Quantum entanglement entropy for SU(3) gauge theories

Etsuko Itou (KEK)

collaboration with K. Nagata (KEK), Y.Nakagawa, A. Nakamura (Hiroshima U.) and V.I.Zakharov (Max Planck Inst.)

cf. arXiv:0911.2596 and 1104.1011:
Y.Nakagawa, A.Nakamura, S.Motoki and V.I.Zakharov

Hadrons and Hadron Interactions in QCD -- Effective theories and Lattice -- (HHIQCD2015)
@ YITP, Kyoto University 2015/3/6

entanglement entropy

=?

a novel order parameter of the confinement

Don’t think....feel....

entanglement entropy

=?

central charge in conformal field theory

Don’t think....feel....

Related works

- H.Casini, M.Muerta and J.A.Rosabal arXiv:1312.1183

  ``Remarks on entanglement entropy for gauge fields”

- D.Radicevic arXiv:1404.1391

  ``Note on Entanglement in Abelian Gauge Theories”

- S.Ghosh, R.M.Soni,S.P.Trivedi arXiv:1501.02593

  ``On the entanglement entropy for gauge theories”

- S.Aoki, T.Iritani, M.Nozaki et.al. arXiv:1502.04267

  ``On the definition of entanglement entropy in lattice gauge theories”

- J.-W. Chen, S.-H. Dai and J.Y. Pang arXiv:1503.01766

  ``Strong coupling expansion of the entanglement entropy of Yang-Mills gauge theories”

Outline

- Introduction (For QCD and conformal theory in 4d)
- Definition of entanglement entropy
- Replica method
- Results for the quenched QCD

Introduction

Basic properties of E.E.

entanglement entropy for quantum system

- how much a quantum state is entangled quantum mechanically
- d.o.f of the system
- quantum properties of the ground state for the system
- in finite T system, it gives a thermal entropy

For QCD theory

A color confinement changes the d.o.f of the system

deconfinement phase		confinement phase	
colorful (gluons)	$\sim O(N_c^2)$	colorless (singlet)	$\sim O(1)$

For conformal theories

Central charge is ...

- an important characteristic of CFT
- roughly the d.o.f. of the conformal system

In 2d CFT,
$$\langle T(z)T(w)\rangle = \frac{1}{(2\pi)^2}\frac{c}{2(z-w)^4}$$

A.A. Belavin, A.M.Polyakov, A.B.Zamolodchikov NPB241(1984)333
A. B. Zamolodchikov, JETP Lett. 43 (1986) 730

In 4d CFT,
$$\langle T_{\mu\nu}(x_1)T_{\sigma\rho}(x_2)T_{\alpha\beta}(x_3)\rangle = \frac{1}{x_{12}^d x_{13}^d x_{23}^d}\Gamma_{\mu\nu,\sigma\rho,\alpha\beta}(x_1,x_2,x_3)$$
$$x_{ij} = |x_i - x_j|$$

J.I.Latorre and H.Osborn: hep-th/9703196
Y. Nakayama arXiv:1302.0884[hep-th]

$$
\begin{aligned}
\Gamma_{\mu\nu,\sigma\rho,\alpha\beta}(x_1,x_2,x_3) =& \mathcal{E}^T_{\mu\nu,\mu'\nu'}\mathcal{E}^T_{\sigma\rho,\sigma'\rho'}\mathcal{E}^T_{\alpha\beta,\alpha'\beta'} \\
& \left[A I_{\nu'\sigma'}(x_{12}) I_{\rho'\alpha'}(x_{23}) I_{\beta'\mu'}(x_{31}) + B I_{\mu'\sigma'}(x_{12}) I_{\nu'\alpha'}(x_{23}) X^2_{\rho'} X^3_{\beta'} (x_2-x_3)^2 + \text{perm}\right] \\
& + C \mathcal{I}^T_{\mu\nu,\sigma\rho}(x_{12}) \left(\frac{X^3_\alpha X^3_\beta}{(X^3)^2} - \frac{1}{d}\delta_{\alpha\beta}\right) + \text{perm} \\
& + D \mathcal{E}^T_{\mu\nu,\mu'\nu'}\mathcal{E}^T_{\sigma\rho,\sigma'\rho'} X^1_{\mu'} X^2_{\sigma'} (x_1-x_2)^2 I_{\nu'\rho'}(x_{12}) \left(\frac{X^3_\alpha X^3_\beta}{(X^3)^2} - \frac{1}{d}\delta_{\alpha\beta}\right) + \text{perm} \\
& + E \left(\frac{X^1_\mu X^1_\nu}{(X^1)^2} - \frac{1}{d}\delta_{\mu\nu}\right) \left(\frac{X^2_\sigma X^2_\rho}{(X^2)^2} - \frac{1}{d}\delta_{\sigma\rho}\right) \left(\frac{X^3_\alpha X^3_\beta}{(X^3)^2} - \frac{1}{d}\delta_{\alpha\beta}\right)
\end{aligned}
$$

$$
\begin{aligned}
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\end{aligned}
$$

According to the conservation law of EMT,
two coefficients are redundant.

The following ``a” is expected to be a monotonical decreasing fn.
along the RG flow.

H.Osborn and A.C.Petkou, : hep-th/9307010

$$
a = \frac{\pi^4}{512 \times 90}(13A - 2B - 40C)
$$

Definition of the entanglement entropy

Entanglement entropy (E.E.)

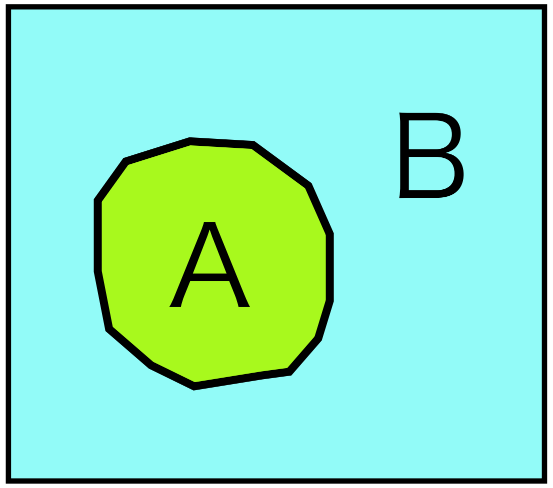
von Neumann entropy $$S_{tot} = -\mathrm{Tr}\rho_{tot} \log \rho_{tot}$$

density matrix $$\rho_{tot} = |\Psi\rangle\langle\Psi|$$ $|\Psi\rangle$:pure ground state

decompose total Hilbert space into two subsystems $$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$$

reduced density matrix $$\rho_A = -\mathrm{Tr}_{\mathcal{H}_B}\rho_{tot}$$

entanglement entropy $$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A$$

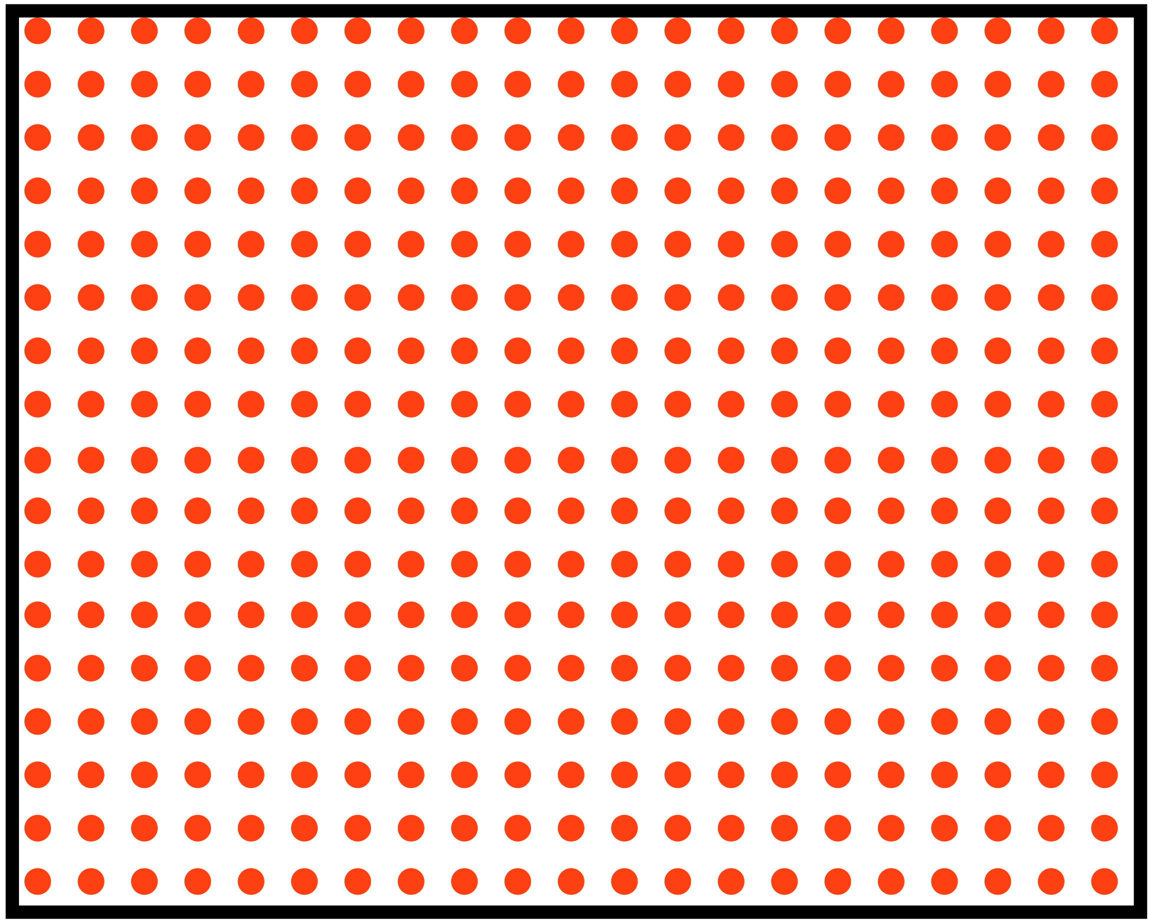


At finite T, it is equivalent to the thermal entropy.

Schematic picture of E.E.

Holzhey,Larsen and Wilczek: NPB424 (1994) 443
Calabrese and Cardy: J.S.M.0406(2004)P06002
Calabrese and Cardy, arXiv:0905.4013

(1+1)-dim. model



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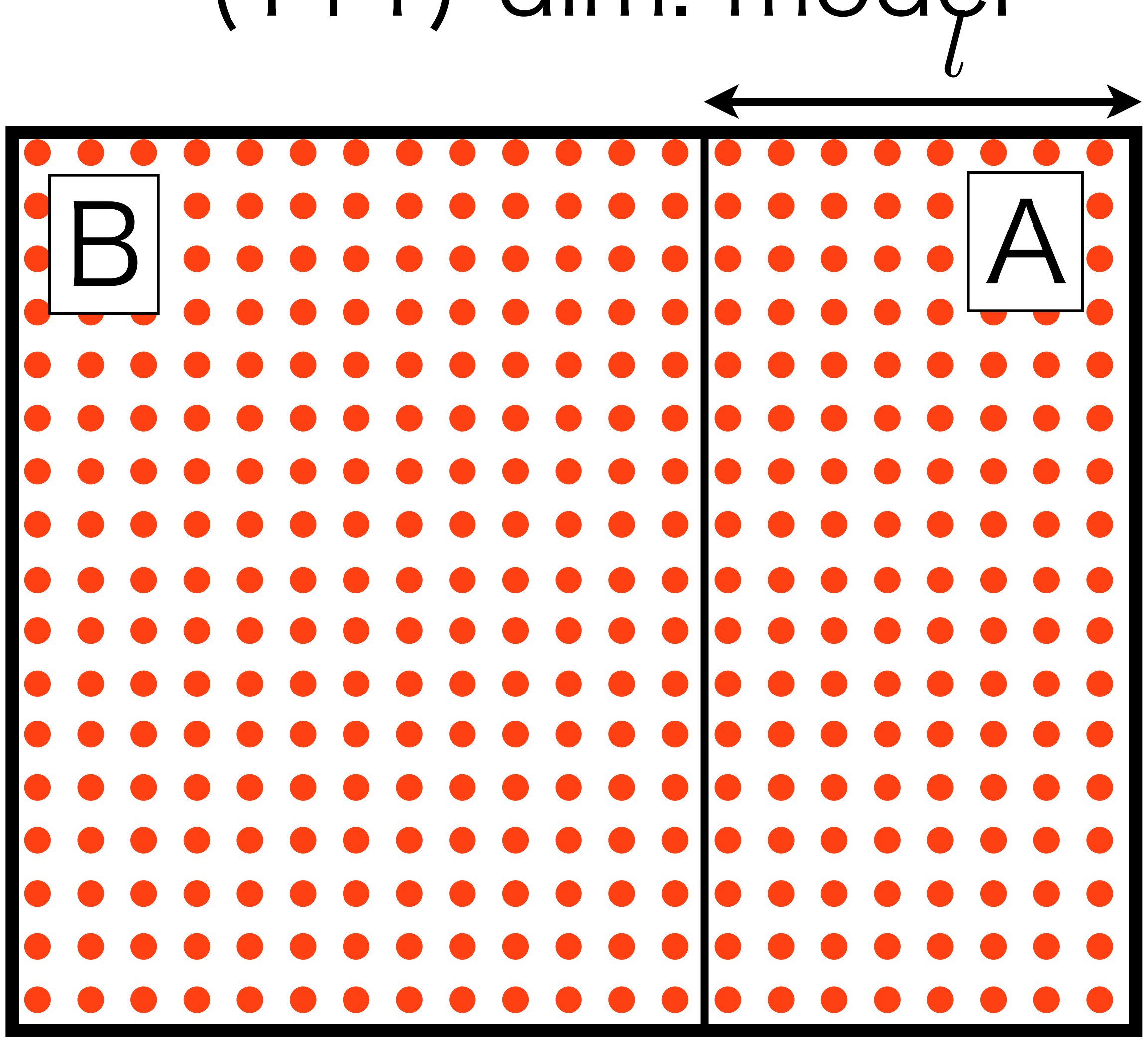

Schematic picture of E.E.

