The ordering of low-lying bound states of three identical particles – revisited; or Can one see the difference between the Δ and Y-string confinement in baryon spectra?

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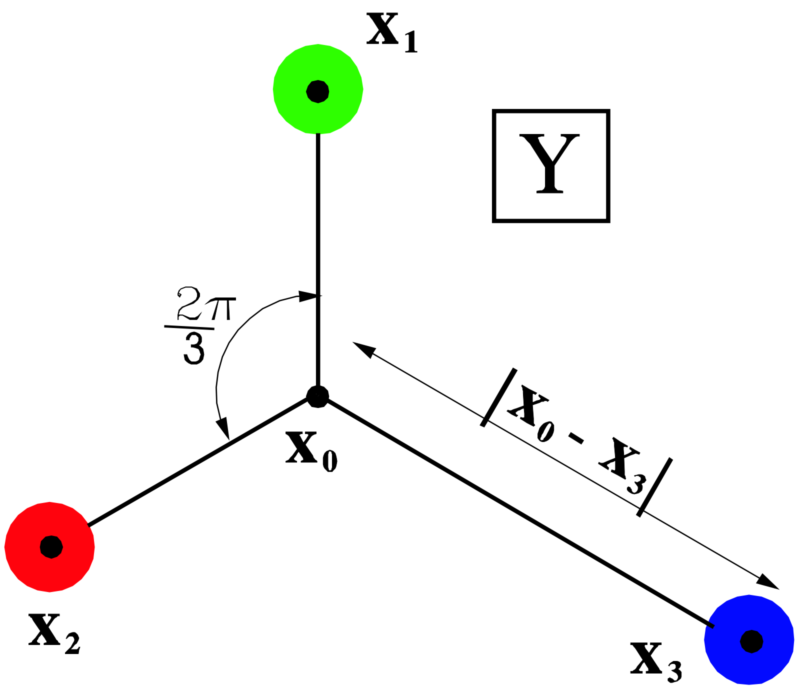
Outline:

- **Open questions: Delta or Y-string in baryons? Ordering of (low-lying) states?**
- **Delta- and Y-string and 3-body hyperspherical coordinates**
- **Dynamical O(2) symmetry of the Y-string**
- **O(6) algebra and spectrum of the 3-body problem in 3 dimensions**
- **Summary and Conclusions**

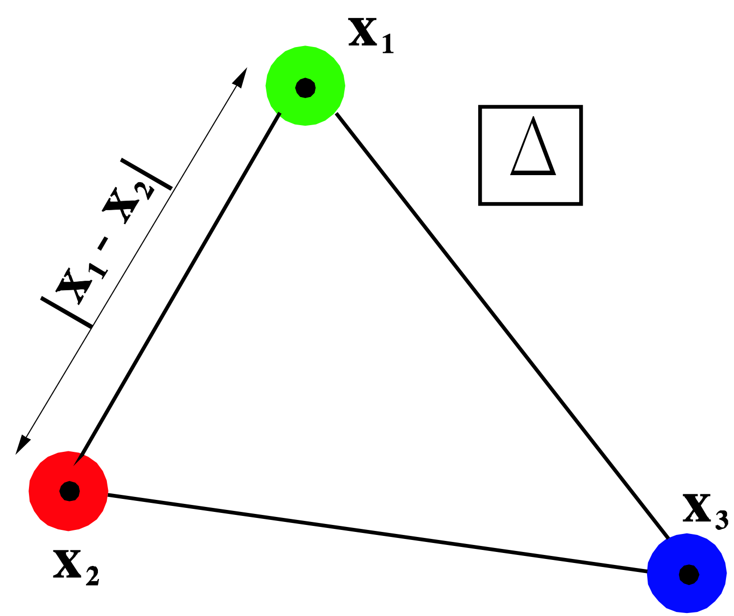
The Y-string

- Defined as the shortest sum of string lengths; this means that the strings pointing to the three quarks form 120 degree angles at the juncture (Fermat-Torricelli-Steiner point)
- Support from lattice QCD Takahashi,Matsufuru, Nemoto and Suganuma, PRL86,18('01); PRD65, 11409 ('02).



The Δ-string

- Sum of two-body potentials
- Also supported by Lattice QCD
- (Alexandrou, deForcrand Tsapalis, PRD65, 054503 ('02)
- How can one distinguish these two kinds of string potentials?



The Y *vs.* Δ-string?

- Bonn group (Metsch, Petry et al.) EJPA 10 ('01) claim equivalence of Δ and Y-strings !?!
- Can one distinguish these two string potentials using (only) the baryon spectra?
- We shall show that, yes, there are clear differences, but only at K=3, 5.
- These differences are related to the dynamical O(2) symmetry of the Y-string.

We would like to note at this stage that we have tested the various radial dependencies (3), (4) and (6) in our Salpeter model. Our investigations, however, clearly showed that the structure of resulting spectra depends only slightly on the various radial dependencies chosen. It turned out that the slope parameter b can always be appropriately rescaled (as, *e.g.*, in eq. (5) with the factor f) to obtain almost the same spectrum for all three choices. We therefore prefer for our model the Δ-shape string potential rising linearly with $r_\Delta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \sum_{i<j} |\mathbf{x}_i - \mathbf{x}_j|$ which, on the one hand, is favored by the most recent lattice studies anyway and, on the other hand, is also much easier to handle numerically. We found, however, that the structure of the resulting spectra depends much more on the Dirac structure chosen, which we shall consider next.

