Gradient flow for chiral effective theories

work in progress with M. Savage

- A peculiar effective theory: chiral effective theory for nucleons & why regularization is interesting
- Gradient flow as regulator
- Nucleons on the brane: regulating interactions in an extra dimension
- Renormalization I: eliminating cutoff dependence for NN scattering
- Chirally covariant gradient flow
- Renormalization II: gradient flow in theories with power divergences
- Current status & future goals for this project

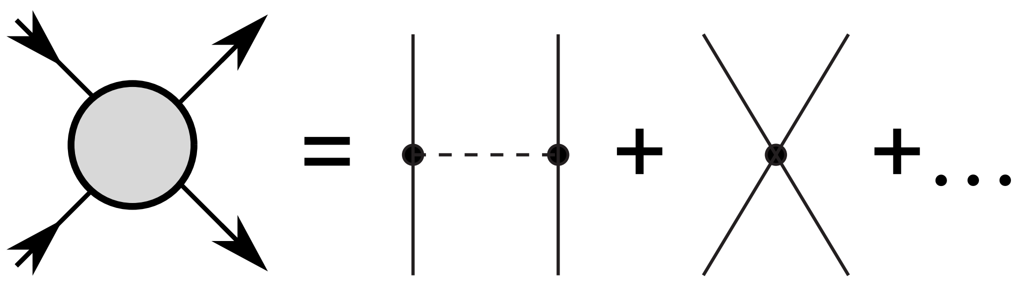
A peculiar effective theory

- Nucleons are bound, so the interaction must be nonperturbative
- Nuclei can be described in terms of nucleons, so we know the interaction is weak.

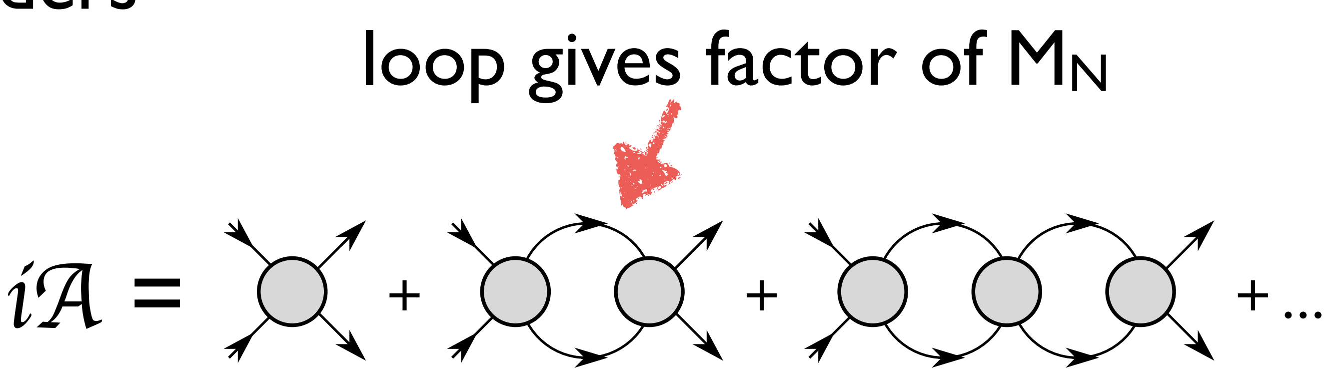
No contradiction: in nonrelativistic quantum mechanics, a weak potential will support a bound state if the particle mass is sufficiently big.

<u>Weinberg</u> (carrying on where Yukawa left off):

- compute the nucleon potential in chiral perturbation theory
- Sum insertions of potential to all orders



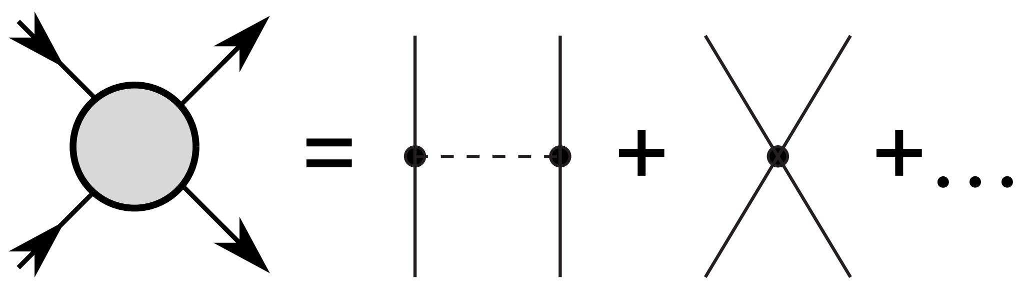
nucleon potential



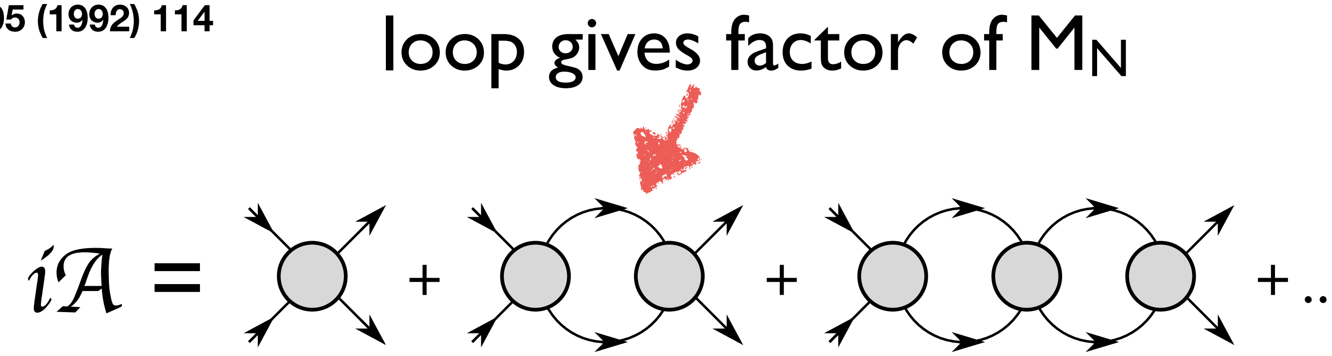
scattering amplitude

Weinberg's expansion for NN scattering

Phys. Lett. B 251 (1990) 258; Nucl. Phys. B363 (1991) 3; Phys. Lett. B295 (1992) 114



nucleon potential V expanded to a given order in χPT



scattering amplitude as sum of ladders (= solving Schrödinger equation with potential V)

Amplitude exhibits divergences requiring counterterms to <u>all</u> orders in chiral expansion in order to be renormalized!

…so cutoff dependence cannot be removed

In principle, Λ-dependent corrections should be higher order in χPT for a range of "reasonable" cutoff Λ.

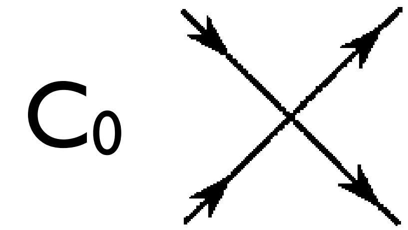
Problems:

- Λ - dependence can hide lack of convergence of χPT when doing numerical fits and regularization scheme dependence on UV physics
- Special counterterms required with momentum cutoff to preserve symmetry

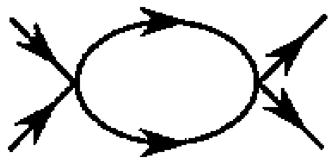
KSW expansion for NN scattering

Phys. Lett. B424 (1998) 390; Nucl. Phys. B534 (1998) 329

Consider contact interaction for non-relativistic nucleons:

