Complex Langevin and Thimbles in Chiral Random Matrix Model

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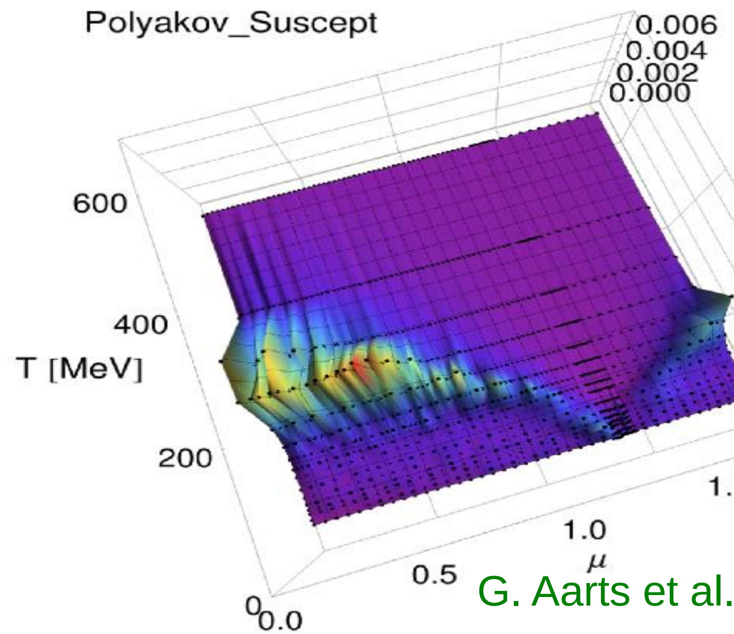
in collaboration with Kikukawa and Sano

Complex Langevin Equation (CLE)

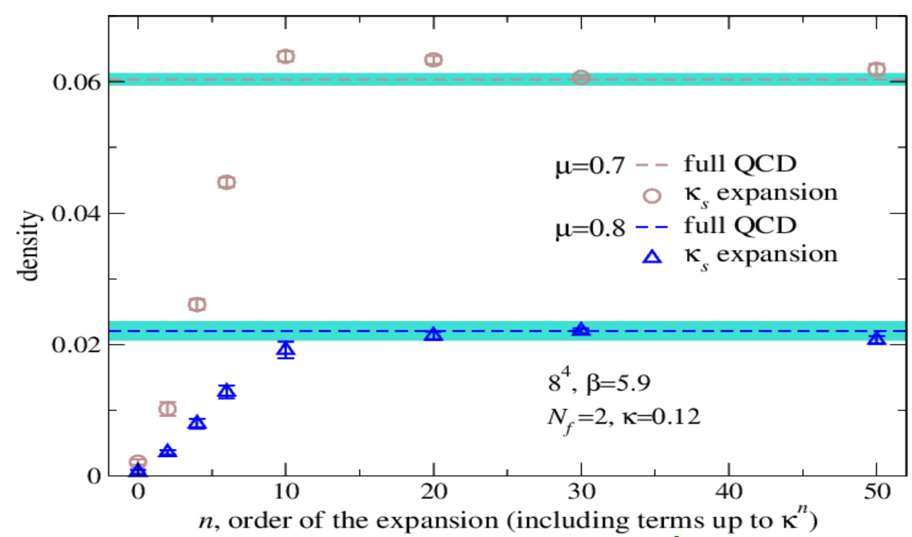
- Lattice simulation – powerful tool for non-perturbative analysis
- Sign problem – importance sampling invalidated by S ∈C
- CLE – may enable sampling w/ complex action

For review, G. Aarts et al, 1412.0847; 1407.2090



HDQCD

G. Aarts et al. 1411.2632



D. Sexty, 1307.7748, G. Aarts et al. 1408.3770

Outline

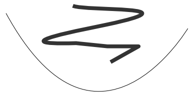
- Complex Langevin equation (CLE); brief review
- Chiral Random Matrix (ChRM) model
- CLE result for ChRM
- Simple N=1/2 model
- Discussions
- Outlook

Langevin dynamics (Brownian motion)

- Statistical sampling = long time average   if ergodicity holds

$$\frac{\partial}{\partial t} v_i(t) = -\gamma v_i(t) + \eta_i(t) \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2kT\gamma\delta(t-t')\delta_{ij}$$

$$= -\gamma \frac{\partial S}{\partial v_i} + \eta_i(t) \qquad S = \frac{1}{2} v^2(t)$$



- Associated Fokker-Planck eqn guarantees equilibration

$$\partial_t \rho(v) = \partial_v(\partial_v + S')\rho(v) \quad \longrightarrow \quad \rho(v) = e^{-S(v)}$$

For $\rho(v,t) = e^{-S(v)/2}\psi(v,t)$

$$\partial_t \psi(v,t) = -H_{\mathrm{FP}}\psi(v,t) \qquad H_{\mathrm{FP}} = Q^\dagger Q = \left(-\partial_v + \frac{S'}{2}\right)\left(\partial_v + \frac{S'}{2}\right) \geq 0$$

$$\boxed{\psi(v,t) \to e^{-S(v)/2}}$$

Complex Langevin dynamics

- Field theory --> "Stochastic quantization"

Parisi-Wu

$$\frac{\partial \phi(x,\theta)}{\partial \theta} = -\frac{\delta S[\phi]}{\delta \phi(x,\theta)} + \eta(x,\theta) \qquad \langle \eta(x,\theta)\eta(x',\theta')\rangle = 2\delta(x-x')\delta(\theta-\theta')$$

