The chiral condense of the Schwinger model at finite temperature with Matrix Product States

H. Saito et al. arXiv:1412.0596

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New numerical approaches in Lattice gauge theory

- Conventional Lattice QCD simulation
  - ✳ based on Monte Carlo
  - ✳ giving a lot hints of non-perturbative properties of QCD
- Alternative numerical approaches for
  - ✳ QCD at finite density
  - ✳ real time dynamics
- Hamiltonian approach with Tensor network method

Tensor network (TN)

- An efficient approximation of quantum many-body state from quantum information
- **Matrix product states (MPS)** - 1d TN

$$|\psi\rangle \approx \sum_{i_0,\cdots,i_{N-1}} \mathrm{Tr}\left[M[0]^{i_0} M[1]^{i_1} \cdots M[N-1]^{i_{N-1}}\right] |i_0 i_1 \cdots i_{N-1}\rangle$$

i_k: physical indices at site k, $M^{i_k}_{mn}$: tensor,
m, n $(=1,\ldots, D)$: indices from this approximation, **D : bond dimension**

- W/ TN, investigating sub-space growing linearly (polynomially in general) ⇔ Hilbert space growing exponentially d^N with system size

Schwinger model for $N_f = 1$

J. Schwinger Phys.Rev. 128 (1962)

- 1+1 dimensional QED model

  not QCD, but **similar to QCD** :

  confinement, chiral symmetry breaking

N. L. Pak and P. Senjanovic, Phys.Let.B71, 2 (1977), K. Johnson Phys.Let. 5, 4(1963)

- Exactly solvable in massless case

  ⇒ a good test case

- Hamiltonian for TN approach

  in spin language

T. Banks, L. Susskind and J. Kogut, PRD13, 4 (1973)

hopping term, mass term, gauge part

$$H = x \sum_{n=0}^{N-2} [\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \frac{\mu}{2} \sum_{n=0}^{N-1} [1 + (-1)^n \sigma_n^z] + \sum_{n=0}^{N-2} \left[l + \frac{1}{2} \sum_{k=0}^{n} ((-1)^k + \sigma_k^z) \right]^2$$

$$= H_{\mathrm{hop}} + H_{\mathrm{mass}} + H_{\mathrm{g}}$$

Gauss law

where inverse coupling $x=1/a^2g^2$, dimensionless mass $\mu=2m/ag^2$ and $l = L(0)$

Hamiltonian of Schwinger model in continuum

T. Byrnes, et al Phys. Rev. D66 (2002), 013002

- Lagrangian of Schwinger model:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu} D_{\mu} - m\right)\psi(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}(x)$$

ψ , $\bar{\psi} \equiv \psi^{\dagger}\gamma^{0}$: 2-component fermion field

$D_{\mu} = \partial_{\mu} + igA_{\mu}$: covariant derivative, $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: gamma matrix

$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$: Field strength

- Hamiltonian via Legendre transformation:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi\right)}\left(\partial_0 \psi\right) + \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_{\mu}\right)}\left(\partial_0 A_{\mu}\right) - \mathcal{L}$$

$$= -i\bar{\psi}\gamma_1\left(\partial^1 + igA^1\right)\psi(x) + m\bar{\psi}\psi(x) + \frac{1}{2}E^2(x)$$

where temporal gauge $A_1 = 0$, $E \equiv -\partial_0 A_1$

- Gauss law: $\partial_1 E(x) = g\bar{\psi}\gamma_0\psi(x)$

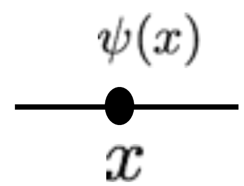
Hamiltonian of Schwinger model on lattice

T. Byrnes, et al Phys. Rev. D66 (2002), 013002

- For fermion, Kogut-Susskind type discretization:

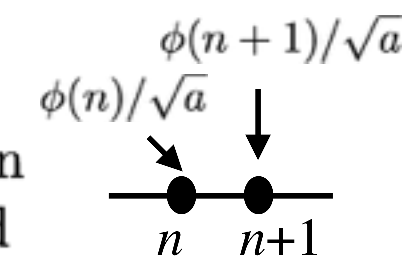
2-component fermion field:

$$\psi(x) = \begin{pmatrix} \psi_{\text{upper}}(x) \\ \psi_{\text{lower}}(x) \end{pmatrix}$$



1-component fermion field:

$$\phi(n)/\sqrt{a} = \begin{cases} \psi_{\text{upper}} & \text{for } n : \text{even} \\ \psi_{\text{lower}} & \text{for } n : \text{odd} \end{cases}$$



- For gauge field, discretization: $E(x) \equiv -\partial_0 A_1(x) \rightarrow gL(n)$

- Hamiltonian on Lattice (still fields theory):

$$\mathcal{H} = -\frac{i}{2a}\sum_{n=1}^{N}\left[\phi^\dagger(n)e^{i\theta}\phi(n+1) - h.s.\right] + m\sum_{n=1}^{N}(-1)^n\phi^\dagger(n)\phi(n) + \frac{g^2 a}{2}\sum_{n=1}^{N}L^2(n)$$

- Jordan-Wigner transformation:

$$\phi(n) = \prod_{l<n}\left[i\sigma_l^z\right]\sigma_n^- \qquad \phi^\dagger(n) = \prod_{l<n}\left[-i\sigma_l^z\right]\sigma_n^+$$

for keeping anti-commutative relation

- Gauss law on Lattice: $L(n+1) - L(n) = \frac{1}{2}\left[(-1)^n + \sigma_n^z\right]$

$\partial_1 E = g\bar{\psi}\gamma_0\psi$ in continuum

- Hamiltonian:

hopping term, mass term, gauge part

$$H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1 + (-1)^n\sigma_n^z\right] + \sum_{n=0}^{N-2}\left[l + \frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z\right)\right]^2$$

zero back ground field

This study

Finite T

1-flavor Schwinger model at finite T

- Chiral symmetry breaking at $T=0$ (via anomaly)

  ⇔ At finite T, the symmetry restoration

- Order parameter : chiral condensate
- Chiral condensate at finite T: $\langle\bar\psi\psi\rangle_\beta = \frac{\mathrm{tr}\left[\bar\psi\psi\ \rho(\beta)\right]}{\mathrm{tr}\left[\rho(\beta)\right]}$ thermal density operator $\rho(\beta)\equiv e^{-\beta H}$
- Thermal density operator $\rho(\beta)$

  ✳ As an imaginary time evolution: $\rho(\beta)\equiv e^{-\beta H} = \underbrace{e^{-\delta H}\cdots e^{-\delta H}}_{N=\beta/\delta}$