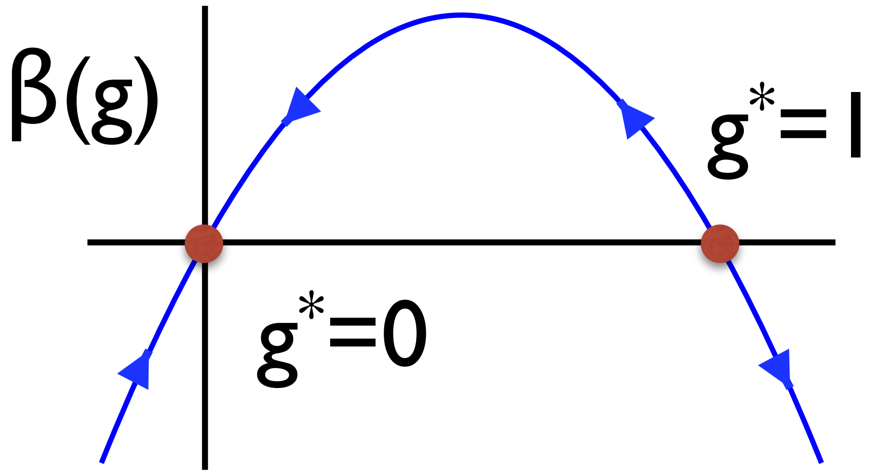


Renormalized by the linearly divergent diagram:



Can compute the β function (PDS scheme):

$$g(\mu) \equiv M\mu\, C_0(\mu)/4\pi, \quad t \equiv \ln\mu \qquad \dot{g} = g(1 - g)$$



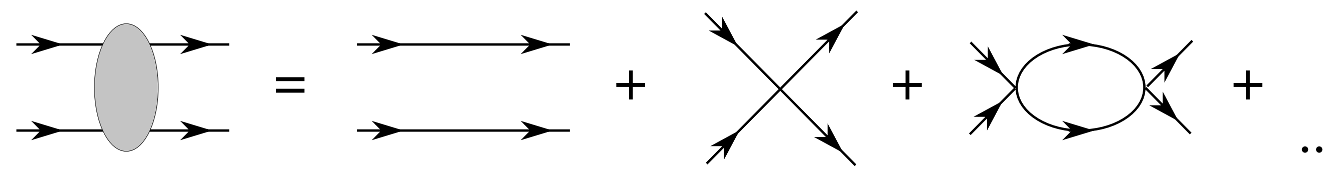
Two fixed points:

- $g^*=0$ is the trivial fixed point corresponding to no interaction
- $g^*=1$ is the nontrivial fixed point corresponding to infinite scattering length ("unitary fermions")

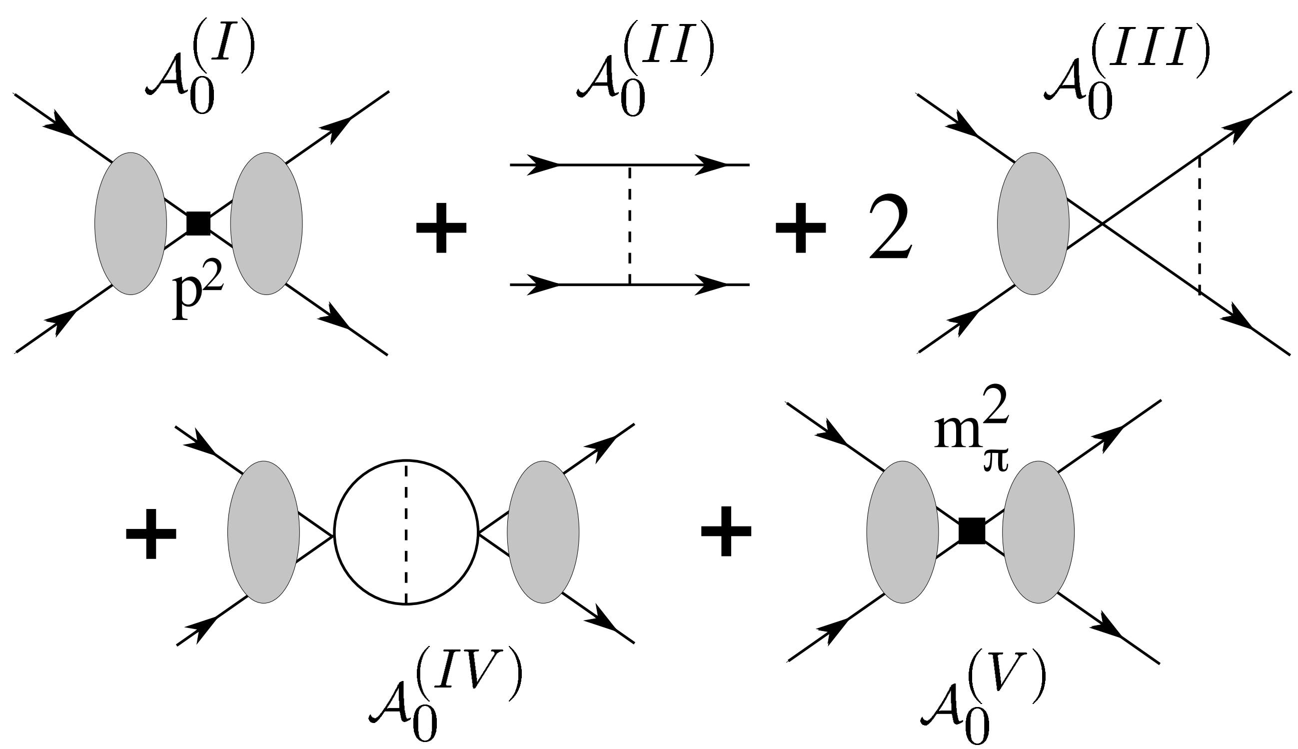
KSW program: perform a χPT expansion of the amplitude about the **nontrivial** fixed pt.

KSW expansion for nuclear effective theory:

LO amplitude
(unitary fermion scattering):



NLO amplitude
with one-pion exchange:



NNLO involves 2-pion exchange, etc.

Advantages of KSW expansion about the unitary fermion limit:

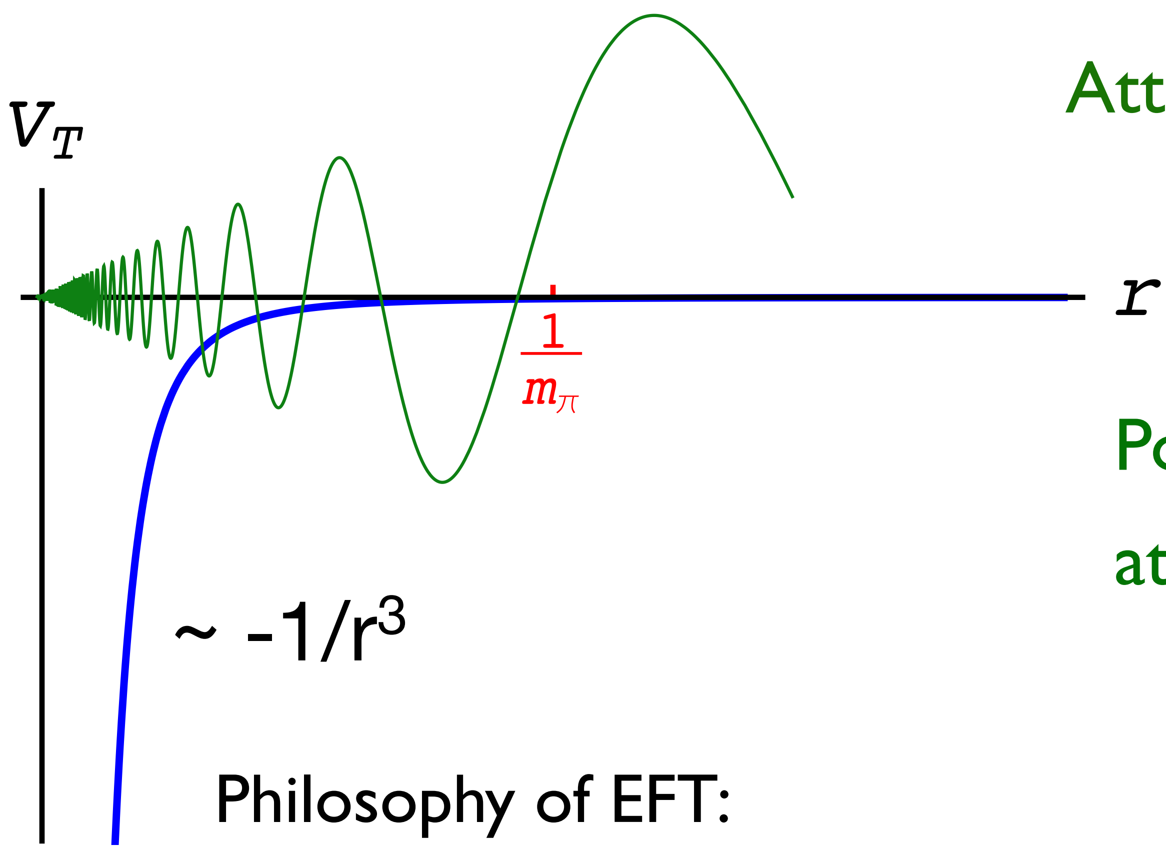
- NN scattering lengths are huge! (eg in $^1S_0$: $a \sim 23$ fm $\sim 17/m_\pi$)
- Anomalous dimensions lead to a consistent power counting that is nonperturbative in NN scattering at leading order.
- By expanding amplitude consistently order by order in χPT, all divergences correspond to operators at the same order, and amplitude can be fully renormalized