Complex Langevin dynamics

- Field theory --> "Stochastic quantization" — Parisi-Wu

$$\frac{\partial\phi(x,\theta)}{\partial\theta} = -\frac{\delta S[\phi]}{\delta\phi(x,\theta)} + \eta(x,\theta) \qquad \langle\eta(x,\theta)\eta(x',\theta')\rangle = 2\delta(x-x')\delta(\theta-\theta')$$

- What if S is *complex* ? config. $x \rightarrow z = x + i\,y$ — Parisi, Klauder

$$\dot{x} = -\mathrm{Re}\frac{\partial S}{\partial z} + \eta(t)$$
$$\dot{y} = -\mathrm{Im}\frac{\partial S}{\partial z} \quad \Rightarrow \quad P(x,y)$$

- [Q]

$$\boxed{\langle O(x)\rangle = \int dx\rho(x)O(x) \overset{?}{=} \int dxdyP(x,y)O(x+iy)}$$

$$\rho(x) = e^{-S(x)}$$

Aarts, James, Seiler, Stamatescu: Eur. Phys. J. C('11) 71;1756

- Formally proven under conditions:
  - holomorphic property, well-localized dist in y direction,

a complex weight from a real distribution?

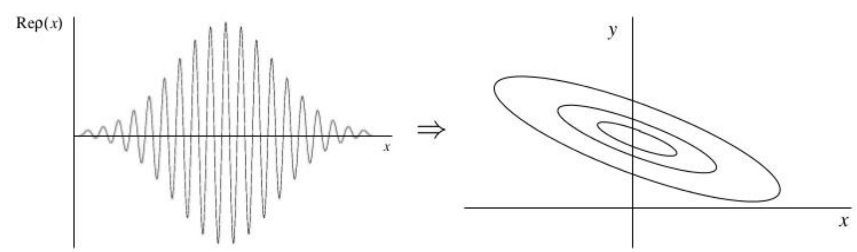
- If $$\langle O(x) \rangle = \int dx \rho(x) O(x) \simeq \int dx dy P(x,y) O(x+iy)$$
  then $$\rho(x) = \int dy P(x-iy, y)$$

- A known (exact) example – Guassian: $S = \frac{\kappa}{2} x^2 \qquad \kappa \in \mathbb{C}$

$$P(x,y) = N \exp[-\alpha x^2 - \beta y^2 - 2\gamma x y] \quad \in \mathbb{R}$$

$$\int dy\, P(x-iy, y) = \sqrt{\frac{\kappa}{2\pi}} \exp[-\frac{\kappa}{2} x^2]$$

e.g., G. Aarts et al, 1412.0847



- N.B. CLE fixes the prefactor

Application to physical systems

- Revived interests since Aarts-Stamatescu (2008~)
- relativistic Bose gas, U(1), SU(2) link model, Thirring model, ...
- ...
- Chiral random matrix (Sano et al., 2011)
- Chiral random matrix (Mollgaard-Splittorff, 2013; Splittorff et al. 2014)
- ...
- see Sexty's talk

Chiral Random Matrix Model

- Introduction of finite t & μ

M. Stephanov

$$Z_N = \int [dW]\, e^{-N\Sigma^2 \operatorname{tr} W^+ W} \det(D+m) = \int [dW] e^{-S}$$

$$D+m = \begin{pmatrix} m & iW+C \\ iW^+ + C & m \end{pmatrix} \qquad C = \begin{pmatrix} (\mu+it)\mathbf{1}_{N/2} & 0 \\ 0 & (\mu-it)\mathbf{1}_{N/2} \end{pmatrix}$$

