Canonical partition functions and Lee-Yang zeros in QCD

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collaboration with

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KN, S. Motoki, Y. Nakagawa, A. Nakamura, T. Saito [PTEP01A103(2012)]
A. Nakamura, KN [arXiv:1305.0760]
A grand canonical partition function

=> n-particle state with a probability $P(n)$
A grand canonical partition function

=> n-particle state with a probability $P(n)$

What is the shape of this probability in QCD?
Contents

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Introduction
Today, I would like to focus on

- Beam energy scan experiment: exp. data at finite density
- Developments of techniques for finite density LQCD
BES experiments

Investigate phase diagram using HIC with different beam energy

- Number of hadrons measured in heavy ion collision
- Success of thermal statistical models [e.g. Andronic, et. al. (2005)]
- Provide information at freeze out point ($\mu_B$ and $T$)

Nayak, Pramana, 79, 719('12)

Cleymans et. al. PRC73, 034905('06)
BES experiments

Grand canonical ?

- Due to the experimental setup, hadrons can be observed for a limited region
  - seeing a part of fireball
  - analogous to grand canonical system

Canonical approach

a method to study the probability distribution

\[ Z(\mu) = \text{tr} e^{-\beta (\hat{H} - \mu \hat{N})} \]

\[ = \sum_{n=-N}^{N} Z_n e^{n\mu / T} \]

\[ Z_n = \langle n | e^{-\beta \hat{H}} | n \rangle \]

= probability for an n-particle state

(references for canonical approach)
Barbour, Davies, Sabeur, PLB215, 567(1988) 2^4
Barbour, Bell NPB372, 385(1992), Barbour et. al.,
arXiv:hep-lat/9705042
A. Hasenfratz, D. Toussaint, NPB371, 539(92) 2^4
Ejiri, PRD78, 074507(2008) 16^3x4
Li, Meng, Alexandru, Liu, 0810.2349, PoS Lat(2008), Li, 1002.4459, PoS, Lat(2009), Li, Alexandru, Liu,
Canonical approach

It can be applied to both theory and experiment.

\[ Z(\mu) = \text{tr} e^{-\beta (\hat{H} - \mu \hat{N})} \]

\[ = \sum_{n=-N}^{N} Z_n e^{n\mu/T} \]

- probability to observe n-particle state
- calculable in LQCD at \( \mu=0 \)

\[ Z_n = \langle n | e^{-\beta \hat{H}} | n \rangle \]

- There may be an opportunity to compare theory with exp.
- Practically, there are controversy
  - difficulty to measure neutron
  - non-equilibrium
What we can learn from distribution

Shape of the distribution ~ signal for CEP

- Higher order moments of the distribution of conserved charges are sensitive to the correlation length

\[ \sigma^2 = \langle (\delta N)^2 \rangle, \quad S = \frac{\langle (\delta N)^3 \rangle}{\sigma^3}, \quad \kappa = \frac{\langle (\delta N)^4 \rangle}{\sigma^4} - 3 \]

- \( S \): skewness (asymmetry)
- \( \kappa \): kurtosis: sharpeness

Hatta, Stephanov, PRL91, 102003 (2003), Stephanov, PRL102, 032301 (2009), Asakawa, Ejiri, Kitazawa, PRL103, 262301 (2009), Stephanov PRL 107, 052301 (2011), etc
What we can learn from the distribution

First order phase transition from Maxwell construction

- Chemical potential \( \mu \) ~ an energy to add one particle

\[
\mu \equiv F(n + 1) - F(n), \quad (Z_n = e^{-F(n)/T})
= -T(\ln Z_{n+1} - \ln Z_n)
\]

(Left) A. Li, PoS Lat09,
(Right) de Forcrand & Kratochvila, NPB Proc. Suppl. 153, 62 (2006),
What we can learn from the distribution

Canonical approach extends data at a given $\mu$ to wide range.

