Yang–Mills gradient flow and the energy–momentum tensor on the lattice

Hiroshi Suzuki

(Kyushu University)

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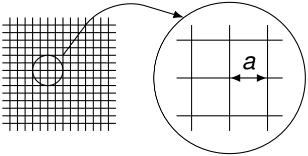
- T. Endo, K. Hieda, D. Miura and H.S., arXiv:1502.01809 [hep-lat]

Lattice gauge theory and the energy–momentum tensor (EMT)

- Lattice gauge theory: the most successful non-perturbative formulation of gauge theory. By discretizing the spacetime. . .



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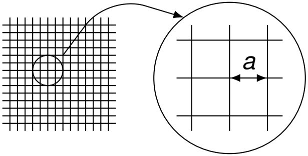

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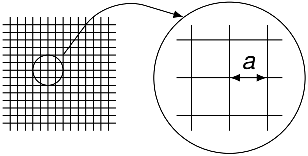

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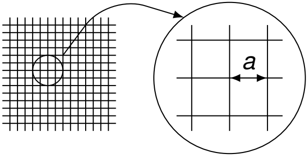

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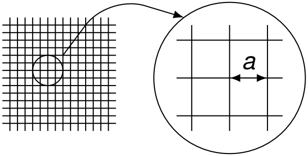

- preserves internal gauge symmetry exactly...
- but incompatible with spacetime symmetries (translation, Poincaré, SUSY, conformal, ...) for $a \neq 0$
- For $a \neq 0$, one cannot define the Noether current associated with the translational invariance, EMT $\{T_{\mu\nu}\}_R(x)$
- Even for the continuum limit $a \to 0$, this is difficult, because EMT is a composite operator which generally contains UV divergences:

$$a \times \frac{1}{a} \stackrel{a\to 0}{\to} 1$$

EMT in lattice gauge theory?

- Is it possible to construct EMT on the lattice, which becomes the correct EMT automatically in the continuum limit $a \to 0$?

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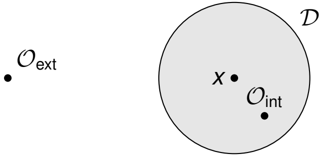
- The correct EMT is characterized by the Ward–Takahashi relation

$$\left\langle \mathcal{O}_{\text{ext}} \int_{\mathcal{D}} \mathrm{d}^D x \, \partial_\mu \left\{T_{\mu\nu}\right\}_R (x) \, \mathcal{O}_{\text{int}} \right\rangle = -\left\langle \mathcal{O}_{\text{ext}} \, \partial_\nu \mathcal{O}_{\text{int}} \right\rangle$$



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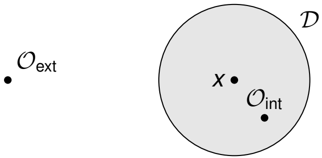

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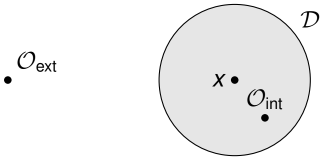

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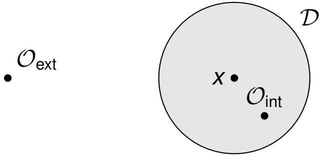

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- If such a construction is possible, we expect wide application: QCD thermodynamics, transport coefficients in gauge theory, momentum/spin structure of baryons, conformal field theory, dilaton physics, ...
- The present work is also an attempt to understand EMT in quantum field theory in the non-perturbative level

EMT on the lattice (Caracciolo et al. (1989–))

- Under the hypercubic symmetry, the operator reproducing the correct EMT of QCD for $a \to 0$ is given by

$$\{T_{\mu\nu}\}_R(x) = \sum_{i=1}^{7} Z_i \mathcal{O}_{i\mu\nu}(x)|_{\text{lattice}} - \text{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x),$$

