Extracting the electro-magnetic pion form factor from QCD in a finite volume

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Plan of Talk

Introduction

- Finite Volume Effects(FVE)
- Chiral Perturbation Theory

Two Point Function (Example)

- Three manipulations to reduce FVE

Three Point Function (Main Target)

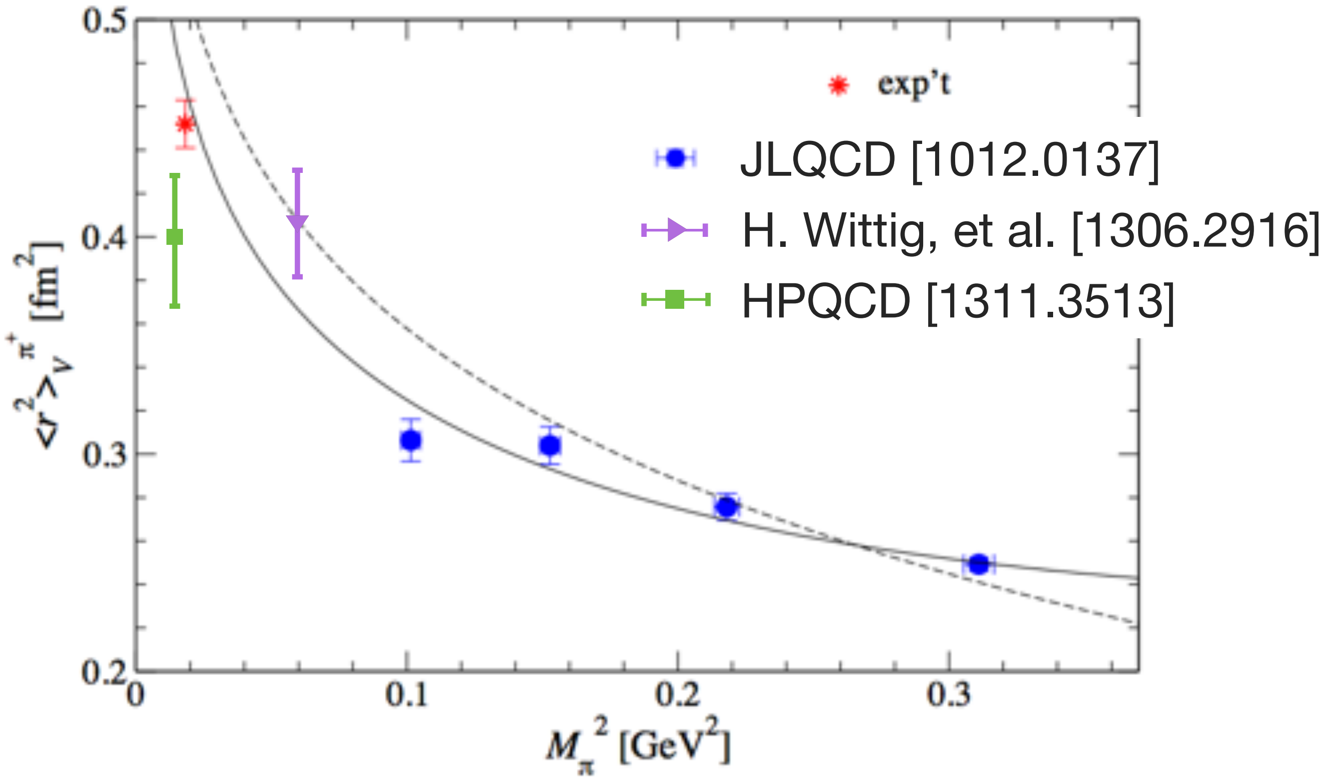
- Pion charge radius and three point function
- Result

Summary


Introduction

Motivation

Charge radius vs Pion mass



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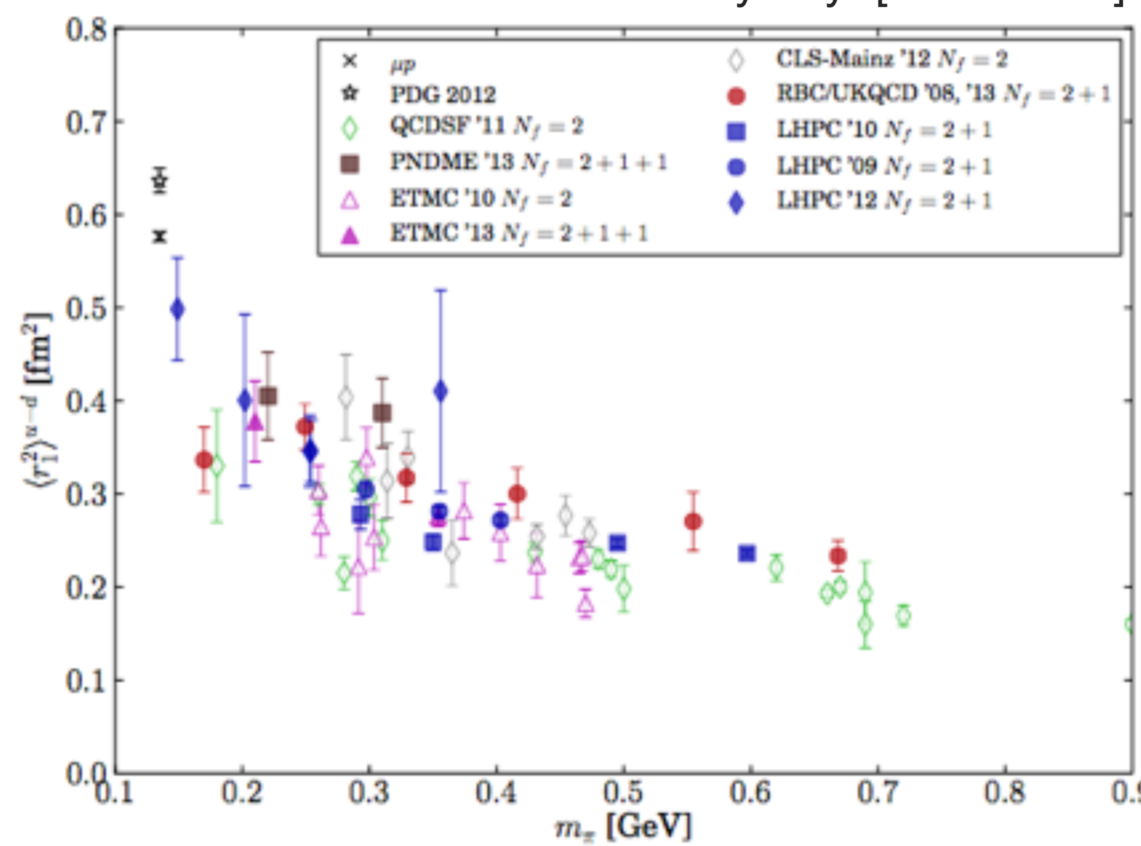