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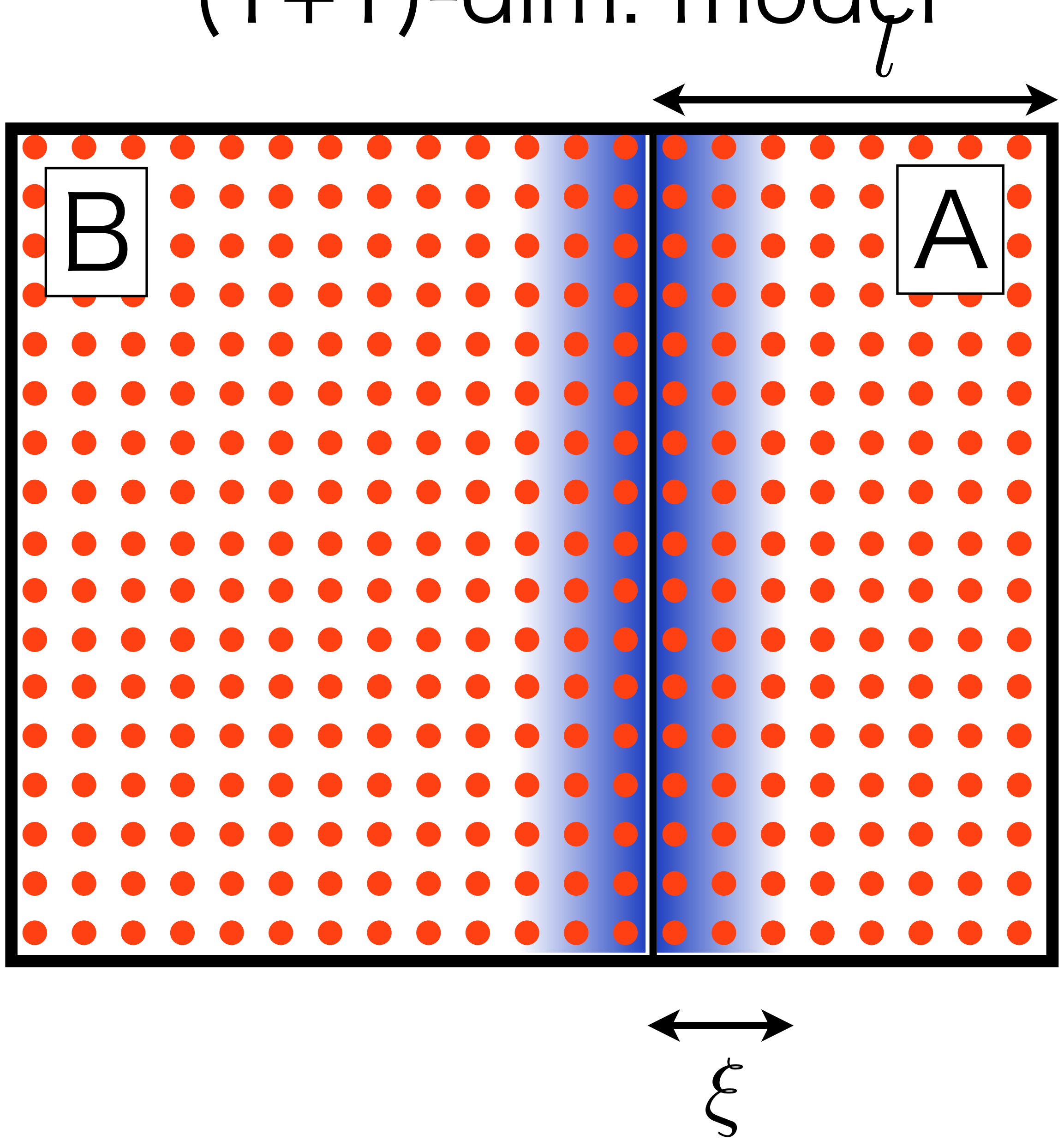


ξ: correlation length of the system

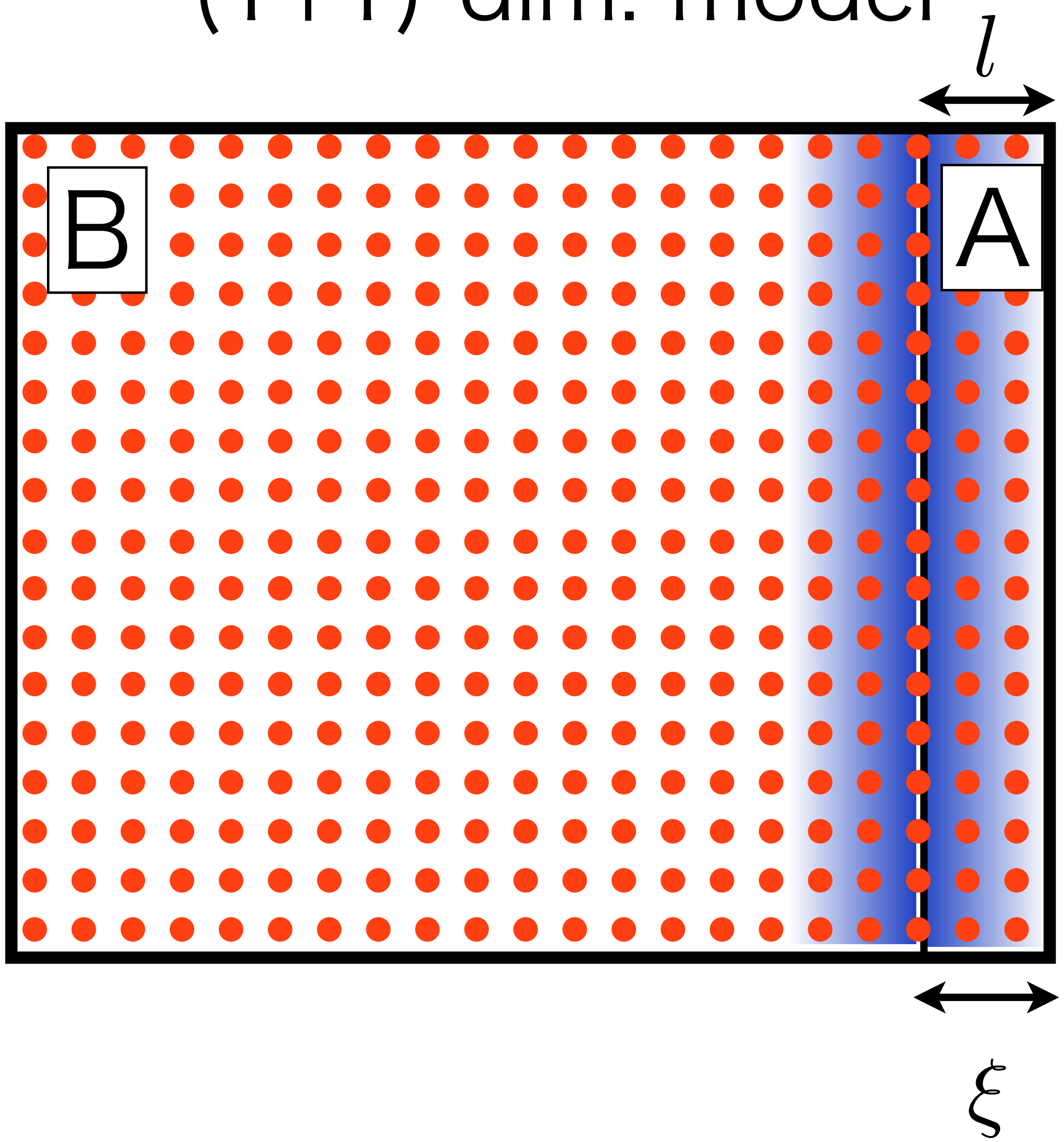
At the critical point,

$$S_A(l) = \frac{c}{3} \log \frac{l}{a} + c_1$$

c is the central charge in 2d CFT.

Schematic picture of E.E.

(1+1)-dim. model



ξ: correlation length of the system

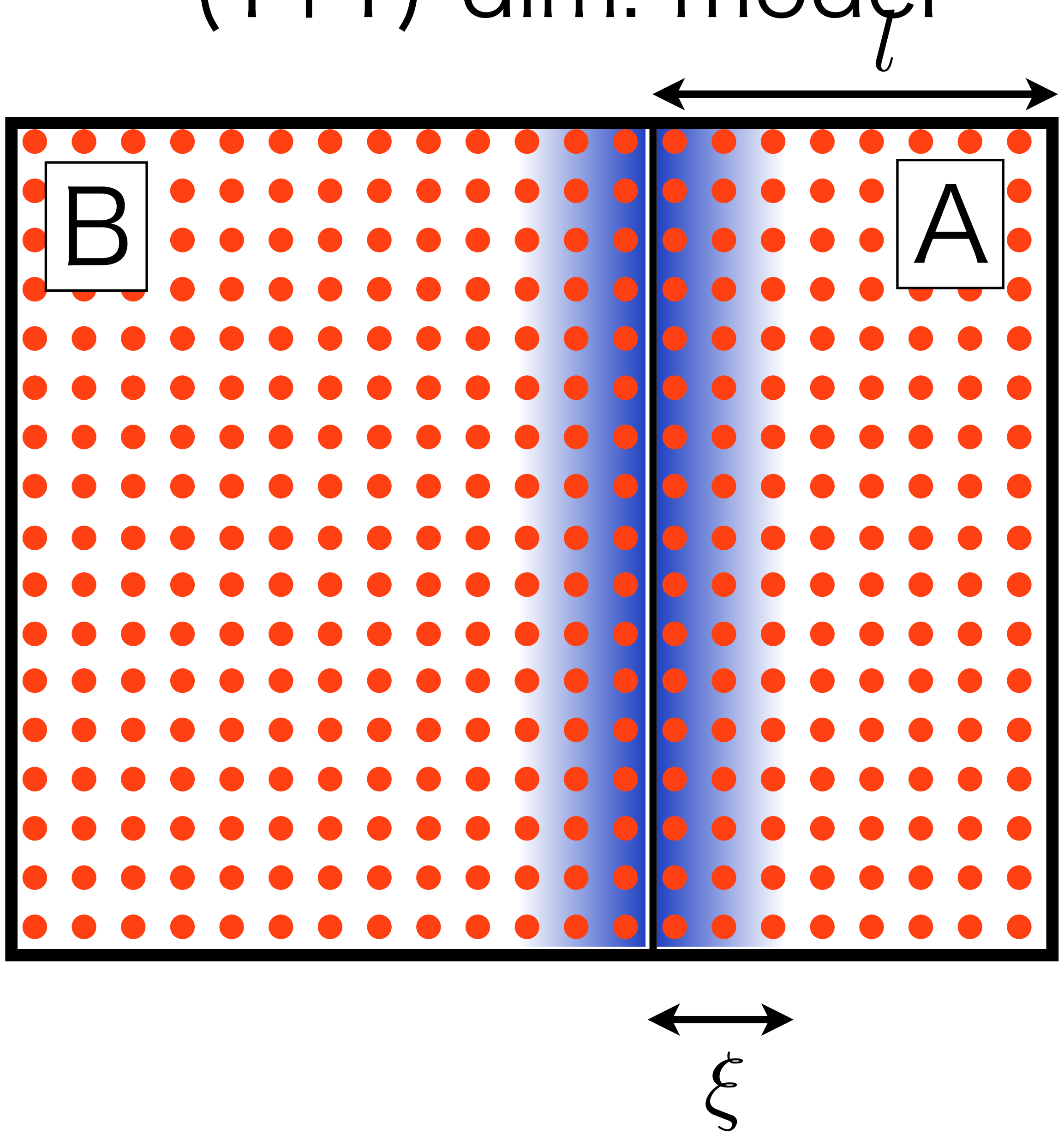
In the non critical system,

$$l \ll \xi$$

$$S_A(l) \sim \frac{c}{3} \log \frac{l}{a}$$

Schematic picture of E.E.

