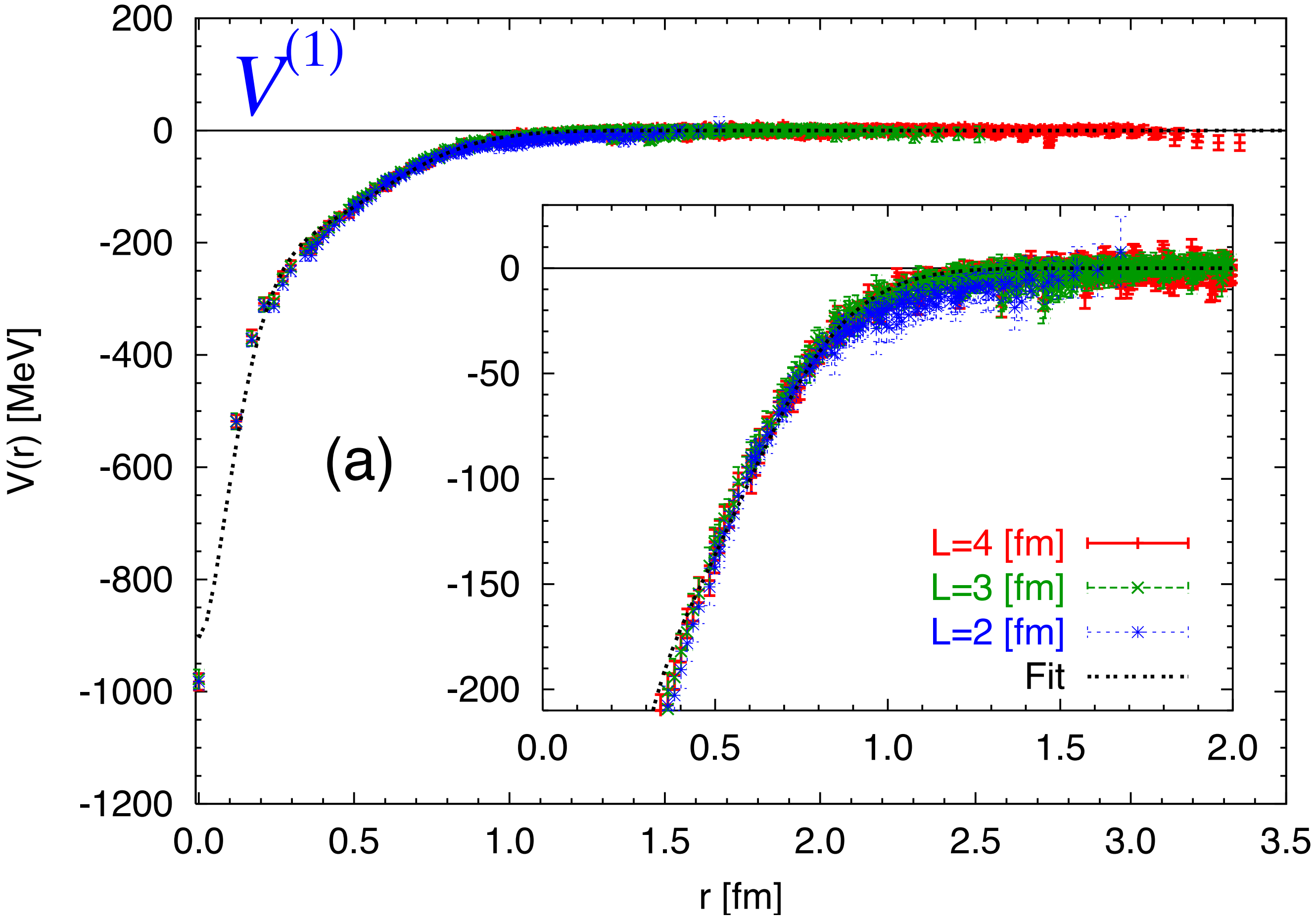
a=0.12 fm

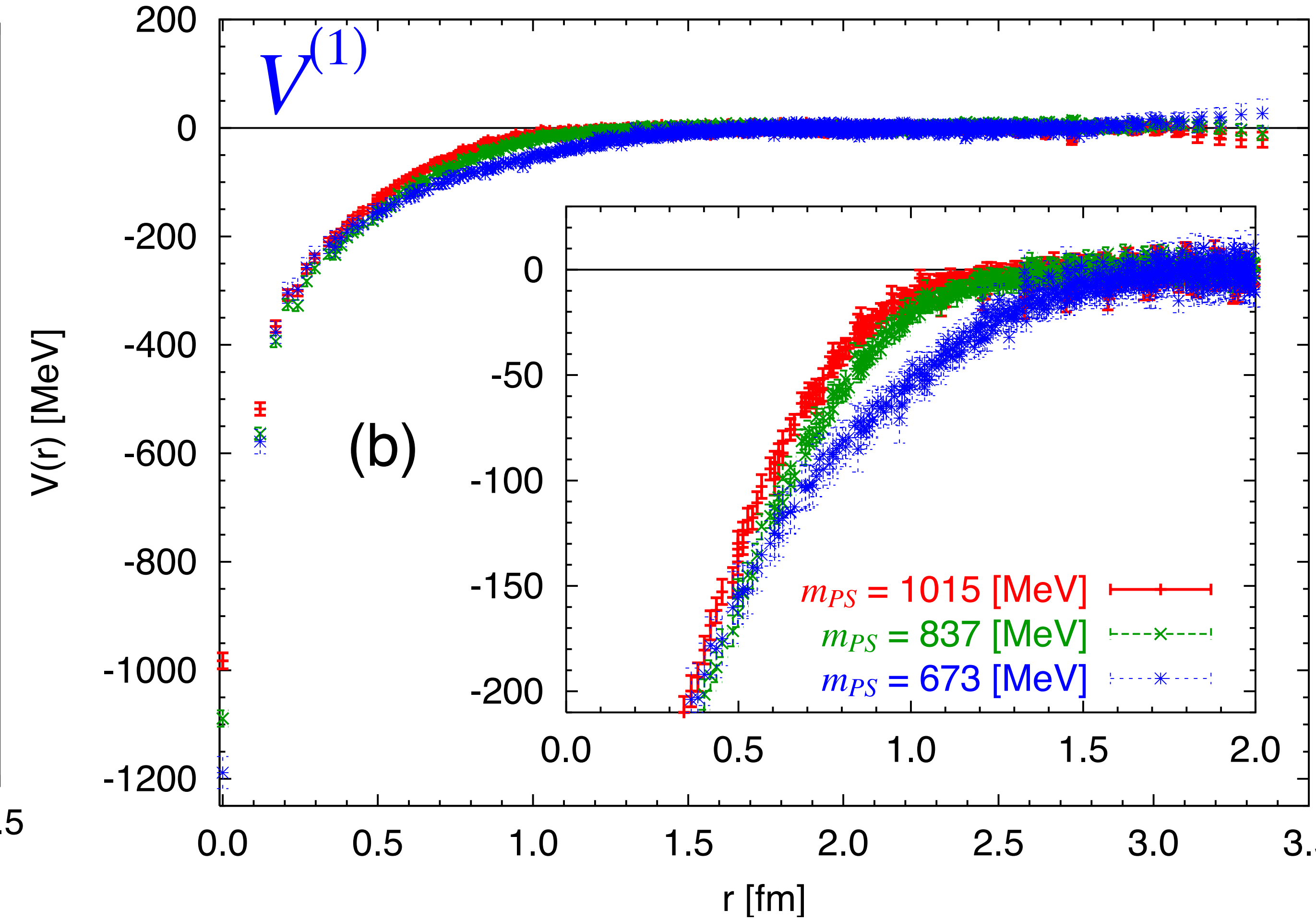
Inoue et al. (HAL QCD Coll.), PRL106(2011)162002