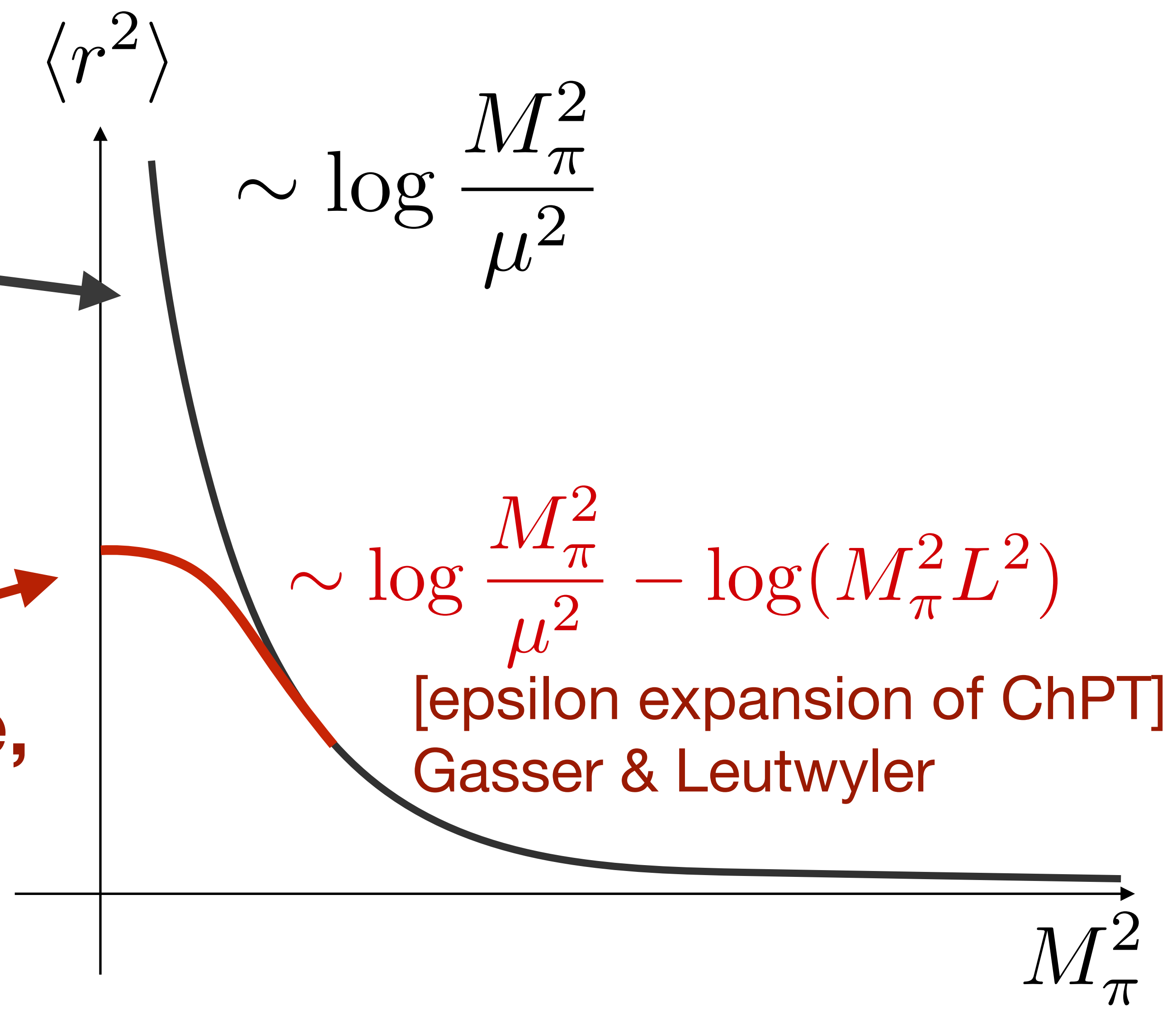
Lattice data tend to be
lower than the experimental values.

Motivation

Charge radius vs Pion mass

It is known that
the pion charge radius shows
a logarithmic divergence
as M_π goes to zero.

In lattice QCD in a finite volume,
infra-red cut-off makes it finite.



Lattice simulations equal what ?

Lattice simulations ≠ Real Physics

Lattice simulations equal what ?

Lattice simulations

= Real Physics + systematic effects

lattice spacing

chiral limit

Finite Volume Effects(FVE)

Lattice simulations equal what ?