(1+1)-dim. model



In the non critical case,

$l \ll \xi$

$$S_A(l) \sim \frac{c}{3} \log \frac{l}{a}$$

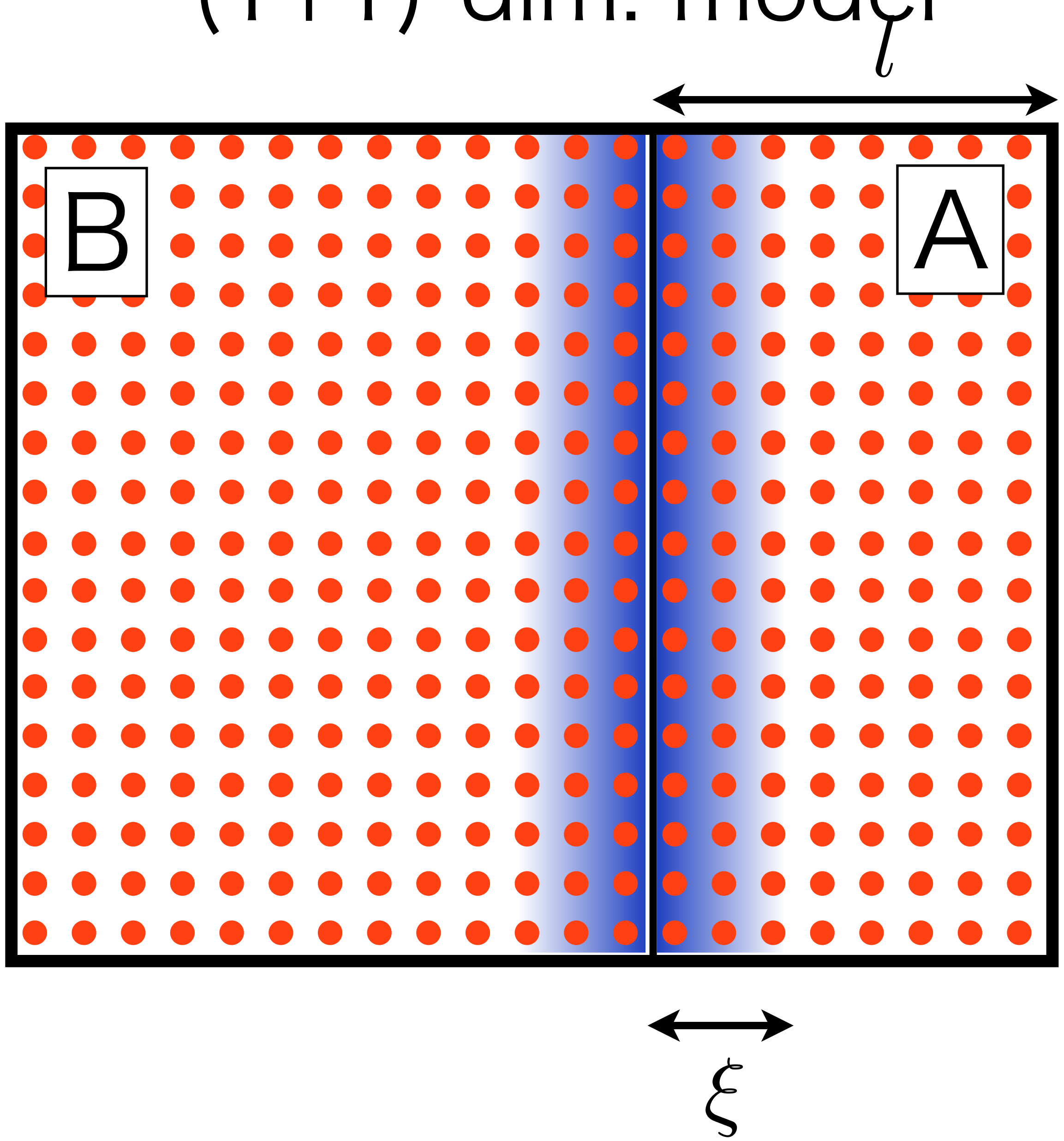
$l \gg \xi$

$$S_A(l) \to \frac{c}{3} \log \frac{\xi}{a}$$

ξ: correlation length of the system

Schematic picture of E.E.