  Ex.) For fixed δ, larger N corresponds to lower T

  starting high T limit using Identity operator

  ✳ Each $e^{-\delta H}$: $e^{-\delta H}\approx\left(1-\frac{\delta}{2}H_g\right)e^{-\delta H_{\mathrm{hop}}}\left(1-\frac{\delta}{2}H_g\right)$, $\quad e^{-\delta H_{\mathrm{hop}}}\approx e^{-\frac{\delta}{2}H_e}e^{-\delta H_o}e^{-\frac{\delta}{2}H_e}$

  ✳ systematic error from step size δ

Global optimization

- MPS approximation for $\rho(\beta)$

$$\rho(\beta) \approx \sum_{\substack{i_0,\cdots,i_{N-1} \\ j_0,\cdots,j_{N-1}}} \mathrm{Tr}\left[M[0]^{i_0 j_0} \cdots M[N-1]^{i_{N-1} j_{N-1}}\right] |i_0 \cdots i_{N-1}\rangle \langle j_0 \cdots j_{N-1}|$$

  systematic error from bond dimension D

- How to obtain elements of tensors $M[0],\cdots, M[N\text{-}1]$

  ✳ Global optimization: Updating each elements w/ fixing the others so that the distance between approximated/ unapproximated operators $\epsilon = |\mathcal{O}_{\mathrm{approx}} - \mathcal{O}|$ is minimum

  ✳ In thermal state calculation, ex. $\rho(\beta) \to \rho'(\beta) \approx \rho(\beta) e^{-\frac{\delta}{2} H_g}$ so that the distance $\epsilon = \left|\rho'(\beta) - \rho(\beta) e^{-\frac{\delta}{2} H_g}\right|$ is minimum

Analytic formula

- Chiral condensate at finite T I. Sachs and A. Wipf, arXiv:1005.1822

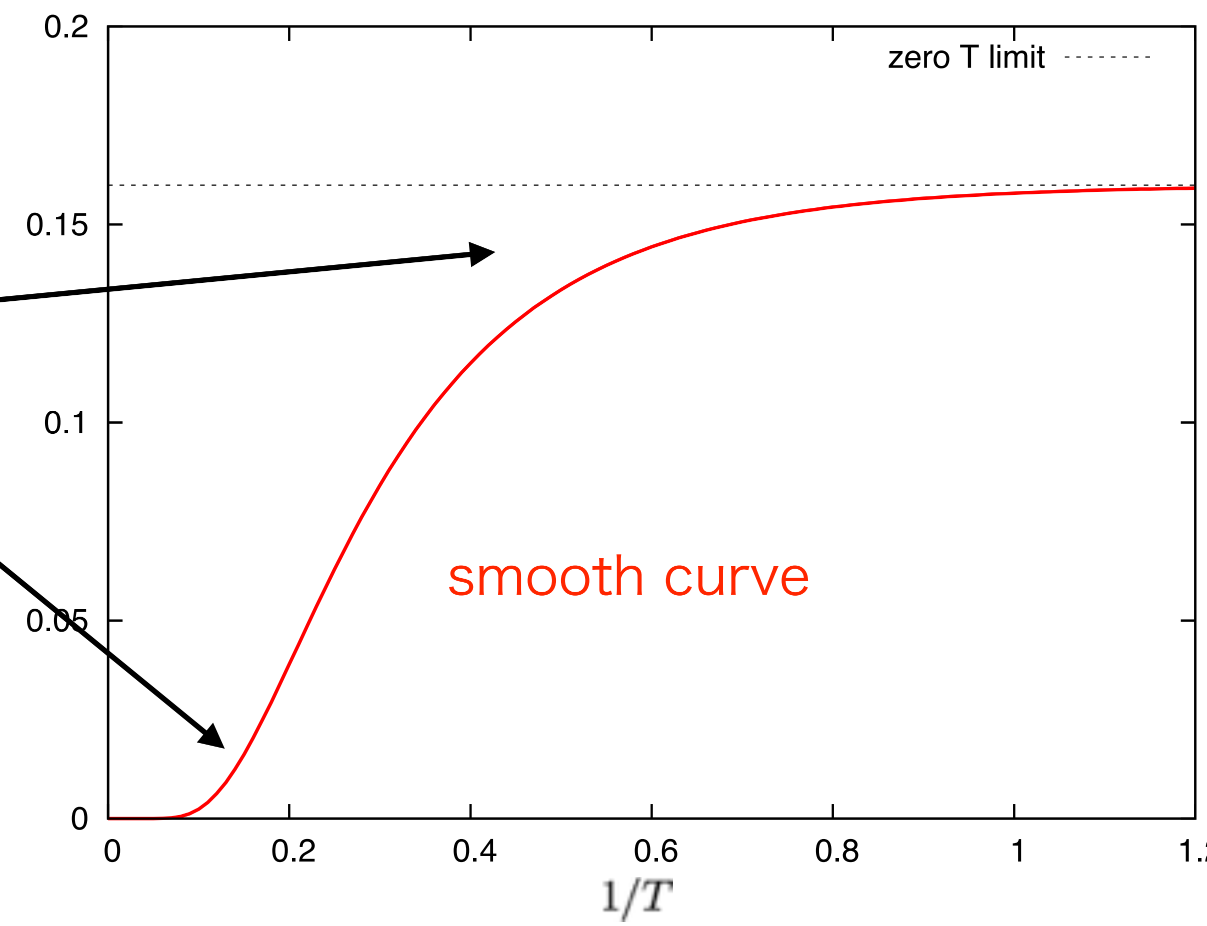
$$\langle \bar{\psi}\psi \rangle = \frac{m_\gamma}{2\pi} e^\gamma e^{2I(\beta m_\gamma)}$$

$$= \begin{cases} \frac{m_\gamma}{2\pi} e^\gamma & \text{for} \quad T \to 0 \\ 2T e^{-\pi T/m_\gamma} & \text{for} \quad T \to \infty \end{cases}$$