Problem:

- Expansion doesn't converge in $^3S_1$ channel! (Fleming, Mehen, Stewart)
  …and presumably in other channels with attractive tensor interaction)

Why might that be? What is special about the attractive tensor interaction?

Failure of perturbative expansion of attractive tensor force?



Attractive tensor interaction from 1-pion exchange

Potential as an unphysical attractive $1/r^3$ behavior at short distance

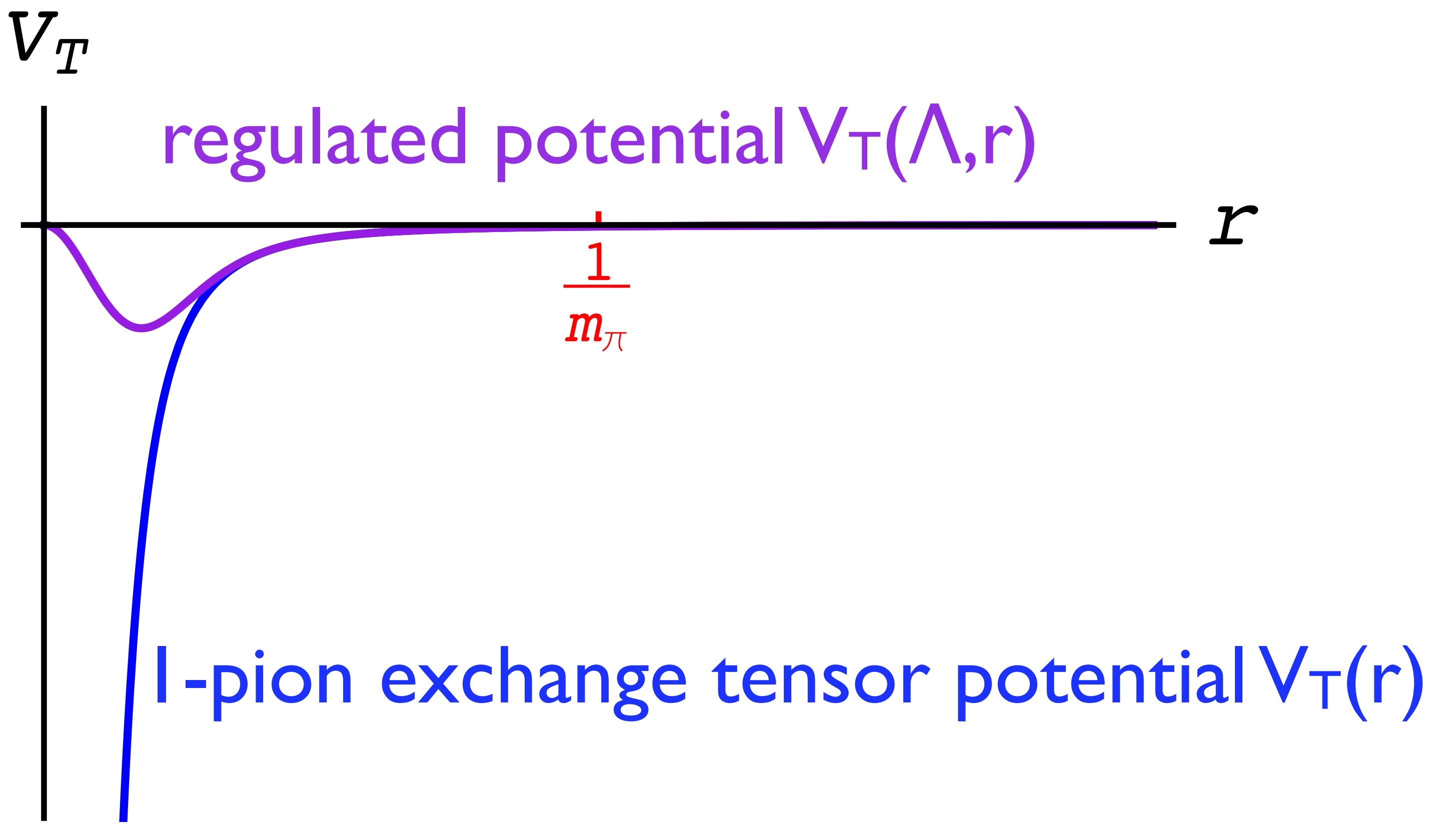
Philosophy of EFT:

- Deform UV physics however one likes to make the calculation easy (eg, dim reg)
- Absorb dependence on unphysical UV in phenomenological coupling constants

The problem:

- (Fake) UV properties ($-1/r^3$ behavior) make tensor interaction inherently nonperturbative - no ground state
- KSW expansion tries to fix this with local counterterms (equivalent to adding δ-functions and their derivatives to $-1/r^3$)…hopeless!

Can KSW expansion be resurrected with a better regularization scheme?

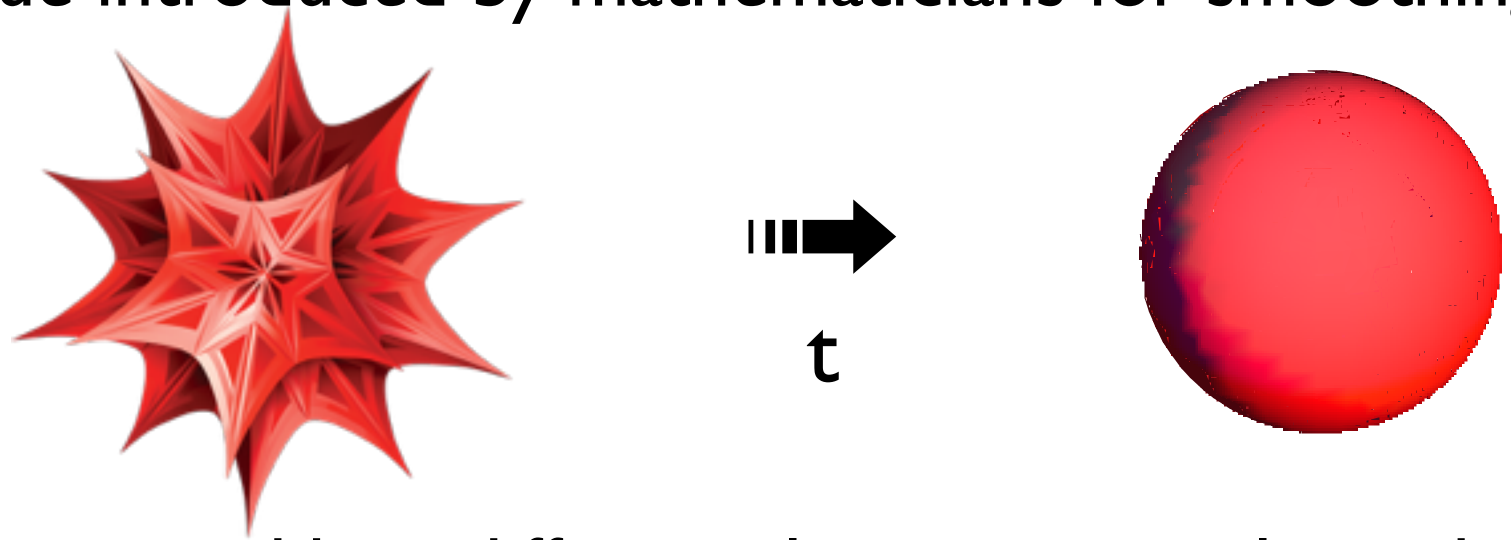


Requirements:

- "Extended" regulator, to cure $-1/r^3$
- Renormalizable (no dependence on UV regulator)
- Preserves chiral symmetry!