(1+1)-dim. model



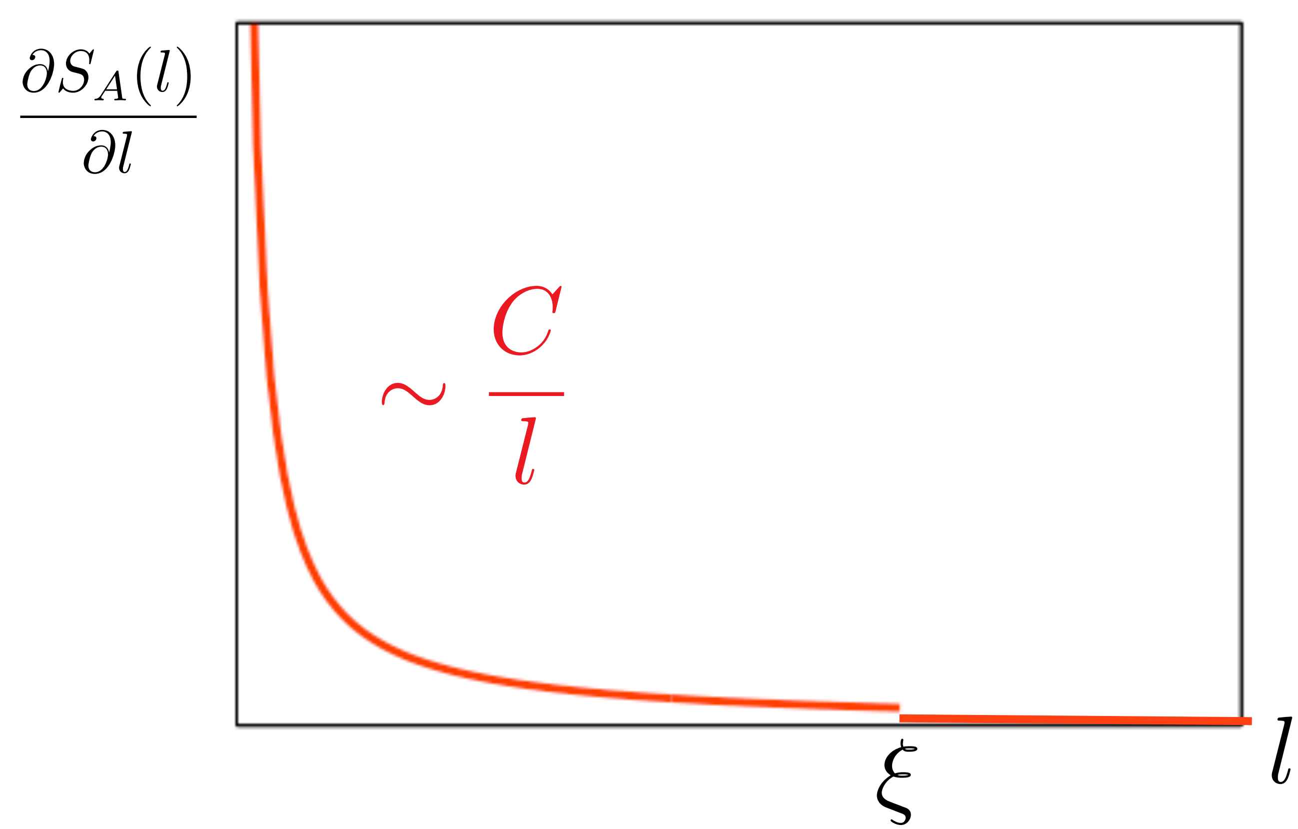
ξ: correlation length of the system

At the critical point or $l \ll \xi$ in the noncritical system

$$S_A(l) = \frac{c}{3}\log\frac{l}{a} + c_1$$

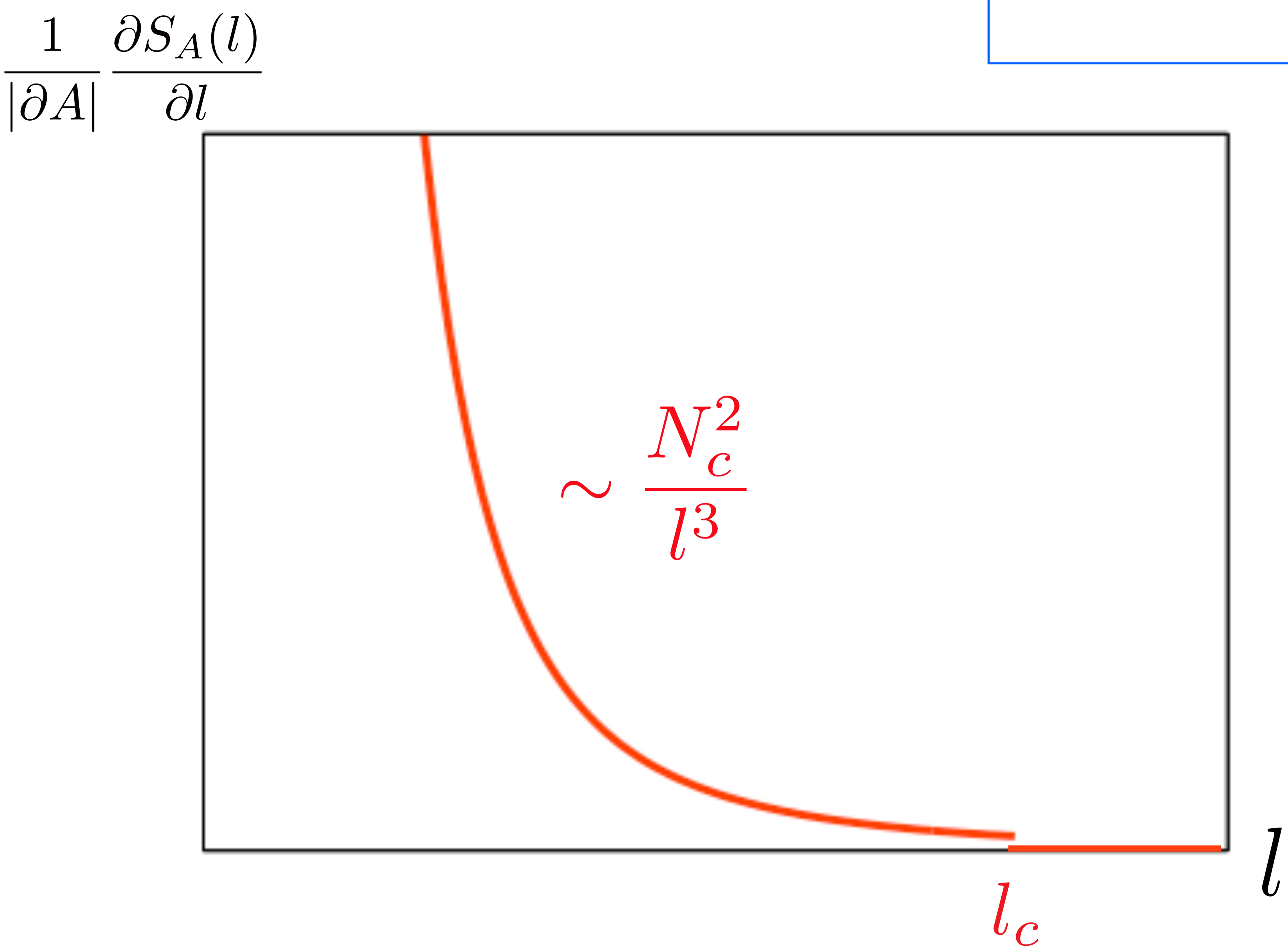
In the non-critical system,

$$S_A(l) \underset{l \gg \xi}{\rightarrow} \frac{c}{3}\log\frac{\xi}{a}$$



4-dimensional CFT and QFT

Ryu and Takayanagi:PRL96(2006)181602
JHEP 0608(2006)045



Holographic approach

(3+1)-dim. $\mathcal{N}=4$ SYM

$$\frac{1}{|\partial A|}S_A(l) = c\frac{N_c^2}{a^2} - c'\frac{N_c^2}{l^2}$$

c' is obtained by AdS and QFT

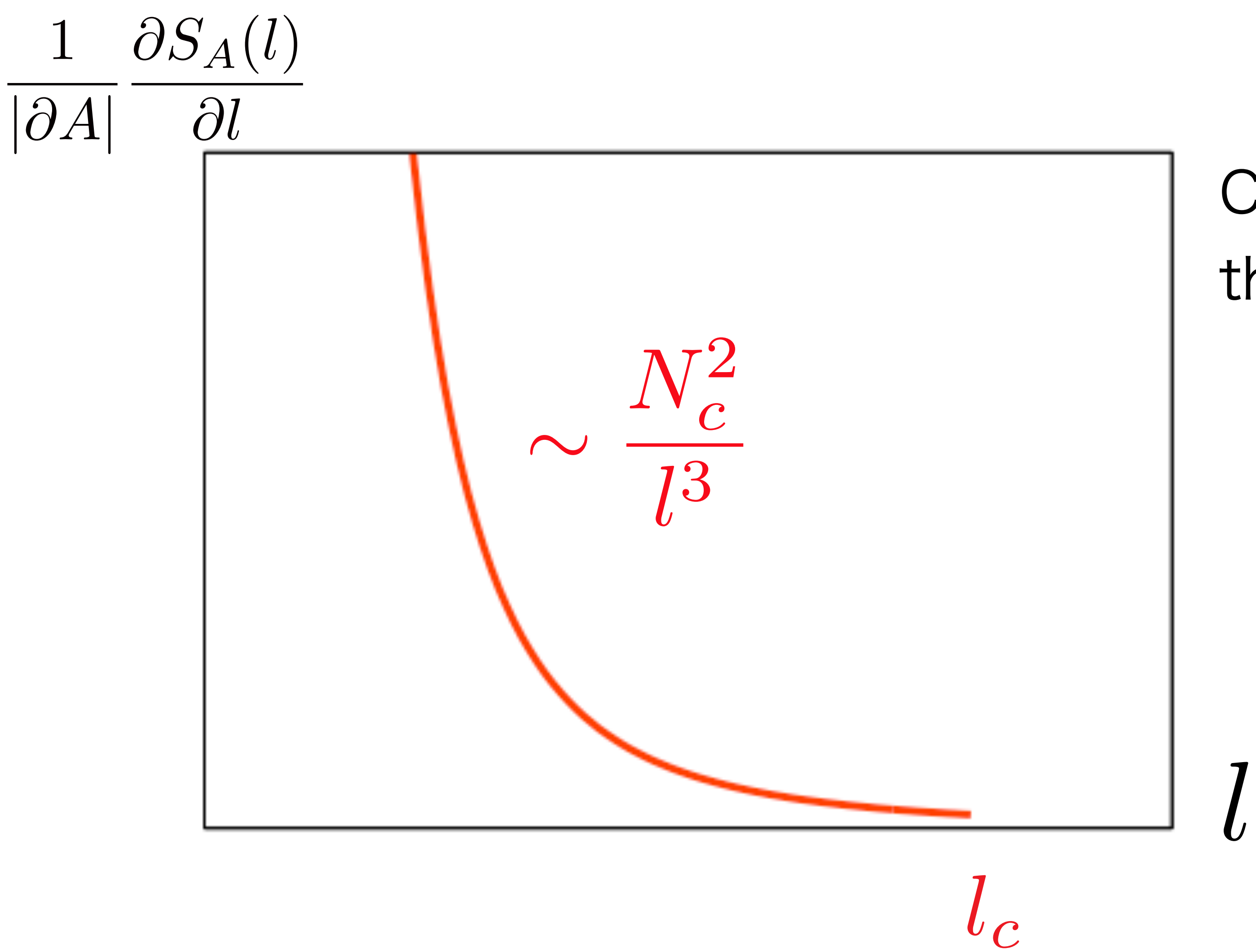
Intuitively...

in short range: gluon dynamics ~ O(Nc^2) and dimensional analysis [mass^3]

in long range: only color singlet ~ O(1) and l-indepent because of confinement

What we want to know?

- Nc dependence in the short l region
- existence of discontinuity
- value of lc (Lambda QCD?)
- UV cutoff (lattice cutoff) dependence



Correction terms exist in non-conformal theory?

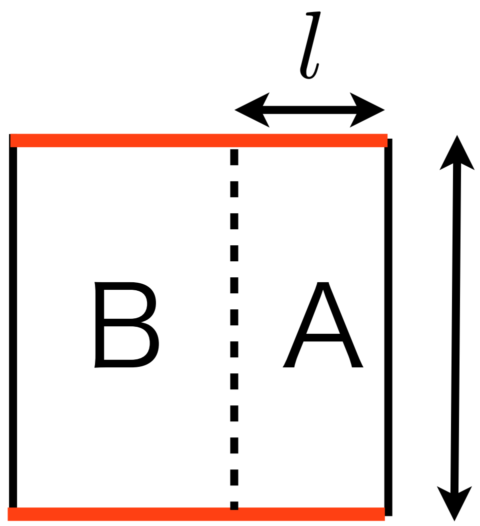
$$\frac{1}{|\partial A|}S_A(l) = c\frac{N_c^2}{a^2} - c'\frac{N_c^2}{l^2} \times (1 + c'' log(a/l))$$

Replica method

$\boxed{\text{Calabrese and Cardy: J.S.M.0406(2004)P06002}}$

Replica method

$Z =$



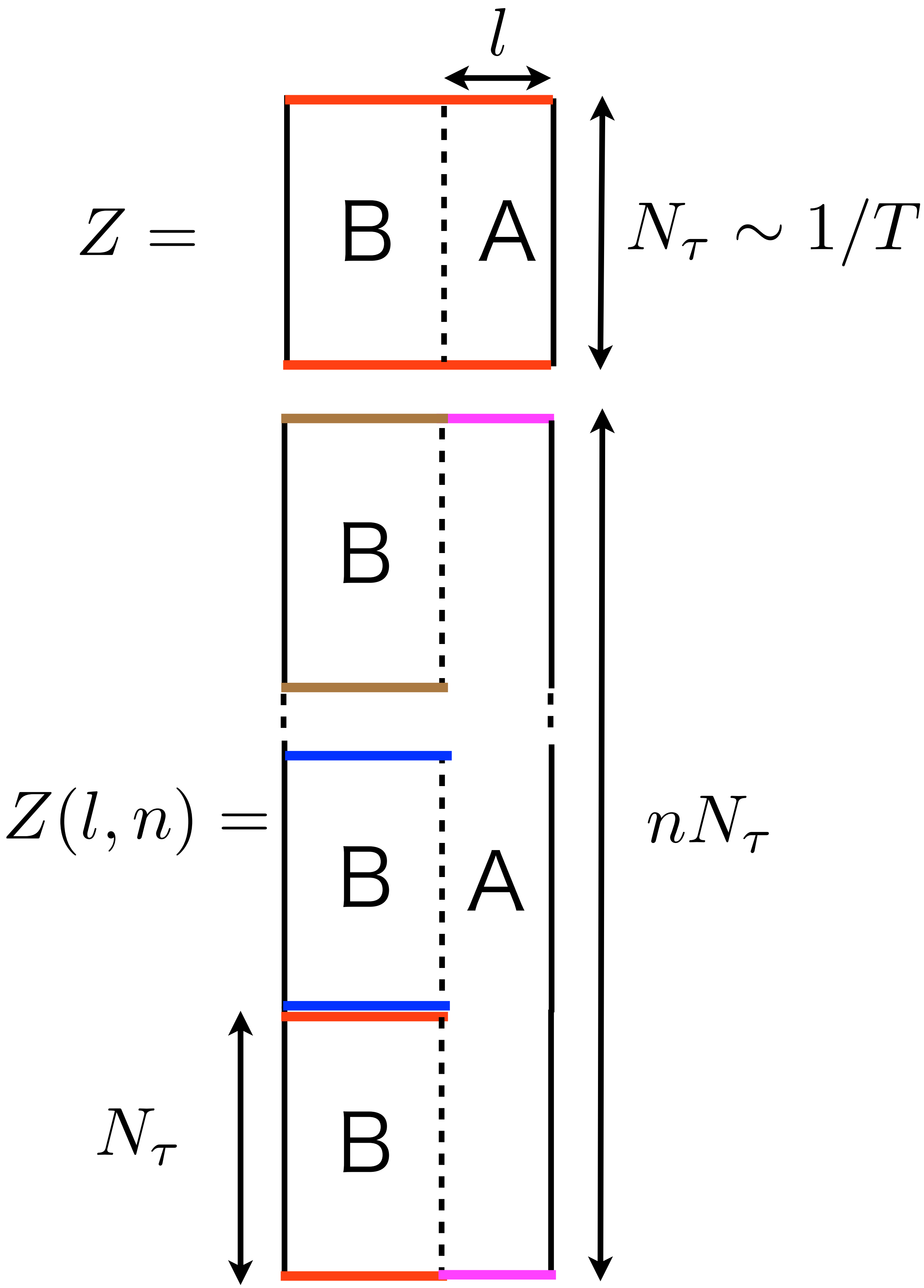
$N_\tau \sim 1/T$

entanglement entropy:

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \ln \mathrm{Tr}_A \rho_A^n$$

$$S_A(l) = -\lim_{n\to 1} \frac{\partial}{\partial n} \ln \left(\frac{Z(l,n)}{Z^n} \right)$$

Replica method



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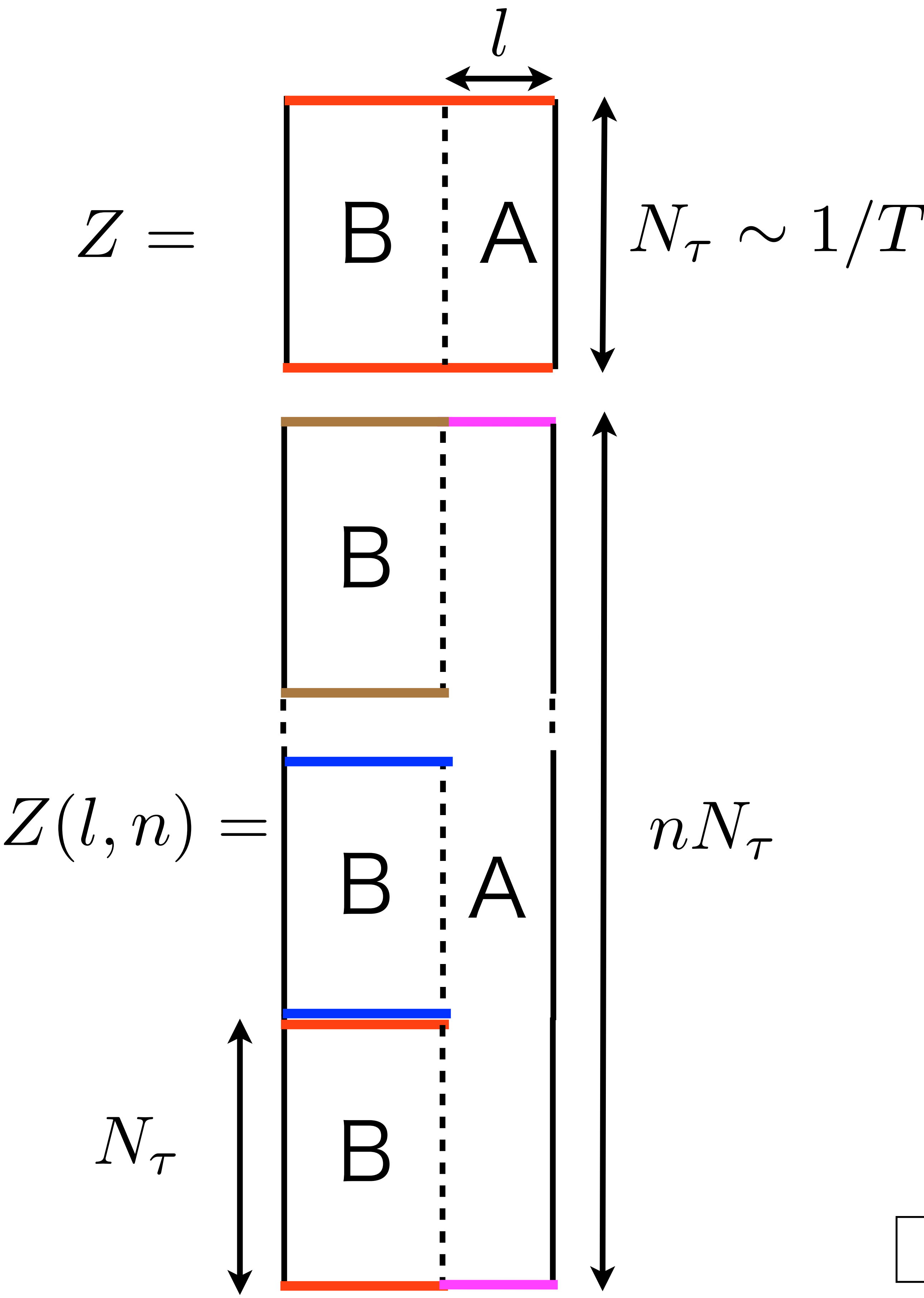


entanglement entropy:

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\ln \mathrm{Tr}_A \rho_A^n$$

$$S_A(l) = -\lim_{n\to 1}\frac{\partial}{\partial n}\ln\left(\frac{Z(l,n)}{Z^n}\right)$$

observable:

$$\frac{\partial S_A(l)}{\partial l} = \lim_{n\to 1}\frac{\partial}{\partial l}\frac{\partial}{\partial n}F[l,n] \quad \text{:free energy}$$

$$\to \frac{F[l+a,n=2]-F[l,n=2]}{a}$$

$$= \int_0^1 d\alpha \langle S_{l+a}[U] - S_l[U]\rangle_\alpha$$

using the interpolation action

$$S_{int} = (1-\alpha)S_l[U] + \alpha S_{l+a}[U]$$

we measure the diff. of the action density

What we want to know?