where $I(x) = \int_0^\infty \frac{1}{1 - e^{x\cosh(t)}} dt$

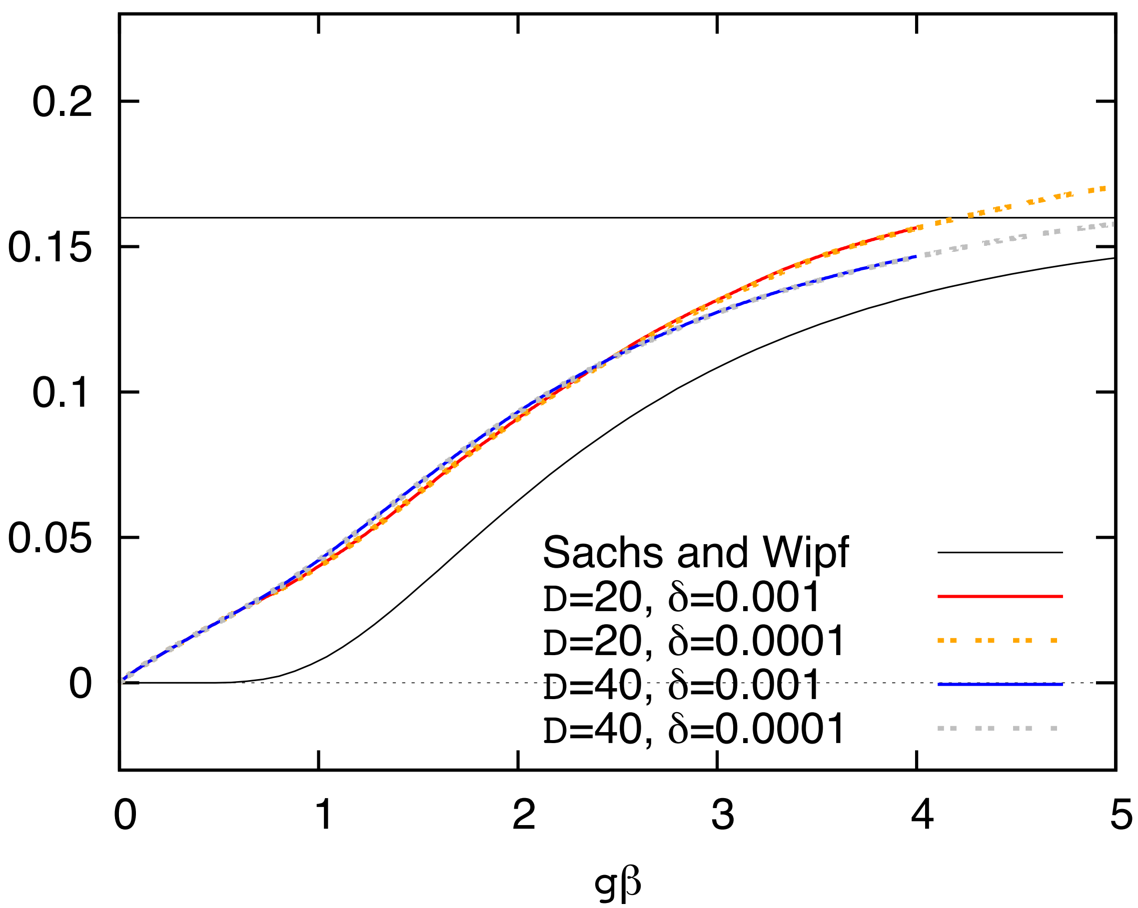
Euler constant $\gamma = 0.57721\ldots$

$m_\gamma = e/\sqrt{\pi}$



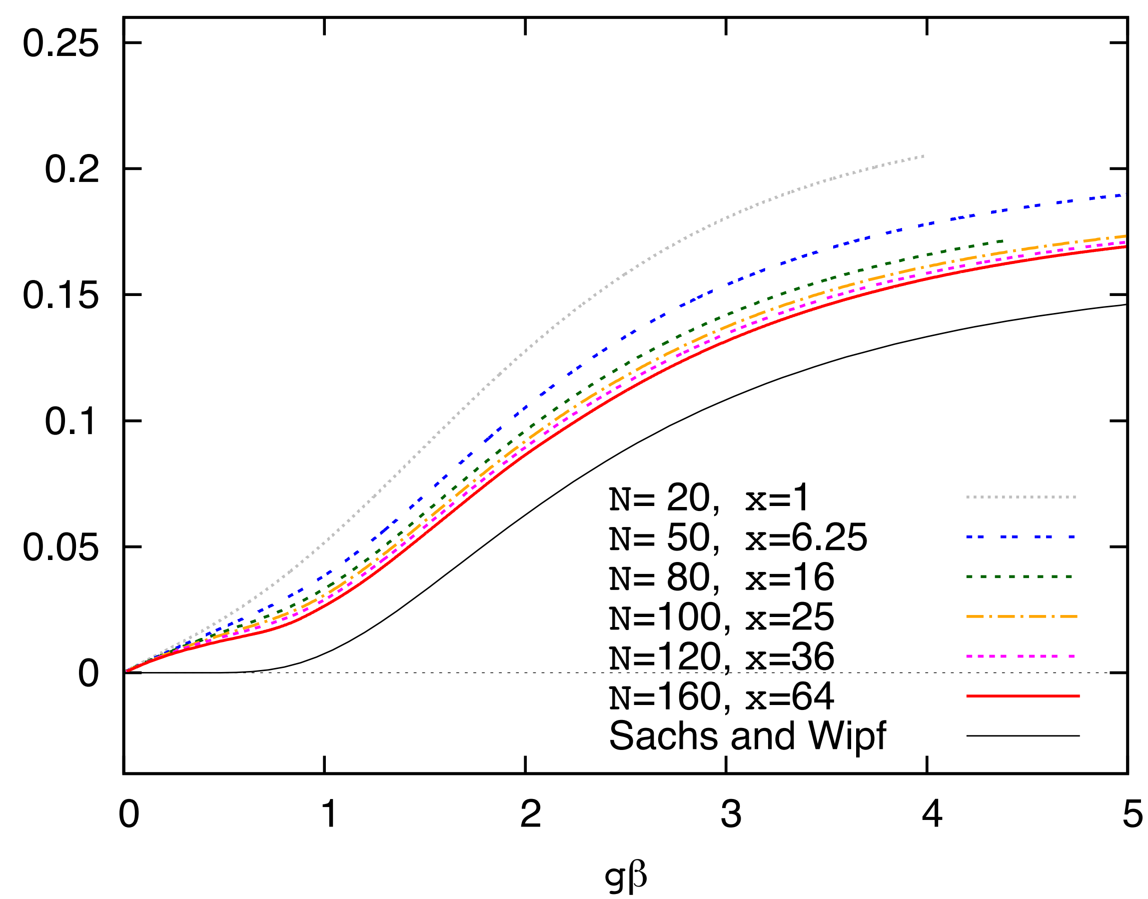
Four systematic errors

<u>From bond dimension D, step size δ</u>