Early results on ordering of states - perturbed harmonic oscillator

- In NPB112, 213 (1976) Gromes and Stamatescu noted, that in first-order perturbation theory, the N=2 shell splits according to a simple **universal pattern**: the energy splittings must be in the ratio 2:2:1 under the influence of any sum of non-harmonic two-body potentials.
- The "Roper" energy/mass is not determined in a "universal" way .
- This is in agreement with the Y-string N=2 spectrum calculated numerically by Cutkosky and Hendrick, PRD16, 786, (1977) and by Carlson, Kogut and Pandharipande PR D27, 242 (1983)
- Does this "theorem" hold for non-harmonic (e.g. linear) potentials?
- Beyond perturbation theory?

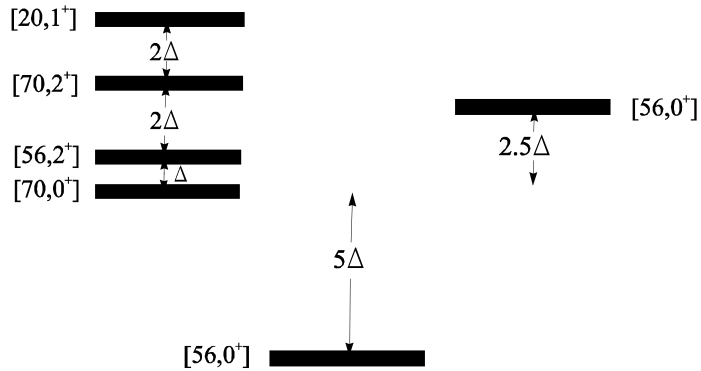


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TABLE VIII. Details of the baryon energies in GeV.

State	$\langle (m^2+p^2)^{1/2} \rangle$	$\langle -2\alpha_s/3r_{ij} \rangle$	$\langle \frac{1}{2}\sqrt{\sigma} r_{ij} \rangle$	$\langle \sqrt{\sigma}(r_{i4}-\frac{1}{2}r_{ij}) \rangle$
$0^+S(G)$	1.80	−0.34	0.86	0.08
1^-M	1.96	−0.27	1.04	0.10
$0^+S(R)$	2.01	−0.30	1.17	0.11
0^+M	2.08	−0.25	1.20	0.08
2^+S	2.06	−0.24	1.22	0.10
2^+M	2.09	−0.23	1.22	0.12
1^+A	2.11	−0.21	1.23	0.15

Model-independent features of spectra of three-identical particles: an old puzzle

- In PRD 24, 197 (1981), Bowler, Corvi, Hey, Jarvis and King, developed a mathematical theory based on the group Sp(12,R), about the 1st order perturbative splittings of the N=2,3 shells of the harmonic oscillator spectrum by two- and three-body permutation symmetric potentials.
- The N=2 shell case of this theory (Bowler-Tynemouth PRD 27, 662 (1982), **agrees well** with earlier results for the 1st order perturbative splittings.
- It also predict the N=3 shell splittings of the harmonic oscillator spectrum by two- and three-body permutation symmetric potentials.
- Compare this with the Delta string potential K=3 spectrum calculated non-perturbatively (with h.s. harmonics) by J.M. Richard

  and Taxil, NPB 329, 310 (1990): a discrepancy even in the ordering of states! Why?
- Which one, if any, of these claims is right?
- Is there a “universal” ordering of three-body bound states, and, if yes, under which conditions?
- Does it hold for the Delta and Y-string?

The Y-string potential

- The minimal Y-string length (potential) (Suganuma&Takahashi) contains two square-roots: classical e.o.m. are feasible, but
- How do you solve this problem in quantum mechanics?

$$L_{\min}=\left[\frac{1}{2}(a^2+b^2+c^2)+\frac{\sqrt{3}}{2}\times\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}\right]^{1/2},$$

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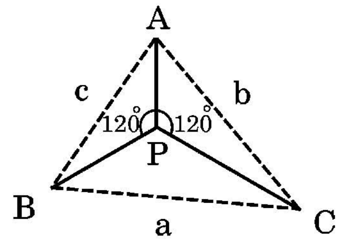


FIG. 1. The flux-tube configuration of the 3Q system with the minimal value of the total flux-tube length. There appears a physical junction linking the three flux tubes at the Fermat point P.

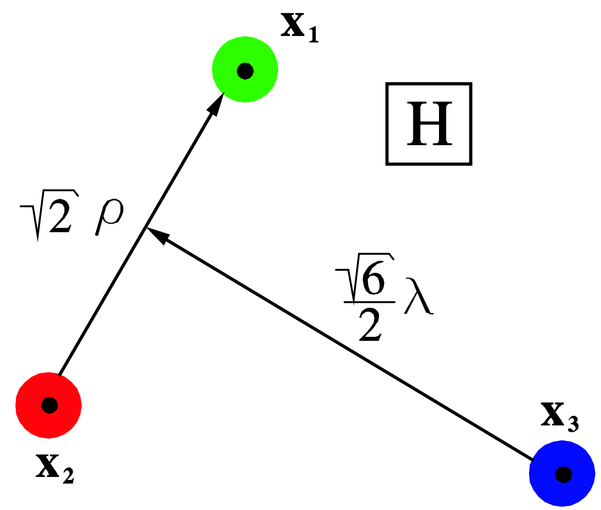
Jacobi and hyper-spherical variables

- Jacobi vectors $(\vec{\rho}, \vec{\lambda})$
- The hyper-radius R and hyper-angles

$$R^2 = \vec{\rho}^{\,2} + \vec{\lambda}^2$$

$$2\chi = \tan^{-1}\left(\frac{2\vec{\rho}\cdot\vec{\lambda}}{\vec{\rho}^{\,2} - \vec{\lambda}^2}\right)$$

$$\theta = \cos^{-1}\left(\frac{\vec{\rho}\cdot\vec{\lambda}}{\rho\lambda}\right)$$



$$\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)$$

The Y-string in terms of permutation asymmetric hyperspherical coordinates

- The Y-string potential in terms of standard hyperspherical coordinates

$$V_Y(R,\chi,\theta)=\sigma R\sqrt{\frac{3}{2}\left(1+\sin(2\chi)|\sin\theta|\right)}$$