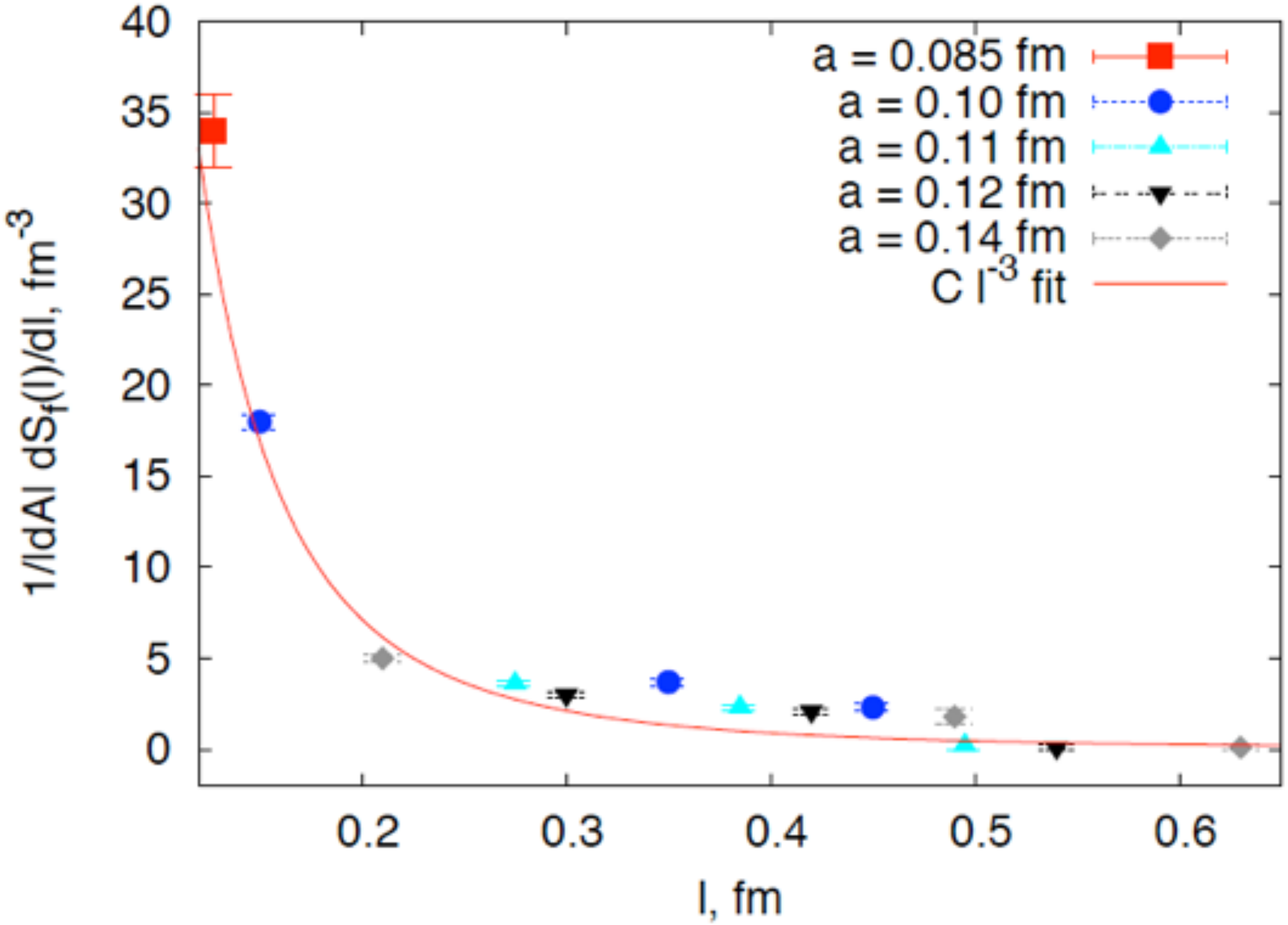
- Nc dependence in the short l region
- existence of discontinuity
- value of lc (Lambda QCD?)
- UV cutoff (lattice cutoff) dependence
- replica number dependence

(add second line)

$$\lim_{n\to 1}\frac{\partial}{\partial n}F[l,n] = (F[l,n=2]-F[l,n=1])/\Delta n|_{\Delta n=1}$$
$$-\frac{\Delta n}{2}(F[l,n=3]-2F[l,n=2]+F[l,n=1])|_{\Delta n=1}$$

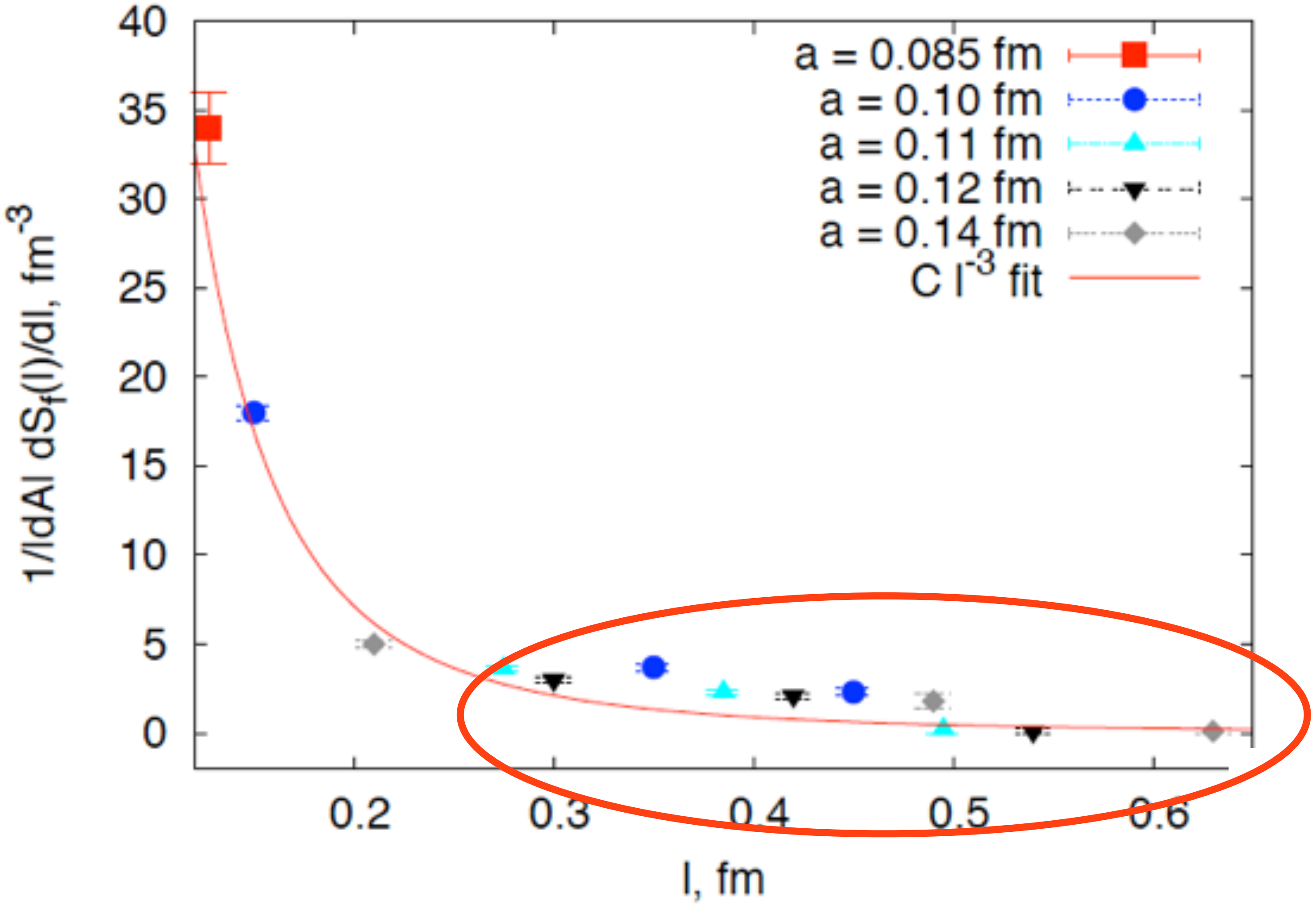
Simulation results

Lattice results for quenched SU(2)



Buividovich and Polikarpov: NPB802(2008)458

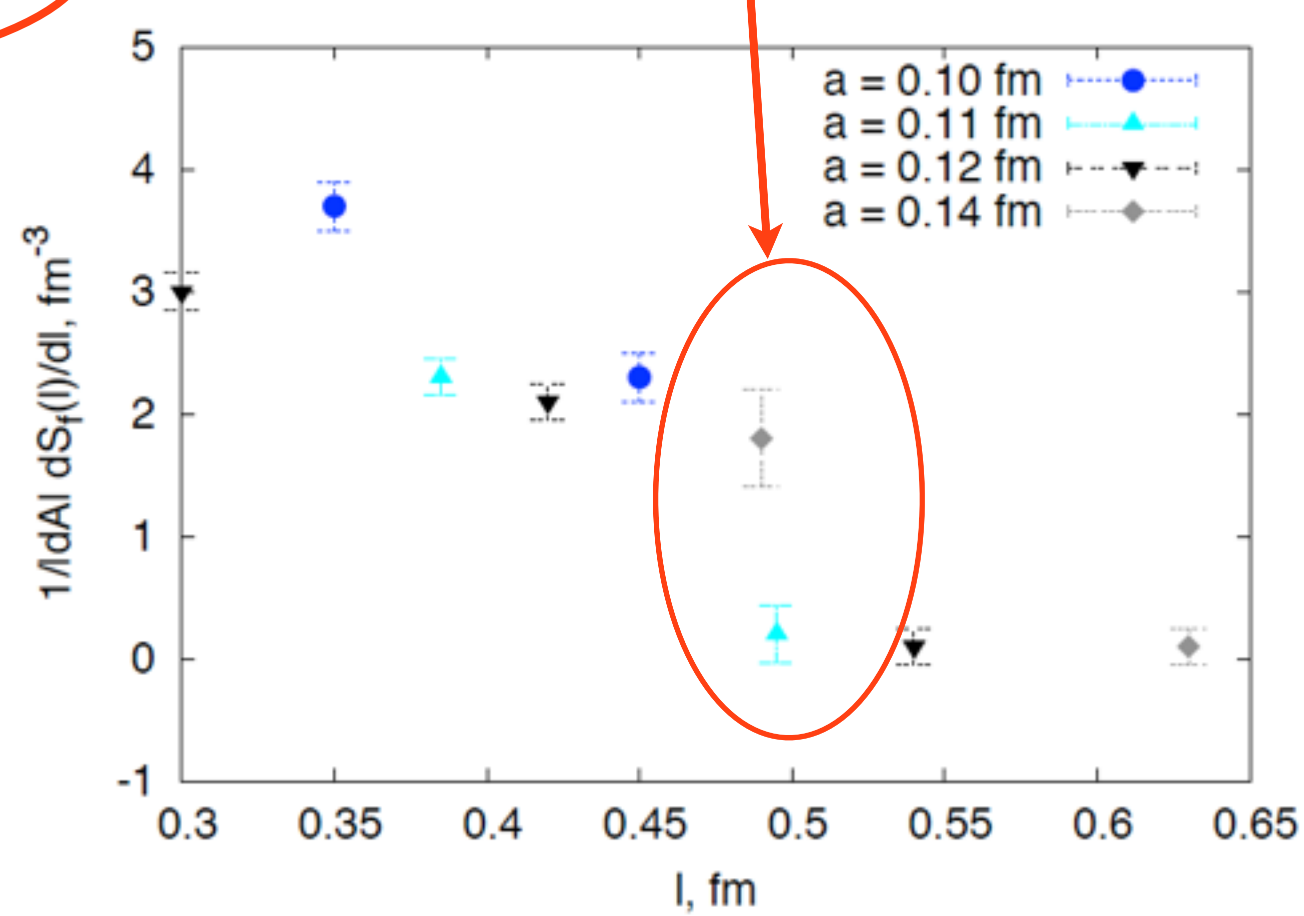
Lattice results for quenched SU(2)



Buividovich and Polikarpov: NPB802(2008)458

- in short range, 1/l^3 scaling
- discontinuity is clear!?