Accessible values of $\mu$ are limited both in theory and experiments
- lattice QCD (Monte Carlo) is possible at $\mu=0$
- HIC data are obtained at chem. freeze-out

$$Z(\mu) = \sum Z_n e^{n\mu/T}$$

$\{Z_n\}$ is given

$Z(\mu)$ for any $\mu$
What can we learn from the distribution

Canonical approach extends data at a given $\mu$ to wide range.

$$P_n \propto Z_n e^{n\mu/T}$$

Application to experimental data of proton number distribution [Nakamura, Nagata(2013)]
Extraction of $Zn$ and $\mu/T$

$\xi = \exp(\mu/T)$

- $\mu/T$ from the CP invariance vs thermal statistical
\[ Z(n_p) \, e^{n_p \mu/T}, \text{sqrts}_{NN}=200 \]

\[ Z(n_p) \, e^{n_p \mu/T}, \text{sqrts}_{NN}=11.5 \]

\[ \mu/T=0.0 \quad \square \]
\[ \mu/T=1.0 \quad \circ \]
\[ \mu/T=2.0 \quad \triangle \]
\[ \mu/T=2.5 \quad \blacksquare \]
Studies of Zn

- Higher order moments can be a signal
- However, it is unclear if freeze out points in experiments hit CEP
- The data would contain information of finite density QCD even if the CEP is achieved.

It is important to study the shape of the distribution theoretically.

- find properties sensitive to the shape
- clarify its physical meaning
Lattice QCD simulations
Lattice simulations

How do we obtain $Z_n$? : reduction formula + reweighing

$$Z(\mu) = \int D U (\det \Delta(\mu))^{N_f} e^{-S_g}$$

- Reduction formula

$$\det \Delta = \xi^{-\frac{N_{\text{red}}}{2}} C_0(\{U\}) \prod (\xi + \lambda_n(\{U\})), \quad \xi = e^{-\mu/T},$$

$$= C_0 \sum_{n=-\frac{N_{\text{red}}}{2}}^{\frac{N_{\text{red}}}{2}} c_n \xi^n$$


- We use a reweighing in $\mu$

$$Z(\mu) = \sum Z_n e^{n\mu/T}, \quad Z_n \propto \langle \frac{C_0^{N_f} d_n}{(\det \Delta(0))^{N_f}} \rangle$$

[Barbour, et. al. ('91).]
Lattice simulations

- gauge configurations are generated at $\mu = 0$ and use reweighing
- volume: $8^3 \times 4, 10^3 \times 4$
- mass: $\text{mps/mv} \approx 0.8$
- action: clover-improved Wilson fermion + renormalization improved gauge
- # of statistics: 400 (20 trajectory-intervals, 3000 therm.)
Result - Zn

- Lines:
  - Gaussian for \( T/T_{pc} = 1.35 \) and 1,
  - \( \exp(-a|n|) \) for \( T/T_{pc} = 0.93 \).

- Volume: \( 8^3 \times 4, 10^3 \times 4 \), mass: mps/mv~0.8
- Gauge configurations are generated at \( \mu = 0 \) and use reweighing
- Action: clover-improved Wilson fermion + renormalization improved gauge
- # of statistics: 400 (20 trajectory-intervals, 3000 therm.)

[KN, S. Motoki, Y. Nakagawa, A. Nakamura, T. saito, PTEP(2012).]
Result - $Z_n \exp(\mu/T)$ at high $T$

If $Z_n$ is the Gaussian, then the baryon number distribution is also Gaussian.

$$P_n \propto Z_n e^{n\mu/T}$$
Result - $Z_n \exp(\mu/T)$ at low $T$

Increasing $\mu$ at low $T$, a non-trivial shape change has been observed.
Baryon number distribution & fluctuations

\[ Z_{N_B} \exp(\frac{N_B \mu}{T})/Z \]

\[ \langle (\delta N)^m \rangle = \sum_{n_B} (\delta N)^m Z_{n_B} e^{n_B \mu_B/T} / Z_{GC} \]

\[ \sigma^2 = \langle (\delta N)^2 \rangle \]

\[ S = \frac{\langle (\delta N)^3 \rangle}{\sigma^3} \]

\[ \kappa = \frac{\langle (\delta N)^4 \rangle}{\sigma^4} - 3 \]
Lee-Yang zeros: from CPF to Phase transition