- It is linear in the hyper-radius R, with a dependence on both hyperangles
- Solve the Schrodinger eq. with linear hyper-radial potential and this hyperangular dependence:

Schroedinger eqn. in standard hyperspherical coordinates

$$\Psi = \frac{1}{\rho^{5/2}} \sum_{K\gamma} Y_{KLM}^{l_x l_y}(\Omega_5^{\kappa}) \sum_{K'\gamma'} Y_{K'L'M'}^{l'_x l'_y}(\Omega_5)\, f_{K\gamma,K'\gamma'}(\kappa\rho)$$

$$\boxed{K = l_x + l_y + 2\nu}$$

$$\left\langle Y_{KL}^{l_x l_y}(\Omega_5) \middle| \sum_{i=1}^{3} \sum_{j>i}^{3} V_{ij}(\rho,\Omega_5) \middle| Y_{K''L''}^{l''_x l''_y}(\Omega_5) \right\rangle$$

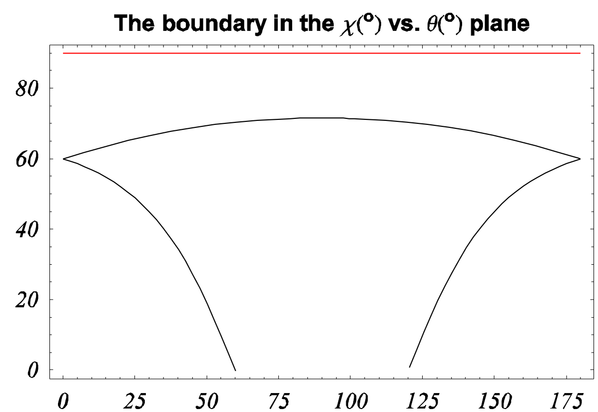
$$\downarrow$$

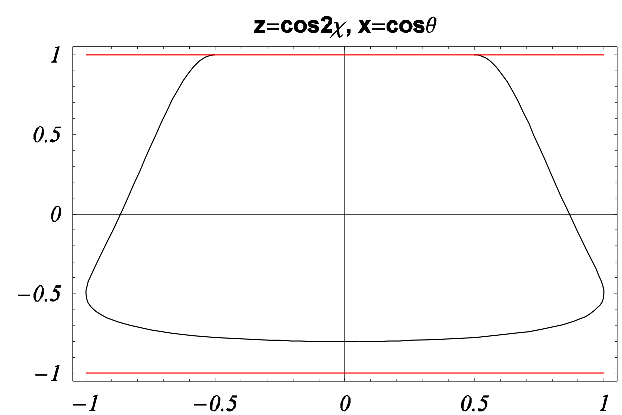
$$\left(-\frac{\hbar^2}{2m}\left[\frac{d^2}{d\rho^2} + \frac{(K+3/2)(K+5/2)}{\rho^2}\right] - E \right) f_{K\gamma,K'\gamma'}(\rho) = -\sum_{K''\gamma''} V_{K''\gamma'',K\gamma}(\rho) f_{K''\gamma'',K'\gamma'}(\rho)$$

- We solved the hyper-radial eqn with linear string potential for K=0,1,2 values of “grand angular momentum”

Standard hyperangular integration region for the matrix elements

- The complete Y-string potential contains 4 parts: the Y-string in the "central" region and three V-strings "in three corners"; the boundary between the four regions is the black line.
- Note the absence of manifest permutation symmetry.
- Cosines of hyper-angles make the boundaries convex, but still not permutation symmetric. Change variables?