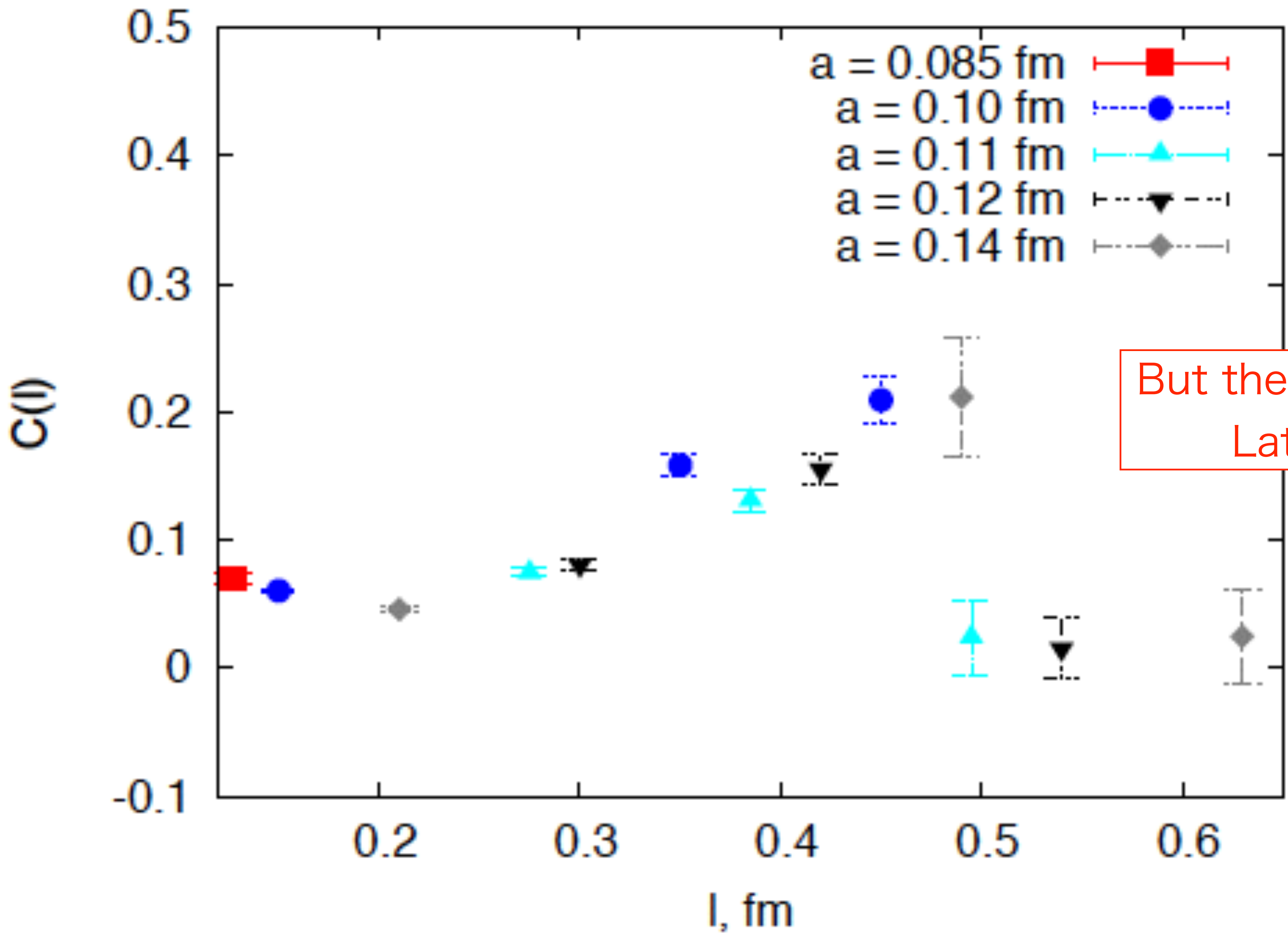
# of configs. is 18 -141



Entropic C-function

$$C(l) = l^3 \frac{1}{|\partial A|} \frac{dS}{dl}$$

should be constant in short l region

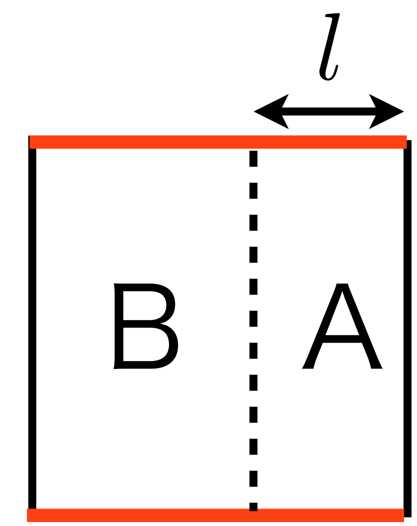


But the data is enhanced
Lattice artifact?

Our result

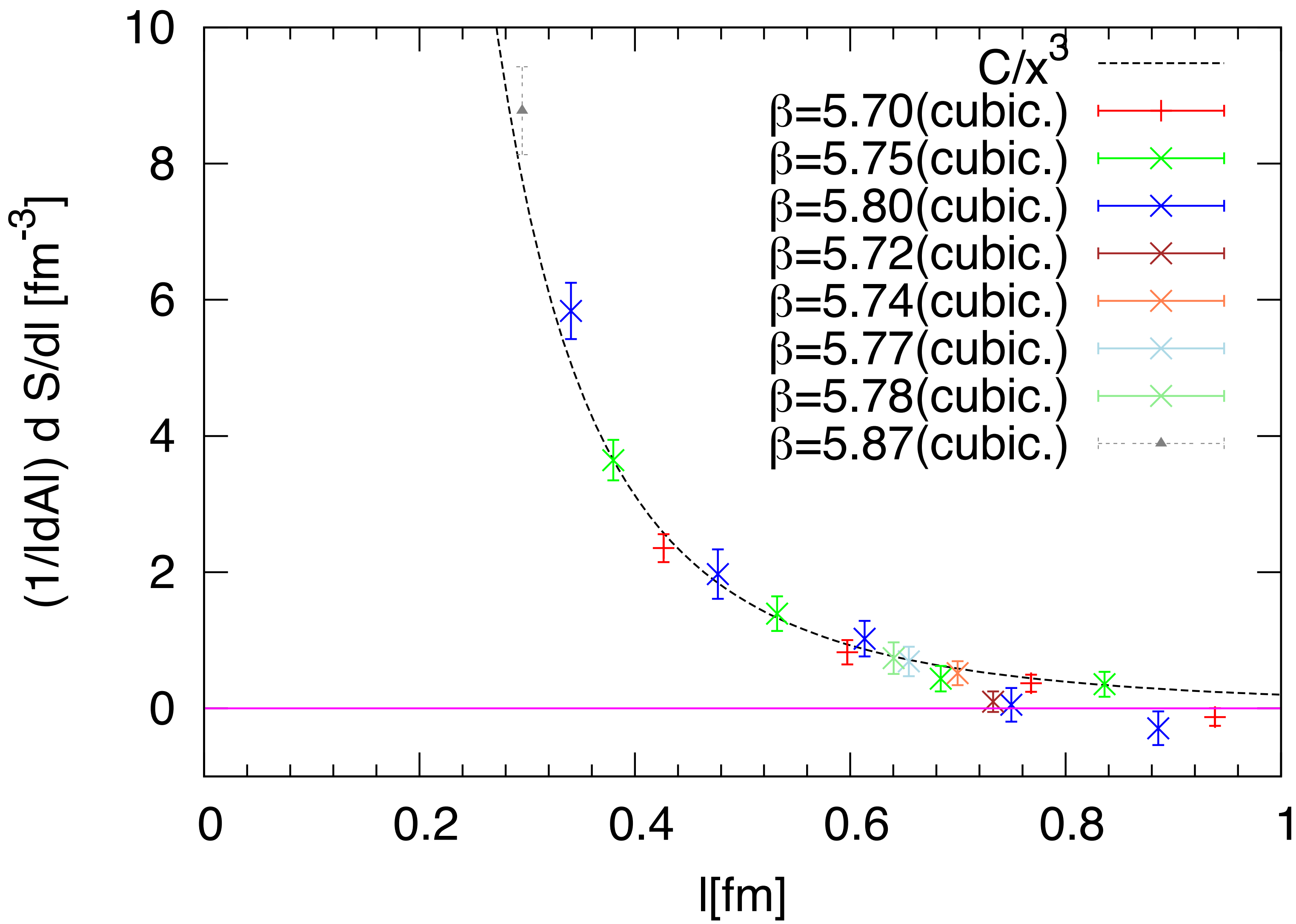
Simulation setup

- Wilson plaquette gauge action
- Ns=Nt=16, 32
- l/a=2,3,4,5,(6)
- beta=5.70 - 5.87
- # of configuration 12,000~30,000
- scale setting r_0 =0.5 fm and ALPHA coll.



Lattice results for quenched SU(3)

T=0, quenched QCD



We measure ~30,000 configs.

- in short range, 1/l^3 scaling
- the coefficient is roughly C=0.2

cf.) C~0.09 in SU(2)

The Nc dependence is

$$\frac{0.2}{0.09} \sim \frac{3^2}{2^2}$$

What we want to know?

- Nc dependence in the short l region

  N_c^2 as expected by AdS/CFT and field theoretical insights

- existence of discontinuity

- value of lc (Lambda QCD?)

- UV cutoff (lattice cutoff) dependence

- replica number (n ->1) dependence

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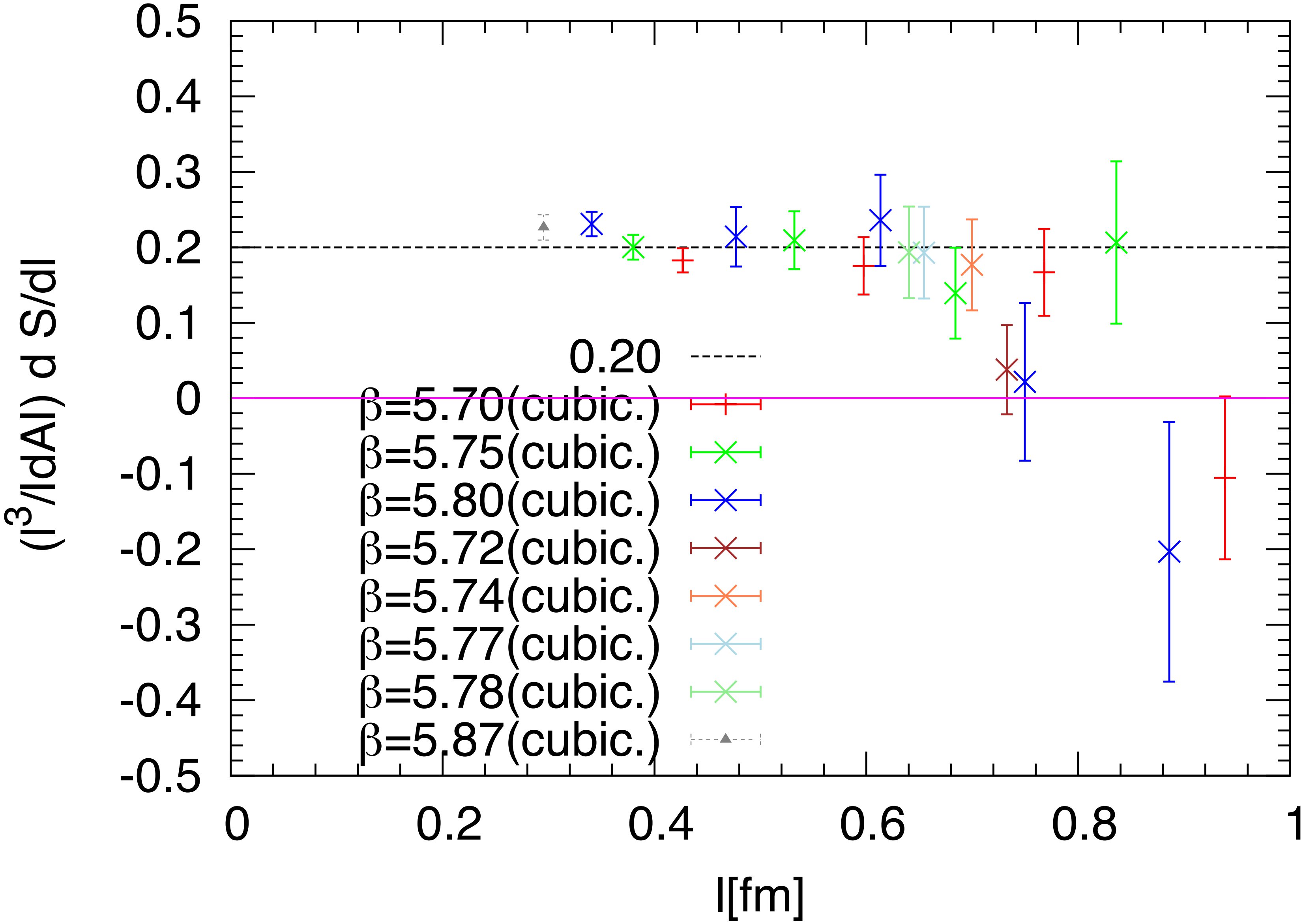


In short l region, C is constant.