Lee-Yang zero Theorem:

zeros of the partition function control the analyticity of the free energy [Lee & Yang 1952]

\[ Z(\mu) = \sum Z_n e^{n\mu/T} \propto \prod (1 - \xi/\xi_i) \]
Result-Lee-Yang zeros

- [Nakamura, Nagata (2013)]
Result-Lee-Yang zeros

- [Nakamura, Nagata (2013)]
Result-Lee-Yang zeros

- [Nakamura, Nagata (2013)]

Lee-Yang zero distribution catches the difference between Tc and TRW
Result - 2
Questions and Subtleties in the calculation

- Lee-Yang zeros are sensitive to $Z_n$; We need careful analysis
- There remains some questions
  - error bars in $Z_n$ : statistical stability
  - truncation of the polynomial : convergence

\[
Z(\mu) = \sum_{n=-\infty}^{\infty} Z_n e^{n\mu/T} \to \sum_{n=-n_0}^{n_0} Z_n e^{n\mu/T} + (|n| > n_0)
\]

We focus on high $T$ region, and perform

- analytic calculation
- reanalysis of lattice data
Analytic calculation of Zn of high T QCD

Properties of high T QCD leads to Gaussian Zn

- Zn can be obtained from the F.T.
  \[ Z_n = \int Z(\theta)e^{in\theta}d\theta, \quad \mu/T = i\theta, (\theta \in \mathbb{R}) \]

- Use properties of high T QCD
  \[ -\frac{f}{T^4} = c_0 + c_2(\mu/T)^2 + c_4(\mu/T)^4 \]
  \[ Z\left(\frac{\mu I}{T}\right) = Z\left(\frac{\mu I}{T} + \frac{2\pi}{N_c}\right). \]

- also use the saddle point approximation
  \[ Z_n = Ce^{-n^2/(4T^3Vc_2)}, \quad (n \equiv 0 \mod 3) \]
Gaussian Zn + RW periodicity leads to Lee-Yang zeros as

\[ \frac{\mu}{T} = \frac{(2l + 1)\pi i}{3} - \frac{3(2k + 1)}{4VT^3c_2} \]
Analytical calculation of LY zeros of high T QCD

Gaussian Zn + RW periodicity leads to Lee-Yang zeros as

$$\frac{\mu}{T} = \frac{(2l + 1)\pi i}{3} - \frac{3(2k + 1)}{4VT^3c_2}$$

High T QCD

$$Z_n = Ce^{-n^2/(4T^3Vc_2)}, \ (n \equiv 0 \mod 3)$$

$$Z(\mu) = C \sum_{n_B=-\infty}^{\infty} e^{-9n_B^2/(4T^3Vc_2)+3n_B\mu/T}$$

Theta function

$$\vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{-\pi in^2 \tau + 2\pi inz}$$

$$i\pi \tau = \frac{9}{(4T^3Vc_2)}$$

$$2\pi iz = \frac{3\mu}{T}$$

$$\vartheta(z, \tau) = 0 \iff z = l + k\tau + \frac{1}{2} + \frac{\tau}{2}, \ (k, l \in \mathbb{Z})$$
Alternative calculation

Cancellation of two types of free energy allows $Z=0$

- Cancellation of free energies [Biskup et al('01)]

$$Z = e^{-V f_I / T} + e^{-V f_{II} / T}$$

$$\begin{cases} 
\text{Re}[f_I - f_{II}] = 0 \\
\frac{V}{T} \text{Im}[f_I - f_{II}] = (2k - 1)\pi, (k \in \mathbb{Z}) 
\end{cases}$$