Attractive potential
in the flavor singlet channel → possibility of a bound state (H-dibaryon)

$$\Lambda\Lambda - N\Xi - \Sigma\Sigma$$

volume dependence / **pion mass dependence**





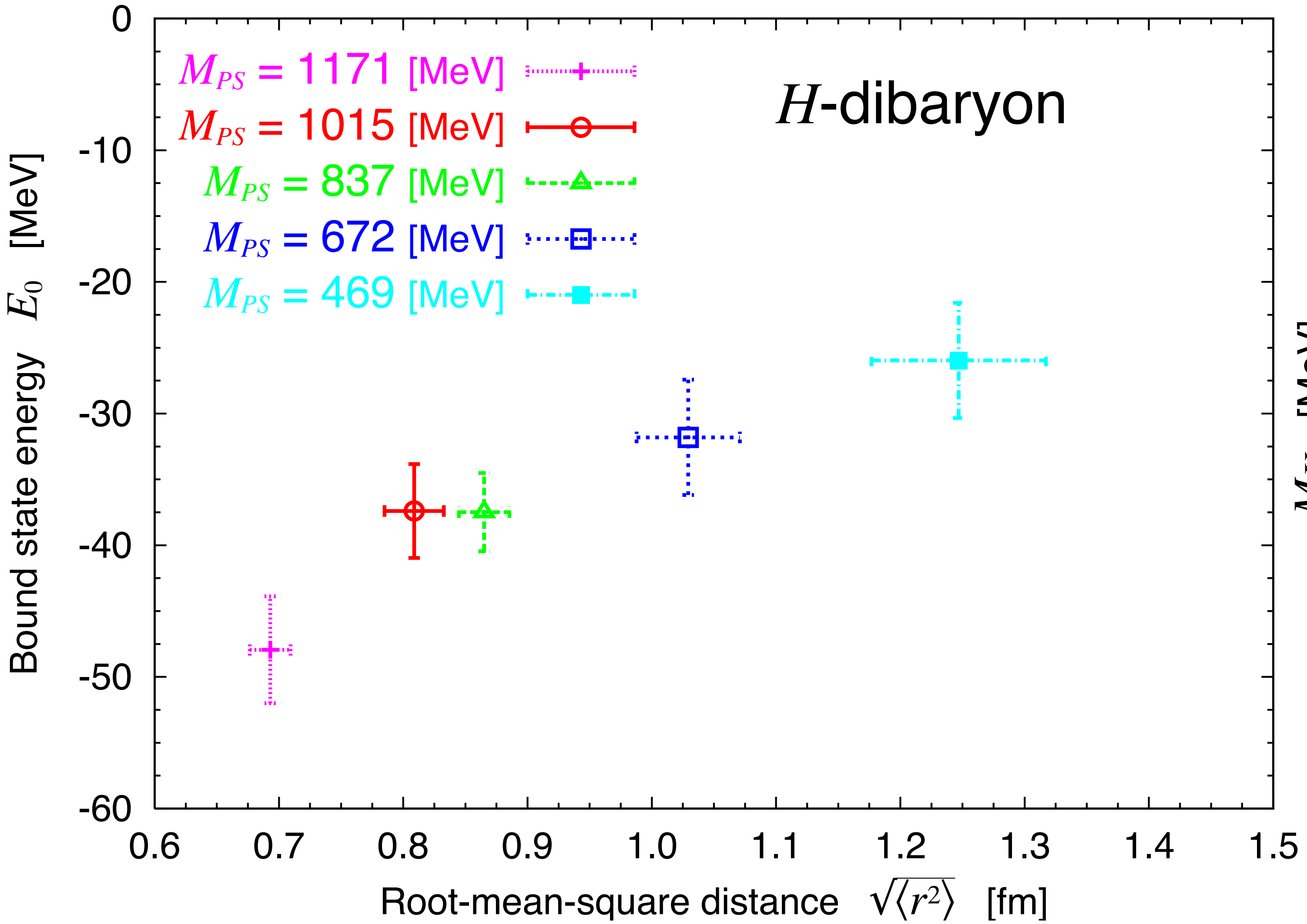
L=3 fm is enough for the potential.

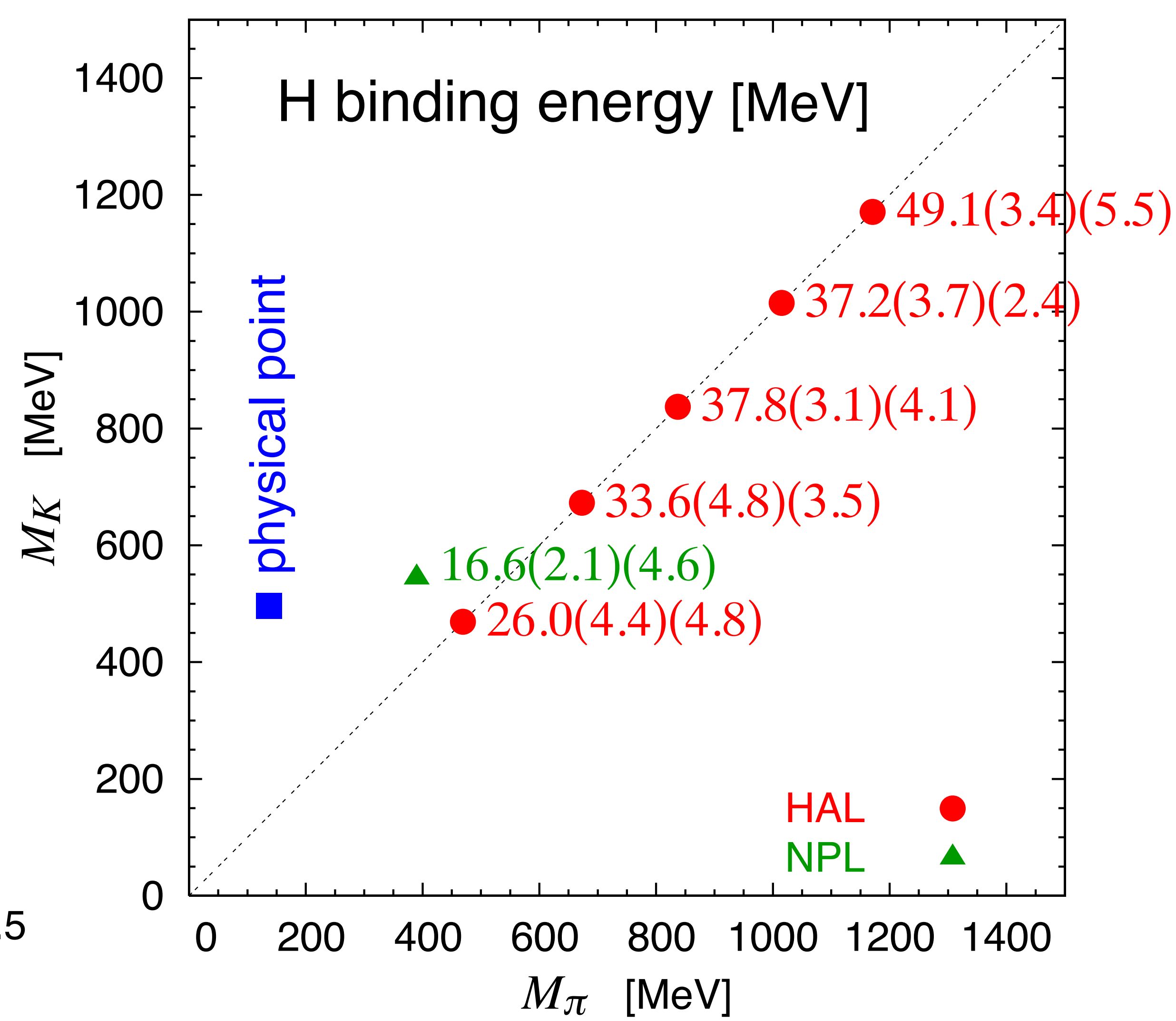
lighter the pion mass, stronger the attraction

fit potentials at L=4 fm by

$$V(r) = a_1 e^{-a_2 r^2} + a_3 \left(1 - e^{-a_4 r^2}\right)^2 \left(\frac{e^{-a_5 r}}{r}\right)^2$$

Solve Schroedinger equation in the infinite volume → One bound state (H-dibaryon) exists.