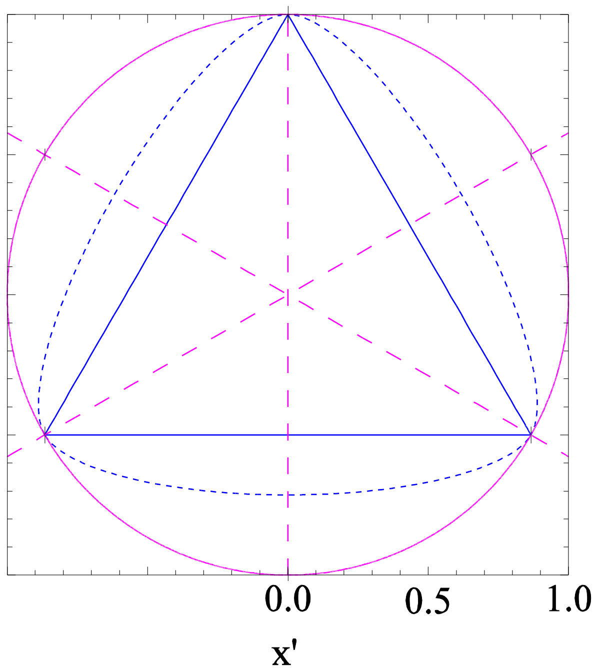


Permutation-symmetric hyper-angles

- Define the new (permutation-symmetric) hyper-angles

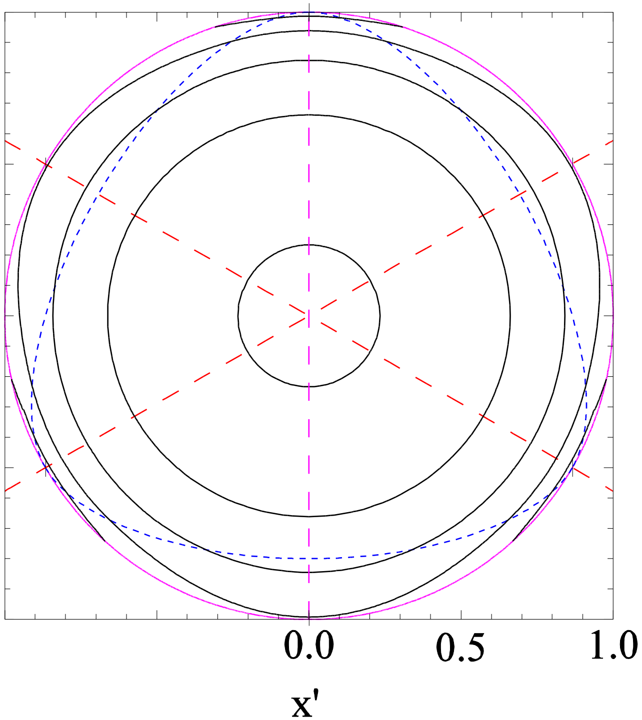
$$z' = z, x' = x\sqrt{1-z^2}$$

- The defining domain changes from the square into a circle (solid pink)
- The convex boundary between the three regions is closed.
- The discrete symmetry group consisting of three reflections (about the vertical (solid black) and the two slanted (red dashes) axes) and two rotations through $2\pi/3$ that correspond to the permutation group S3 of three quarks.



The Y-string in terms of new hyper-angles

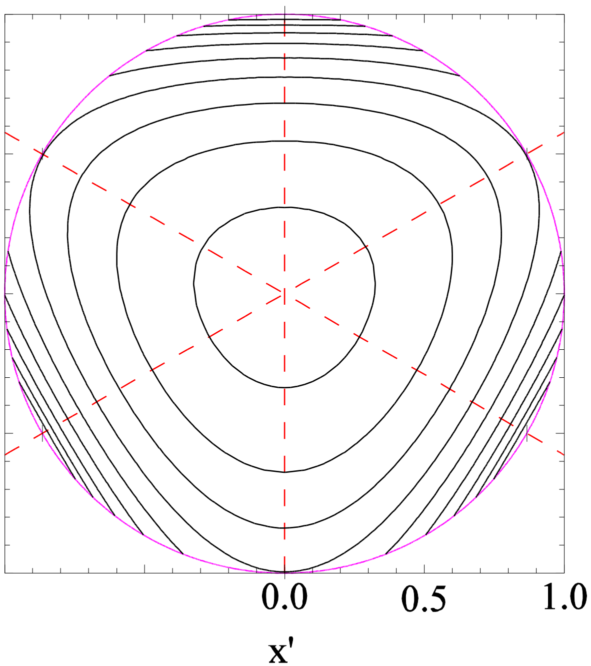
- The contour plot of the Y-string potential consists of concentric circles (solid black)
- The Y-string potential is axially symmetric under rotations: not a function of the (new) hyperangle ***Φ***



$$V_Y(R,\alpha,\varphi) = \sigma R\sqrt{\frac{3}{2}\left(1+|\cos\alpha|\right)}$$

The Delta-string potential in terms of new hyper-angles

- This O(2) symmetry is not shared by sums of two-body potentials, like the Delta string.
- The Delta-string potential has a different form in the new permutation-symmetric hyper-angles
- Other (sums of) two-body potentials have similar contour plots
- There must be a difference between the Delta and Y-strings, but where in the spectrum is it?

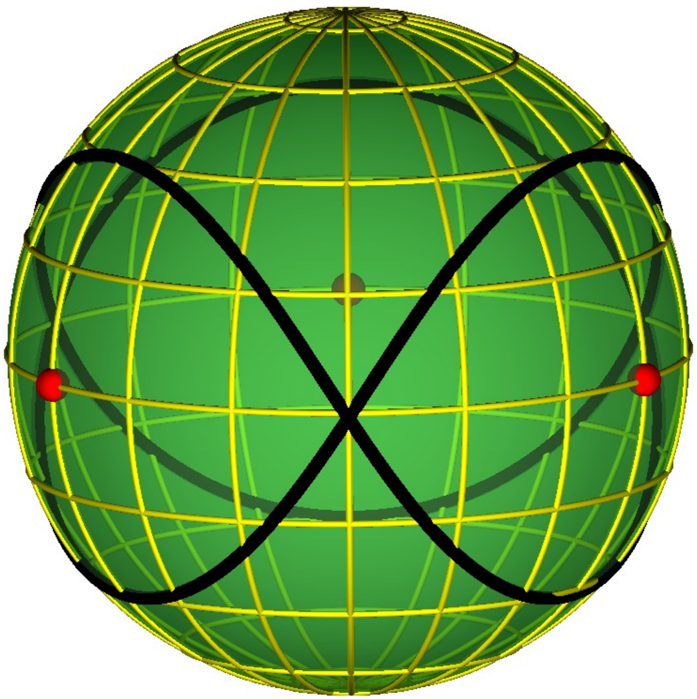


The shape-space sphere

$$X = \left(\frac{2\vec{\rho}\cdot\vec{\lambda}}{\vec{\rho}^2 + \vec{\lambda}^2} \right)$$