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- Seven non-universal coefficients Z_i must be determined by lattice perturbation theory or by a non-perturbative method
- Determination of Z_i's using the gradient flow! (Del Debbio's talk)

Yang–Mills gradient flow (Lüscher, (2009–)) (cf. Kaplan's talk)

- Yang–Mills gradient flow is an evolution of the gauge field $A_\mu(x)$ wrt a fictitious time $t \in \mathbb{R}$, according to

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\text{YM}}}{\delta B_\mu(t,x)} = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots ,$$

where the initial value is the conventional gauge field

$$B_\mu(t=0,x) = A_\mu(x)$$

and

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)], \qquad D_\mu = \partial_\mu + [B_\mu, \cdot]$$

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- But, why this can be relevant to lattice EMT???
- The key is the UV finiteness of the gradient flow

Perturbative expansion of the gradient flow

- Yang–Mills gradient flow

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x),$$

where the term with α_0 is introduced to suppress the gauge mode. This can be formally solved as

$$B_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(s,y) \right],$$

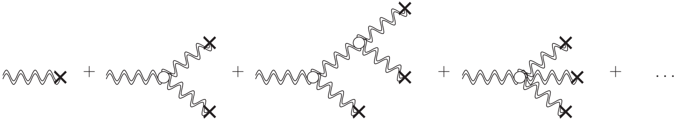
where K is the heat kernel (for $\alpha_0 = 1$)

$$K_t(x)_{\mu\nu} = \delta_{\mu\nu} \int_p e^{ipx} e^{-tp^2} = \delta_{\mu\nu} \frac{1}{(4\pi t)^{D/2}} e^{-\frac{x^2}{4t}},$$

and R denotes non-linear terms

$$R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]]$$

Pictorially, (double lines: K, crosses: A_μ, white circles: R) (cf. Kaplan's talk),



Perturbative expansion of the gradient flow

- Quantum correlation function of the flowed gauge field

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \, B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \, e^{-S_{\mathrm{YM}}}$$

is obtained by taking the quantum expectation value of the initial value $A_\mu(x)$. For example, the contraction of two A_μ's

produces the free propagator of the flowed field (in the Feynman gauge)

$$\left\langle B^a_\mu(t, x) B^b_\nu(s, y) \right\rangle_0 = \delta^{ab} g_0^2 \delta_{\mu\nu} \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2}$$

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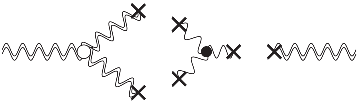
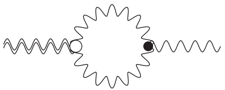
- Similarly, for (black circle: Yang–Mills vertex)



we have the loop flow-line Feynman diagram



backup: Gauge invariance of the gradient flow

- Under the infinitesimal gauge transformation (no $B_t(t,x)$; in 4D sense),

$$B_\mu(t,x) \to B_\mu(t,x) + D_\mu\omega(t,x),$$

the flow equation

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$

changes to

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- Choosing $\omega(t,x)$ as

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α_0 can be changed accordingly

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- Gauge invariant quantity (in 4D sense) is independent of α_0

- Correlation function of the flowed gauge field

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n)\rangle , \qquad t_1 > 0, \ldots, t_n > 0,$$

when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization

UV finiteness of the gradient flow I (Lüscher–Weisz (2011))

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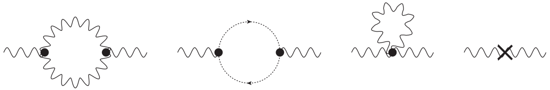
- One-loop corrections (consisting only from Yang–Mills vertices)



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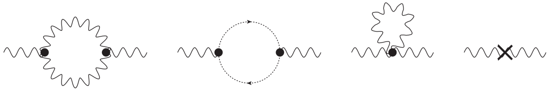

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- Usually, for the two-point function to become UV finite, further wave function renormalization ($A^a_\mu = Z^{1/2} Z_3^{1/2} (A_R)^a_\mu$) is required