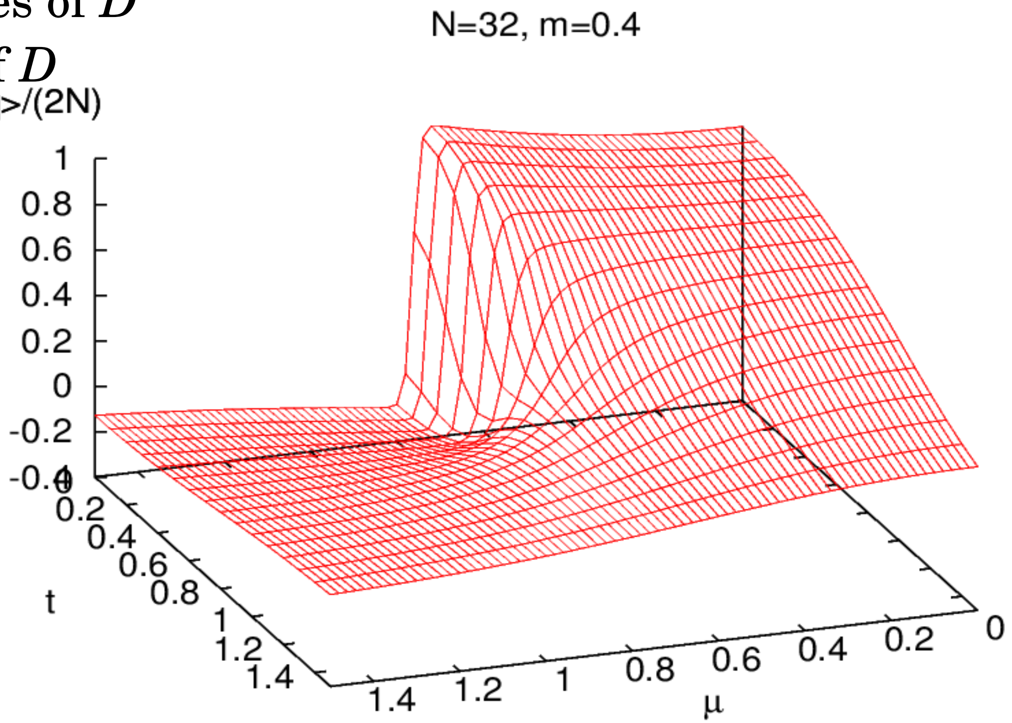
$W=W_1+i\,W_2$; N-by-N complex matrix

fields t&μ suppress small eigenvalues of D

nonzero μ breaks anti-Hermiticity of D

- Quark condensate

$$-\langle q\bar{q}\rangle = \frac{\partial \log Z_N}{\partial m}$$



Chiral Random Matrix Model

- CLE

$$S = N\Sigma^2 \mathrm{tr}\, W^+ W - \log\left[\det\left(D+m\right)\right]$$

$$W = W_1 + i\, W_2 ; \quad W_{1,2} \in \mathbb{C}$$

$$W_1(\theta+\epsilon) = W_1(\theta) + \epsilon K_1(\theta) + \sqrt{\epsilon}\eta_1(\theta)$$
$$W_2(\theta+\epsilon) = W_2(\theta) + \epsilon K_2(\theta) + \sqrt{\epsilon}\eta_2(\theta)$$

- Force term $K_{1,2}$ have a pole
- Simulation is straightforward

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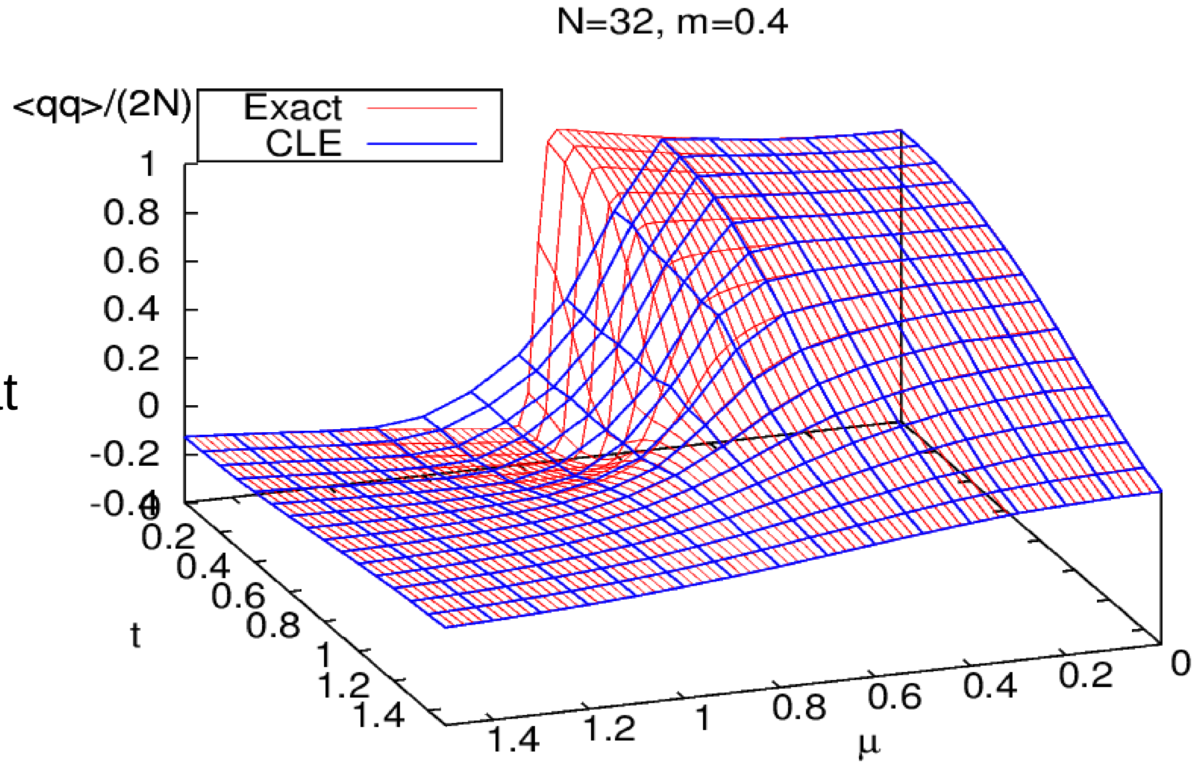
- Force term $K_{1,2}$ have a pole
- Simulation is straightforward
- Phase transition is only qualitatively reproduced
- Incorrect in transition region at low t ...