- Approximate solution for $c_2 \gg c_4$

$$\left(\mu_R, \mu_I\right) = \left(\frac{3(2k - 1)}{4VT^3c_2}, -\pi/3\right)$$

- It is also possible to solve it in the presence of $c_4$

- $f_I$ and $f_{III}$, and $f_{II}$ and $f_{III}$. 
Analytical calculation of LY zeros of high T QCD

\[ \frac{\mu}{T} = \frac{(2l + 1) \pi i}{3} - \frac{3(2k + 1)}{4VT^3 c_2} \]

\[ \text{angular} \quad \text{radial} \]

\[ \xi = \exp\left(-\frac{\mu}{T}\right) \]
Analytical calculation of LY zeros of high T QCD

\[
\frac{\mu}{T} = \frac{(2l + 1) \pi i}{3} - \frac{3(2k + 1)}{4VT^3c_2}
\]

\[\xi = \exp(-\mu/T)\]
Analytical calculation of LY zeros of high T QCD

\[ \frac{\mu}{T} = \frac{(2l + 1) \pi i}{3} - \frac{3(2k + 1)}{4VT^3 c_2} \]

- angular
- radial

\[ \xi = \exp(-\mu/T) \]
Lee-Yang zeros : Ising vs free fermion gas

a) Ising

\[ \angle e^h \]

b) high T QCD

\[ \angle e^{\mu_B}/T \]

c) high T QCD

\[ \angle e^{\mu_q}/T \]
Reanalysis of Lattice data & analytic result

- We reanalyze previous lattice data.
  - errors of Zn: bootstrap analysis (1000 BS samples.)
  - estimate the convergence
  - finite size scaling

\[ |\text{Re}[\xi]|^{VT^3} = \exp\left(-\frac{(3k + 1)}{(4c_2)}\right) \]

---

a) \( N_s=10 \) \( n_0=37 \)
b) \( N_s=8 \) \( n_0=32 \)
c) \( N_s=8 \) \( n_0=19 \)
Reanalysis of Lee-Yang zeros for high T QCD

Analytic and lattice calculation are consistent!

Solid Line: $|\text{Re}[\xi]|^{VT^3} = \exp\left(\frac{-3k + 1}{4c_2}\right)$

- Lee-Yang zeros near the unit circle are statistically stable
- b)-c): convergence ok
- a)-c): volume scaling consistent with the analytic one.
- underestimation: saddle point approximation
Completion of deconfinement

Gaussian behavior

Roberge-Weiss phase transition

c2-dominance

RW periodicity

Theta function

implication

 completion

c2-dominance

RW periodicity

Theta function

Gaussian behavior

Roberge-Weiss phase transition
Summary

Temperature

Early Universe
Future LHC Experiments
Current LHC Experiments

Crossover

Critical Point

Vacuum

Hadron Gas

$Z_{n_B}$

$n_B$

$N_s=10$

$N_s=8$

$p$-Bosons

$n_0=19$

$N_s=10$, $n_0=37$

$p$-Bosons

$p$-Bosons

$T/T_\text{pc}=0.99$

$T/T_\text{pc}=0.93$

$p$-Bosons

$p$-Bosons

$p$-Bosons

$p$-Bosons

Joint Institute for Computational Fundamental Science
Summary

- Canonical approach
  - provides a way to extend data obtained at a certain $\mu/T$ to other values of $\mu/T$.
  - can be applied not only to lattice QCD but also to experimental data.

- We showed that the net baryon number follows
  - Gaussian distribution at high temperatures
  - Gaussian shape is an indication of RW phase transition (high T)
  - Lee-Yang zeros are sensitive to a shape of the canonical partition functions.
Applications to Beam energy scan experiment
Number of hadrons
- fixed at a time when inelastic process ceases
  - chemical freeze-out

Number of hadrons provide information at the freeze-out time
Hadron yields and thermal statistical model

\[ n_i = \frac{N_i}{V} = -\frac{T}{V} \frac{\partial \ln Z_i}{\partial \mu} = \frac{g_i}{2\pi^2} \int_{0}^{\infty} \frac{p^2 dp}{\exp[(E_i - \mu_i)/T] + 1} \]