The discontinuity is not clear yet.

cf.) $\frac{1}{\Lambda_{QCD}} \sim 0.7[\mathrm{fm}]$

What we want to know?

- Nc dependence in the short l region

  N_c^2 as expected by AdS/CFT and field theoretical insights

- existence of discontinuity

  still unclear, but there is a signal

- value of lc (Lambda QCD?)

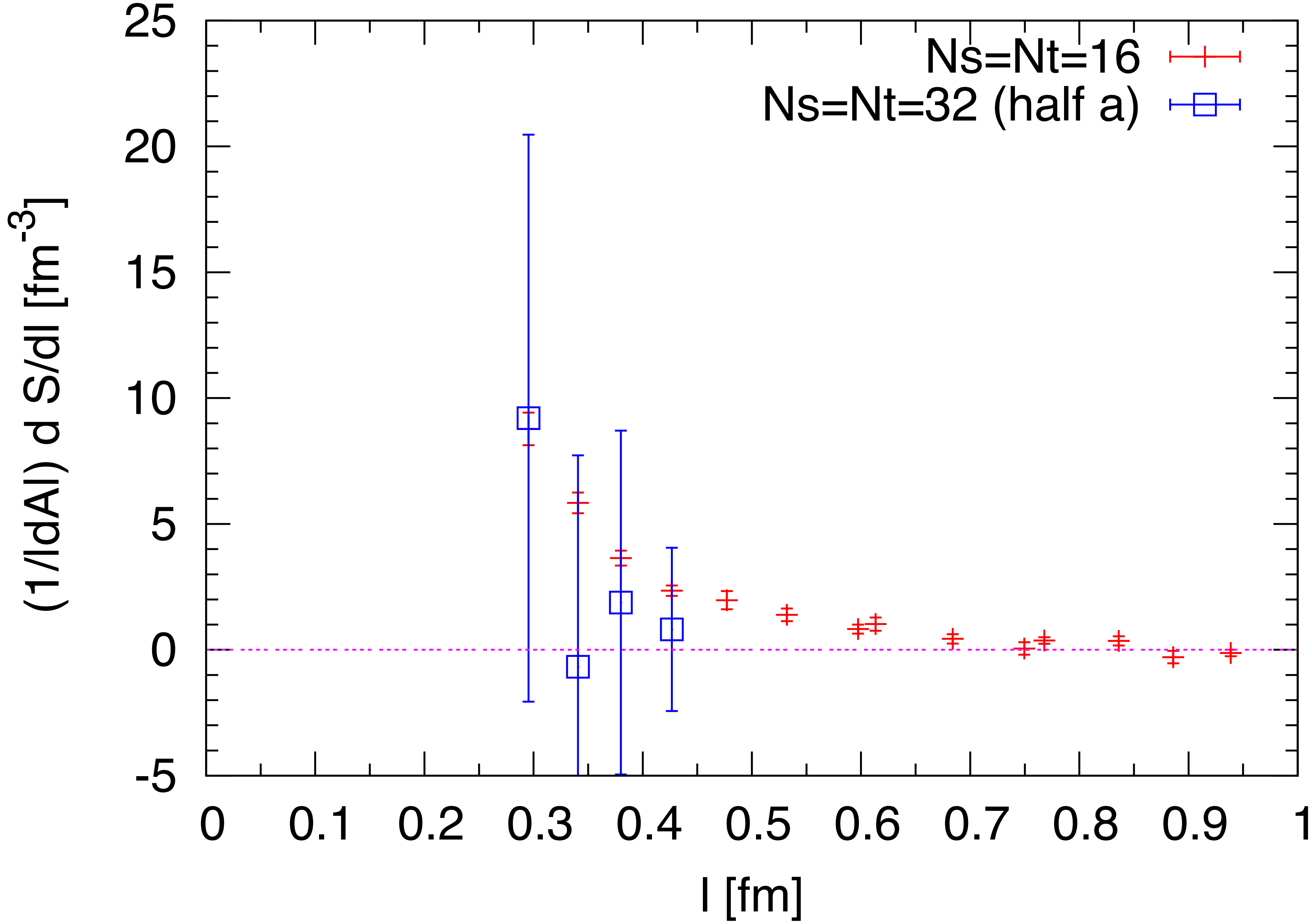
  seems to be the same with Lambda QCD

- UV cutoff (lattice cutoff) dependence

- replica number (n ->1) dependence

UV cutoff dependence

$$N_s = 16, l/a = 2 \to N_s = 32, l/a' = 4$$
$$a = 2a'$$



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- replica number (n ->1) dependence

Replica number dependence

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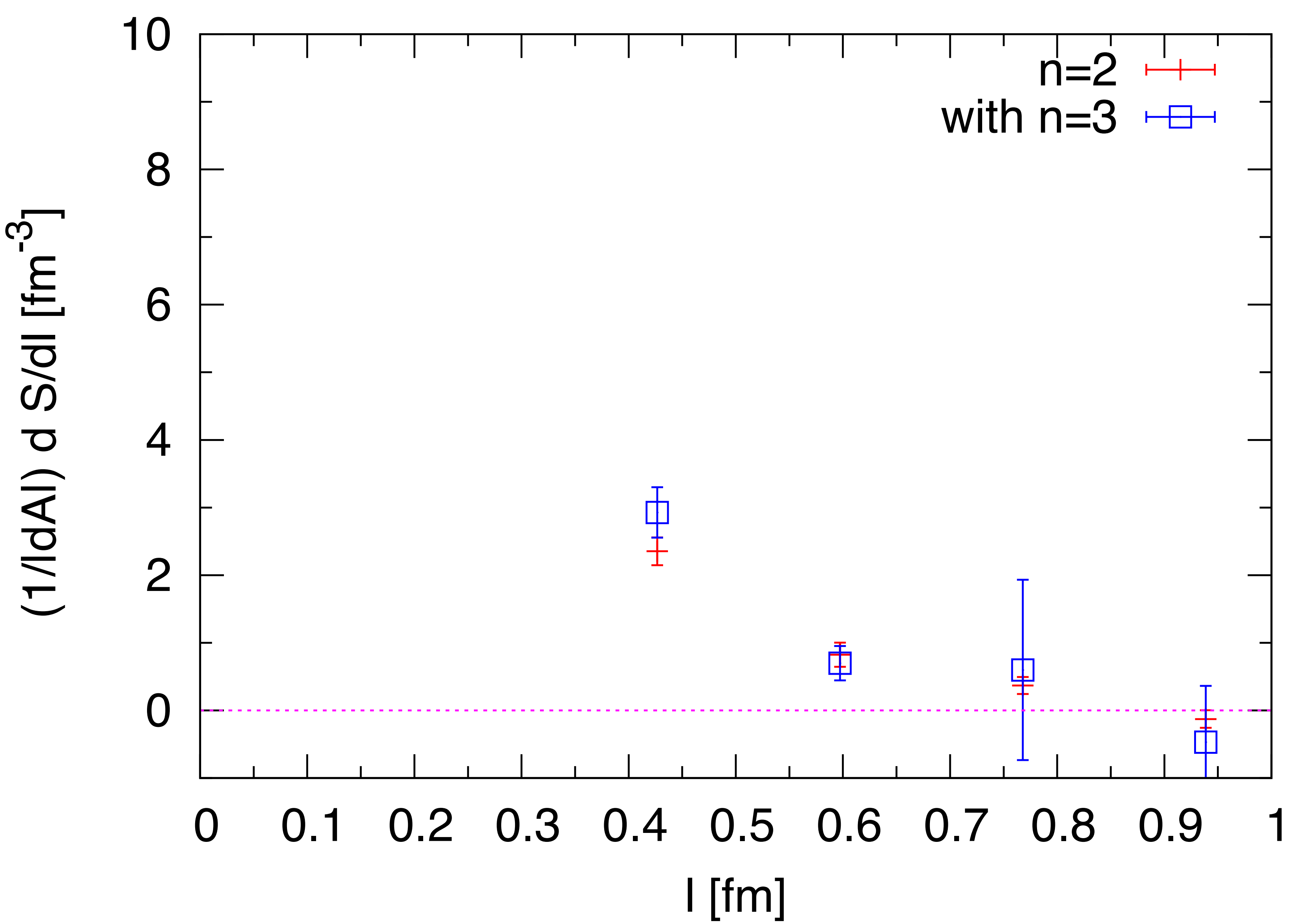


What we found…

- Nc dependence in the short l region

  N_c^2 as expected by AdS/CFT and field theoretical insights

- existence of discontinuity

  still unclear, but there is a signal

- value of lc (Lambda QCD?)

  seems to be the same with Lambda QCD

- UV cutoff (lattice cutoff) dependence

  still unclear

- replica number (n ->1) dependence

  seems to be negligible

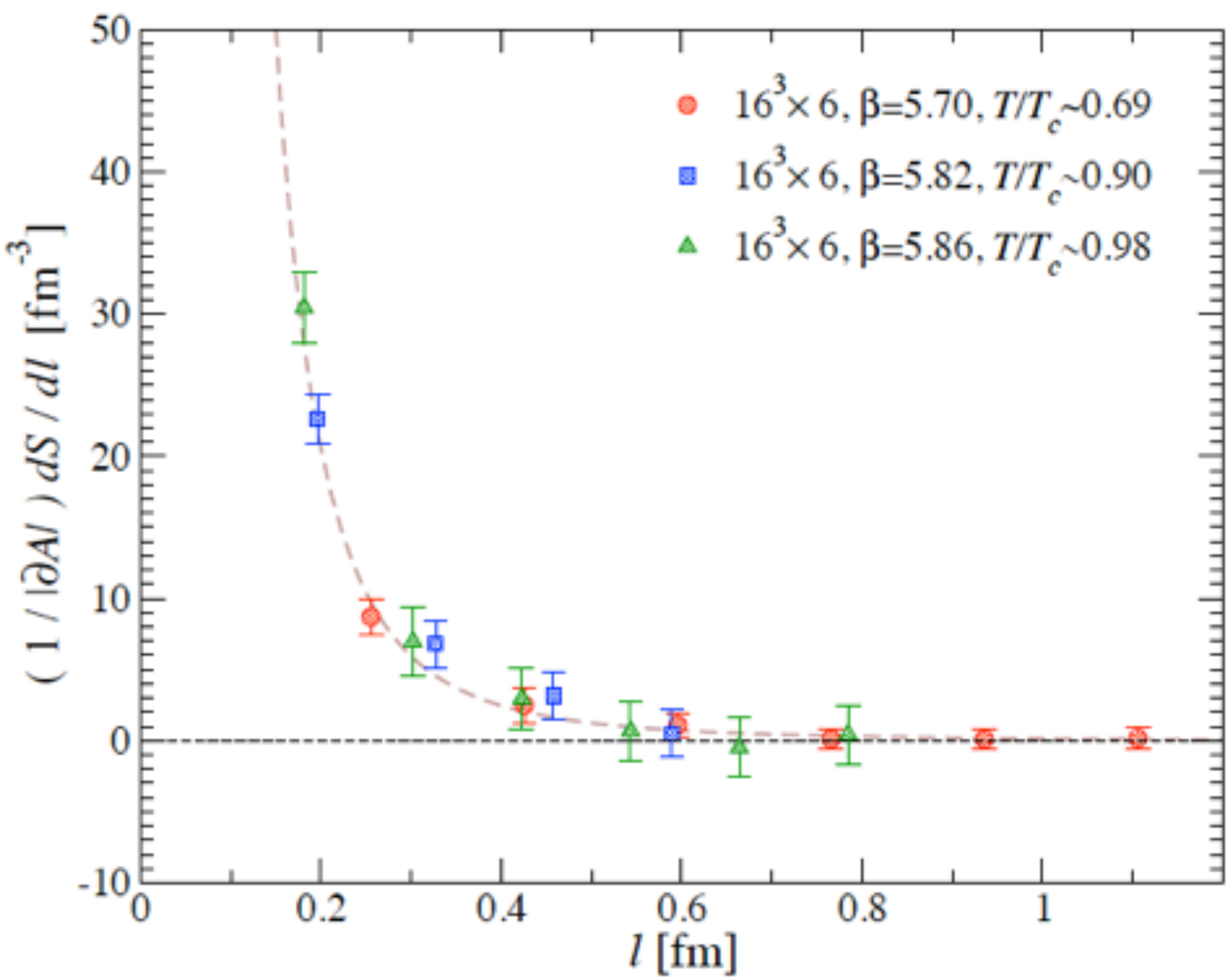
Future directions for E.E. using the lattice