$$Y = \left(\frac{\vec{\rho}^2 - \vec{\lambda}^2}{\vec{\rho}^2 + \vec{\lambda}^2} \right)$$

$$Z = \left(\frac{2\left(\vec{\rho}\times\vec{\lambda}\right)_z}{\vec{\rho}^2 + \vec{\lambda}^2} \right)$$



- Define a unit sphere with (X,Y,Z) coordinates
- What we showed before was a projection form infinity above the North Pole.
- Red points correspond to equi-distant collinear configurations ("Euler" points)
- The solid black line is the boundary between integrations regions (plus its "reflection" for triangles of opposite orientation).

Dynamical O(2) symmetry of the Y-string

- Consequently there is a new constant of motion G that is associated with "hyper-rotations" in the $\left(\vec{\rho}, \vec{\lambda}\right)$ plane
- This G is the "hyper-angular momentum" conjugate to the new hyper-angle Φ.
- Permutation group S3 is a discrete subgroup of this O(2) -> G must be a good quantum number of h.s. harmonics
- How and where does this O(2) symmetry fit in?

$$G = \boldsymbol{\lambda} \cdot \mathbf{p}_{\rho} - \boldsymbol{\rho} \cdot \mathbf{p}_{\lambda},$$

$$\delta \boldsymbol{\rho} = \varepsilon \boldsymbol{\lambda}$$

$$\delta \boldsymbol{\lambda} = -\varepsilon \boldsymbol{\rho}.$$

$$G = \frac{L}{2} \cos \alpha - \frac{m}{4} \left(R \sin \alpha \right)^2 \dot{\varphi}$$

$$\varphi = \tan^{-1} \left(\frac{2 \vec{\rho} \cdot \vec{\lambda}}{\vec{\rho}^2 - \vec{\lambda}^2} \right)$$

$$\alpha = \cos^{-1} \left(\frac{2 \left(\vec{\rho} \times \vec{\lambda} \right)_z}{\vec{\rho}^2 + \vec{\lambda}^2} \right)$$

The O(2D) algebra of the three-body problem in D dimensions

- The D-dimensional n.r. kinetic en. T has an O(2D) symmetry.

$$T = \frac{m}{2}\left(\dot{\boldsymbol{\rho}}^2 + \dot{\boldsymbol{\lambda}}^2\right) = \frac{m}{2}\left(\dot{\boldsymbol{X}}_\mu\right)^2 = \frac{1}{2m}\left(\mathbf{p}_\rho^2 + \mathbf{p}_\lambda^2\right) = \frac{1}{2m}\left(\boldsymbol{P}_\mu\right)^2 .$$

- It can be written as a function of hyper-radial kin. en. and the "grand angular momentum" tensor K squared

$$T = \frac{m}{2}\dot{R}^2 + \frac{K_{\mu\nu}^2}{2mR^2}$$

$$\begin{aligned} K_{\mu\nu} &= m\left(\mathbf{X}_\mu \dot{\mathbf{X}}_\nu - \mathbf{X}_\nu \dot{\mathbf{X}}_\mu\right) \\ &= \left(\mathbf{X}_\mu \mathbf{P}_\nu - \mathbf{X}_\nu \mathbf{P}_\mu\right) \end{aligned}$$

- Generator G of O(2) symmetry is a part of the K tensor → O(2) is a subgroup of O(2D).
- The 3-body problem in D dimensions can be reduced to a hyper- radial Schrodinger equation with a different effective potential for each permutation-symmetric h.s. harmonic.
- Need to know O(2D) h.s. harmonics in D-dimensions with good quantum number G.

Permutation symmetric O(6) harmonics

- Labels of SO(6) perm.-symmetric hyper-spherical harmonics

SO(6)

U(1)

multiplicity

~~SU(3)~~

$$\mathcal{Y}^{KQ\,v}_{J,m}(x)$$

SO(3)

□SO(2)

$$U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$$

Igor Salom explicitly calculated the O(6) h.s. harmonics (using commercially available symbolic manipulator software)

$$\mathcal{Y}_{0,0}^{0,0,0}(X) = \frac{1}{\pi^{3/2}}$$

$$\mathcal{Y}_{1,1}^{1,1,-1}(X) = \frac{\sqrt{\frac{3}{2}}X_+^+}{\pi^{3/2}R} = \frac{\sqrt{\frac{3}{2}}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)}{\pi^{3/2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2}}$$

$$\mathcal{Y}_{1,1}^{2,0,0}(X) = \frac{\sqrt{3}\left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{\pi^{3/2}R^2} = \frac{2\sqrt{3}\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{2,0,0}(X) = \frac{\sqrt{3}X_+^+ X_+^-}{\pi^{3/2}R^2} = \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{0,0}^{2,2,0}(X) = \frac{\sqrt{2}\left|X^+\right|^2}{\pi^{3/2}R^2} = \frac{\sqrt{2}\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{2,2,-3}(X) = \frac{\sqrt{\frac{3}{2}}\left(X_+^+\right)^2}{\pi^{3/2}R^2} = \frac{\sqrt{\frac{3}{2}}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