Hadron yields and thermal statistical model

\[ n_i = \frac{N_i}{V} = -\frac{T}{V} \frac{\partial \ln Z_i}{\partial \mu} = \frac{g_i}{2\pi^2} \int_0^\infty \frac{p^2 \, dp}{\exp[(E_i - \mu_i)/T] \pm 1} \]

collision energy \(\sim \mu, T\)

(freeze-out parameters)
Energy dependence of freeze-out parameters

Freeze-out points are located on a line ~ freeze-out line

Cleymans et. al.
Energy dependence of freeze-out parameters

Cleymans et. al.
Probability distribution for some hadrons are obtained. This is used to obtain event-by-event fluctuation at a freeze-out point.
Extension of data to wide range of $\mu$

The probability distribution of net baryon number

(here we use the proton number as an approximation)

$$P_n \propto Z_n e^{n\mu/T}$$

$\mu/T$ is determined from CP invariance : $Z(n) = Z(-n)$

\[\sqrt{s_{NN}} = 39\text{ GeV}\]

\[\text{sqrt}(s) = 62.4\]
Extension of data to wide range of $\mu$

- $\mu/T$ obtained from CP invariance agree with those obtained from thermal statistical model for wide range of collision energies.

\[ \xi = \exp(\mu/T) \]
$Z(n_p) \ e^{n_p \mu/T}$, $\text{sqrts}_{NN}=200$

$\mu/T=0.0$  
$\mu/T=1.0$  
$\mu/T=2.0$  
$\mu/T=2.5$

$Z(n_p) \ e^{n_p \mu/T}$, $\text{sqrts}_{NN}=11.5$

$\mu/T=0.0$  
$\mu/T=1.0$  
$\mu/T=2.0$  
$\mu/T=2.5$
$Z(n_p) \ e^{n_p \mu/T}$, $\sqrt{s_{NN}}=200$

$Z(n_p) \ e^{n_p \mu/T}$, $\sqrt{s_{NN}}=11.5$

$\lambda_n \propto (\partial/\partial \mu)^n \ln Z(\mu)$
RHIC Data

Kurtosis $\frac{\lambda_4}{\lambda_2}$ as a function of $\frac{\mu}{T}$
Buck up slides
How to achieve LY zeros?

- Calculation of $Z_n$: truncation is inevitable

$$Z(\mu) = \sum_{n=-\infty}^{\infty} Z_n e^{n\mu/T}$$

$$\rightarrow \sum_{n=-n_0}^{n_0} Z_n e^{n\mu/T}$$
Calculation of Lee-Yang zeros
- Zeros of the fugacity polynomial

\[ Z(\mu) = \sum_n Z_n e^{n\mu/T} \]

- Cancellation of the free energy

\[ Z = e^{-V f_I/T} + e^{-V f_{II}/T} \]
1. Cancellation of free-energy

Free energy is different in different RW phases

- RW phases are distinguished by the argument of Pol.
- The \( \arg(\text{Pol}) \) is translated into \( A_4 \).

\[
\mathcal{L}_4 = \bar{\psi} (\gamma_4 (igA_4 + \mu)) \psi
\]

\[
\mu \rightarrow \mu + igA_4 = \mu + i\omega T
\]

- This modifies the free energy as

\[
f_I = -T^4 (c_0 + c_2 (\mu/T)^2 + c_4 (\mu/T)^4),
\]

\[
f_{II} = -T^4 (c_0 + c_2 (\mu'/T)^2 + c_4 (\mu'/T)^4), \mu' = \mu + i\omega T
\]

\( \mu, \mu' \in C \)
1. Cancellation of free-energy

Cancellation of two types of free energy allows $Z=0$

- Cancellation of free energies [Biskup et al.'01)]

$$Z = e^{-V f_1/T} + e^{-V f_{II}/T}$$

\[
\begin{align*}
\text{Re}[f_1 - f_{II}] &= 0 \\
\frac{V}{T} \text{Im}[f_1 - f_{II}] &= (2k - 1)\pi, \quad (k \in \mathbb{Z})
\end{align*}
\]