Real Physics = Lattice simulations - FVE

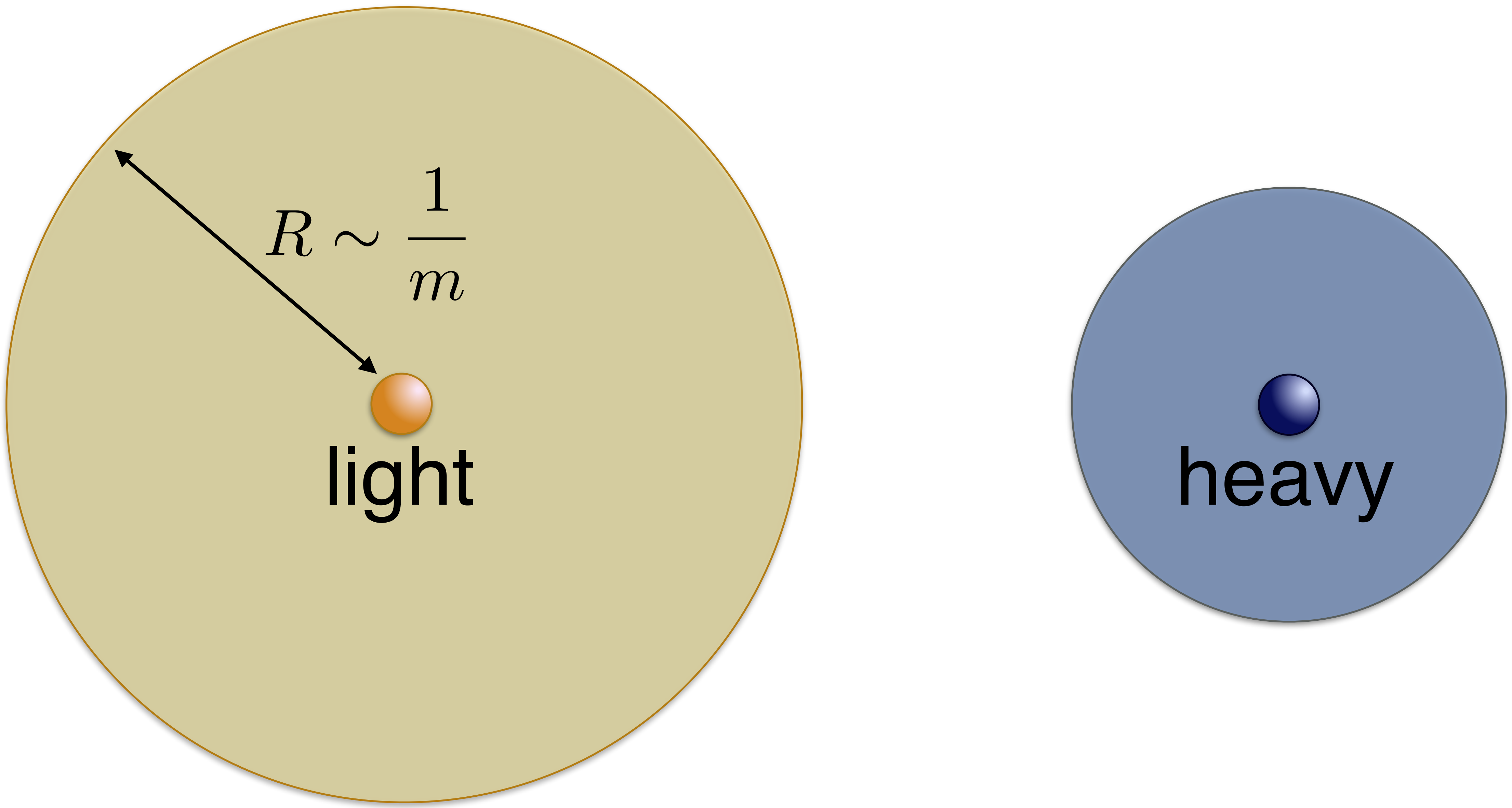
To obtain information of real physics
from lattice simulations,
we must remove or reduce FVE.

FVE is low energy effect.

⇒ We can study FVE
by a low energy effective theory.

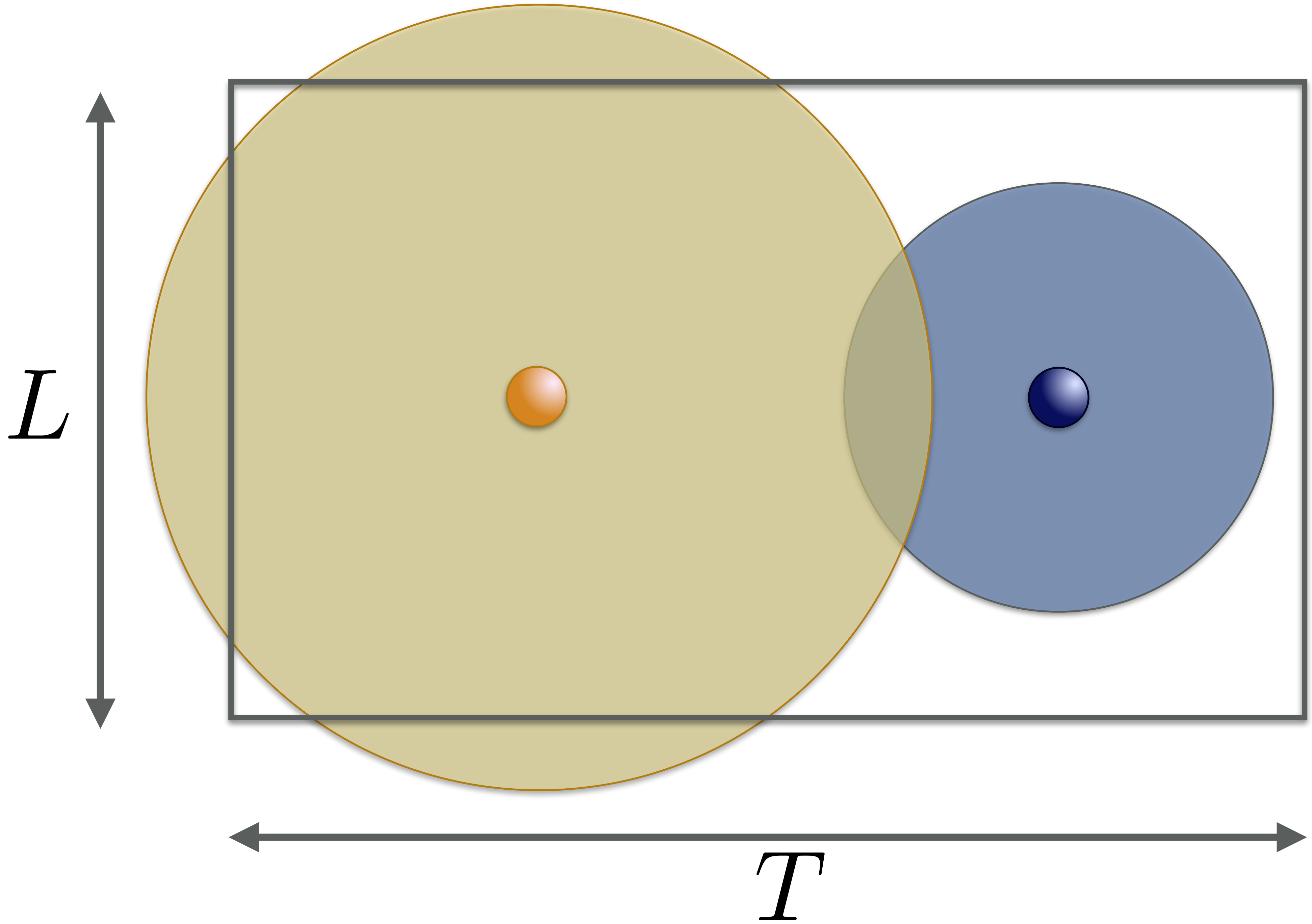
Then, we may find a way to reduce FVE.