and

$$\mathcal{Y}^{3,1,3}_{1,1}(X) = \frac{\sqrt{6}\left(X^-_+ \left|X^+\right|^2 - \frac{1}{2}R^2 X^+_+\right)}{\pi^{3/2}R^3}$$

$$= \frac{\sqrt{6}\left((\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2)\left((\lambda_1 + i\rho_1)^2 + (\lambda_2 + i\rho_2)^2 + (\lambda_3 + i\rho_3)^2\right) - \frac{1}{2}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right.}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,1,-5}_{2,2}(X) = \frac{\sqrt{5}X^+_+\left(X^-_+X^+_0 - X^+_+X^-_0\right)}{\pi^{3/2}R^3} = \frac{2\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,1,-2}_{3,3}(X) = \frac{\sqrt{15}\left(X^+_+\right)^2 X^-_+}{2\pi^{3/2}R^3} = \frac{\sqrt{15}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,3,-1}_{1,1}(X) = \frac{\sqrt{3}X^+_+\left|X^+\right|^2}{\pi^{3/2}R^3} = \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,3,-6}_{3,3}(X) = \frac{\sqrt{5}\left(X^+_+\right)^3}{2\pi^{3/2}R^3} = \frac{\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^3}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{4,0,0}_{0,0}(X) = -\frac{\sqrt{3}\left(R^4 - 2\left|X^-\right|^2\left|X^+\right|^2\right)}{\pi^{3/2}R^4}$$

The O(6) approach to the three-body problem in 3 dimensions

- The sum of products of O(6) hyper-angular matrix elements and three-body potential's expansion coefficients yields the spectrum

$$C_{[K'],[K]} = \delta_{[K'],[K]} + \pi\sqrt{\pi} \sum_{K>0,Q,\nu}^{\infty} \frac{v_{K,Q,\nu}^{3-\mathrm{body}}}{v_{000}^{3-\mathrm{body}}} \langle \mathcal{Y}_{[K']}(\alpha,\phi,\Phi_i) | \mathcal{Y}_{00}^{K,Q,\nu}(\alpha,\phi) | \mathcal{Y}_{[K]}(\alpha,\phi,\Phi_i) \rangle$$

- The hyper-radial equation with linear potential yields the Airy functions: just need these O(6) matrix elements and the problem is solved!
- Igor Salom constructed these O(6) h.s. harmonics and their m.e.s.; we can now look at the splitting of different K-bands in the spectrum!
- K=0,1,2 are well known
- K=3,4,5 are new and interesting - many SU(6) multiplets

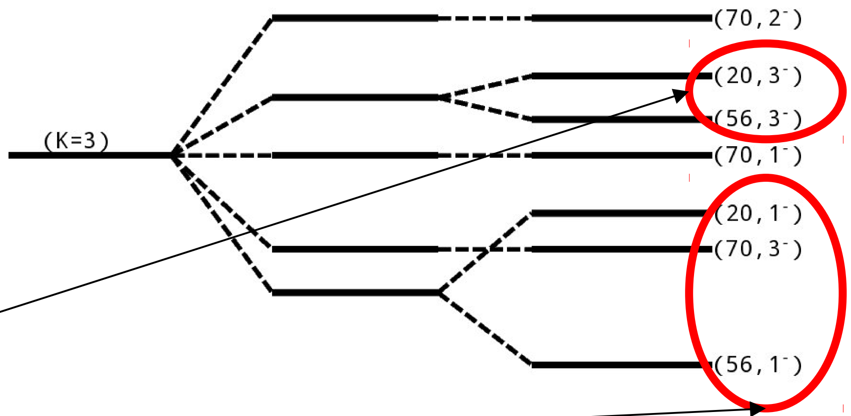
K=3 shell results in 3 dimensions

- There are 50 states that fall into 7 SU(6) multiplets; energy levels depend on 3 parameters: v_{00}, v_{40}, v_{66}
- Confirms Richard &Taxil's NP B329, 310 (1990) results

$$
\begin{array}{lll}
[20,1^-] & \frac{1}{\pi\sqrt{\pi}} & \left(v_{00} + \frac{1}{\sqrt{3}} v_{40} - \frac{2}{7} v_{66} \right) \\
[56,1^-] & \frac{1}{\pi\sqrt{\pi}} & \left(v_{00} + \frac{1}{\sqrt{3}} v_{40} + \frac{2}{7} v_{66} \right) \\
[70,1^-] & \frac{1}{\pi\sqrt{\pi}} & \left(v_{00} \right) \\
[70,2^-] & \frac{1}{\pi\sqrt{\pi}} & \left(v_{00} - \frac{1}{\sqrt{3}} v_{40} \right) \\
[70,3^-] & \frac{1}{\pi\sqrt{\pi}} & \left(v_{00} - \frac{1}{\sqrt{3}} v_{40} \right) \\
[20,3^-] & \frac{1}{\pi\sqrt{\pi}} & \left(v_{00} - \frac{\sqrt{3}}{7} v_{40} - v_{66} \right) \\
[56,3^-] & \frac{1}{\pi\sqrt{\pi}} & \left(v_{00} - \frac{\sqrt{3}}{7} v_{40} + v_{66} \right),
\end{array}
$$