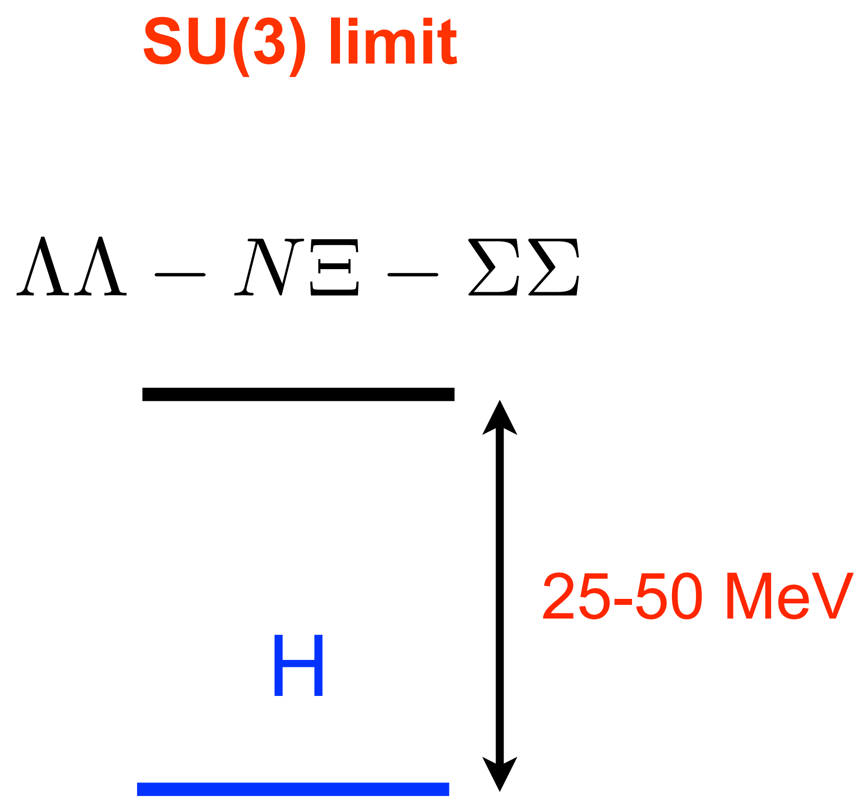
An H-dibaryon exists in the flavor SU(3) limit.

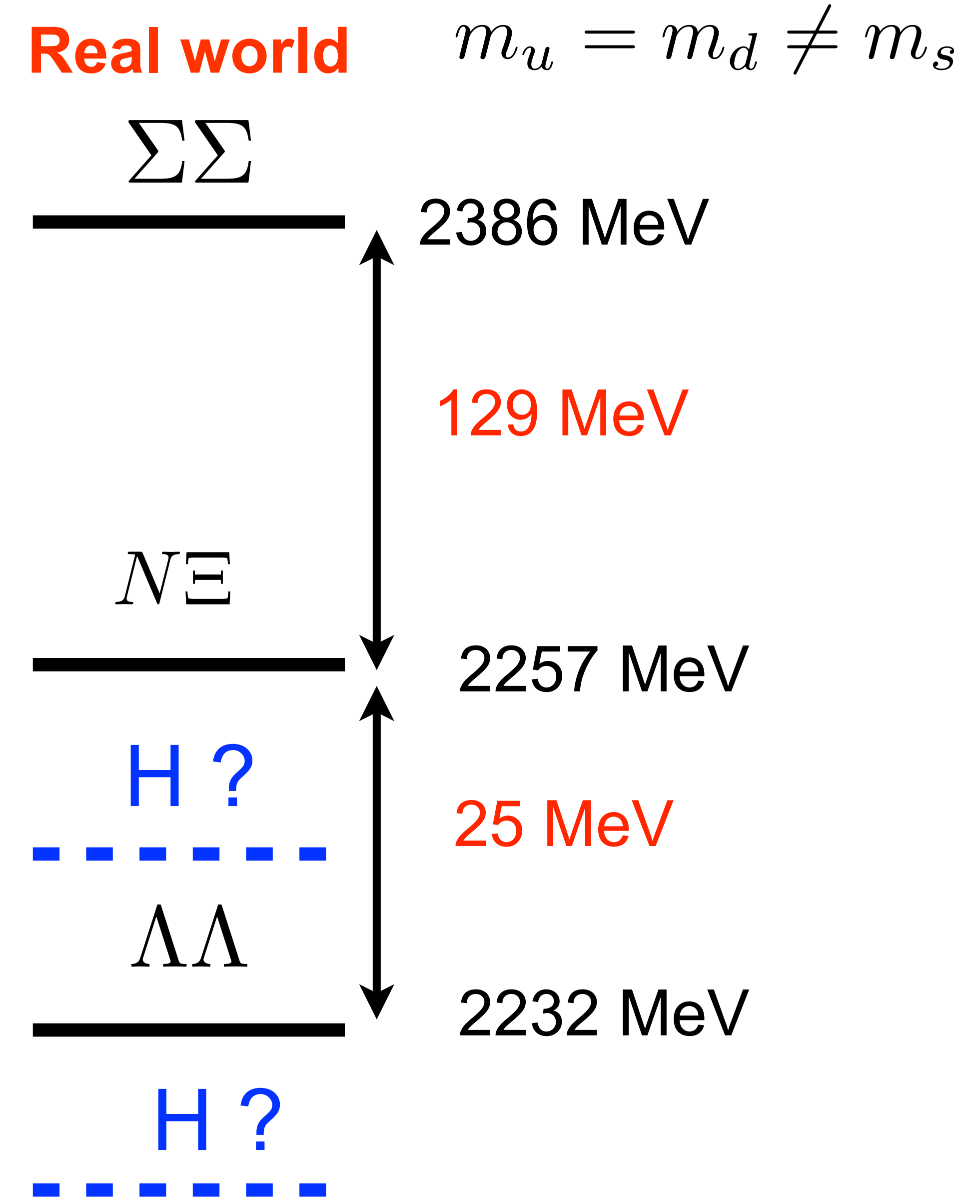
Binding energy = 25-50 MeV at this range of quark mass.

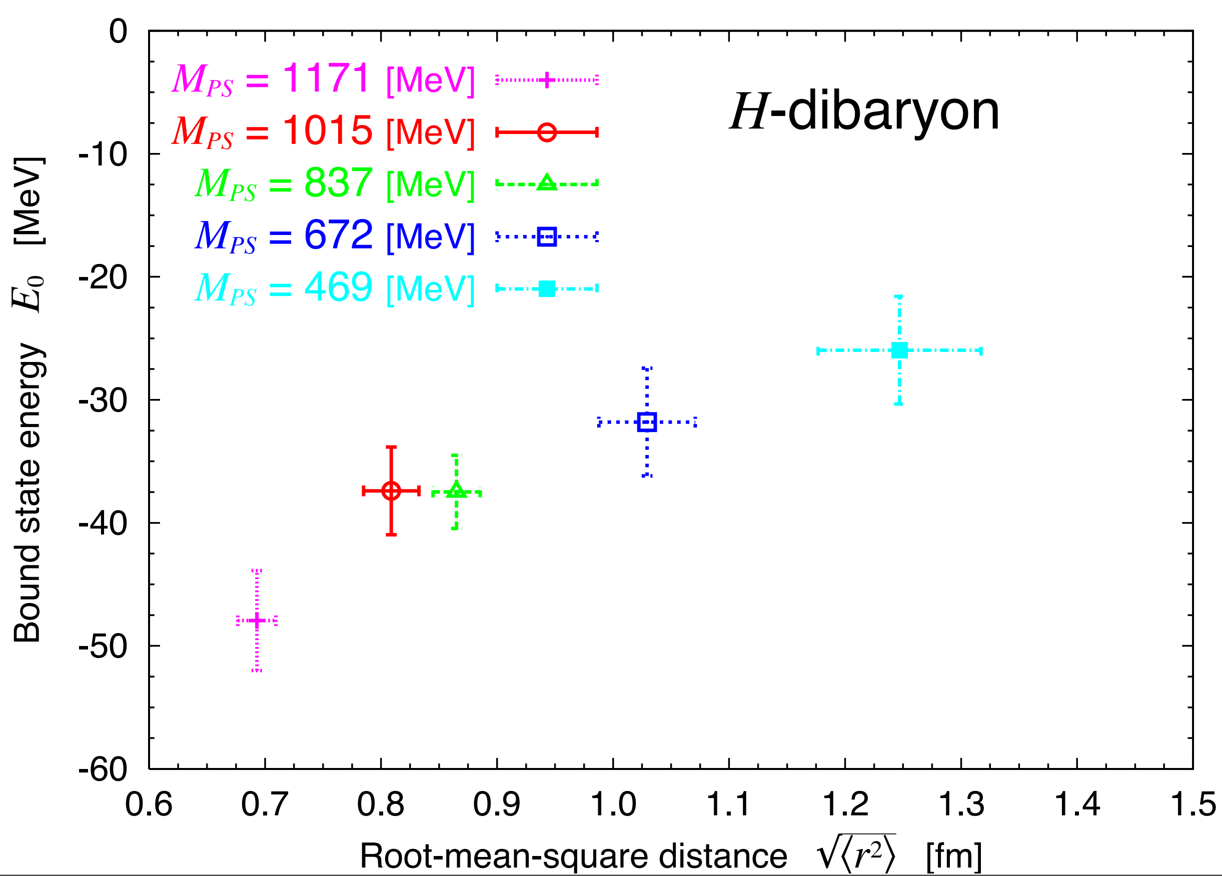
A mild quark mass dependence.