CLE result of ChRM at t=0

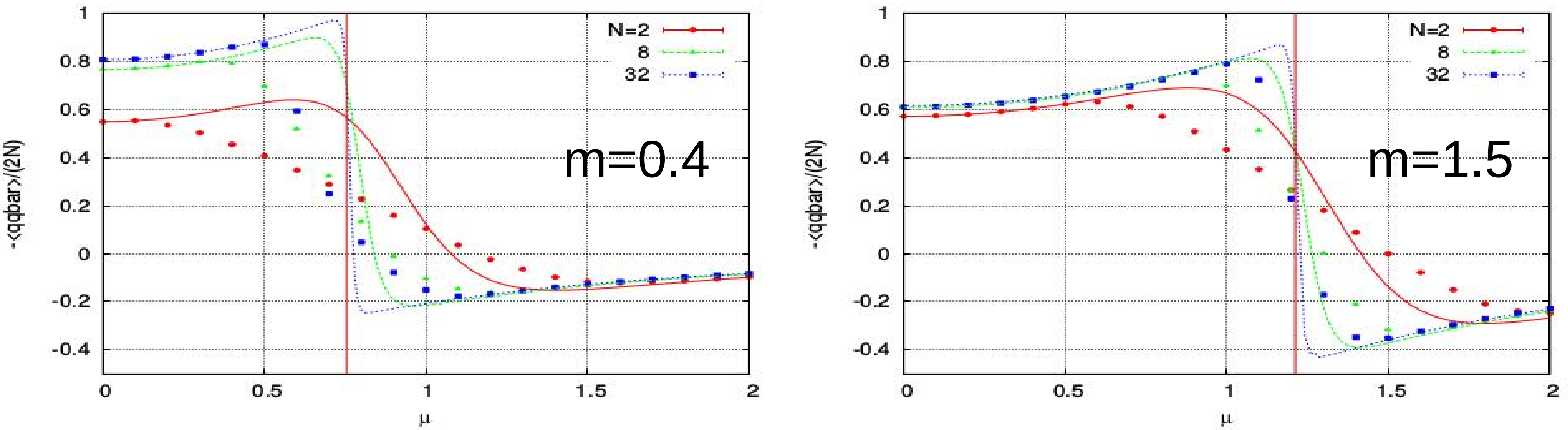


FIG. 1: Chiral condensate $\langle \bar{q}q \rangle$ obtained with CLE Simulation (points) for $N = 2$, 8, and 32 as a function of the chemical potential μ ($m = 0.4$ (left) and 1 (right)). The exact solutions are drawn in curves.

- CLE fails in transition region
- Difference is larger for smaller *N* and smaller m
- Better works for larger *N*? - not clear yet

Reweighting from RLE

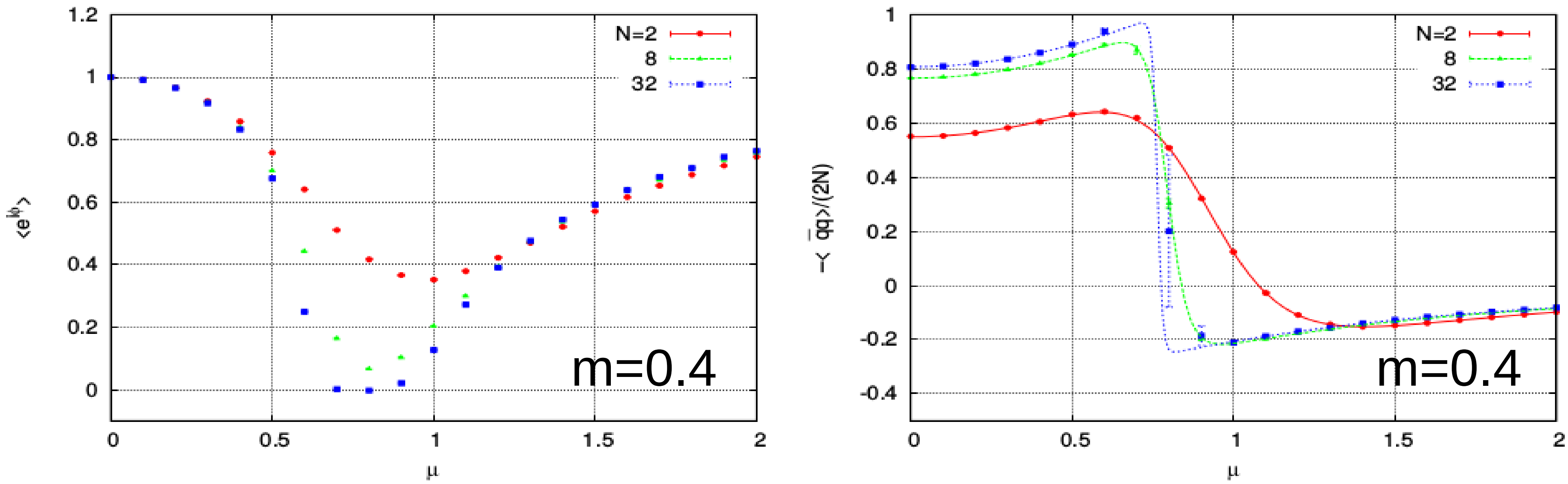


FIG. 3: Phase quenched simulation and reweighting. Average phase (left) and reweighted chiral condensate (right) for at $m = 0.4$ for $N = 2, 8$ and 32.

- <e^iϕ> becomes smaller for larger N (as expected)
- For *N* = 2 & 8, sign problem is very mild, but CLE fails
- Failure of CLE is not directly related to overlap problem.