Gradient flow as regulator

Technique introduced by mathematicians for smoothing manifolds



Mapping governed by a differential equation similar to heat equation

map from minimization of "energy functional"

J. Eells and J. H. Sampson, American Journal of Mathematics pp. 109–160 (1964).
R. S. Hamilton et al., Journal of Differential Geometry 17, 255 (1982).
G. Perelman, arXiv preprint math/0211159 (2002).
G. Perelman, arXiv preprint math/0303109 (2003).
G. Perelman, arXiv preprint math/0307245 (2003).

Introduced "Ricci flow"

Solved the Poincaré Conjecture

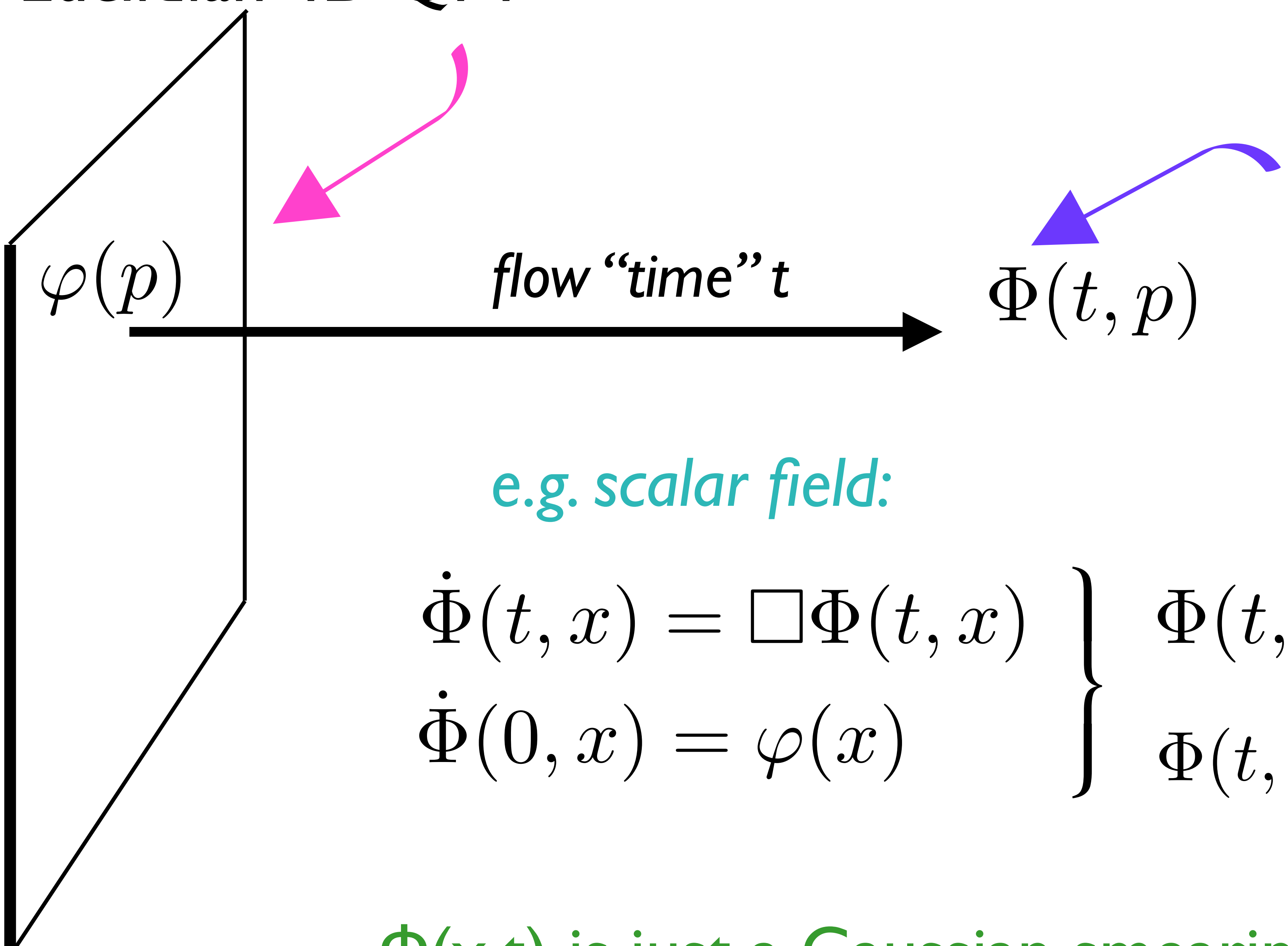
Ricci flow: $$\dot{g}_{ij} = -2R_{ij}$$

- Behaves like heat equation for smooth manifolds
- smooths out bumps
- **diffeomorphism covariant**

Gradient flow applied to quantum field theories by Lüscher

M. Luscher, JHEP 1008, 071 (2010), 1006.4518.
M. Luscher and P. Weisz, JHEP 1102, 051 (2011), 1101.0963.

Euclidian 4D QFT



e.g. scalar field:

$$\left.\begin{aligned}\dot\Phi(t,x) &= \Box\Phi(t,x)\\ \dot\Phi(0,x) &= \varphi(x)\end{aligned}\right\}\quad \begin{aligned}&\Phi(t,p) = e^{-tp^2}\varphi(p)\\ &\Phi(t,x) \propto \int_y e^{\frac{(x-y)^2}{t}}\varphi(y)\end{aligned}$$

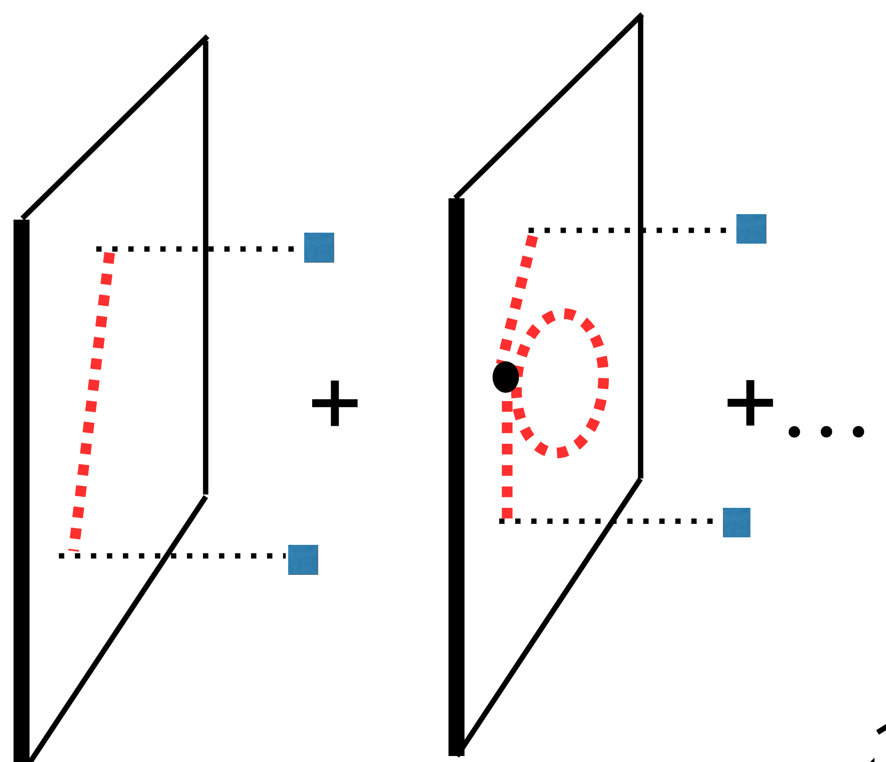
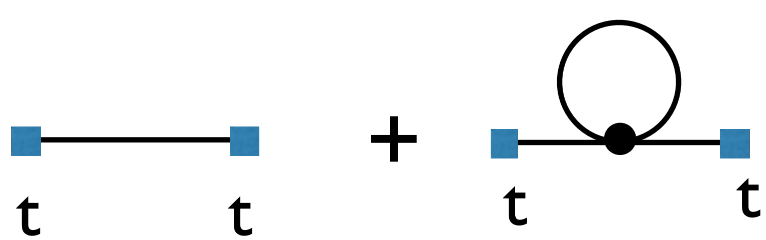
$\Phi(x,t)$ is just a Gaussian smearing of $\varphi(x)$
$1/t$ has dimension mass^2 and serves as cutoff