Physics of FVE

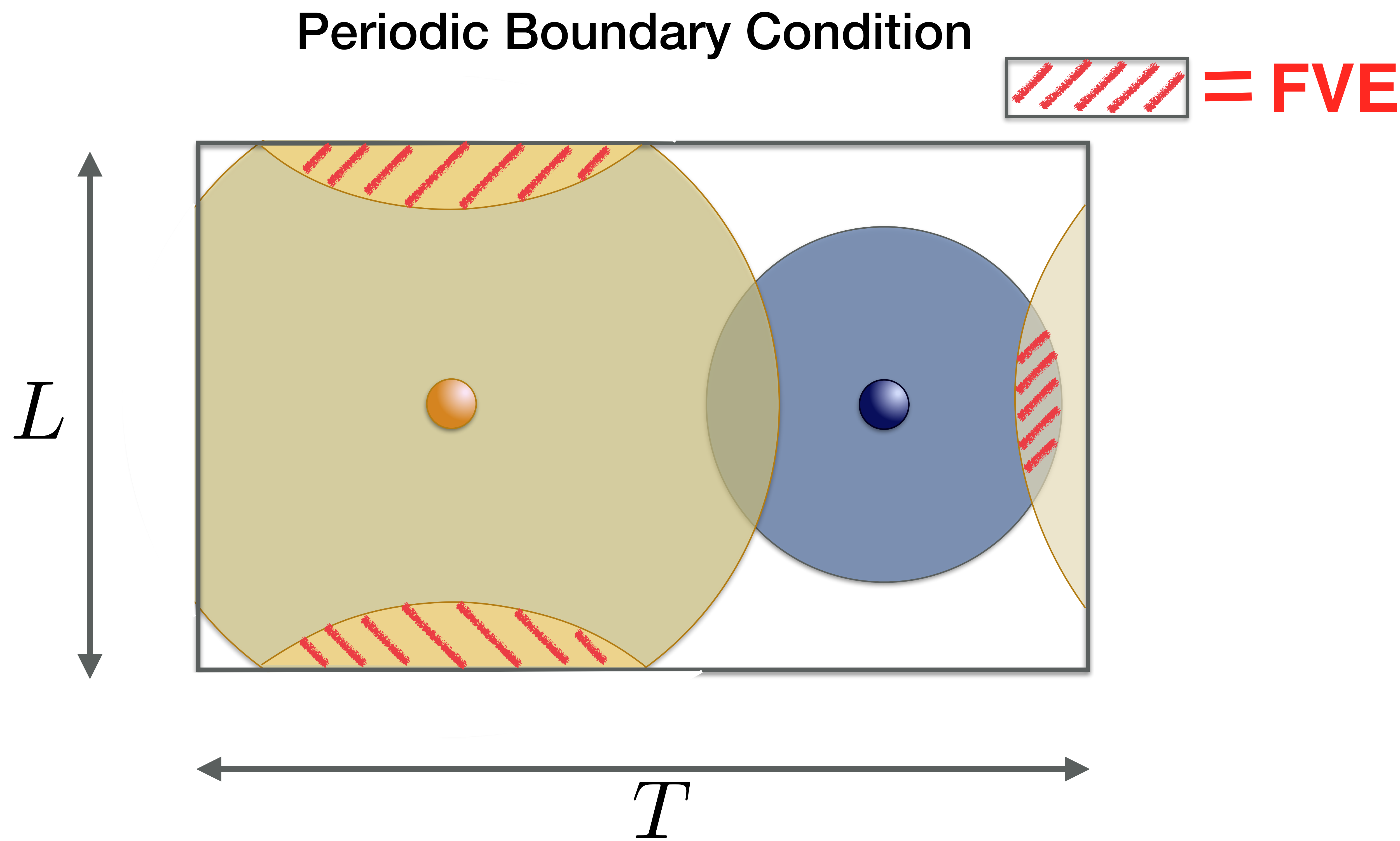


Physics of FVE

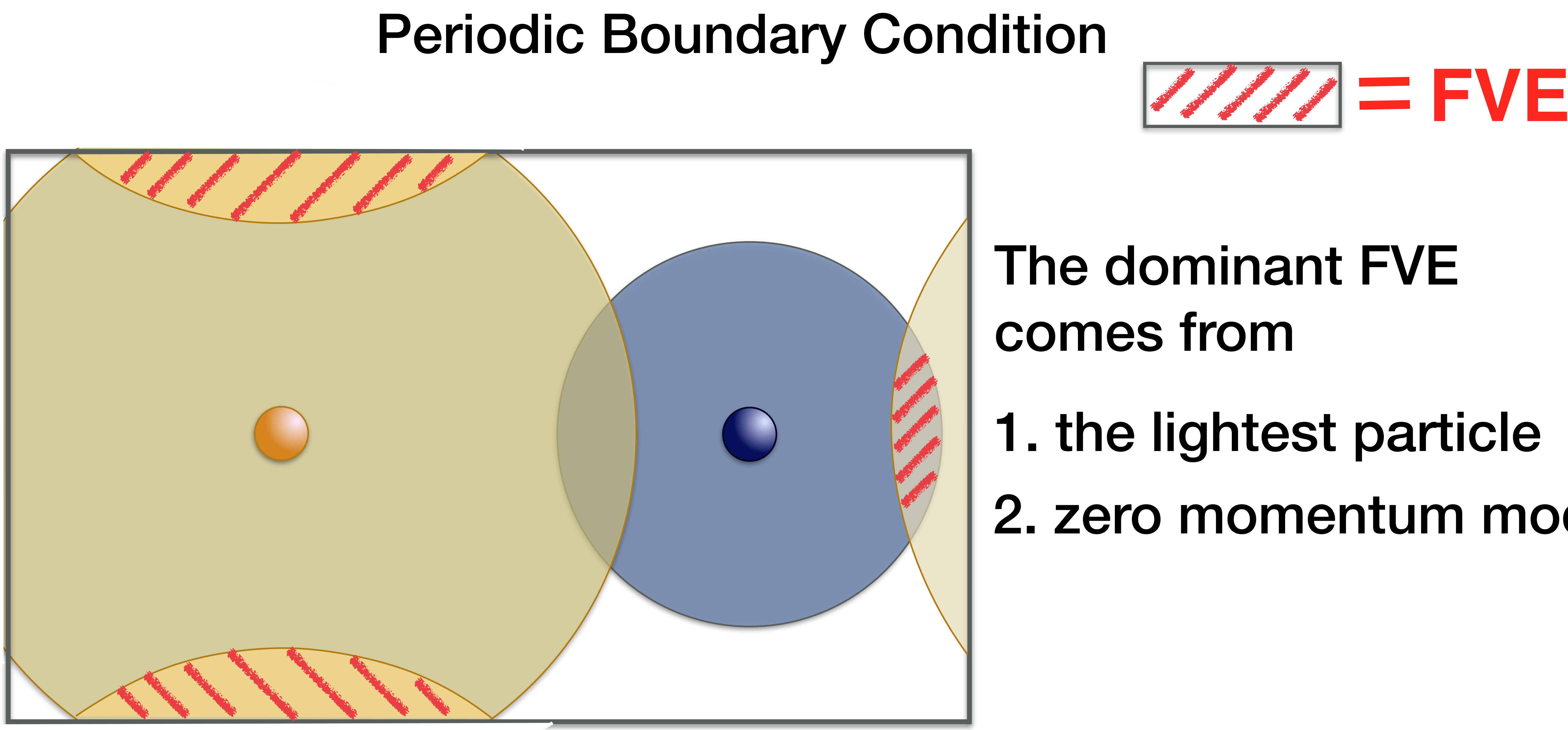
Periodic Boundary Condition



Physics of FVE



Physics of FVE



The dominant FVE comes from

1. the lightest particle
2. zero momentum mode

FVE = Pion physics

1. the lightest particle
2. zero momentum mode

In QCD, the lightest particle is the Pion.

Sensitivity to a finite volume:

$$e^{-m_\pi L} \sim 0.3 \qquad e^{-m_K L} \sim 0.007 \qquad L = 2\,\mathrm{fm}$$

FVE can be analyzed by considering the Pion physics.

Pion physics is described by
Chiral Perturbation Theory (ChPT).

FVE = Pion physics = ChPT

Epsilon regime

The worst case,
the so-called epsilon regime: $m_\pi \ll \frac{1}{L}$

$$\text{FVE} \sim \mathcal{O}(100\,\%)$$

Can we reduce this quietly large FVE ?

YES, we can with our method.

Chiral Perturbation Theory(ChPT)

Epsilon expansion of ChPT J. Gasser and H. Leutwyler 1984

$$\mathcal{L}_{\mathrm{ChPT}} = \frac{F^2}{4}\mathrm{Tr}[(\partial_\mu U(x))^\dagger(\partial^\mu U(x))] - \frac{\Sigma}{2}\mathrm{Tr}[\mathcal{M}^\dagger U(x) + U(x)^\dagger \mathcal{M}] + \cdots$$

$$U(x) = U_0 \exp\left(\frac{i\sqrt{2}}{F}\xi(x)\right) \qquad U(x) \in SU(N_f)$$

zero momentum mode

epsilon expansion

$U_0 \sim \mathcal{O}(1)$: non-perturbative

$\partial_\mu \sim \frac{1}{V^{1/4}} \sim m_\pi^{1/2} \sim m^{1/4} \sim \xi(x) \sim \mathcal{O}(\epsilon)$: perturbative

Summary of our work

What we done

We study FVE in the epsilon regime with ChPT
and focus on how to reduce the zero-mode's contributions.

As a result

- FVE can be reduced by three manipulations
    1. Non-zero momentum insertion
    2. Some subtraction
    3. Taking appropriate ratio

  even in the epsilon regime where FVE ~O(100%)

- Our analysis is also useful for p regime.