- QCD at zero T
- would give the Lambda QCD
- even in full QCD case
- QCD at finite T
- gives the thermal entropy and the correlation length in QGP phase
- conformal window in 4dim many flavor QCD
- would give the a-function and central charge

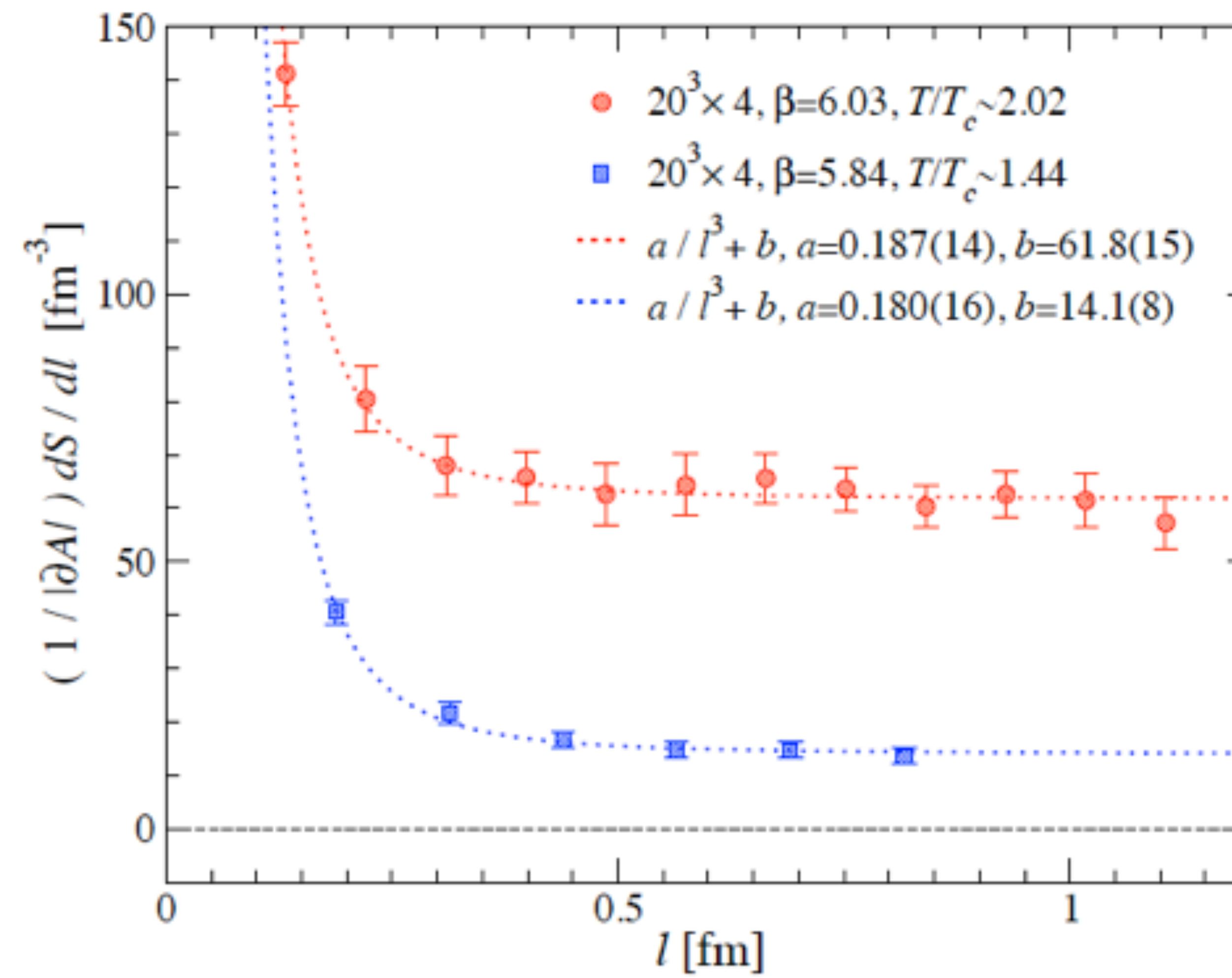
finite T for quenched SU(3)

arXiv:1104.1011: Y.Nakagawa et al.

T< Tc



T >Tc

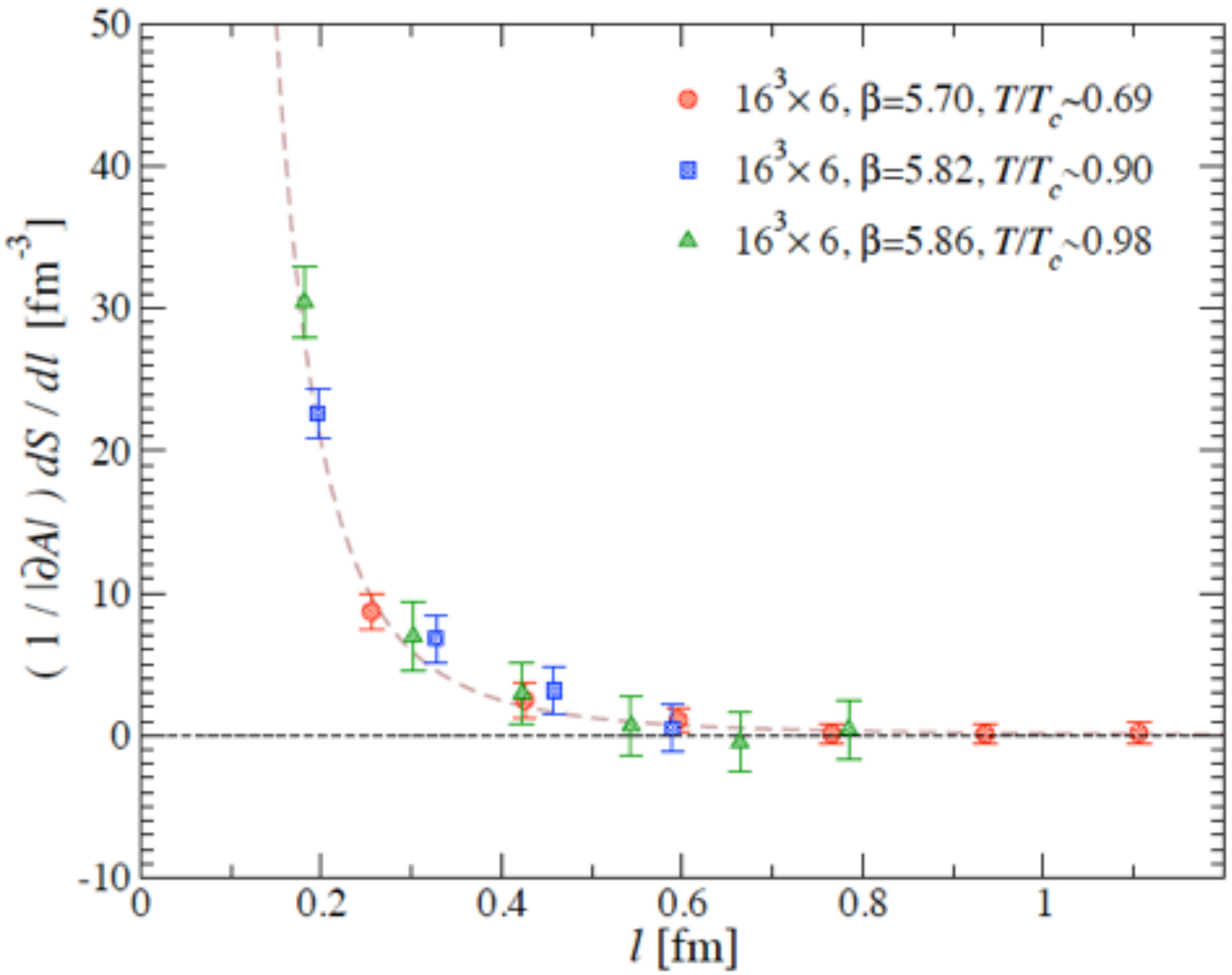


thermal entropy	replica method	integration method (2% error)
		Boyd et al.:NPB469,419(1996)
$T \sim 1.44 T_c$	$s \sim 14.1(8)$	17
$T \sim 2.02 T_c$	$s \sim 61.8(15)$	56

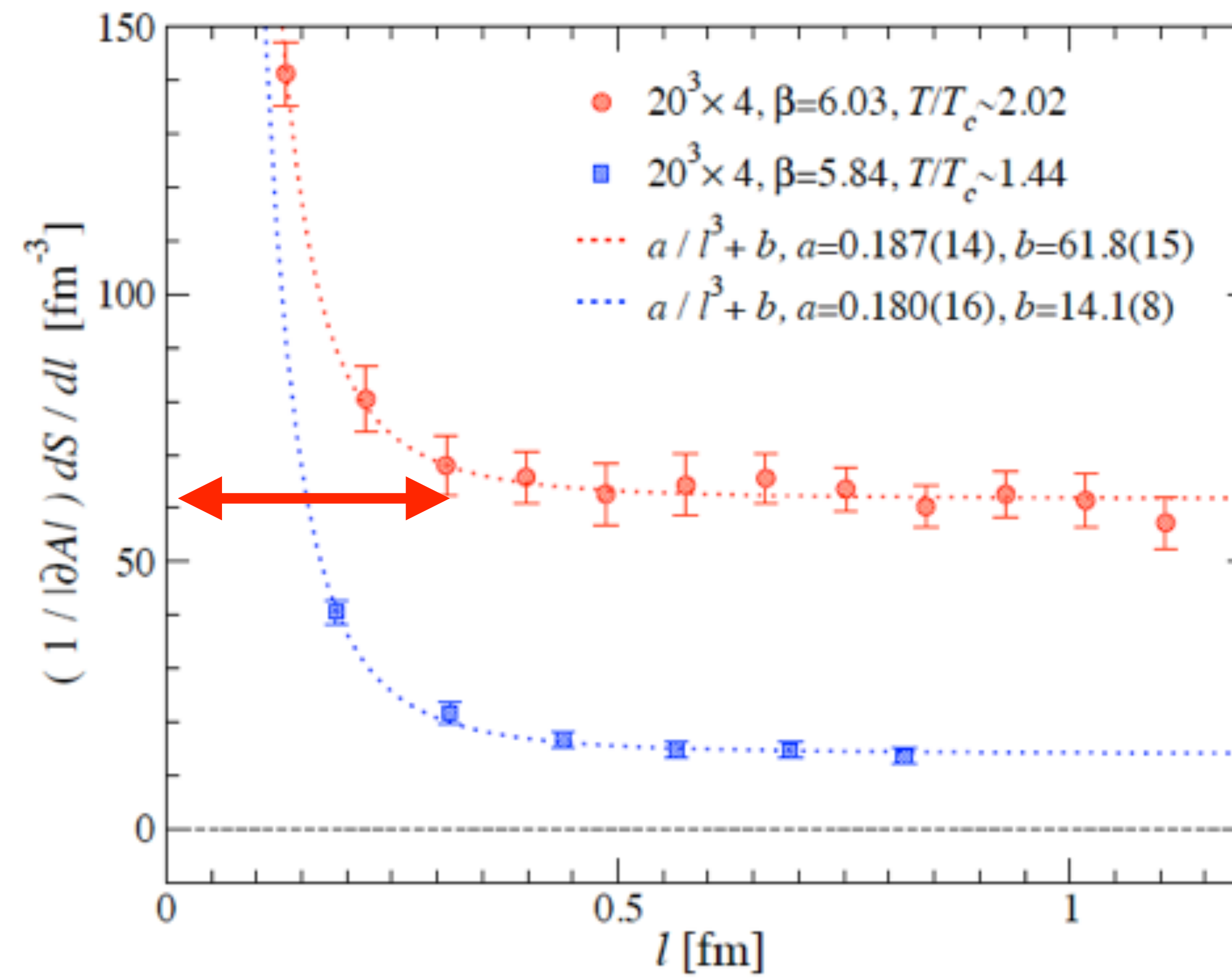
finite T for quenched SU(3)

arXiv:1104.1011: Y.Nakagawa et al.

T< Tc



T >Tc



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