K=3 shell results in 3 dimensions

- At K=3 with only v_{40} non-zero coefficient, i.e. for the Y-string, there is still some degeneracy left:
- The [20,3] and [56,3] are degenerate, and[20,1] is degenerate with [56,1]
- With both v_{40}, v_{66} non-zero coefficients, i.e. for the Delta-string there is complete lifting of all degeneracies



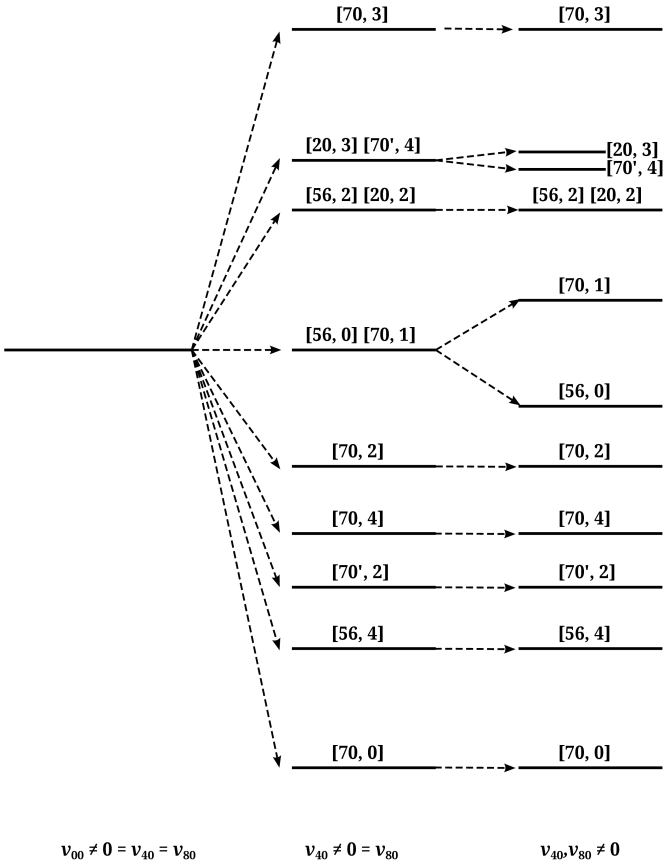
K=4 shell results in 3 dimensions

- There are 105 h.s. harmonics that fall into 12 SU(6) multiplets
- Energy levels depend on 3 parameters, one of which v_{80} is different from the v_{66} (in the K=3 case).
- The Y and the Delta string are very similar.

$$
\begin{array}{rl}
[70,0^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{\sqrt{3}}{2} v_{40} + \frac{1}{2\sqrt{5}} v_{80} \right) \\
[56,0^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{2}{\sqrt{5}} v_{80} \right) \\
[70,1^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{\sqrt{5}} v_{80} \right) \\
[70,2^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{4\sqrt{3}}{35} v_{40} + \frac{\sqrt{5}}{7} v_{80} \right) \\
[70',2^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{2\sqrt{3}}{7} v_{40} - \frac{1}{7\sqrt{5}} v_{80} \right) \\
[56,2^+], [20,2^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{6\sqrt{3}}{35} v_{40} - \frac{1}{7\sqrt{5}} v_{80} \right) \\
[20,3^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{3\sqrt{3}}{14} v_{40} - \frac{\sqrt{5}}{14} v_{80} \right) \\
[70,3^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{5\sqrt{3}}{14} v_{40} + \frac{1}{14\sqrt{5}} v_{80} \right) \\
[56,4^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{5\sqrt{3}}{14} v_{40} + \frac{3}{14\sqrt{5}} v_{80} \right) \\
[70,4^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{3\sqrt{3}}{14} v_{40} - \frac{\sqrt{5}}{42} v_{80} \right) \\
[70',4^+] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{3\sqrt{3}}{14} v_{40} + \frac{1}{42\sqrt{5}} v_{80} \right),
\end{array}
$$

K=4 shell results in 3 dimensions

- There are 105 h.s. harmonics that fall into 12 SU(6) multiplets;
- energy levels depend on 3 parameters: v_{00}, v_{40}, v_{80}



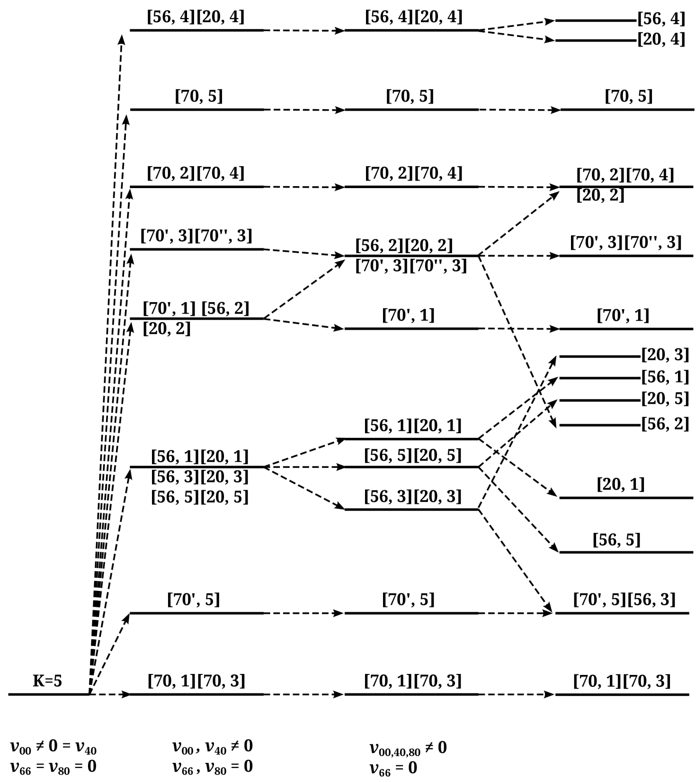
K=5 shell results in 3 dimensions

- -There are 196 states (h.s. harmonics) that fall into 19 SU(6) multiplets in the K=5 shell.
- - Eigen-energies depend on all 4 parameters v_00, v_40, v_66, v_80
- - The values of these parameters differ from one potential to another.
- - Generally they have the same signs, but unequal values for the Delta and Y-string potentials, with the exception of the v_66 which vanishes in the Y-string case.