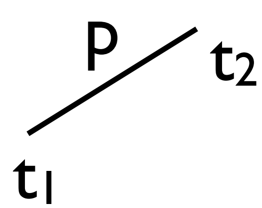
4d Lagrangian: $$\mathcal{L}_4 = \frac{1}{2}\varphi(-\Box + m^2)\varphi + \frac{\lambda}{4!}\varphi^4$$

5d flow:
$$\left.\begin{aligned} \dot\Phi(t,x) &= \Box\Phi(t,x) \\ \dot\Phi(0,x) &= \varphi(x) \end{aligned}\right\} \quad \Phi(t,p) = e^{-tp^2}\varphi(p)$$

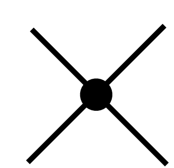
5d 2-pt function:
$$\langle\Phi(t,x)\Phi(t,y)\rangle = \frac{1}{(16\pi^2 t^2)^2}\int_{x',y'} e^{-(x-x')^2/4t}e^{-(y-y')^2/4t}\,\langle\varphi(x')\varphi(y')\rangle$$

$$\langle\Phi(t,p)\Phi(t,q)\rangle = e^{-t(p^2+q^2)}\,\langle\varphi(p)\varphi(q)\rangle$$

=  + … ≡  + …

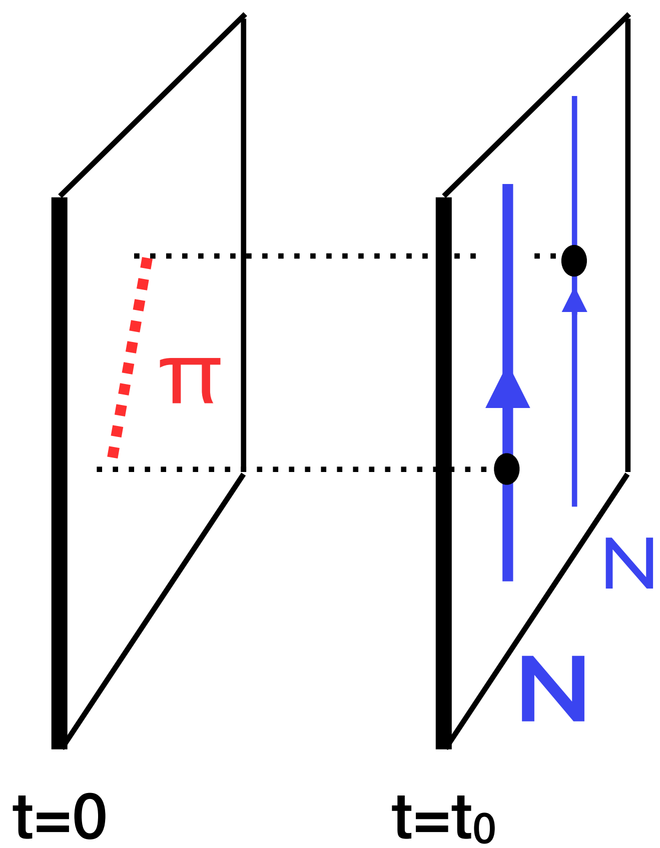


$$= \frac{e^{-(t_1+t_2)p^2}}{p^2+m^2}$$

 $= -\lambda\delta(t)$

Nucleons on a brane

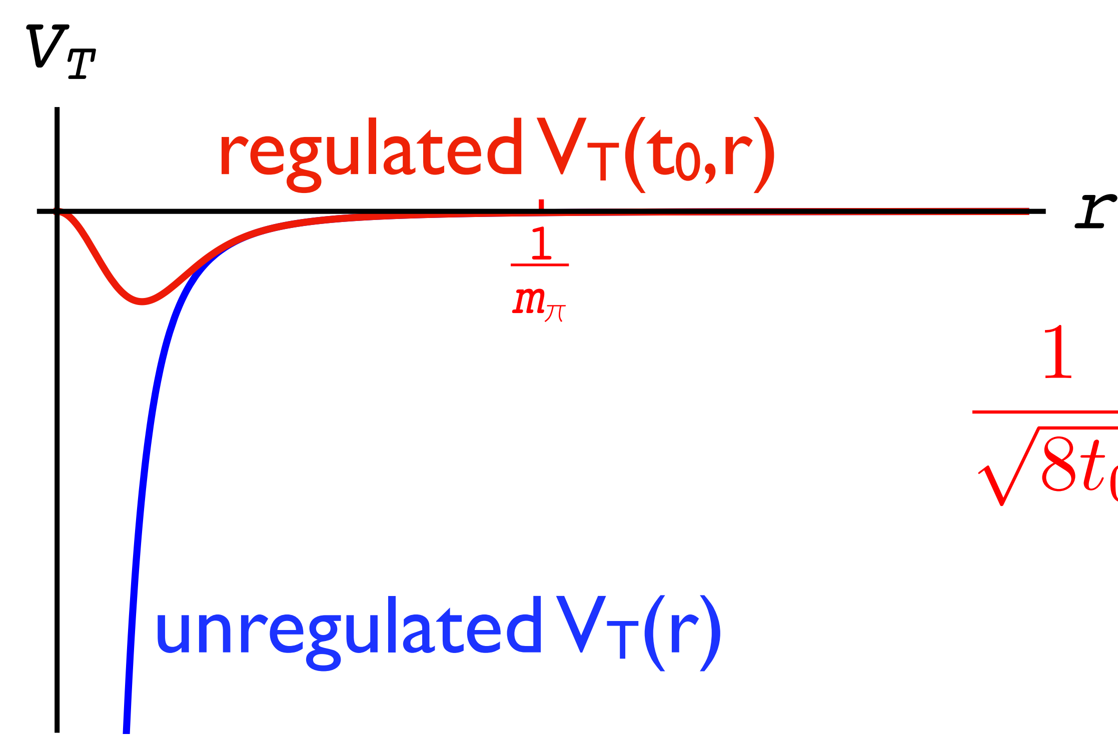
This simple example suggests a way to regulate the nucleon-nucleon interaction:



one pion exchange potential:

$$\tau_1 \cdot \tau_2 \frac{q\cdot\sigma_1\, q\cdot\sigma_2}{q^2+m_\pi^2} \Longrightarrow \tau_1 \cdot \tau_2 \frac{q\cdot\sigma_1\, q\cdot\sigma_2}{q^2+m_\pi^2} \times e^{-2t_0 q^2}$$