UV finiteness of the gradient flow I (Lüscher–Weisz (2011))

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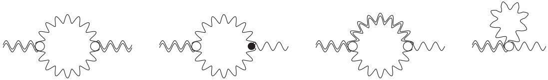

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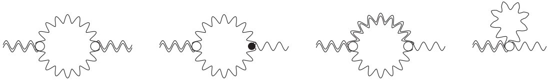

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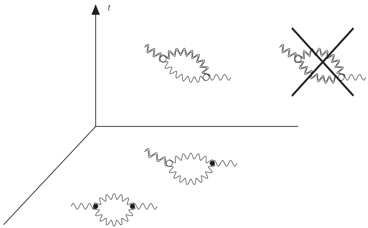

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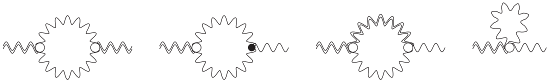

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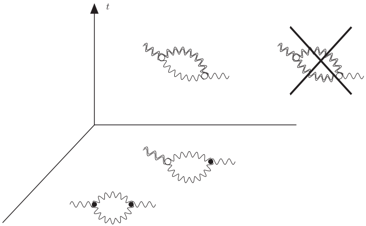

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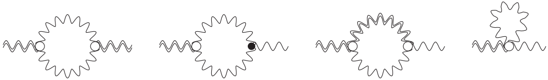

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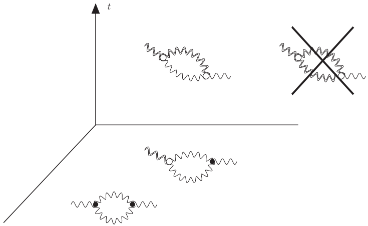


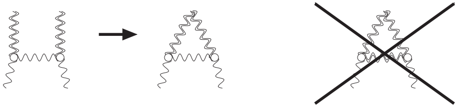
- No bulk ($t>0$) counterterm: because of the Gaussian damping factor $\sim e^{-tp^2}$ in the propagator
- No boundary ($t=0$) counterterm besides Yang–Mills ones: because of a BRS symmetry

- Correlation function of the flow gauge field

$$\langle B_{\mu_1}(t_1, x_1) B_{\mu_2}(t_2, x_2) \cdots B_{\mu_n}(t_n, x_n) \rangle , \qquad t_1 > 0, \ldots, t_n > 0,$$

remains finite even for the equal-point product

$$t_1 \longrightarrow t_2, \qquad x_1 \longrightarrow x_2,$$



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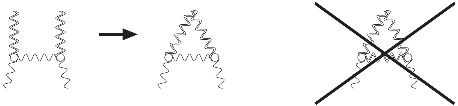

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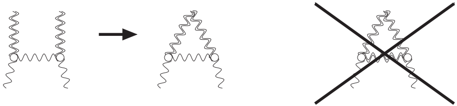

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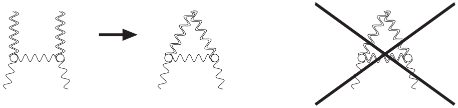

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- Such UV finite quantities must be independent of the regularization

Our strategy for lattice EMT

- We try to bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed composite operator as an intermediate tool

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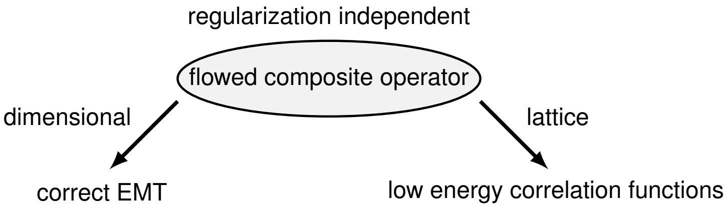
- We try to bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed composite operator as an intermediate tool
- Schematically,