Two point function

Two point function

$$\langle P(x)P(0)\rangle = \mathcal{A} + \mathcal{B}\frac{1}{V}\sum_{q\neq 0}\frac{e^{iqx}}{q^2} + \cdots$$

zero mode contributions

$\mathcal{A}$ — Constant (x-independent part)

Non-zero momentum insertion
Subtraction at different time-slice

$\mathcal{B}$ — Overall factor on x-dependent part

Taking a ratio

Remove zero-mode contributions

Non-zero momentum insertion

$$P(x_0 : \mathbf{p}) \equiv \int d^3x \, e^{-i\mathbf{p}\cdot\mathbf{x}} P(x) \qquad \mathbf{p} \neq \mathbf{0}$$

$$\langle P(x_0 : \mathbf{p}) P(0) \rangle = \mathcal{A}\, \delta_{\mathbf{p},\mathbf{0}} + \mathcal{B} \frac{1}{T} \sum_{q^0} \frac{e^{iq^0 x^0}}{(q^0)^2 + \mathbf{p}^2} + \cdots$$

$$= 0 + \mathcal{B} \frac{\cosh(E(\mathbf{p})(x_0 - T/2))}{2E(\mathbf{p}) \sinh(E(\mathbf{p})T/2)} + \cdots$$

Constant part $\mathcal{A}$ has been removed.

Remove zero-mode contributions

Subtraction at different time-slice

$$\Delta P(x_0 : \mathbf{0}) \equiv P(x_0 : \mathbf{0}) - P(x_0^{\mathrm{ref}} : \mathbf{0})$$

$$\langle \Delta P(x_0 : \mathbf{0}) P(0) \rangle = \mathcal{A} + \mathcal{B} \frac{1}{V} \sum_{q \neq 0} \frac{e^{iqx}}{q^2} - (x_0 \to x_0^{\mathrm{ref}})$$

$$= 0 + \mathcal{B} T \left[h_1(x_0/T) - h_1(x_0^{\mathrm{ref}}/T) \right]$$

$$h_1(\tau) \equiv \tfrac{1}{2} \left(\tau - \tfrac{1}{2} \right)^2 - \tfrac{1}{24}$$

Even for zero-momentum, $\mathcal{A}$ can be removed.

Remove zero-mode contributions

Constant (x-independent) $\mathcal{A}$ have been removed.

$$\langle P(x_0:\mathbf{p})P(0)\rangle = \mathcal{B}\frac{\cosh(E(\mathbf{p})(x_0-T/2))}{2E(\mathbf{p})\sinh(E(\mathbf{p})T/2)}$$

$$\langle \Delta P(x_0:\mathbf{0})P(0)\rangle = \mathcal{B}T[h_1(x_0/T)-h_1(x_0^{\mathrm{ref}}/T)]$$

Taking a ratio of these

$$\frac{\langle P(x_0:\mathbf{p})P(0)\rangle}{\langle \Delta P(x_0:\mathbf{0})P(0)\rangle} = \frac{\frac{\cosh(E(\mathbf{p})(x_0-T/2))}{2E(\mathbf{p})\sinh(E(\mathbf{p})T/2)}}{T[h_1(x_0/T)-h_1(x_0^{\mathrm{ref}}/T)]}$$

$\mathcal{A}$, $\mathcal{B}$ have been removed!!

Remove zero-mode contributions

Three manipulations

- **Inserting non-zero momentum**

$$\langle P(x_0 : \mathbf{p}) P(0) \rangle = \mathcal{B} \frac{\cosh(E(\mathbf{p})(x_0 - T/2))}{2E(\mathbf{p}) \sinh(E(\mathbf{p})T/2)}$$

- **Subtracting by different time-slice one**

$$\langle \Delta P(x_0 : \mathbf{0}) P(0) \rangle = \mathcal{B} T [h_1(x_0/T) - h_1(x_0^{\mathrm{ref}}/T)]$$

- **Taking a ratio of these**

$$\frac{\langle P(x_0 : \mathbf{p}) P(0) \rangle}{\langle \Delta P(x_0 : \mathbf{0}) P(0) \rangle} = \frac{\frac{\cosh(E(\mathbf{p})(x_0 - T/2))}{2E(\mathbf{p}) \sinh(E(\mathbf{p})T/2)}}{T[h_1(x_0/T) - h_1(x_0^{\mathrm{ref}}/T)]}$$

The dominant FVE have been removed !


Three point function

Pion Form factor

We calculate Pseudoscalar-Vector-Pseudoscalar three point function <PVP> which is related to the Pion charge darius:

$$\langle r^2 \rangle = 6 \left. \frac{dF_V(q^2)}{dq^2} \right|_{q^2=0}$$

$$\langle \pi^+(p_2) | J_\mu^{EM}(x) | \pi^+(p_1) \rangle = (p_1 + p_2)_\mu F_V(t)$$

$$\int d^4x e^{ip_2 x} \int d^4z e^{-ip_1 z} \langle P^{12}(x) J_\mu^{EM}(y) P^{21}(z) \rangle$$

$$= \frac{\langle 0 | P^{12}(0) | \pi^+(p_2) \rangle \langle \pi^+(p_1) | P^{21}(0) | 0 \rangle}{(p_1^2 + m_\pi^2)(p_2^2 + m_\pi^2)} \langle \pi^+(p_2) | J_\mu^{EM}(x) | \pi^+(p_1) \rangle$$