Simplified model

- N=1 model

$$Z_1 = \int dx_1 dx_2\, e^{-\beta(x_1^2+x_2^2)} \left(m^2+(x_1-i\mu)^2+x_2^2\right)$$

- N=1/2 model

$$Z_{1/2} = \int dx\, e^{-\beta x^2} \left(m^2+(x-i\mu)^2\right) = \sqrt{\frac{\pi}{\beta}}\left(m^2+\frac{1}{2\beta}-\mu^2\right)$$

Simplified model

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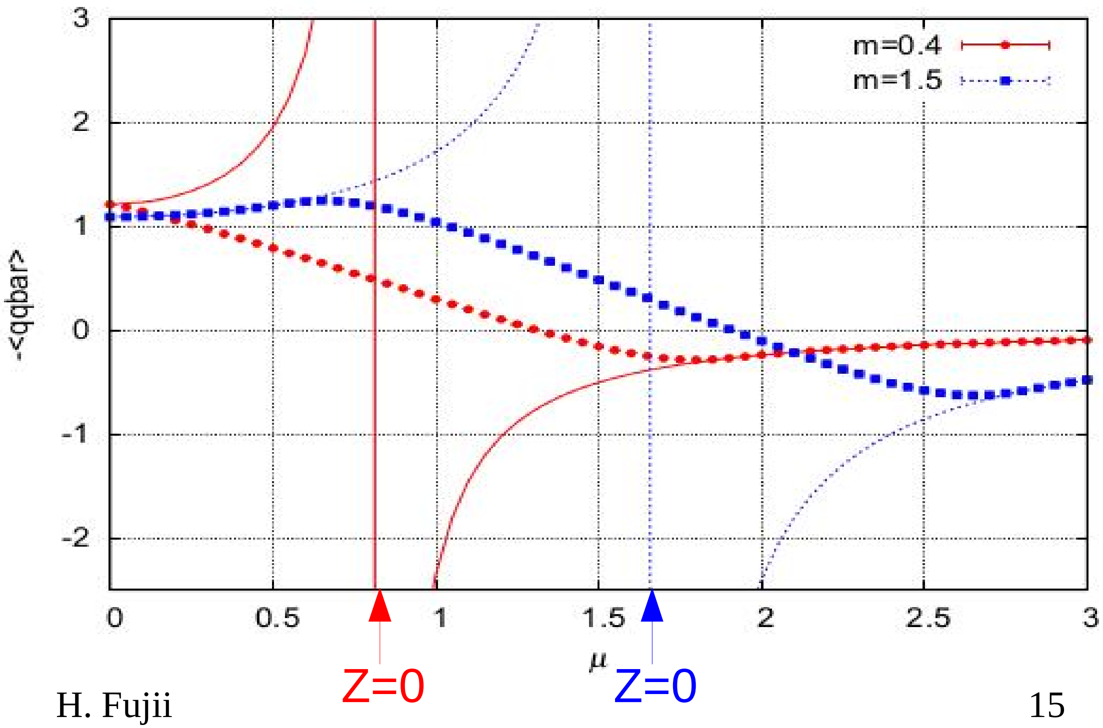
- Quark condensate

$$-\langle\bar{q}q\rangle=\frac{d\ln Z_{1/2}}{dm}=\frac{2m}{m^2+\frac{1}{2\beta}-\mu^2}$$

- both small and large μ ends are reproduced
- Good region at small μ becomes wider for larger m
- Zero of Z cannot be generated