EMT in dimensional regularization

- EMT in dimensional regularization is simple and explicit, because it preserves the translational invariance:

$$\{T_{\mu\nu}\}_R(x)$$
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where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_\rho F^a_{\mu\rho}(x)F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu}\sum_{\rho,\sigma} F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x),$$
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- We want to find a composite operator of the flowed fields which reproduces this combination…

Small flow-time expansion

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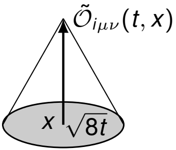

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \mathrm{VEV} \right] + O(t)$$

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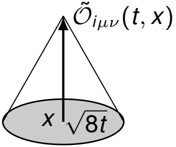

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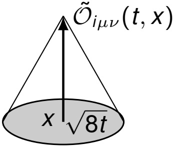

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- So, if we know the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$, the 4D operator in the LHS can be extracted

Renormalization group argument

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- When $\tilde{\mathcal{O}}_{j\mu\nu}(t,x)$ are indep. of renormalized parameters,

$$\left(\mu \frac{\partial}{\partial \mu}\right)_0 \zeta_{ij}(t) = 0,$$

and $\zeta_{ij}(t)$ are indep. of the renormalization scale q, when expressed in terms of running parameters. We may take, for example, $q = 1/\sqrt{8t}$, and

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- For $t \to 0$, $\bar{g}(1/\sqrt{8t}) \to 0$ because of the asymptotic freedom; perturbation theory is justified!

Flow of fermion fields

- A possible choice (Lüscher (2013))

$$\partial_t \chi(t,x) = [\Delta - \alpha_0 \partial_\mu B_\mu(t,x)]\,\chi(t,x), \qquad \chi(t=0,x) = \psi(x),$$
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$$\Delta = D_\mu D_\mu, \qquad D_\mu = \partial_\mu + B_\mu,$$
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- Unfortunately, the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2}\chi(t,x), \qquad \bar{\chi}_R(t,x) = Z_\chi^{1/2}\bar{\chi}(t,x),$$
$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} C_2(R) 3 \frac{1}{\epsilon} + O(g^4),$$

although composite operators of $\chi_R(t,x)$ are UV finite

Ringed fermion fields

- To avoid the complication associated with Z_χ, we introduce

$$\mathring{\chi}(t,x) = \mathcal{C}\frac{\chi(t,x)}{\sqrt{t^2\left\langle\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle}} = \mathcal{C}\frac{\chi_R(t,x)}{\sqrt{t^2\left\langle\bar{\chi}_R(t,x)\overleftrightarrow{\not{D}}\chi_R(t,x)\right\rangle}} = \chi_R(t,x) + O(g^2),$$

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$$\mathcal{C} \equiv \sqrt{\frac{-2\dim(R)N_{\mathrm{f}}}{(4\pi)^2}},$$

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  and similarly for $\bar{\chi}(t,x)$
- Since Z_χ is cancelled out in $\mathring{\chi}(t,x)$, composite operators of $\mathring{\chi}(t,x)$ and $\mathring{\bar{\chi}}(t,x)$ are UV finite

EMT from the gradient flow

- Small flow-time expansion:

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \langle \mathcal{O}_{j\mu\nu}(x) \rangle \right] + O(t)$$

We consider following composite operators of flowed fields:

$$\begin{aligned}
\tilde{\mathcal{O}}_{1\mu\nu}(t,x) &\equiv G^a_{\mu\rho}(t,x) G^a_{\nu\rho}(t,x), \\
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\end{aligned}$$