Potential in the $^3S_1$ channel:



$$\frac{1}{\sqrt{8t_0}} = 1\ \text{GeV}$$

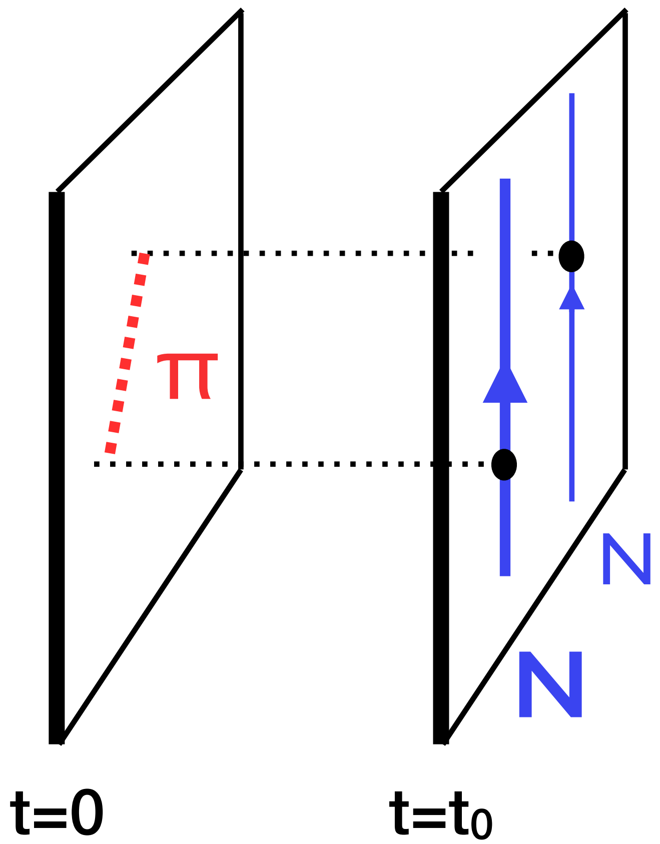
Making the substitution $$\tau_1 \cdot \tau_2 \frac{q \cdot \sigma_1 \, q \cdot \sigma_2}{q^2 + m_\pi^2} \Longrightarrow \tau_1 \cdot \tau_2 \frac{q \cdot \sigma_1 \, q \cdot \sigma_2}{q^2 + m_\pi^2} \times e^{-2t_0 q^2}$$

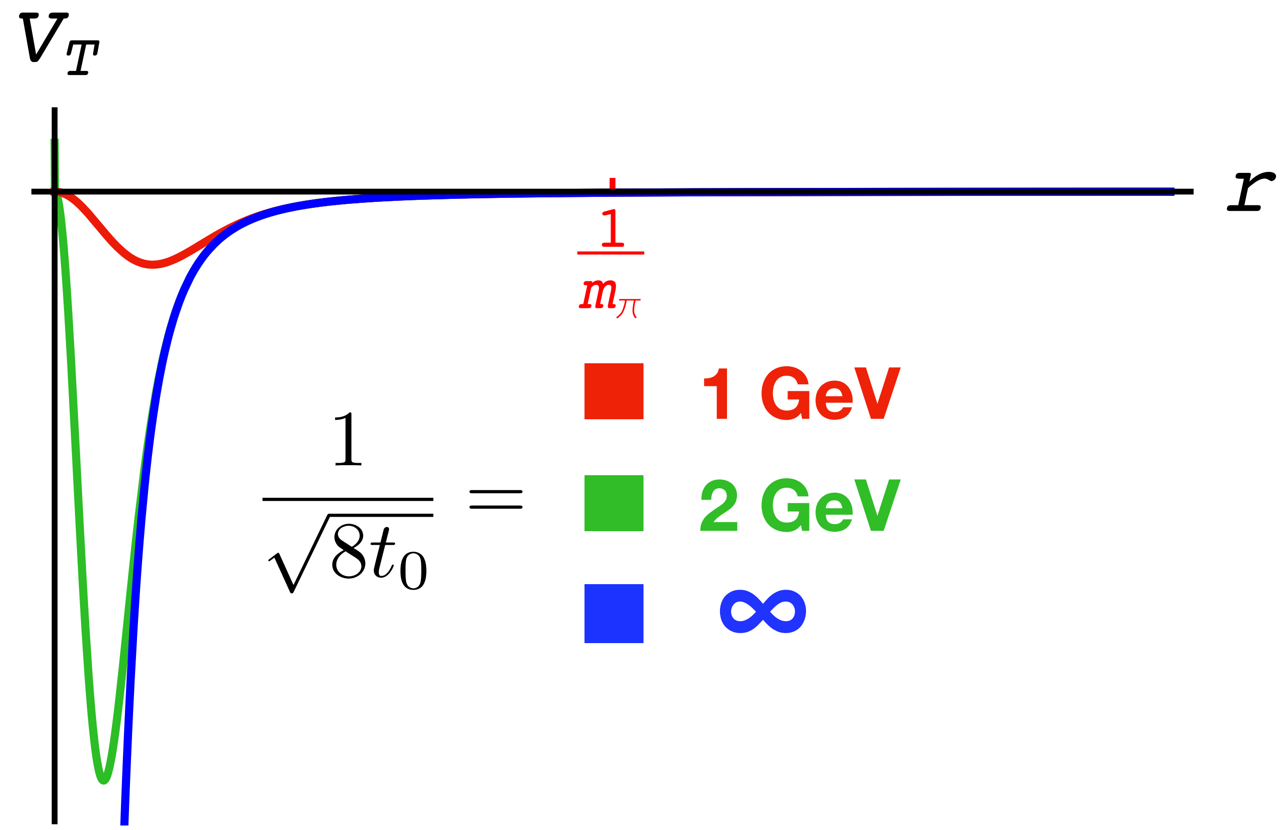
- Is very simple! A gaussian cutoff! No need for gradient flow machinery?!
- Heat equation is too simple… we will see that it violates chiral symmetry 👎
- …but that the gradient flow machinery *can* preserve chiral symmetry 👏

But first:
sketch how renormalization could eliminate dependence on arbitrary choice of t_0