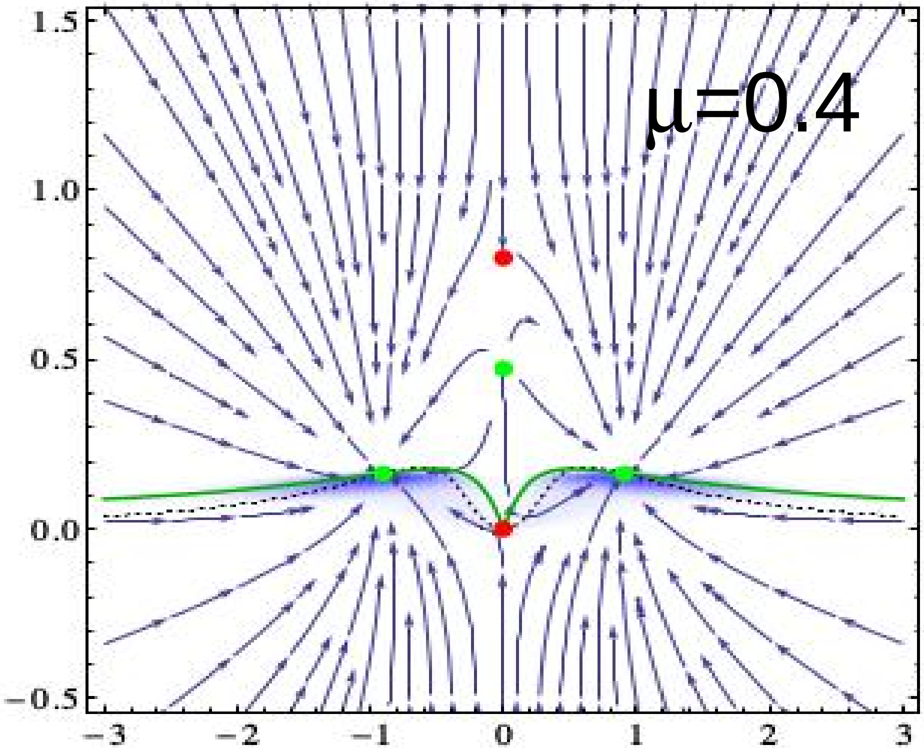
Flow and P(x,y) in N=1/2 model

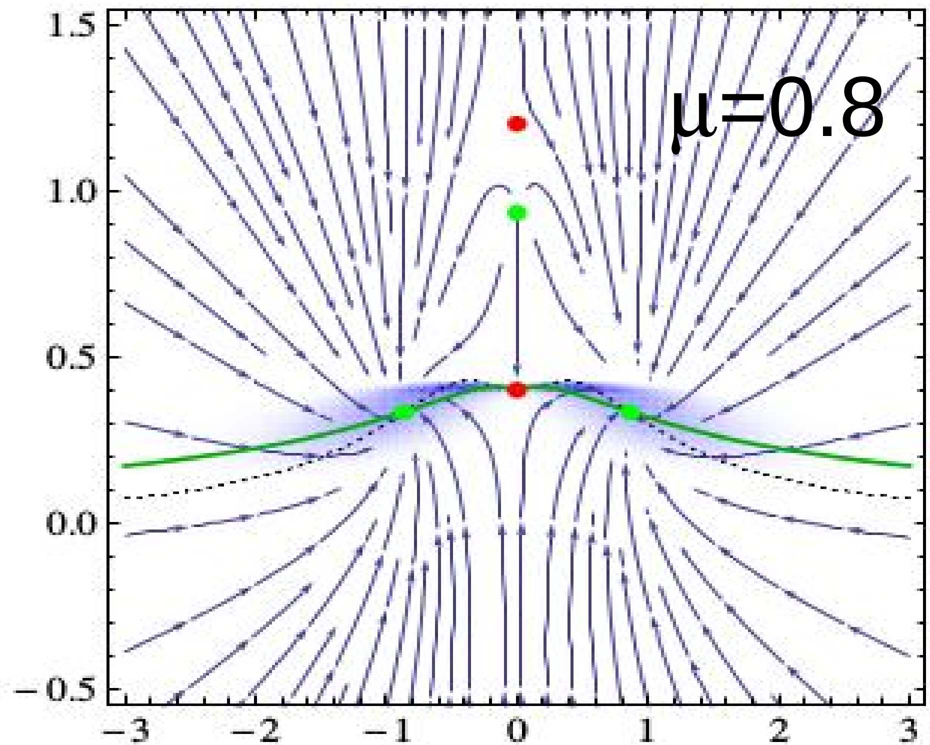
- Classical Langevin flow

$$-K(z) = \frac{\partial S}{\partial z} = 2\beta z - \frac{2(z-i\mu)}{m^2+(z-i\mu)^2}$$

Two attractive, one repulsive C.P.
P(x,y) is well-localised in y-drection

m=0.4





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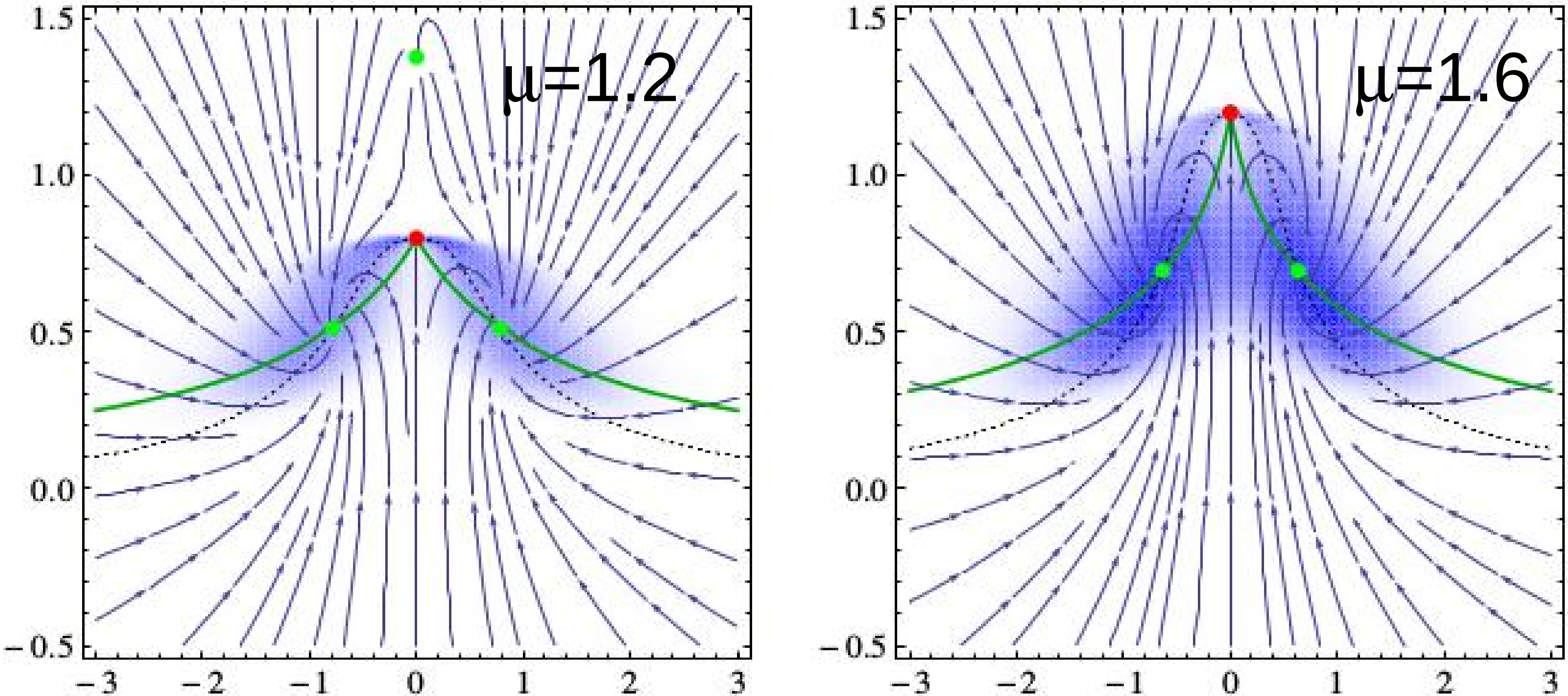
m=0.4