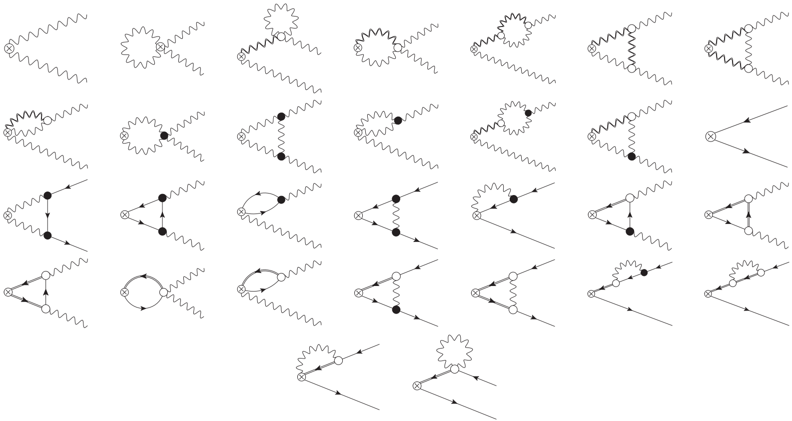
- We compute $\zeta_{ij}(t)$ to the one-loop order and substitute

$$\mathcal{O}_{i\mu\nu}(x) - \langle \mathcal{O}_{i\mu\nu}(x)\rangle = \lim_{t\to 0}\left\{\sum_j \left(\zeta^{-1}\right)_{ij}(t)\left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle \tilde{\mathcal{O}}_{j\mu\nu}(t,x)\right\rangle\right]\right\},$$

  in the expression of EMT in dimensional regularization

Calculation of $\zeta_{ij}(t)$

- To the one-loop order, we have to evaluate following flow-line Feynman diagrams:



Master formula

- Gathering all the above elements, we have

$$\{T_{\mu\nu}\}_R(x) = \lim_{t\to 0}\Bigg\{c_1(t)G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) + \left[c_2(t) - \frac{1}{4}c_1(t)\right]\delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x)$$
$$+ c_3(t)\mathring{\bar{\chi}}(t,x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x)$$
$$+ [c_4(t) - 2c_3(t)]\,\delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x) + c_5'(t)\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x) - \text{VEV}\Bigg\},$$

where (for the MS scheme; $\ln\pi \to \gamma_E - 2\ln 2$ for $\overline{\text{MS}}$)

$$c_1(t) = \frac{1}{\bar{g}(1/\sqrt{8t})^2} - b_0\ln\pi - \frac{7}{8}\frac{1}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{12}{7}T(R)N_{\mathrm{f}}\right],$$

$$c_2(t) = \frac{1}{8}\frac{1}{(4\pi)^2}\left[\frac{11}{3}C_2(G) + \frac{11}{3}T(R)N_{\mathrm{f}}\right],$$

$$c_3(t) = \frac{1}{4}\left\{1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(R)\left[\frac{3}{2} + \ln(432)\right]\right\},$$

$$c_4(t) = \frac{1}{8}d_0\bar{g}(1/\sqrt{8t})^2,$$

$$c_5'(t) = -\bar{m}(1/\sqrt{8t})\left\{1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(R)\left[3\ln\pi + \frac{7}{2} + \ln(432)\right]\right\},$$

Master formula

- and

$$b_0 = \frac{1}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}T(R)N_{\mathrm{f}}\right], \qquad d_0 = \frac{1}{(4\pi)^2}6C_2(R)$$

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- Correlation functions of the RHS of the master formula can be computed non-perturbatively by using lattice regularization

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Application to thermodynamics of $SU(3)$ pure Yang–Mills theory