$$
\begin{array}{ll}
[70,1^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{\sqrt{3}}{2} v_{40} + \qquad + \frac{1}{2\sqrt{5}} v_{80} \right) \\
[70',1^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \qquad + \qquad + \frac{1}{\sqrt{5}} v_{80} \right) \\
[56,1^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}} v_{40} - \frac{2}{5} v_{66} - \frac{1}{2\sqrt{5}} v_{80} \right) \\
[20,1^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}} v_{40} + \frac{2}{5} v_{66} - \frac{1}{2\sqrt{5}} v_{80} \right) \\
[70,2^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{2\sqrt{3}} v_{40} + \qquad - \frac{1}{2\sqrt{5}} v_{80} \right) \\
[56,2^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \qquad + \frac{3}{5} v_{66} - \frac{1}{\sqrt{5}} v_{80} \right) \\
[20,2^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \qquad - \frac{3}{5} v_{66} - \frac{1}{\sqrt{5}} v_{80} \right) \\
[70,3^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{\sqrt{3}}{2} v_{40} + \qquad - \frac{1}{3\sqrt{5}} v_{80} \right) \\
[70',3^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1687 - 215\sqrt{241} + 8\sqrt{482\,(133 - 5\sqrt{241})}}{8676\sqrt{3}} v_{40} + \frac{-3133 + 137\sqrt{241} + 16\sqrt{482\,(133 - 5\sqrt{241})}}{8676\sqrt{5}} v_{80} \right) \\
[70'',3^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1687 + 215\sqrt{241} + 8\sqrt{482\,(133 + 5\sqrt{241})}}{8676\sqrt{3}} v_{40} + \frac{-3133 - 137\sqrt{241} + 16\sqrt{482\,(133 + 5\sqrt{241})}}{8676\sqrt{5}} v_{80} \right) \\
[56,3^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}} v_{40} + \frac{14}{15} v_{66} + \frac{7}{6\sqrt{5}} v_{80} \right) \\
[20,3^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}} v_{40} - \frac{14}{15} v_{66} + \frac{7}{6\sqrt{5}} v_{80} \right) \\
[70,4^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{2\sqrt{3}} v_{40} + \qquad - \frac{1}{2\sqrt{5}} v_{80} \right) \\
[56,4^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{7}{6\sqrt{3}} v_{40} - \frac{2}{15} v_{66} + \frac{1}{6\sqrt{5}} v_{80} \right) \\
[20,4^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{7}{6\sqrt{3}} v_{40} + \frac{2}{15} v_{66} + \frac{1}{6\sqrt{5}} v_{80} \right) \\
[70,5^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{5}{6\sqrt{3}} v_{40} + \qquad + \frac{\sqrt{5}}{66} v_{80} \right) \\
[70',5^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{7}{6\sqrt{3}} v_{40} + \qquad + \frac{17}{66\sqrt{5}} v_{80} \right) \\
[56,5^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}} v_{40} + \frac{8}{15} v_{66} - \frac{19}{66\sqrt{5}} v_{80} \right) \\
[20,5^-] & \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}} v_{40} - \frac{8}{15} v_{66} - \frac{19}{66\sqrt{5}} v_{80} \right) .
\end{array}
$$

K=5 shell results in 3 dimensions

- -There are 196 states (h.s. harmonics) that fall into 19 SU(6) multiplets in the K=5 shell.
- Here plotted with the Delta-string values of the four parameters
- The Y-string differs qualitatively from the Delta, in so far as the last splitting is equal (or close) to zero: v_66 = 0!
- This spectrum is not likely ever to be fully explored experimentally



Summary

- The Y-string potential has a hidden dynamical O(2) symmetry that distinguishes it from the Delta string, but only at K=3,5,...
- This is the first systematic study of the O(2) dynamical symmetry in K=0,1,2,3,4,5 bands with the Y-and Delta string potentials in 3D baryons.
- Compare with older D=3 results: we confirmed Richard and Taxil's K=3 results and rebutted Bowler et al..
- We predict K=4,5 states' energies for permutation symmetric, but otherwise arbitrary potentials: this holds in atomic, molecular, nuclear physics
- Can be extended to arbitrarily high finite K.

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Publications

- **V.D., T. Sato and M. Šuvakov, Eur. J. Phys. C 62, 383 (2009)**
- **V.D., T. Sato and M. Šuvakov, Phys. Rev. D 80, 054501 (2009)**
- **V.D. and I. Salom, J. Math. Phys. 55, 082105 (2014); Acta Phys. Polon. Supp. 6, 905 (2013)**
- **M. Šuvakov and V.D., Phys. Rev. E 83, 056603 (2011), PRL 110, 114301 (2013); with Ana Hudomal PRL 113, 101102 (2014)**