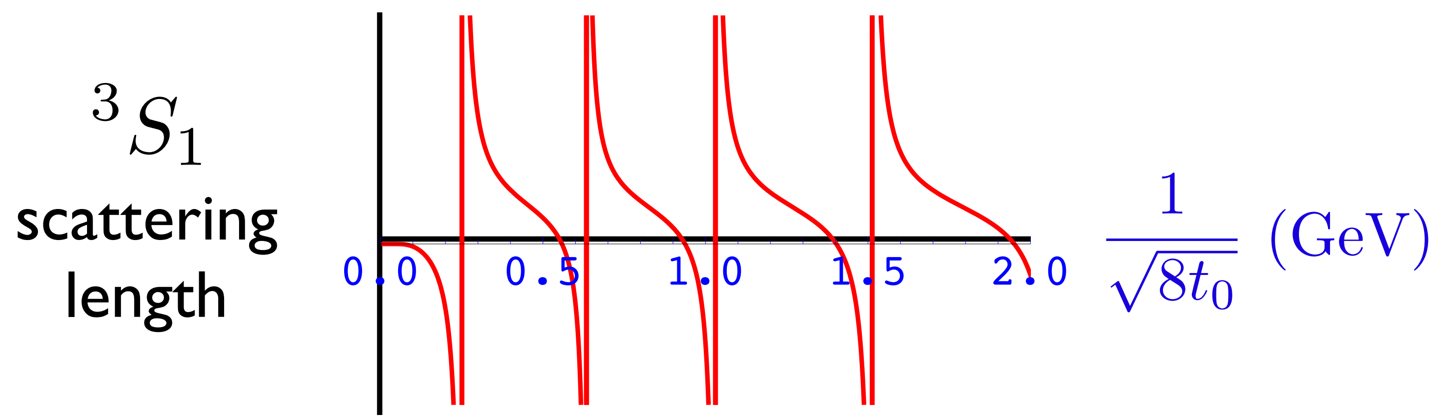
Renormalization I

How to eliminate the arbitrary t_0 dependence in NN scattering

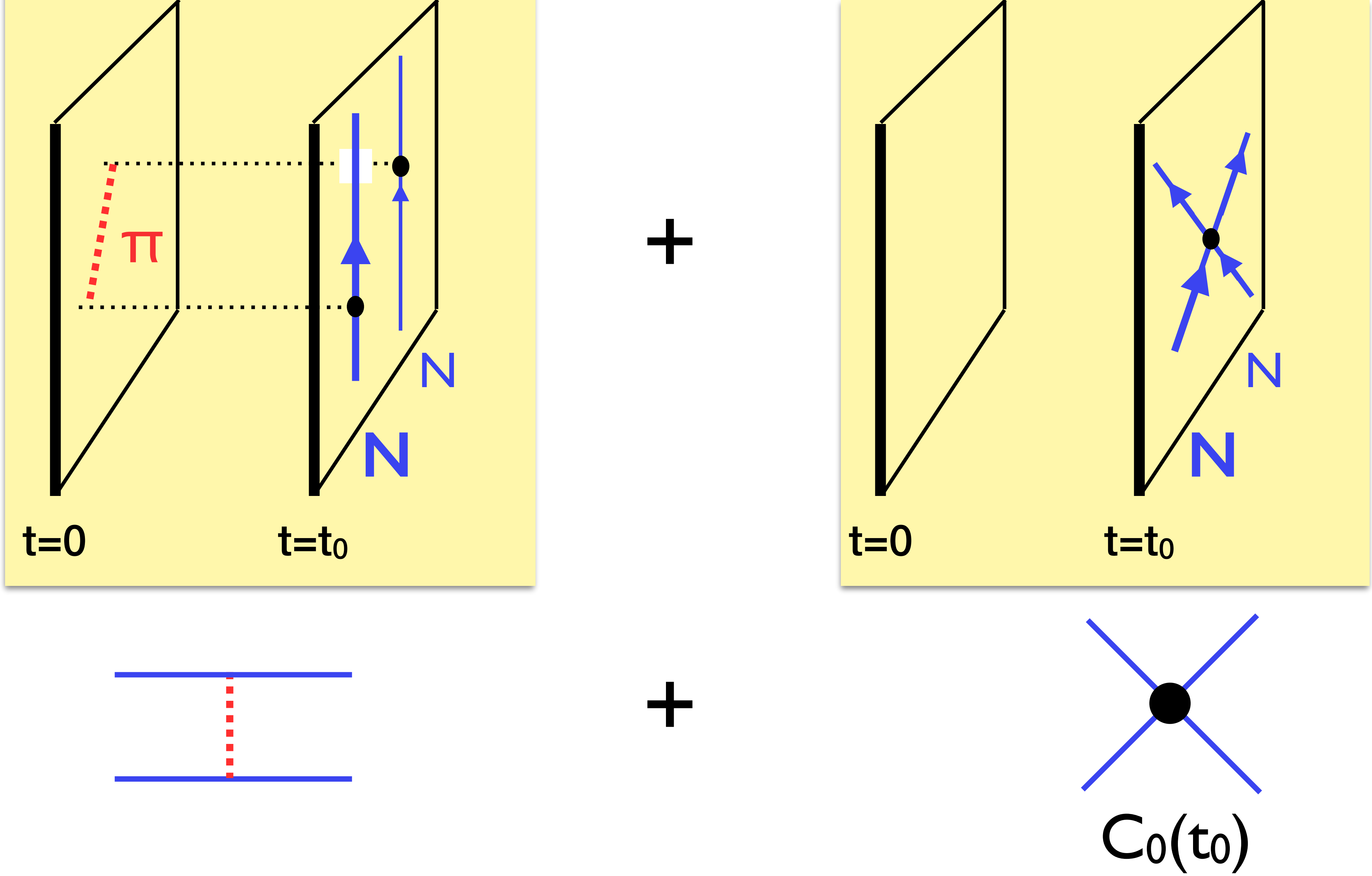




First: consider what happens to the scattering length for NN scattering with this potential, as a function of t_0:



As the cutoff is removed, an increasing number of bound states are trapped (points where scattering length diverges)



2-nucleon potential = 1-pion exchange + contact interaction

- Introducing a contact interaction (δ-function potential) allows one to absorb the scattering length dependence on t_0
- C_0 will exhibit limit cycle behavior as a function of the cutoff $\Lambda = 1/\sqrt{8t_0}$ to counteract the t0 dependence of the pion potential

Chirally covariant gradient flow

The gaussian cut-off described violates chiral symmetry

SU(2) x SU(2) can be written in terms of the SU(2) unitary matrix field Σ which transforms linearly under the chiral symmetry:

$\Sigma(x) \Rightarrow L\,\Sigma(x)R^\dagger$, $L \in SU(2)_L$, $R \in SU(2)_R$

Σ can be written in terms of the pion field, which transform nonlinearly:

$$\Sigma = e^{i\pi^a(x)\sigma_a/f}$$

f = 93 MeV is the pion decay constant

The leading term in the chiral Lagrangian is

σ-model metric

$$\mathcal{L}_0 = \frac{f^2}{4}\partial_\mu\Sigma^\dagger\partial_\mu\Sigma \equiv \frac{1}{2}g_{ab}\partial_\mu\pi^a\partial_\mu\pi^b$$

$$\mathcal{L}_0 = \frac{f^2}{4}\partial_\mu \Sigma^\dagger \partial_\mu \Sigma \equiv \frac{1}{2}\, g_{ab}\partial_\mu \pi^a \partial_\mu \pi^b$$

A natural candidate for a covariant flow equation in the chiral limit is

$$\dot{\phi}^a = -g^{ab}\frac{\partial S_0}{\partial \phi^b} = \Box\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c\ , \qquad \phi^a(0,x) = \pi^a(x)$$

B.C.

heat eq. term

nonlinear terms required by chiral symmetry

Can compute the metric and Christoffel symbol:

$$g_{ab} = \delta_{ab} + \frac{-1+2\theta^2+\cos 2\theta}{2\theta^4}\left(\theta^a\theta^b - \theta^2\delta_{ab}\right)$$

$$\theta^a = \frac{\phi^a}{f}$$

$$g_{xb,c} = \frac{\partial g_{xb}}{\partial \phi^c}$$

$$\begin{aligned}\Gamma^a_{bc} &= \frac{1}{2}g^{ax}\left[g_{xb,c} + g_{xc,b} - g_{bc,x}\right] \\ &= \frac{1}{f}\left[\frac{2}{3}\left(\delta_{bc}\delta_{ad} - (\delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd})\right)\theta^d + O(\theta^3)\right]\end{aligned}$$

Three technical remarks:

☆ *Easy to generalize flow equation to include explicit chiral symmetry breaking*

$$\mathcal{L}_0 = \frac{f^2}{4}\partial_\mu \Sigma^\dagger \partial_\mu \Sigma - Bf^2\left(M\Sigma + \text{h.c.}\right) = \left(\frac{1}{2} g_{ab}(\phi)\partial_\mu\phi^a\partial_\mu\phi^b + m_\pi^2 f^2 \cos\phi/f\right)$$

quark mass matrix

so a convenient gradient flow equation becomes:

$$\dot\phi^a = -g^{ab}\frac{\partial S_0}{\partial\phi^b} = \left(\Box - m_\pi^2\right)\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c + m_\pi^2\phi^a\left(1 - \frac{\sin\phi/f}{\phi/f}\right)$$

This form is convenient for reading off interactions, but not for seeing chiral symmetry…

★ *A flow equation transforming linearly under SU(2) x SU(2)*

$$\dot{\phi}^a = \left(\Box - m_\pi^2\right)\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c + m_\pi^2\phi^a\left(1 - \frac{\sin\phi/f}{\phi/f}\right)$$

transforms as a covariant vector under diffeomorphisms

$$v'^a = v^b\frac{\partial\phi'^a}{\partial\phi^b}$$

where Φ'(Φ) is the chirally transformed pion field. A nicer and **equivalent** equation is

$$\Sigma^\dagger\partial_t\,\Sigma = \partial_\mu\left(\Sigma^\dagger\partial_\mu\Sigma\right) + B\left(\left[M^\dagger\Sigma - \frac{1}{2}\mathrm{Tr}M^\dagger\Sigma\right] - \mathrm{h.c.}\right)$$

which transforms linearly under SU(2) x SU(2) like a RH current:

$$J' = R^\dagger J R$$

A 5d formulation for the path integral

Following Lüscher and Weisz: can formulate theory as a 5d path integral with a c

to ensure Φ obeys the classical flow equation in the bulk (here: in chiral limit)

Chirally invariant measure: $\int \Pi_{a=1}^{3} \left[d\phi^a\right] \sqrt{g}$

Chirally invariant constraint:

$$\delta\left[-\Sigma^\dagger \partial_t \Sigma + \partial_\mu \left(\Sigma^\dagger \partial_\mu \Sigma\right)\right]$$

$$\frac{1}{\sqrt{g}} \Pi_{a=1}^{3}\, \delta\left[-\dot{\phi}^a + \Box\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c\right]$$