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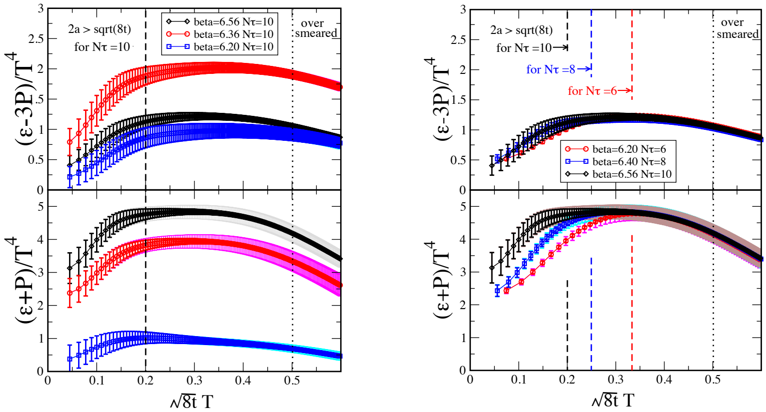
- Experiment setting
  - Wilson plaquette action
  - $N_s^3 \times N_\tau = 32^3 \times (6, 8, 10, 32)$, $\beta = 5.89$–6.56, ~ 300 configurations
  - Wilson flow: 4th order Runge–Kutta with $\epsilon/a^2 = 0.025$
  - Scale setting: $\beta \leftrightarrow a\Lambda_{\overline{\text{MS}}}$ from ALPHA Collaboration, aT_c at $\beta = 6.20$ from Boyd et al.
  - 4-loop running coupling in the $\overline{\text{MS}}$ scheme
  - Clover field strength $G_{\mu\nu}^a(x)$

- Thermal expectation values versus the flow time $\sqrt{8t}$



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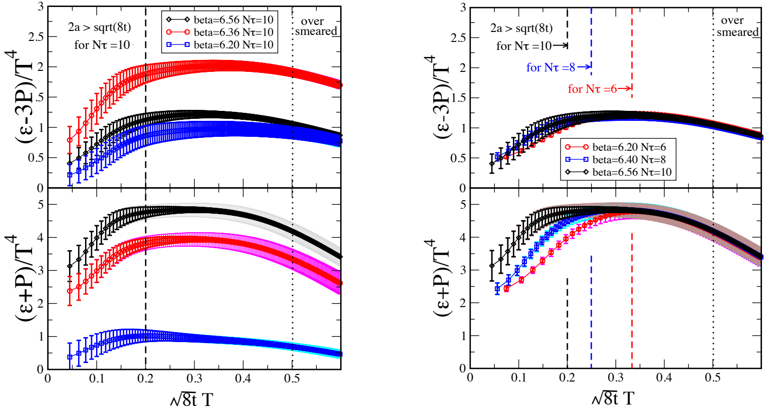
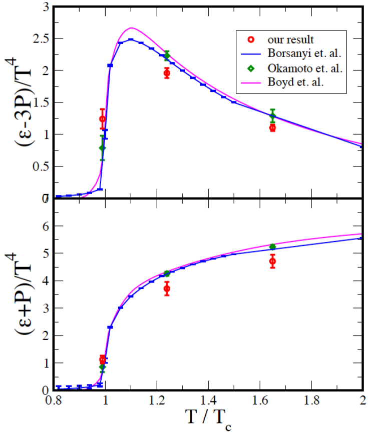


- We observe stable behavior for $2a < \sqrt{8t} < 1/(2T)$) which indicates (!!!) the $t \to 0$ limit

- Continuum limit (from values at $\sqrt{8t}T = 0.40$)



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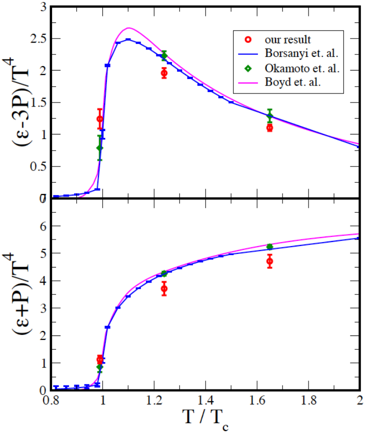

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- That our simple method produces results being consistent with past comprehensive studies (within 2σ) indicates that our reasoning is correct. This finding encouraged us very much!

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- We developed a formula that relates a correctly-normalized conserved EMT and composite operators defined through the gradient flow:

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