<u>From chain length N, inverse coupling x</u>

continuum limit with fixed physical length $N/\sqrt{x} = 20$



Extrapolations

at $g\beta = 0.4$

preliminary results

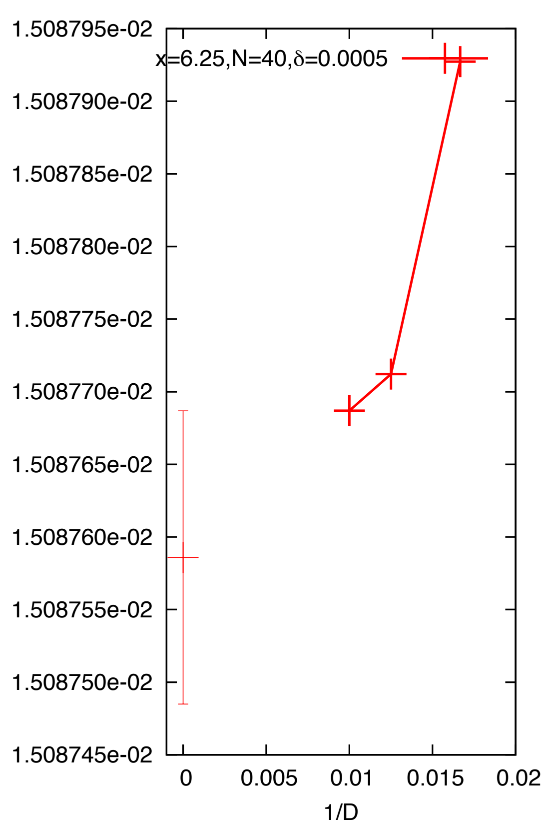
$D \to \infty$ with fixed δ, N, x

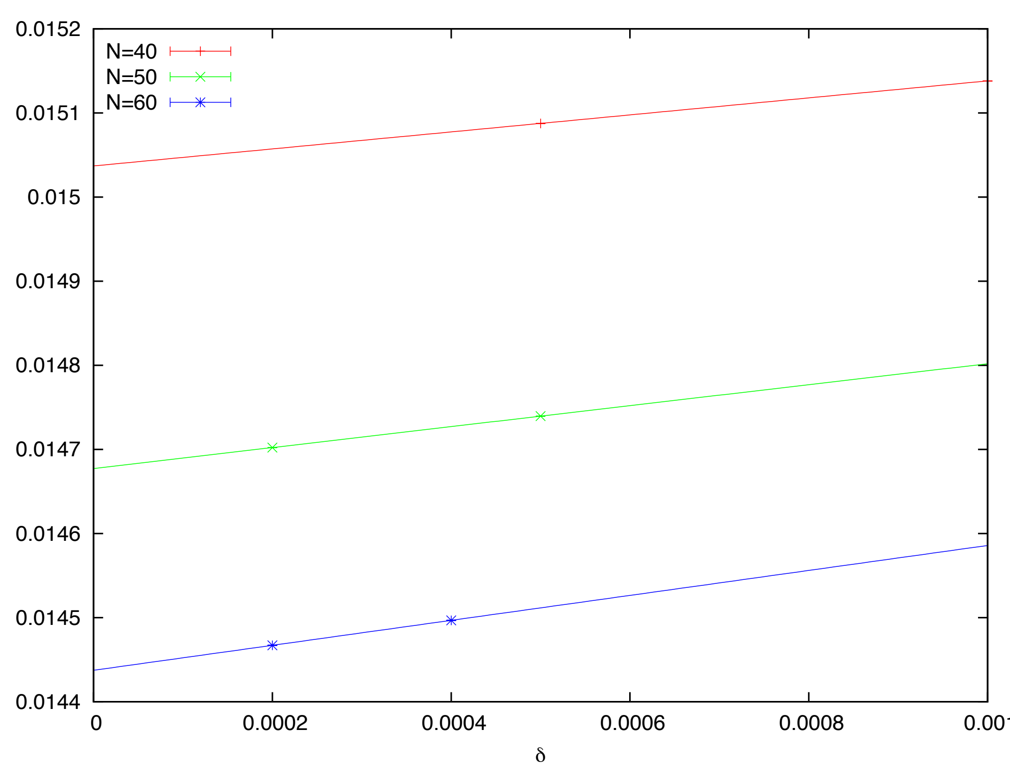
from a mathematical proof, convergence in D