Saddle point approximation

- When S can be approximated as a sum of two Gaussians
  - (separated by a zero of e^{-S} in the simple model)

$$e^{-S(x)} \sim e^{-S(x_+)+\frac{1}{2}\kappa_+(x-x_+)^2} + e^{-S(x_-)+\frac{1}{2}\kappa_-(x-x_-)^2}$$

- CLE will result approximately (after y-integration)

$$e^{-S(x)} = \sqrt{\frac{\kappa_+}{2\pi}}\, e^{-\frac{1}{2}\kappa_+(x-x_+)^2} + \sqrt{\frac{\kappa_-}{2\pi}}\, e^{-\frac{1}{2}\kappa_-(x-x_-)^2}$$

- CLE seems to miss the "global phase" $e^{-S(x_+)}$

  and adds a relative phase of √(κ±)

- C.f., a well-known example:

for $e^{-S(x)} = x\, e^{-\frac{1}{2}x^2}$, CLE simulates $e^{-S(x)} \sim |x| e^{-\frac{1}{2}x^2}$

Integration on “Lefschetz” thimbles

Witten, Aurola coll., Komaba coll., ...

- Z can be evaluated effectively by deforming integration path – the line (space) of the most descent: “Thimbles” J
- determined by the flow $dz = \overline{dS/dz}\, d\xi$ from a C.P.
- on which Im(S) = const and Re(S) is increasing away from a C.P.

$$\int dx e^{-s(x)} = \int_{\mathcal{J}} dz e^{-S(z)} = e^{-i\mathrm{Im}[S(z(\xi))]} \int_{\mathcal{J}} d\xi \left| \frac{dz}{d\xi} \right| e^{-\mathrm{Re}[S(z(\xi))]}$$

- Imag part appears as “global” Im(S) and “resigual” Jacobian phases
- When Z=0 at some param (e.g., LY zero, etc.), thimble integrals give a cancellation thanks to these phases.
- CLE samples cannot give zero result for Z

Complex phase seems very important

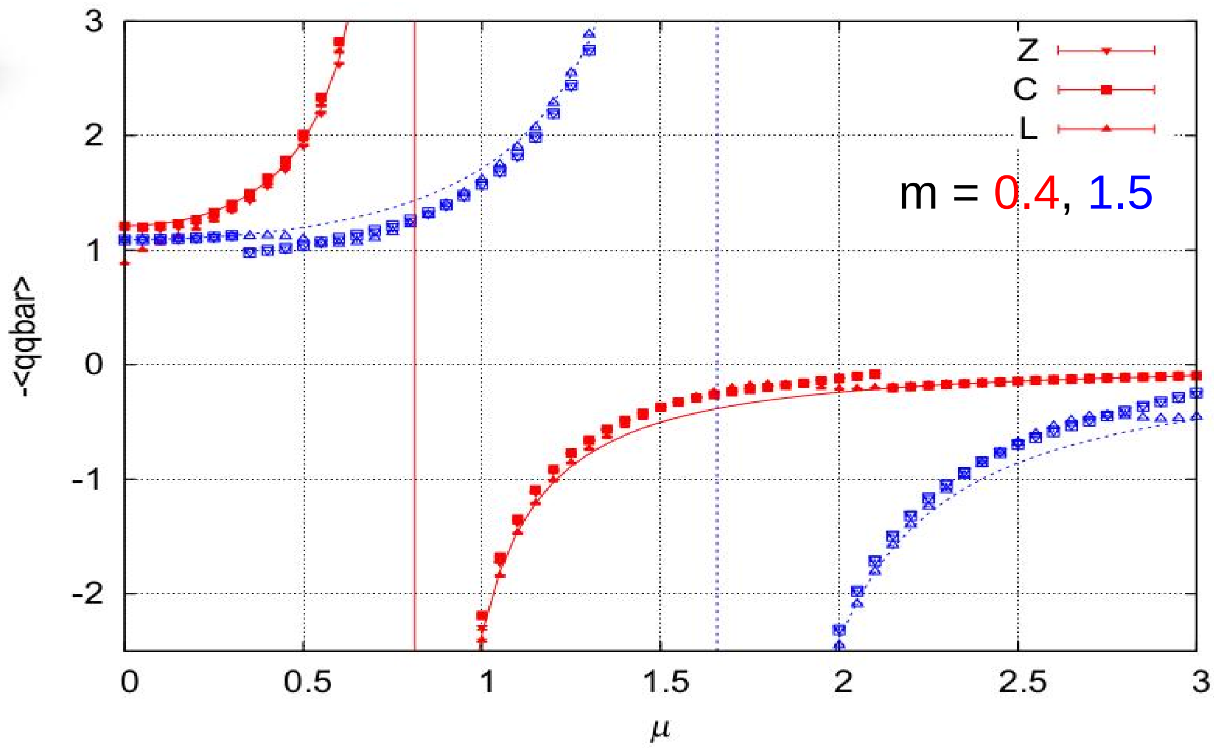
Modified CLE sampling in N=1/2 model

- Let us try the proper phase to be put on CLE samples:
  - (Z) Phase of exact $Z_\pm$ (corresponding to each thimble contrib)
  - (C) Phase of $\mathrm{Im}(S(z_\pm))$ at the C.P. and $\arg(\kappa_\pm)/2$
  - (L) Phase of $\mathrm{Im}(S(z))$ and $\arg(\kappa)/2$ locally at each sampling point z