Real world ?

2-3. H-dibaryon with the flavor SU(3) breaking







S=-2 "Inelastic" scattering

$$m_N = 939 \text{ MeV}, \; m_\Lambda = 1116 \text{ MeV}, \; m_\Sigma = 1193 \text{ MeV}, \; m_\Xi = 1318 \text{ MeV}$$

S=-2 System(I=0)

$$M_{\Lambda\Lambda} = 2232 \text{ MeV} < M_{N\Xi} = 2257 \text{ MeV} < M_{\Sigma\Sigma} = 2386 \text{ MeV}$$

The eigen-state of QCD in the finite box is a mixture of them:

$$|S=-2, I=0, E\rangle_L = c_1(L)|\Lambda\Lambda, E\rangle + c_2(L)|\Xi N, E\rangle + c_3(L)|\Sigma\Sigma, E\rangle$$

$$E = 2\sqrt{m_\Lambda^2 + \mathbf{p}_1^2} = \sqrt{m_\Xi^2 + \mathbf{p}_2^2} + \sqrt{m_N^2 + \mathbf{p}_2^2} = 2\sqrt{m_\Sigma^2 + \mathbf{p}_3^2}$$

In this situation, we can not directly extract the scattering phase shift in lattice QCD.

Extended method

S. Aoki, *et al.* , Proc. Jpn. Acad. Ser. B, 87 (2011) 509.

Consider 3x3 coupled channel potential matrix.

$$\begin{pmatrix} V_{\Lambda\Lambda,\Lambda\Lambda}(\mathbf{x}) & V_{\Lambda\Lambda,N\Xi}(\mathbf{x}) & V_{\Lambda\Lambda,\Sigma\Sigma}(\mathbf{x}) \\ V_{N\Xi,\Lambda\Lambda}(\mathbf{x}) & V_{N\Xi,N\Xi}(\mathbf{x}) & V_{N\Xi,\Sigma\Sigma}(\mathbf{x}) \\ V_{\Sigma\Sigma,\Lambda\Lambda}(\mathbf{x}) & V_{\Sigma\Sigma,N\Xi}(\mathbf{x}) & V_{\Sigma\Sigma,\Sigma\Sigma}(\mathbf{x}) \end{pmatrix}$$

Preliminary results from HAL QCD Collaboration

Sasaki for HAL QCD Collaboration

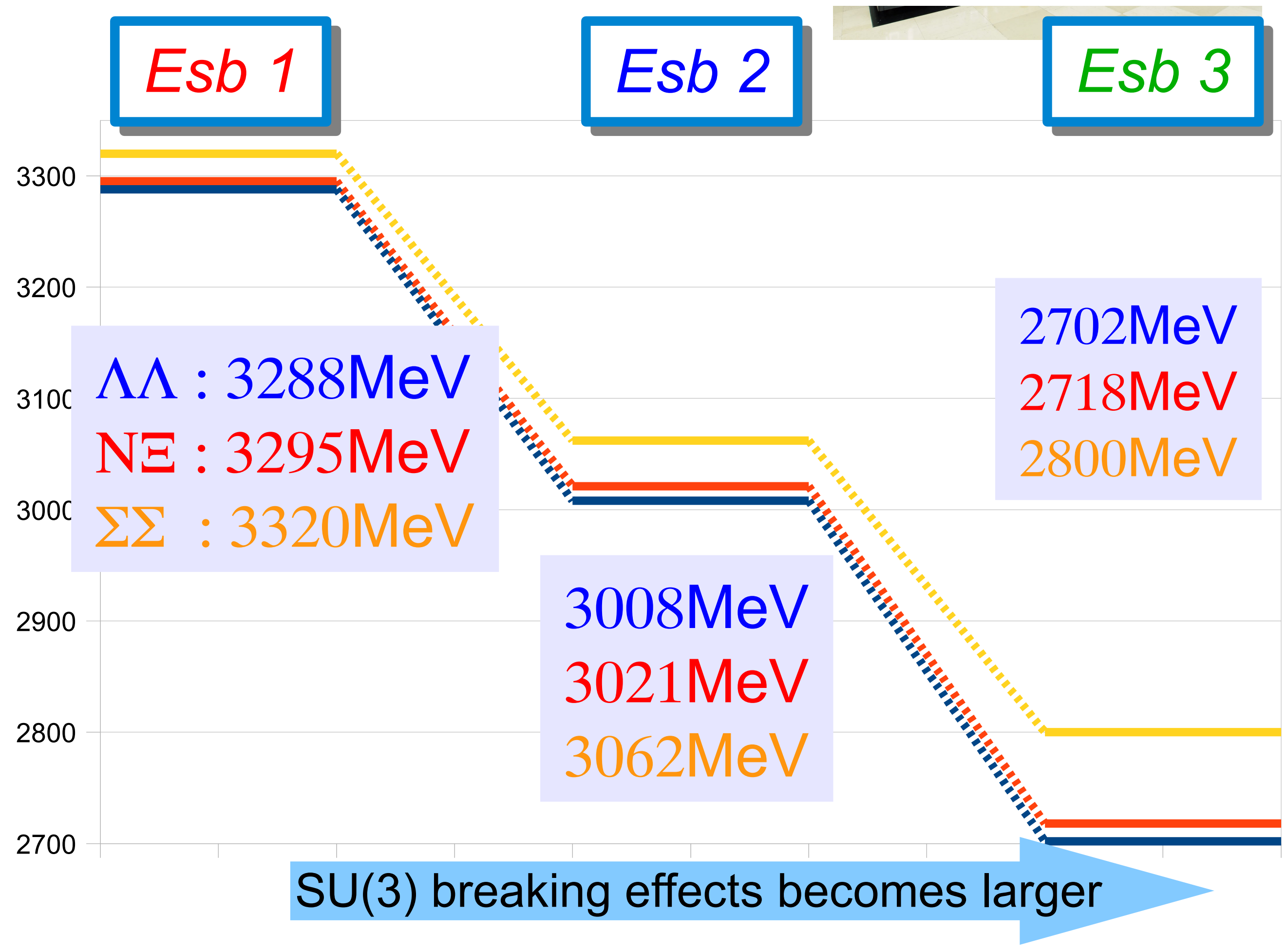
$$N_f = 2+1 \text{ full QCD with } L = 2.9 \text{ fm}$$

Gauge ensembles

In unit of MeV	Esb 1	Esb 2	Esb 3
π	701±1	570±2	411±2
K	789±1	713±2	635±2
m_π/m_K	0.89	0.80	0.65
N	1585±5	1411±12	1215±12
Λ	1644±5	1504±10	1351± 8
Σ	1660±4	1531±11	1400±10
Ξ	1710±5	1610± 9	1503± 7