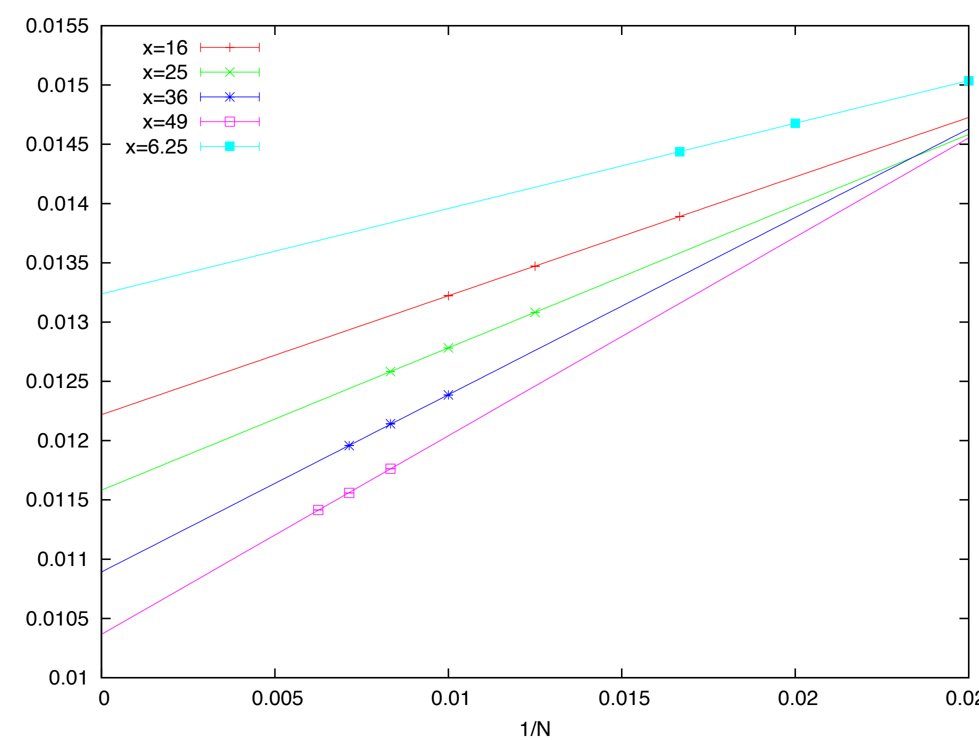
$\delta \to 0$ with fixed N, x

linear convergence in δ theoretically predicted

$N \to \infty$ with fixed x



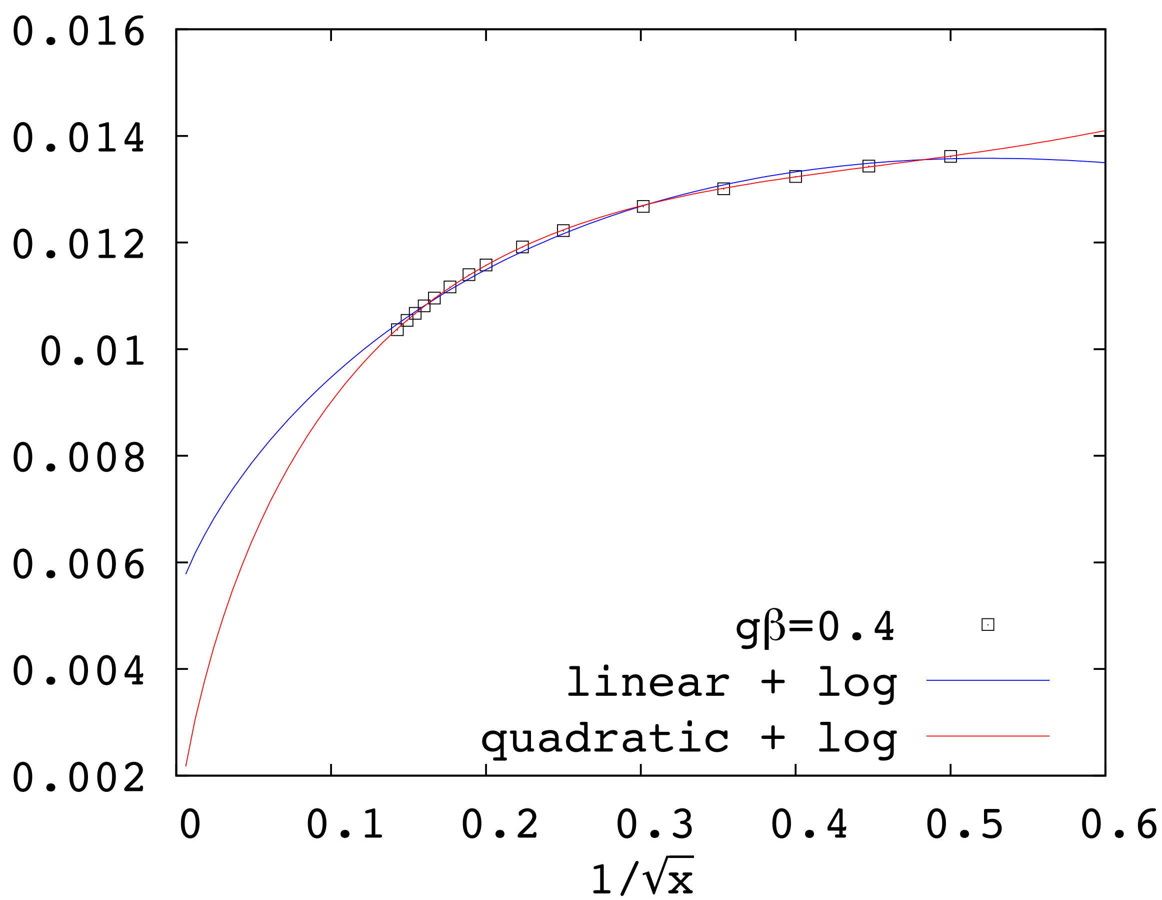


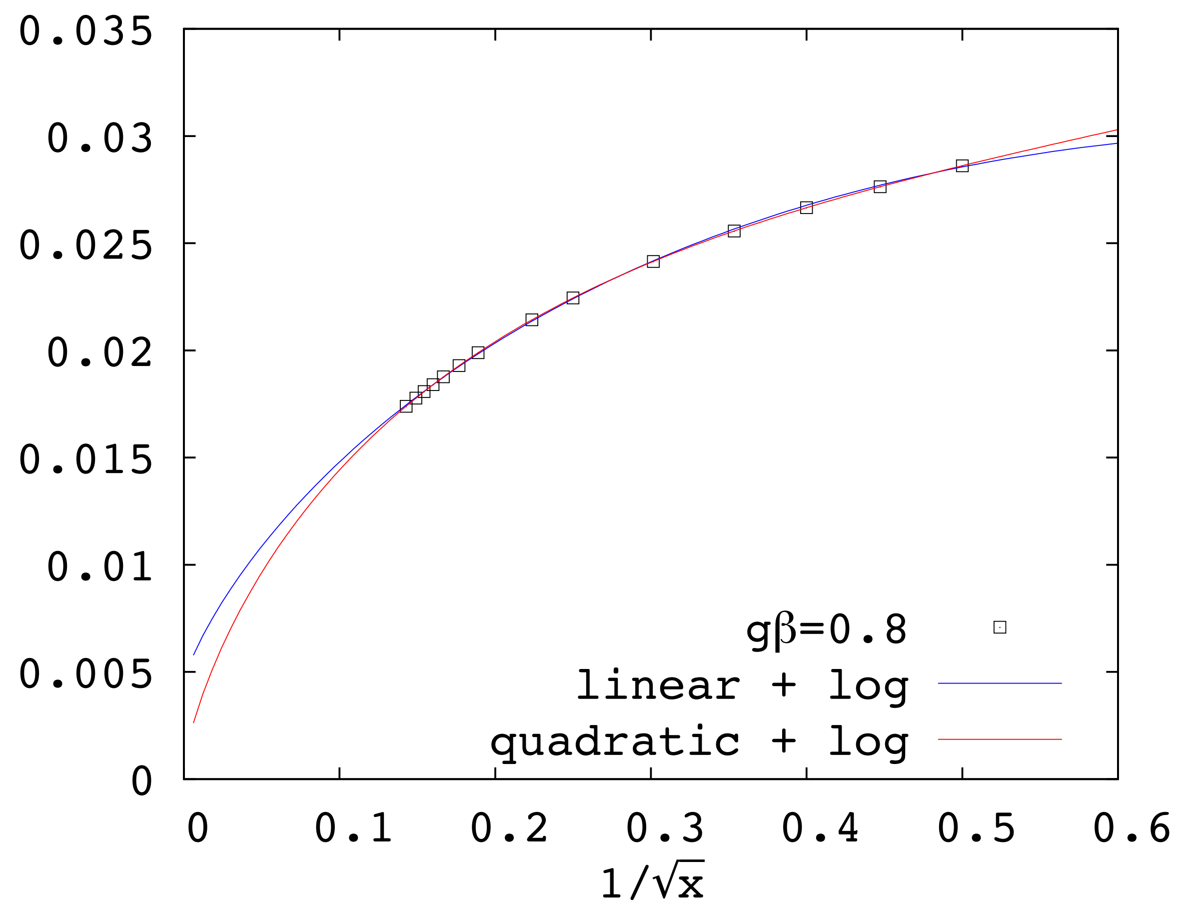


Continuum extrapolation

- continuum limit extrapolation $1/\sqrt{x} \to 0$

preliminary results





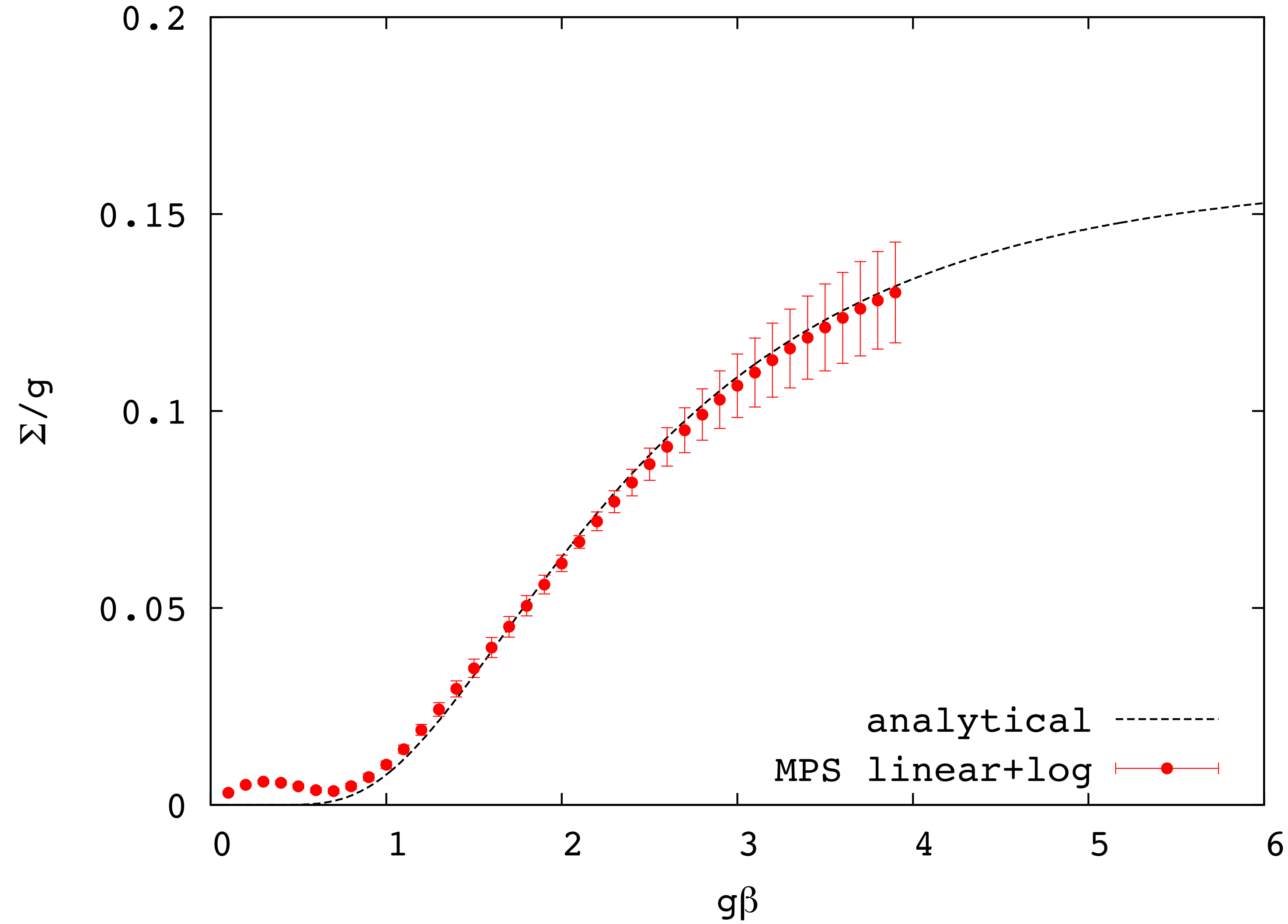
linear + log:

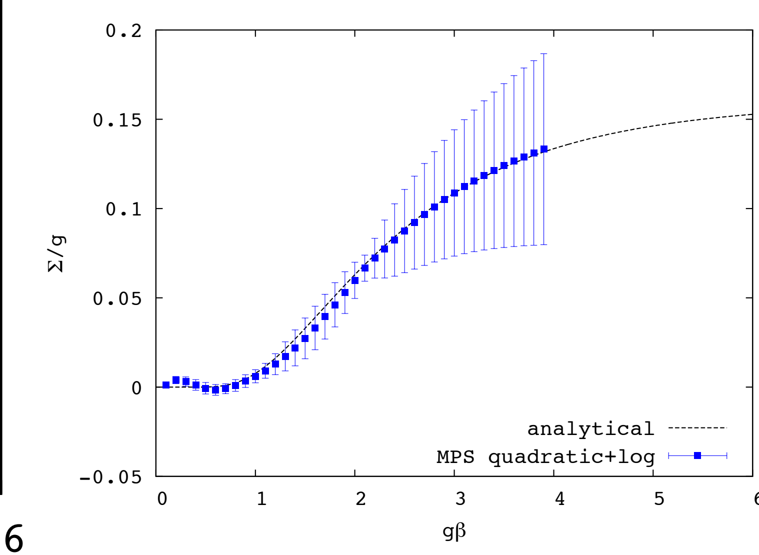
$ay + by\log(y) + c$ where $y = 1/\sqrt{x}$

Chiral condensate at high T

After eliminating those systematic errors …

preliminary results





Summary

- Computing chiral condensate at finite T in Hamiltonian formalism with tensor network methods
- Evaluating dependence of bond dimension/step size
- As a preliminary result, by taking continuum limit, we obtain results consistent with the analytic formula.