Modified CLE sampling in N=1/2 model

- Let us try the proper phase to be put on CLE samples:
  - (Z) Phase of exact Z_± (corresponding to each thimble contrib)
  - (C) Phase of Im(S(z±)) at the C.P. and arg(κ±)/2
  - (L) Phase of Im(S(z)) and arg(κ)/2 locally at each sampling point z

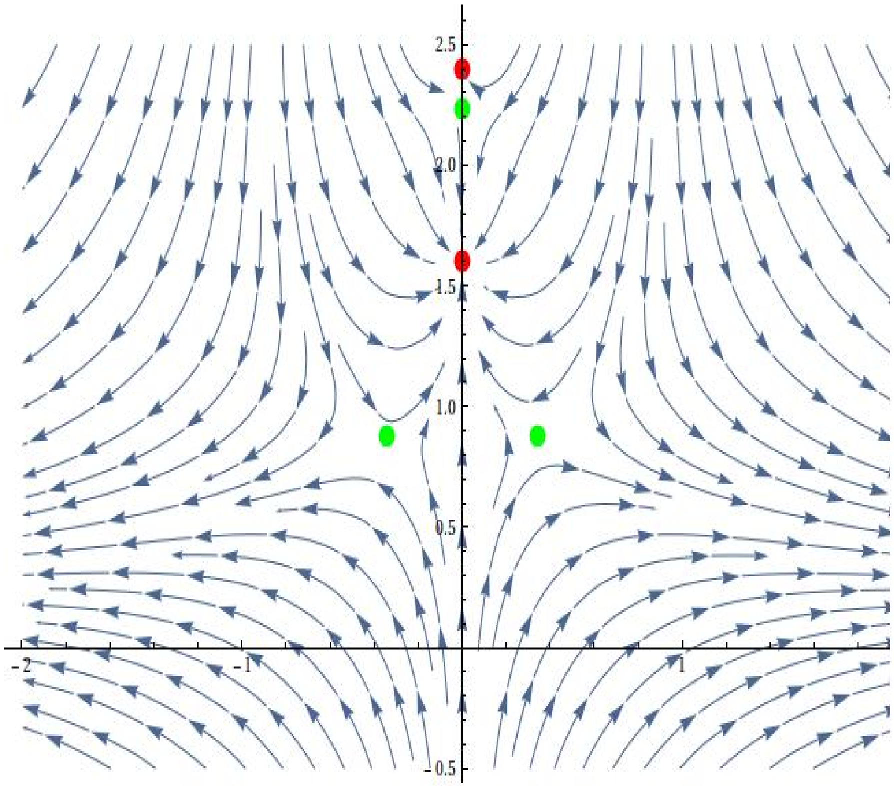
- Not perfect
- but still suggesting



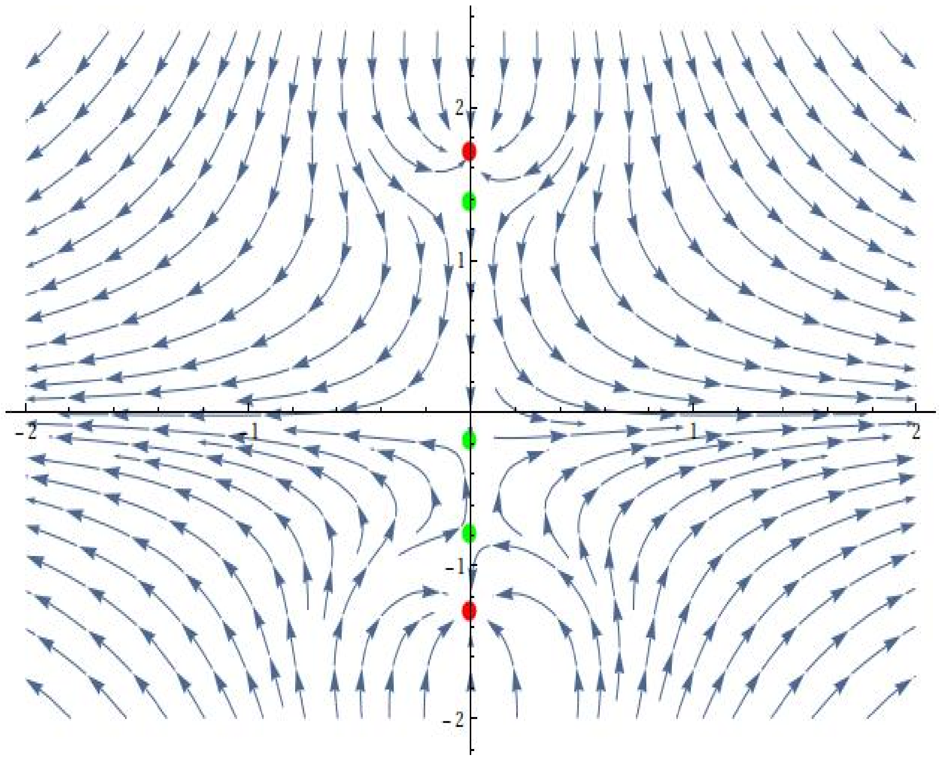
When CLE works?

- at large μ,
- at large m

m=0.4, μ=2.0



m=1.5, μ=0.2



Outlook

- Complex Langevin simulation applied to Chiral Random Matrix at finite t and μ
- CLE fails in chiral transition region of the model, but works at larger t or μ or m.
- Distribution P(x,y) localizes around two attractive CP in the transition region
- With additional phases motivated by Thimble idea, the result of CLE can be “improved”
- Models with N>1 have more complicated saddles – work in progress

- Like to understand CLE with Fermion determinant, and how it works