Together: $\int \Pi_{a=1}^{3} \left[d\phi^a\right] \delta\left[-\dot{\phi}^a + \Box\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c\right]$

$$\int [d\phi]\,\delta\left[-\dot\phi^a + \Box\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c\right]$$

$$= \int [d\phi][d\omega]e^{\int dtdx\, i\omega_a\left[-\dot\phi^a + \Box\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c\right]}$$

- 5d action = Lagrange multiplier ω times flow eq.
- No √g factor in measure
- Φ obeys BC Φ(0,x) = π(x)

In addition, have the 4d action:

$$Z = \int [d\pi]\sqrt{g}\, e^{-\int dx\,\mathcal{L}_\chi(\pi)} \int_{\phi(0,x)=\pi(x)} [d\phi][d\omega]e^{\int dtdx\, i\omega_a\left[-\dot\phi^a + \Box\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c\right]}$$

Feynman rules for flow diagrams

$$\dot{\phi}^a = \left(\Box - m_\pi^2\right)\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial_\mu\phi^c + m_\pi^2\phi^a\left(1 - \frac{\sin\phi/f}{\phi/f}\right)$$

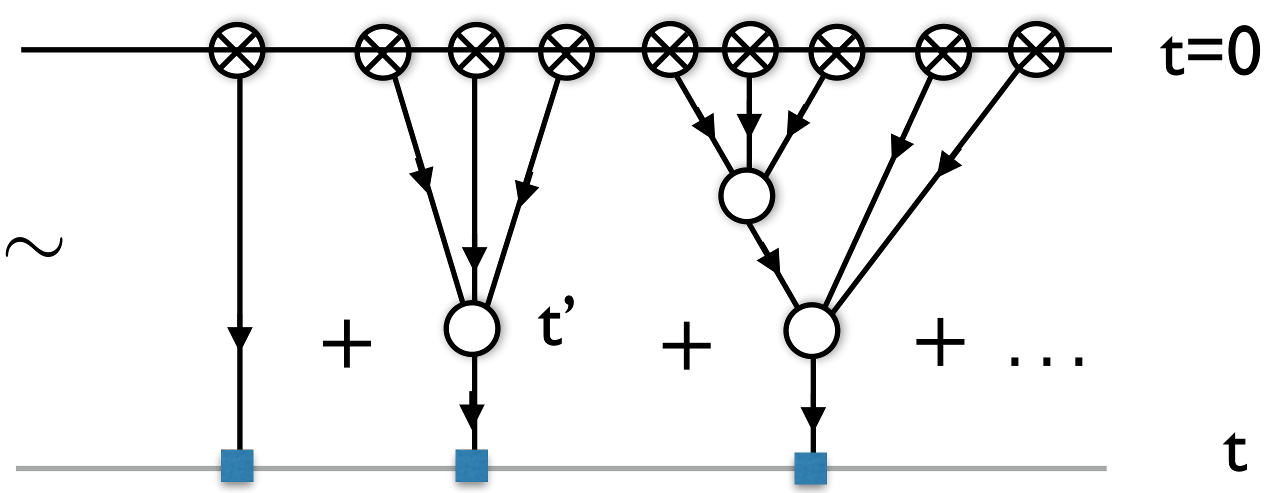
perturbative solution in powers of 1/f (usual chiral expansion)

$$\phi(t,p) \sim K_t(p)\pi(p) + \eta\int_0^t dt'\, K_{t-t'}(p)\int\prod_{i=1}^{3} dq_i\delta(p-q_{\rm tot})\prod_{i=1}^{3}K_{t'}(q_i)\pi(q_i) + O(\pi^5)$$

$$K_t(p) = e^{-t(p^2+m_\pi^2)}$$

quantum π fields

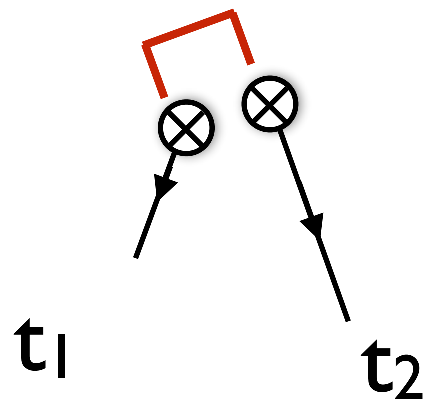
$\phi(t,p) \sim$



$K_t(p) = e^{-t(p^2+m_\pi^2)}$

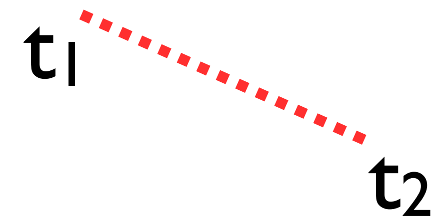
Quantum π fields get contracted according to 4d chiral Lagrangian Feynman rules

Propagator:

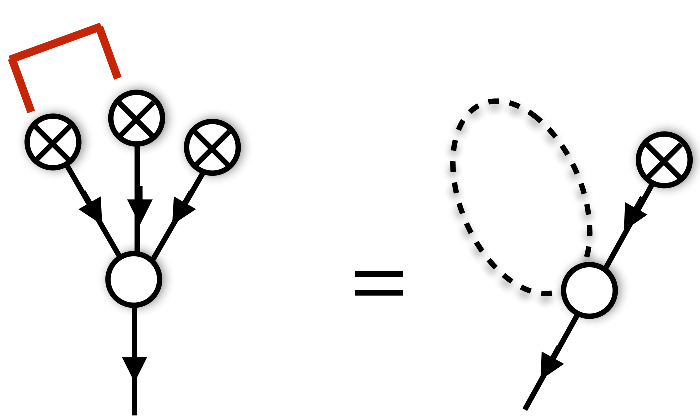


$$= e^{-t_1(p^2+m_\pi^2)} \frac{1}{p^2+m_\pi^2} e^{-t_2(p^2+m_\pi^2)}$$

=



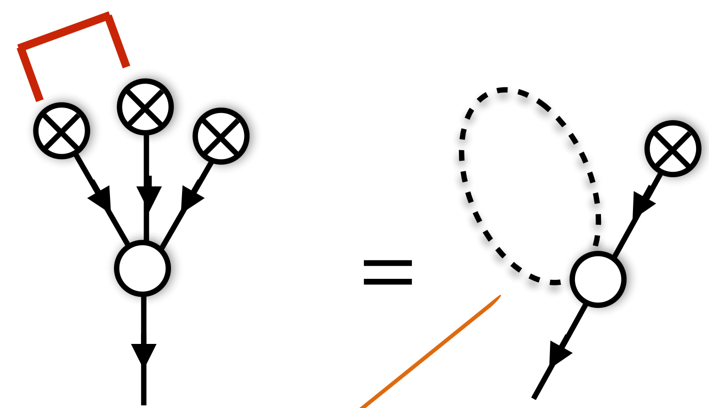
Same propagator we saw from the simple heat equation
…but now there are also nontrivial loops



Renormalization II

Loops in flow diagrams

e.g:



$$\int_0^t dt' \int \frac{d^d q}{(2\pi)^d} \frac{e^{-2t'(q^2+m_\pi^2)}}{q^2+m_\pi^2} = \int \frac{d^d q}{(2\pi)^d} \left(\frac{1-e^{-2t(q^2+m_\pi^2)}}{2(q^2+m_\pi^2)^2} \right)$$

finite

flow loops are 2 powers more convergent than usual loops

QCD only has log divergences, so no new divergence from flow loops

χPT has power law divergences, so get new counterterms proportional to δ(t)

Counterterm to BC or counter term to the flow eq?

Current status…

- Have defined a chirally covariant gradient flow with well-defined Feynman rules
- Divergence structure not fully understood yet
- Searching for a more tractable formulation
- Need to study higher order corrections to KSW expansion for NN amplitudes

…future goals

- Believe that this framework will help regulate singular tensor interaction between nucleons
- This could revive KSW expansion
  - renormalizable
  - analytic expansion, perturbative in pion exchange

After 80 years in 4 dimensions the pion is getting restless