I. Sachs and A. Wipf, arXiv:1005.1822

- Future plan: Schwinger model for $N_{\mathrm{f}} = 2$ at finite chemical potential

Towards to QCD

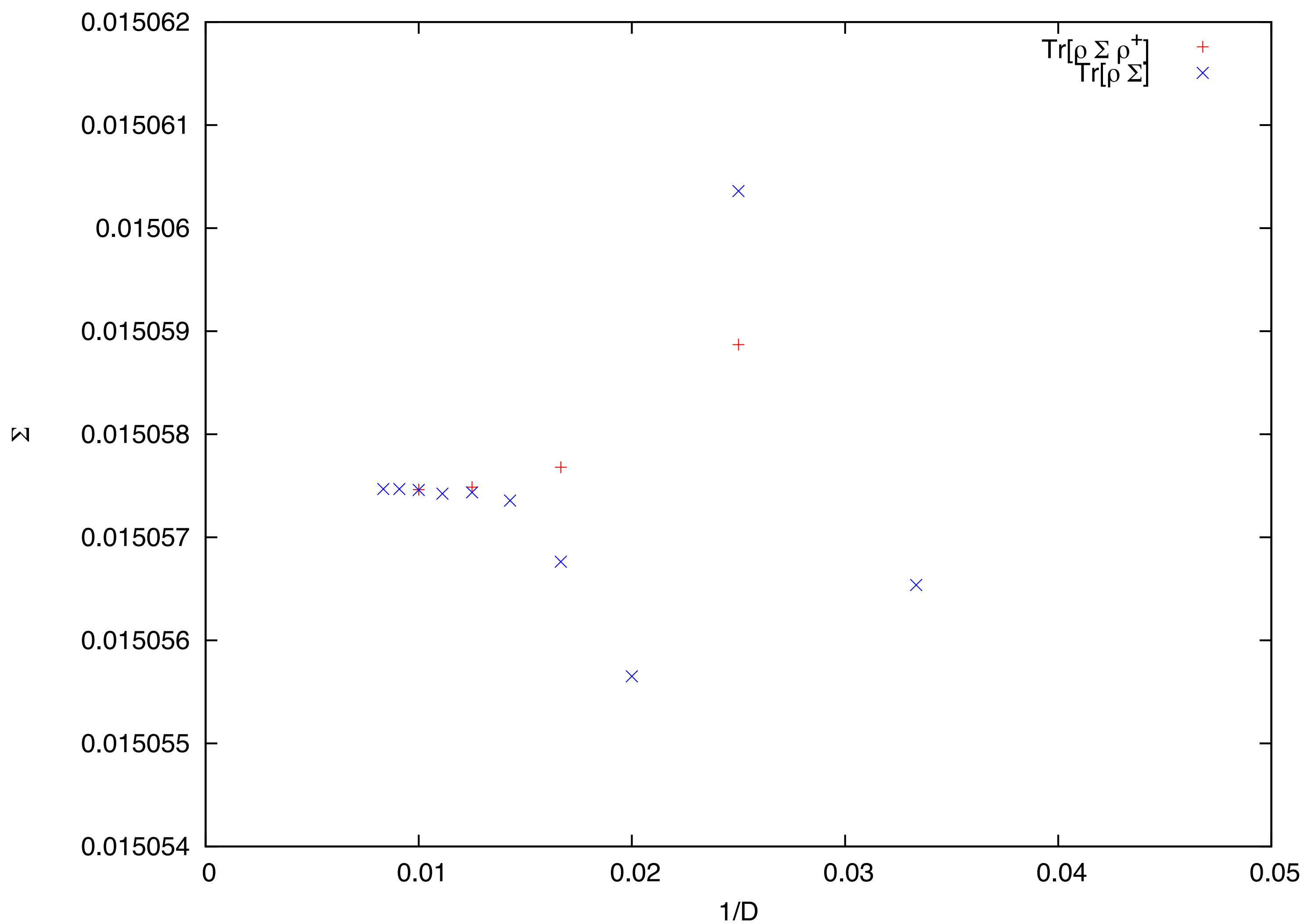
dense tasks left…

- ✳ Hamiltonian formulation of non-Abelian case
- ✳ Technical developments of TN for higher dimensional system