Three point function

Pseudoscalar-Vector-Pseudoscalar three point function

$$\langle P^{12}(x_0 : -\mathbf{p}_f) V_0^{ii}(y_0 : \mathbf{q}) P^{21}(z_0 : \mathbf{p}_i)\rangle$$
$$= 0 + \frac{L^3\Sigma^2}{4F} \langle \mathcal{C}(U_0)\rangle_{U_0} \delta_{\mathbf{q},\mathbf{p}_f-\mathbf{p}_i} F_V(q_0, \mathbf{q})$$
$$\times [iE(\mathbf{p}_i)c(\mathbf{p}_f, t)s(\mathbf{p}_i, t') + iE(\mathbf{p}_f)s(\mathbf{p}_f, t)c(\mathbf{p}_i, t')]$$

$$c(\mathbf{p}, t) = \frac{\cosh[E(\mathbf{p})(t - T/2)]}{2E(\mathbf{p})\sinh[E(\mathbf{p})(t - T/2)]} \quad s(\mathbf{p}, t) = \frac{\sinh[E(\mathbf{p})(t - T/2)]}{2E(\mathbf{p})\sinh[E(\mathbf{p})(t - T/2)]}$$

$$E(\mathbf{p}) = \sqrt{M^2 + \mathbf{p}^2} \qquad M^2 = 2m\Sigma/F$$

Three point function

$$\frac{\langle P^{12}(x_0:-\mathbf{p}_f)V_0^{ii}(y_0:\mathbf{p}_f-\mathbf{p}_i)P^{21}(z_0:\mathbf{p}_i)\rangle}{\langle \Delta P^{12}(x_0:\mathbf{0})V_0^{ii}(y_0:\mathbf{0})\Delta P^{21}(z_0:\mathbf{0})\rangle}$$

$$=F_V(q_0,\mathbf{p}_f-\mathbf{p}_i)\frac{iE(\mathbf{p}_i)c(\mathbf{p}_f,t)s(\mathbf{p}_i,t')+iE(\mathbf{p}_f)s(\mathbf{p}_f,t)c(\mathbf{p}_i,t')}{iE(\mathbf{0})\Delta c(\mathbf{0},t)\Delta s(\mathbf{0},t')+iE(\mathbf{0})\Delta s(\mathbf{0},t)\Delta c(\mathbf{0},t')}$$

$$\frac{\langle \Delta P^{12}(x_0:\mathbf{0})V_0^{ii}(y_0:-\mathbf{p}_i)P^{21}(z_0:\mathbf{p}_i)\rangle}{\langle \Delta P^{12}(x_0:\mathbf{0})V_0^{ii}(y_0:\mathbf{0})\Delta P^{21}(z_0:\mathbf{0})\rangle}$$

$$=F_V(q_0,-\mathbf{p}_i)\frac{iE(\mathbf{p}_i)\Delta c(\mathbf{0},t)s(\mathbf{p}_i,t')+iE(\mathbf{0})\Delta s(\mathbf{0},t)c(\mathbf{p}_i,t')}{iE(\mathbf{0})\Delta c(\mathbf{0},t)\Delta s(\mathbf{0},t')+iE(\mathbf{0})\Delta s(\mathbf{0},t)\Delta c(\mathbf{0},t')}$$

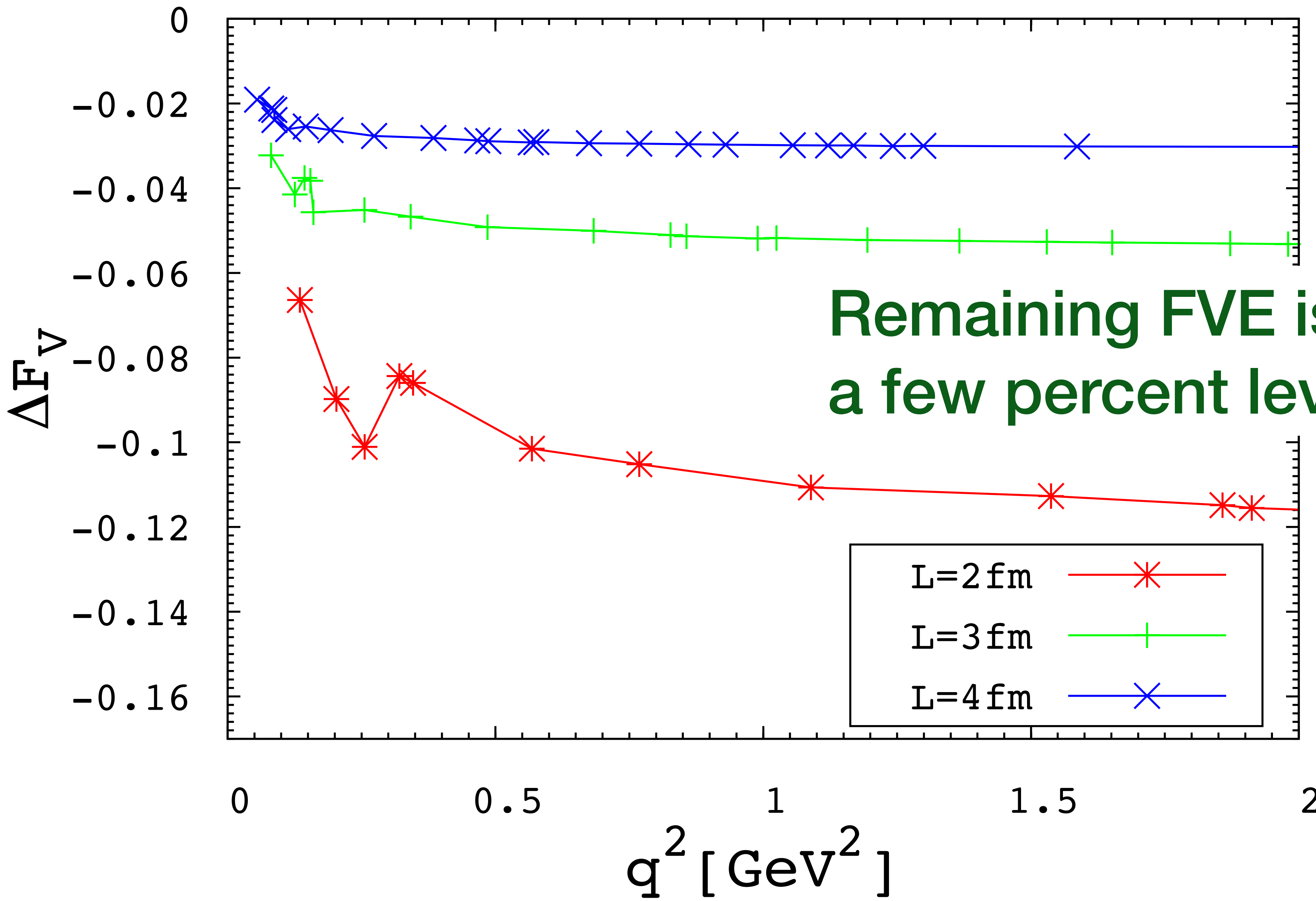
Zero-mode coefficients don't appear any more.