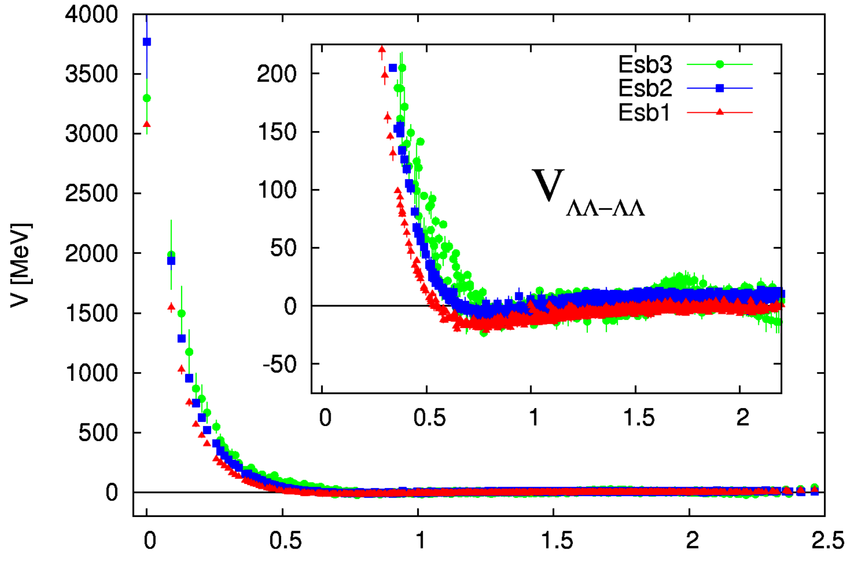
u,d quark masses lighter

thresholds

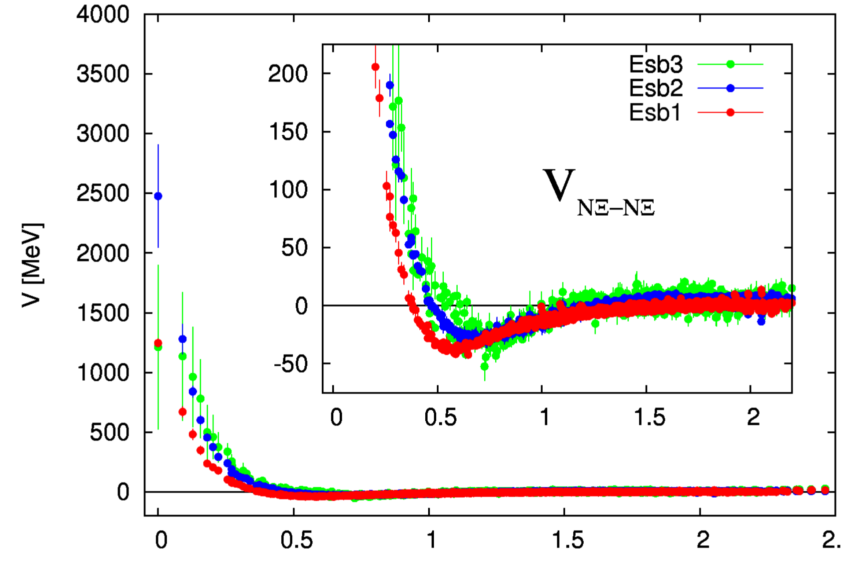


coupled channel 3x3 potentials

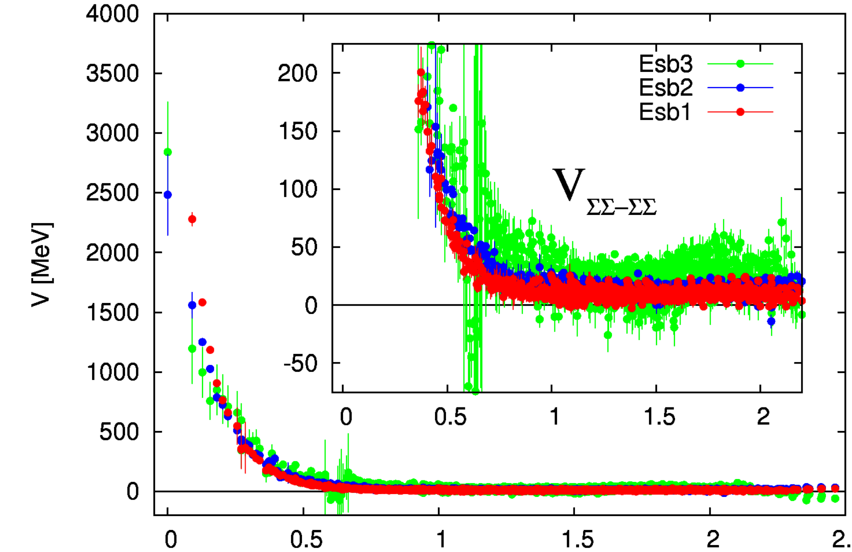
Diagonal elements



shallow attractive pocket



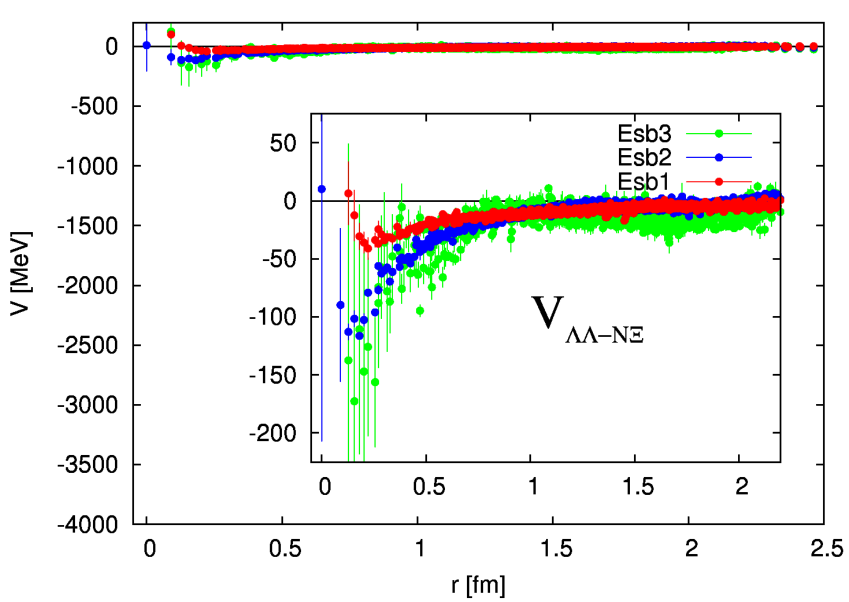
Deeper attractive pocket

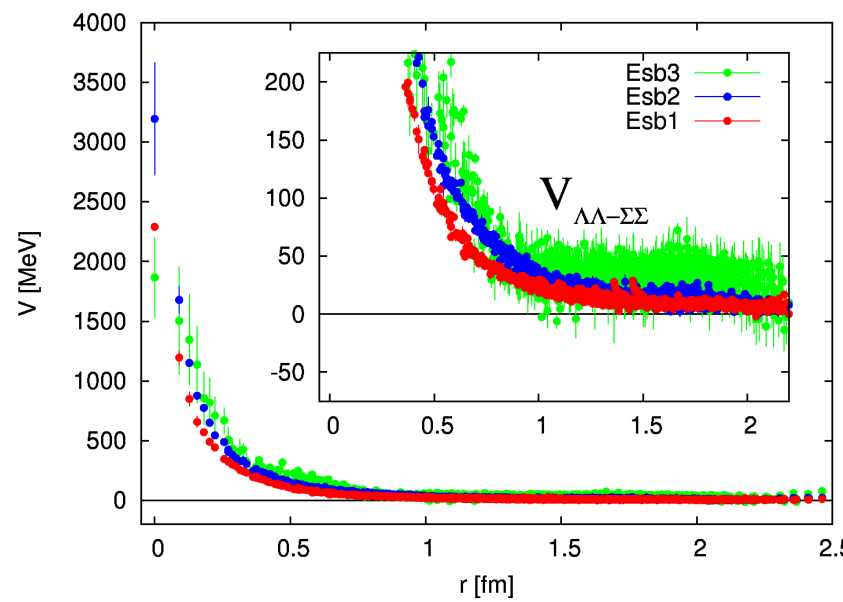


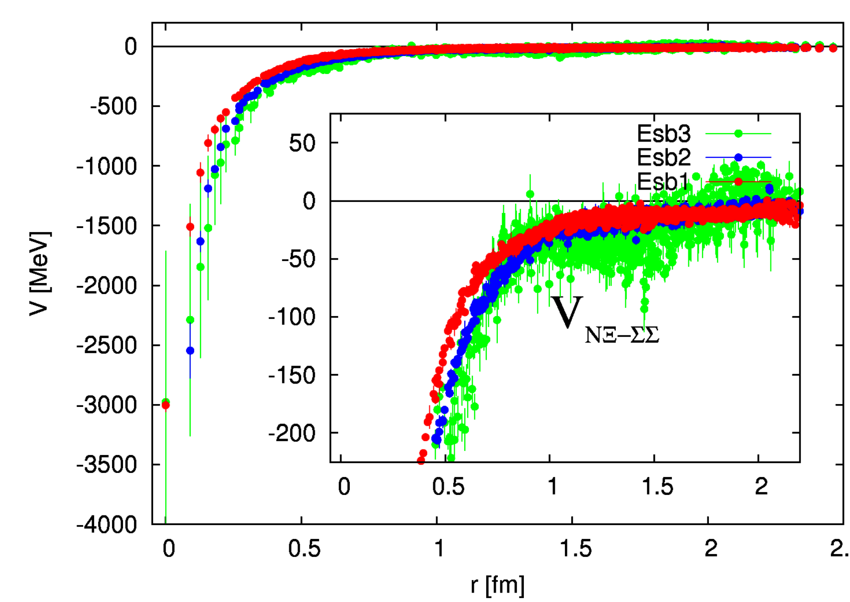
Strongly repulsive

All channels have repulsive core

Off-diagonal elements



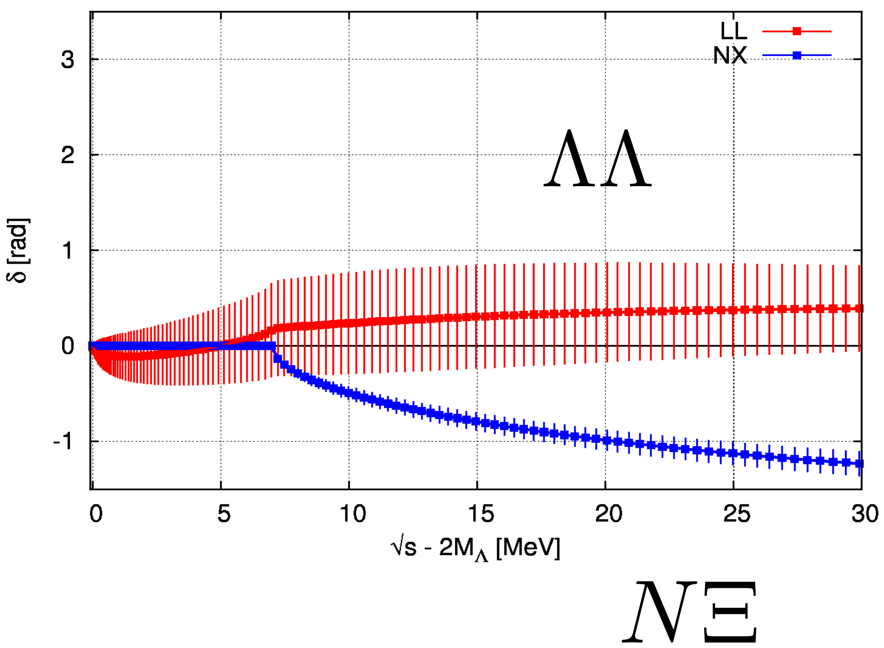




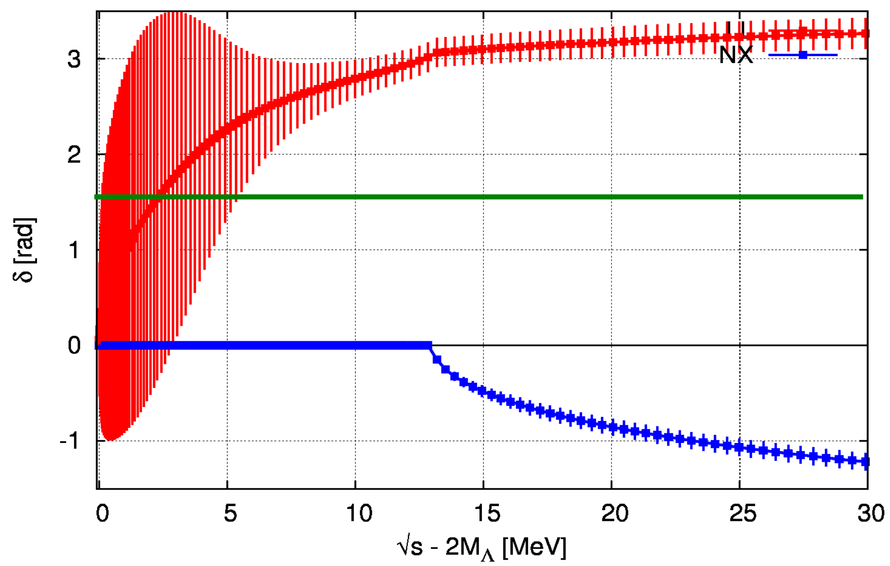
$\Lambda\Lambda$ and $N\Xi$ phase shift

Preliminary !

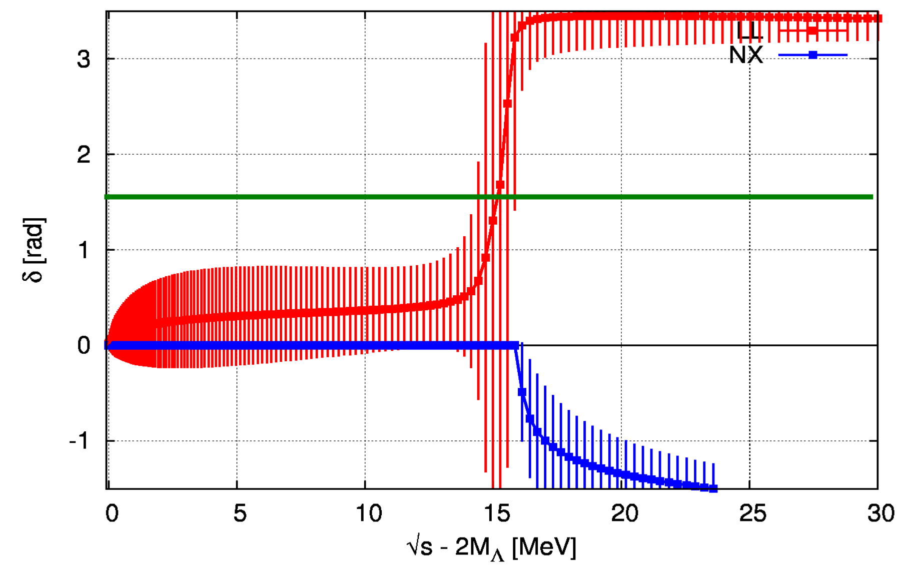
mπ = 700 MeV



mπ = 570 MeV



mπ = 410 MeV



Bound H-dibaryon
coupled to NΞ

H as resonace
near ΛΛ threshold
(H as bound NΞ)

H as resonance near
NΞ threshold
(H as bound NΞ)

This suggests that H-dibaryon becomes resonance at physical point.
Below or above NΞ ? Need simulation at physical point.

Physically, it is essential that H-dibaryon is a bound state in the flavor SU(3) limit.

3. Conclusion

- HAL QCD approach is a promising method to extract hadronic interactions in lattice QCD.
  - LS force, Antisymmetric LS
  - D-D(Ikeda), D-K, Omega-Omega, Omega-N
  - Comparison of HAL and Luescher: NN, pipi
- Extensions of the HAL QCD method to inelastic/multi-particle scatterings
  - Asymptotic behavior of the NBS wave functions
  - Existence of non-local but energy-independent coupled channel potentials
  - 3 nucleon force (T.Doi), coupled channel
- A treatment of bound-states ?