Backup

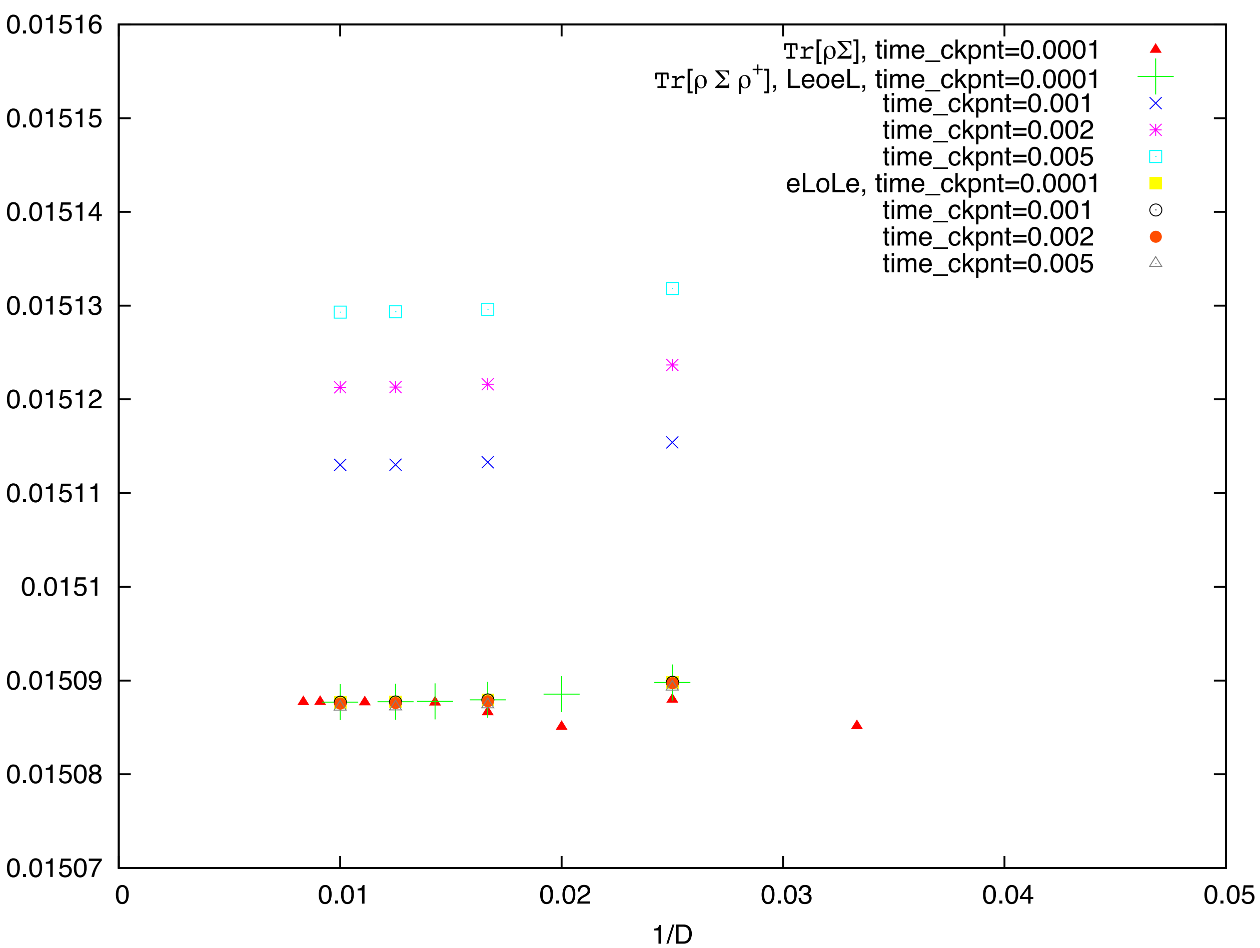
Technical development 1

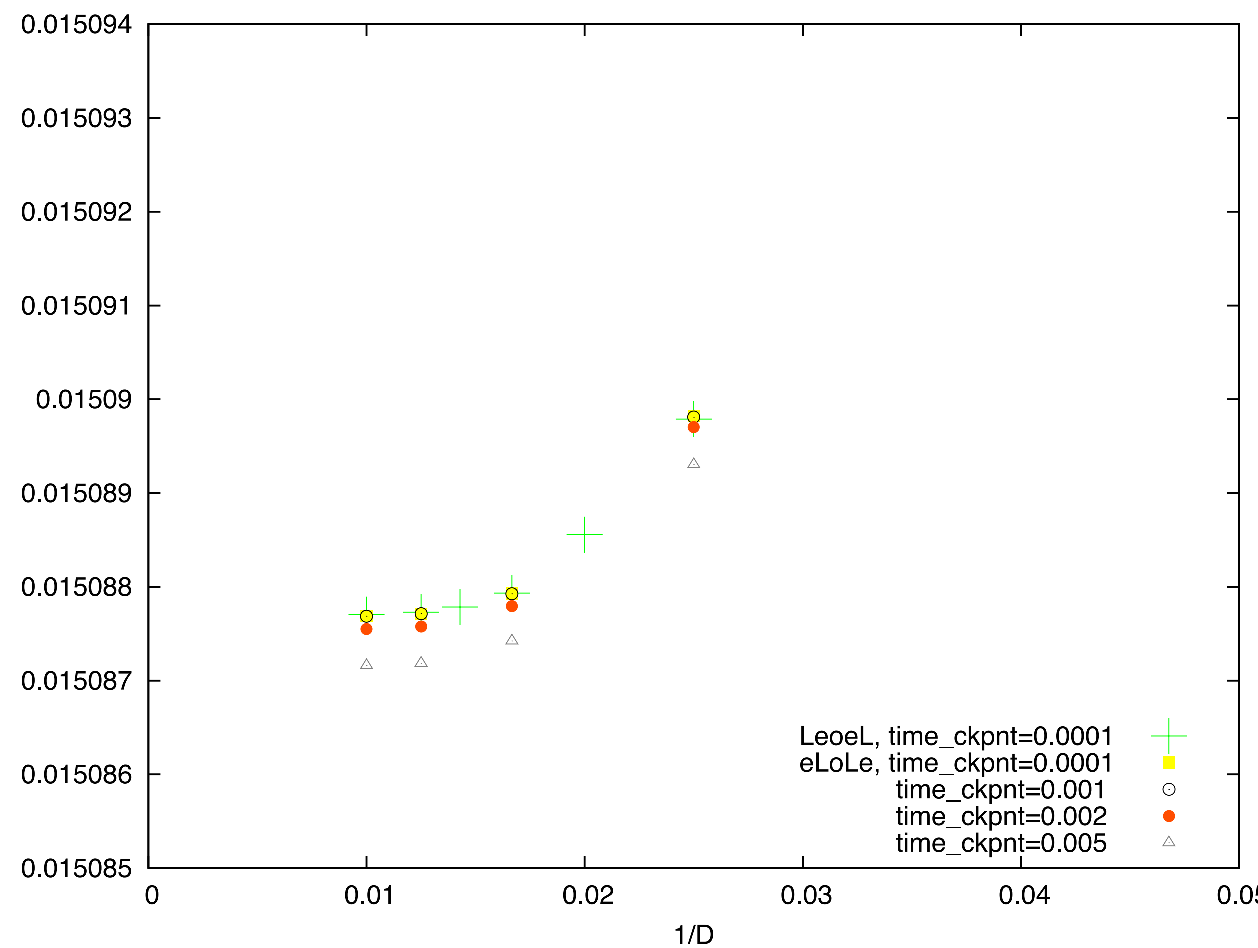
$$\langle \bar{\psi}\psi \rangle_\beta = \frac{\mathrm{tr}\left[\rho(\frac{\beta}{2})^\dagger \, \bar{\psi}\psi \, \rho(\frac{\beta}{2})\right]}{\mathrm{tr}\left[\rho(\frac{\beta}{2})^\dagger \rho(\frac{\beta}{2})\right]}$$



Technical development 2

$$\rho(\delta) \approx e^{-\frac{\delta}{2}H_e}\left(1-\frac{\delta}{2}H_g\right)e^{-\delta H_o}\left(1-\frac{\delta}{2}H_g\right)e^{-\frac{\delta}{2}H_e}$$





Lattice gauge theory (LGT) with TN approach

- Earlier Study: critical behavior of Schwinger model with Density Matrix Renormalization Group

T. Byrnes, et al. PRD.66.013002 (2002)

- Nowadays: various branches

✳ Our previous studies

M. C. Banuls et al JHEP 1311, 158, LAT2013, 332 (2013)

✳ Strong coupling exp.

K. Cichy, et al. Comput.Phys.Commun. 184 1666 (2013)

✳ TN rep. of LGT with continuous group

L. Tagliacozzo, et al. arXiv:1405.4811

✳ LGT with TN on higher dimension

✳ Real time evolution

B. Buyens, et al. arXiv:1312.6654

✳ Quantum link model

D. Banerjee, et al. PRL 109 175302 (2012)
D. Banerjee, et al. PRL 110 125303 (2013)
Rico, et al. PRL112, 201601 (2014)

✳ Tensor Renormalization Group

Y. Shimizu, Y. Kuramashi arXiv:1403.0642 (With Lagrangian)