Only perturbative FVE is remained in F_V

Remaining FVE

Remaining FVE vs Energy

Phys.Rev. D90, 114508



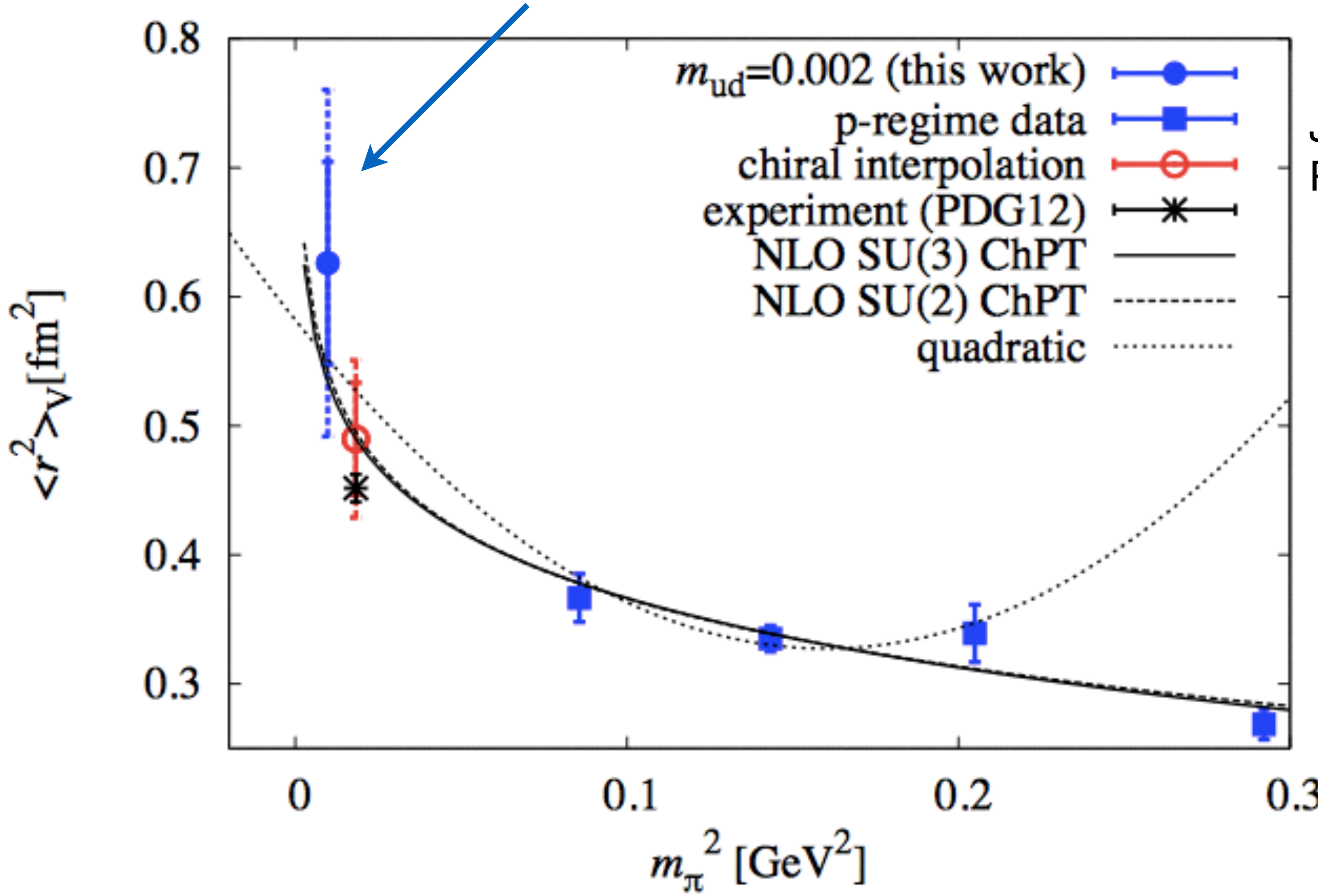
$$\Delta F_V = F_V^{\mathrm{finite}} - F_V^{\infty}$$

$$F_V(0) = 1$$

Remaining FVE is suppressed to a few percent level already at L = 3fm.

Pion charge radius

Our result has been used.



JLQCD arXiv:1405.4077
Phys.Rev. D90 (2014) 3, 034506

First result higher than experimental value.

Summary

We have calculated Pseudoscalar-Vector-Pseudoscalar three point function within the epsilon expansion of ChPT.

Conclusion

- Even when Mpi L < 1, FVE is suppressed to O(10%) with our method:
  1. **Momentum insertion**
  2. **Subtraction at different space-time**
  3. **Taking an appropriate ratio of 1's and 2's**
- Our result can be applied for p regime.

Future Work

- Applications to other hadron's form factors, 4pt functions would be interesting.


Thank you for
your kind attention.