- “Potentials” at physical point on 京 $m_\pi \simeq 150\mathrm{MeV}, L \simeq 9\mathrm{fm}$

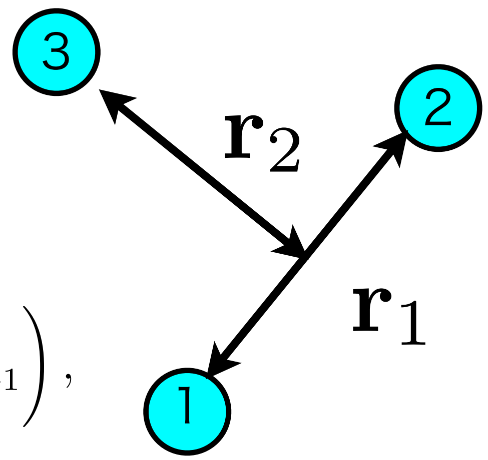
Back up

Unitarity constraint

$$T^\dagger - T = iT^\dagger T.$$

parametrization

$$ {}_0\langle [\boldsymbol{p}^A]_n | T | [\boldsymbol{p}^B]_n \rangle_0 \equiv \delta(E^A - E^B)\delta^{(3)}(\boldsymbol{P}^A - \boldsymbol{P}^B) T(\underline{[\boldsymbol{q}^A]_n, [\boldsymbol{q}^B]_n}) $$



(modified) Jacobi coordinates and momenta

$$\boldsymbol{r}_k = \sqrt{\frac{k}{k+1}} \times \boldsymbol{r}^J_k, \qquad \boldsymbol{q}_k = \sqrt{\frac{k+1}{k}} \times \boldsymbol{q}^J_k \qquad \boldsymbol{r}^J_k = \frac{1}{k}\sum_{i=1}^{k} \boldsymbol{x}_i - \boldsymbol{x}_{k+1}, \quad \boldsymbol{q}^J_k = \frac{k}{k+1}\left(\frac{1}{k}\sum_{i=1}^{k} \boldsymbol{p}_i - \boldsymbol{p}_{k+1}\right),$$

$$\begin{aligned} T([\boldsymbol{q}^A]_n, [\boldsymbol{q}^B]_n) &\equiv T(\boldsymbol{Q}_A, \boldsymbol{Q}_B) \\ &= \sum_{[L],[K]} T_{[L][K]}(Q_A, Q_B) Y_{[L]}(\Omega_{\boldsymbol{Q}_A}) \overline{Y_{[K]}(\Omega_{\boldsymbol{Q}_B})} \end{aligned}$$

$$\boldsymbol{Q}_X = (\boldsymbol{q}^X{}_1, \boldsymbol{q}^X{}_2, \cdots, \boldsymbol{q}^X{}_{n-1})$$ momentum in D=3(n-1) dim.

hyper-spherical harmonic function

$$\hat{L}^2 Y_{[L]}(\Omega_{\boldsymbol{s}}) = L(L+D-2) Y_{[L]}(\Omega_{\boldsymbol{s}})$$

solution to the unitarity constraint with non-relativistic approximation

$$T_{[L][K]}(Q,Q) = \sum_{[N]} U_{[L][N]}(Q) T_{[N]}(Q) U^{\dagger}_{[N][K]}(Q),$$

↑

$$T_{[L]}(Q) = -\frac{2n^{3/2}}{mQ^{3n-5}} e^{i\delta_{[L]}(Q)} \sin\delta_{[L]}(Q),$$

"phase shift" $\delta_{[L]}(Q)$

Lippmann-Schwinger equation in QFT

$$|\alpha\rangle_{\rm in} = |\alpha\rangle_0 + \int d\beta \frac{|\beta\rangle_0 T_{\beta\alpha}}{E_\alpha - E_\beta + i\varepsilon}, \qquad \underline{T_{\beta\alpha} = {}_0\langle\beta|V|\alpha\rangle_{\rm in}}, \qquad \underline{{}_0\langle\beta|T|\alpha\rangle_0} = 2\pi\delta(E_\alpha - E_\beta)\underline{T_{\alpha\beta}}.$$

off-shell on-shell off-shell

$$(H_0 + V)|\alpha\rangle_{\rm in} = E_\alpha|\alpha\rangle_{\rm in}, \quad \text{full}$$

$$H_0|\alpha\rangle_0 = E_\alpha|\alpha\rangle_0. \quad \text{free}$$

NBS wave functions

n-scalar fields with different flavors

$$\Psi_\alpha^n([\boldsymbol{x}]) = {}_{\rm in}\langle 0|\varphi^n([\boldsymbol{x}],0)|\alpha\rangle_{\rm in}, \qquad \varphi^n([\boldsymbol{x}],t) = T\{\prod_{i=1}^n \varphi_i(\boldsymbol{x}_i,t)\},$$

$$\Psi_\alpha^n([\boldsymbol{x}]) = \frac{1}{Z_\alpha}{}_0\langle 0|\varphi^n([\boldsymbol{x}],0)|\alpha\rangle_0 + \int d\beta \frac{1}{Z_\beta}\frac{{}_0\langle 0|\varphi^n([\boldsymbol{x}],0)|\beta\rangle_0 T_{\beta\alpha}}{E_\alpha - E_\beta + i\varepsilon}.$$

$${}_0\langle 0|\varphi^n([\boldsymbol{x}],0)|[\boldsymbol{k}]_n\rangle_0 = \left(\frac{1}{\sqrt{(2\pi)^3}}\right)^n \prod_{i=1}^n \frac{1}{\sqrt{2E_{k_i}}} e^{i\boldsymbol{k}_i\boldsymbol{x}_i}$$

D-dimensional hyper-coordinates

$$\Psi^n(\boldsymbol{R}, \boldsymbol{Q}_A) = C\left[e^{i\boldsymbol{Q}_A\cdot R} + \frac{2m}{2\pi n^{3/2}}\int d^DQ\,\frac{e^{i\boldsymbol{Q}\cdot R}}{Q_A^2 - Q^2 + i\varepsilon}T(\boldsymbol{Q}, \boldsymbol{Q}_A)\right]$$

Expansion in terms of hyper-spherical harmonic function

$$e^{i\boldsymbol{Q}\cdot\boldsymbol{R}} = (D-2)!!\frac{2\pi^{D/2}}{\Gamma(D/2)}\sum_{[L]} i^L\, \underline{j_L^D(QR)}\, Y_{[L]}(\Omega_{\boldsymbol{R}})\,\overline{Y_{[L]}(\Omega_{\boldsymbol{Q}})},$$

hyper-spherical Bessel function

$$\Psi^n(\boldsymbol{R}, \boldsymbol{Q}_A) = \sum_{[L],[K]} \Psi^n_{[L],[K]}(R, Q_A)Y_{[L]}(\Omega_{\boldsymbol{R}})\overline{Y_{[K]}(\Omega_{\boldsymbol{Q}_A})},$$

Asymptotic behavior of NBS wave functions $\quad R \to \infty$

$$\Psi^n_{[L],[K]}(R, Q_A) \simeq C i^L \frac{(2\pi)^{D/2}}{(Q_AR)^{\frac{D-1}{2}}}\sum_{[N]} U_{[L][N]}(Q_A)e^{i\delta_{[N]}(Q_A)}U^\dagger_{[N][K]}(Q_A)$$

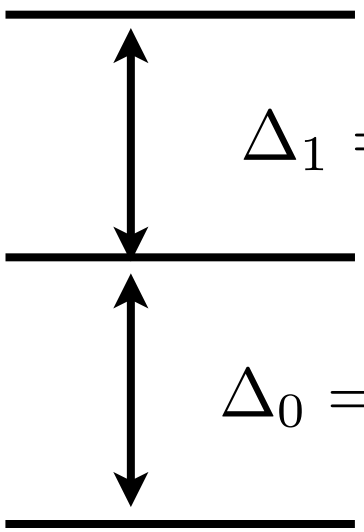
$$\times\sqrt{\frac{2}{\pi}}\;\underline{\sin\left(Q_AR - \Delta_L + \delta_{[N]}(Q_A)\right)}$$

$$\Delta_L = \frac{2L_D - 1}{4}\pi.$$

scattering wave with “phase shift” !

Let us consider $NN \to NN, NN\pi$

energy $W \in \Delta_1$



$W_{\rm th}^2 = 2m_N + 2m_\pi$

$\Delta_1 = [W_{\rm th}^1, W_{\rm th}^2)$

$W_{\rm th}^1 = 2m_N + m_\pi$

$\Delta_0 = [W_{\rm th}^0, W_{\rm th}^1)$

$W_{\rm th}^0 = 2m_N$

2 operators		2 states
$N(x)N(y)$	$\times$	$\vert NN, W, c_0\rangle$
$N(x)N(y)\pi(z)$		$\vert NN+\pi, W, c_1\rangle$

other quantum numbers (c_0, c_1)

4 NBS wave functions

$$
\begin{aligned}
Z_N \varphi^{00}_{W,c_0}(\boldsymbol{x}_0) &= \langle 0 | T\{N(\boldsymbol{x},0)N(\boldsymbol{x}+\boldsymbol{x}_0,0)\}|NN,W,c_0\rangle_{\rm in}, \\
Z_N Z_\pi^{1/2} \varphi^{10}_{W,c_0}(\boldsymbol{x}_0,\boldsymbol{x}_1) &= \langle 0 | T\{N(\boldsymbol{x},0)N(\boldsymbol{x}+\boldsymbol{x}_0,0)\pi(\boldsymbol{x}+\boldsymbol{x}_1,0)\}|NN,W,c_0\rangle_{\rm in}, \\
Z_N \varphi^{01}_{W,c_1}(\boldsymbol{x}_0) &= \langle 0 | T\{N(\boldsymbol{x},0)N(\boldsymbol{x}+\boldsymbol{x}_0,0)\}|NN+\pi,W,c_1\rangle_{\rm in}, \\
Z_N Z_\pi^{1/2} \varphi^{11}_{W,c_1}(\boldsymbol{x}_0,\boldsymbol{x}_1) &= \langle 0 | T\{N(\boldsymbol{x},0)N(\boldsymbol{x}+\boldsymbol{x}_0,0)\pi(\boldsymbol{x}+\boldsymbol{x}_1,0)\}|NN+\pi,W,c_1\rangle_{\rm in},
\end{aligned}
$$

$\varphi^{ij}_{W,c_j}([\mathbf{x}]_i)$ $\quad i(j)$: number of π's in the operator(state) $\qquad [\boldsymbol{x}]_0 = \boldsymbol{x}_0 \quad [\boldsymbol{x}]_1 = \boldsymbol{x}_0, \boldsymbol{x}_1$

coupled channel equation

$$(E_W^k - H_0^k)\varphi_{W,c_i}^{ki} = \sum_{l=0,1}\int \prod_{n=0}^{l} d^3y_n\, \underline{U^{kl}([\boldsymbol{x}]_k, [\boldsymbol{y}]_l)}\varphi_{W,c_i}^{li}([\boldsymbol{y}]_l), \quad k, i \in (0,1)$$

$$E_W^n = \frac{\boldsymbol{p}_1^2}{2m_N} + \frac{\boldsymbol{p}_2^2}{2m_N} + \sum_{i=1}^{n}\frac{\boldsymbol{k}_i^2}{2m_\pi}$$

kinetic energy

non-relativistic approx. for n=1

$$W = \sqrt{m_N^2 + \boldsymbol{p}_1^2} + \sqrt{m_N^2 + \boldsymbol{p}_2^2} + \sum_{i=1}^{n}\sqrt{m_\pi^2 + \boldsymbol{k}_i^2}$$

total energy

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 + \sum_{i=1}^{n}\boldsymbol{k}_i = 0.$$

Proof of existence for U

Define a vector of NBS wave functions as

state index

$$\varphi_{W,c_i}^{i} \equiv \left(\varphi_{W,c_i}^{0i}([\boldsymbol{x}]_0), \varphi_{W,c_i}^{1i}([\boldsymbol{x}]_1)\right)^T, \quad i = 0, 1, \qquad W \in \Delta_1$$

$$\varphi_{W,c_0}^{0} \equiv \left(\varphi_{W,c_0}^{00}([\boldsymbol{x}]_0), \varphi_{W,c_0}^{10}([\boldsymbol{x}]_1)\right)^T, \qquad W \in \Delta_0$$

Norm kernel

$$\mathcal{N}^{ij}_{W_1c_i,W_2d_j} = \left(\varphi^i_{W_1,c_i},\varphi^j_{W_2,d_j}\right) \equiv \sum_{k=0,1}\int\prod_{l=0}^{k} d^3x_l\,\overline{\varphi^{ki}_{W_1,c_i}([\boldsymbol{x}]_k)}\varphi^{kj}_{W_2,d_j}([\boldsymbol{x}]_k).$$

Inverse

$$\sum_{W\in\Delta_0+\Delta_1}\sum_{h\in I(W),e_h}(\mathcal{N}^{-1})^{ih}_{W_1c_i,We_h}\mathcal{N}^{hj}_{We_h,W_2d_j} = \delta^{ij}\delta_{W_1,W_2}\delta_{c_i,d_j}$$

Structure

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}^{00}(\Delta_0,\Delta_0), & \mathcal{N}^{00}(\Delta_0,\Delta_1), & \mathcal{N}^{01}(\Delta_0,\Delta_1) \\ \mathcal{N}^{00}(\Delta_1,\Delta_0), & \mathcal{N}^{00}(\Delta_1,\Delta_1), & \mathcal{N}^{01}(\Delta_1,\Delta_1) \\ \mathcal{N}^{10}(\Delta_1,\Delta_0), & \mathcal{N}^{10}(\Delta_1,\Delta_1), & \mathcal{N}^{\underline{11}}(\Delta_1,\Delta_1) \end{pmatrix}$$

energy state

bra, ket

$$\langle[\boldsymbol{x}]_k|\varphi^i_{W,c_i}\rangle = \varphi^{ki}_{W,c_i}([\boldsymbol{x}]_k),$$

$$\langle\psi^i_{W,c_i}|[\boldsymbol{x}]_k\rangle = \sum_{W_1\in\Delta_0\cup\Delta_1}\sum_{j\in I(W_1),d_j}(\mathcal{N}^{-1})^{ij}_{Wc_i,W_1d_j}\overline{\varphi^{kj}_{W_1,d_j}([\boldsymbol{x}]_k)}$$

orthogonality

$$\langle\psi^i_{W_1,c_i}|\varphi^j_{W_2,d_j}\rangle = \sum_{k=0,1}\int\prod_{l=0}^{k}d^3x_l\,\langle\psi^i_{W_1,c_i}|[\boldsymbol{x}]_k\rangle\langle[\boldsymbol{x}]_k|\varphi^j_{W_2,d_j}\rangle = (\mathcal{N}^{-1}\cdot\mathcal{N})^{ij}_{W_1c_i,W_2d_j}$$

$$= \delta^{ij}\delta_{W_1,W_2}\delta_{c_i,d_j}.$$

Abstract operators

$$\langle [\boldsymbol{x}]_k | (E_W - H_0) | [\boldsymbol{y}]_l \rangle \equiv \delta_{kl} (E_W^k - H_0^k) \prod_{n=0}^{k} \delta^{(3)}(\boldsymbol{x}_n - \boldsymbol{y}_n)$$

$$\langle [\boldsymbol{x}]_k | U | [\boldsymbol{y}]_l \rangle \equiv U^{kl}([\boldsymbol{x}]_k, [\boldsymbol{y}]_l),$$

Abstract coupled channel equation

$$(E_W - H_0) | \varphi^i_{W,c_i} \rangle = U | \varphi^i_{W,c_i} \rangle.$$

construction of non-local coupled channel potential

$$U = \sum_{W \in \Delta_0 \cup \Delta_1} \sum_{i \in I(W)} \sum_{c_i} (E_W - H_0) | \varphi^i_{W,c_i} \rangle \langle \psi^i_{W,c_i} |,$$

$\because$

$$U | \varphi^i_{W,c_i} \rangle = \sum_{W_1 \in \Delta_0 \cup \Delta_1} \sum_{j \in I(W_1)} \sum_{d_j} (E_W - H_0) | \varphi^j_{W_1,d_j} \rangle \langle \psi^j_{W_1,d_j} | \varphi^i_{W,c_i} \rangle = (E_W - H_0) | \varphi^i_{W,c_i} \rangle$$

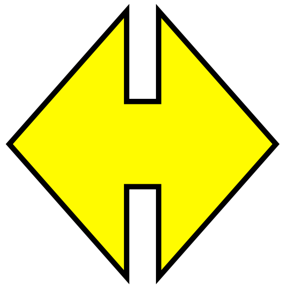
Energy independent (coupled channel) potential exists above the inelastic threshold.

The construction of U can easily be generalized to $NN + n\pi \rightarrow NN + k\pi$

or to $\Lambda\Lambda \rightarrow \Lambda\Lambda, N\Xi, \Sigma\Sigma$

QCD at given energy

$W_{\rm total}$



Quantum mechanics with coupled channel potentials for stable particles

$N, \bar{N}, \pi, \cdots$

$\Delta, \rho, \cdots$ resonance $N\pi, \pi\pi, \cdots$

deuteron, H,... bound-state ? $NN, \Lambda\Lambda, \cdots$

$D, H, \cdots$