- Approximate solution for $c_2>>c_4$

$$\left(\mu_R, \mu_I\right) = \left(\frac{3(2k - 1)}{4VT^3c_2}, -\pi/3\right)$$

- Possible to solve it with $c_4$

- $f_1$ and $f_{III}$, and $f_{II}$ and $f_{III}$. 
2. Zeros of the fugacity polynomial

We show that the fugacity polynomial of high T QCD is well approximated by a well-known polynomial.

\[ Z(\mu) = \sum_{n=-N}^{N} Z_n e^{n\mu/T} \]

- We need \( Z_n \): Fourier integral with the free energy as input (analytic)
- (We use the fugacity expansion in lattice simulation.)
First, we derive $Z_n$ using the Fourier transformation.

$$Z(\mu) = \sum_{n=-N}^{N} Z_n e^{n\mu/T}$$

$Z_n$ from the Fourier transformation

$$Z_n = \int Z(\theta) e^{in\theta} d\theta, \quad \mu/T = i\theta, (\theta \in \mathbb{R})$$

$$f = -\frac{T}{V} \ln Z(\mu)$$

- We use the quartic form of $f(\mu)$ as input.
- $f(\mu)$ is obtained for real $\mu$. On the other hand, the Fourier integral requires complex $\mu$. 
Next, we use the RW periodicity

\[ Z_n = \int Z(\theta) e^{in\theta} d\theta, \quad \mu/T = i\theta, (\theta \in \mathbb{R}) \]

We decompose the integral into three domains.

\[ Z_n = \int_{-\pi/3}^{\pi/3} e^{-V f(\theta)/T} e^{in\theta} \frac{d\theta}{2\pi} + \int_{\pi/3}^{\pi} e^{-V f(\theta-2\pi/3)/T} e^{in\theta} \frac{d\theta}{2\pi} \]

\[ + \int_{\pi}^{5\pi/3} e^{-V f(\theta-4\pi/3)/T} e^{in\theta} \frac{d\theta}{2\pi}, \]

Shift of \( \theta \) leads to

\[ Z_n = \frac{3}{2\pi} \int_{-\pi/3}^{\pi/3} e^{-V f(\theta)/T} e^{in\theta} d\theta, \quad n \equiv 0 \pmod{3} \]

(This relation holds for any temperature, regardless of the RW phase transition)
Now, we use the quartic expression of $f(\mu)$ and SPA

$$Z_n = \frac{3}{2\pi} \int_{-\pi/3}^{\pi/3} e^{VT^3(c_0-c_2\theta^2+c_4\theta^4)} e^{in\theta} d\theta, \quad n \equiv 0 \mod 3$$

At $T/T_c > 1.1 \sim 1.2$, $c_2/c_4 \sim 10$. We use the saddle point approximation.

$$Z_n = C e^{-n^2/(4T^3 V c_2)}, \quad (n \equiv 0 \mod 3)$$

- We assume the Gaussian $Z_n$ is valid for large $n$.
- SPA is valid for small Re[$\mu$]
If Zn is Gaussian, then $Z(\mu)$ is a Jacobi-theta function.

$$Z(\mu) = C \sum_{n_B = -\infty}^{\infty} e^{-9n_B^2/(4T^3Vc_2) + 3n_B\mu/T}$$

**Theta function**

$$\theta(z, \tau) = \sum_{n = -\infty}^{\infty} e^{-\pi in^2\tau + 2\pi inz}$$

$$i\pi\tau = 9/(4T^3Vc_2),$$
$$2\pi iz = 3\mu/T$$

**Zeros of theta function**

$$\theta(z, \tau) = 0 \leftrightarrow z = l + k\tau + \frac{1}{2} + \frac{\tau}{2}, \ (k, l \in \mathbb{Z})$$

$$\frac{\mu}{T} = \frac{(2l + 1)\pi i}{3} - \frac{3(2k + 1)}{4VT^3c_2} \cdot (\mu_R, \mu_I) = \left(\frac{3(2k - 1)}{4VT^3c_2}, -\pi/3\right)$$
Lee-Yang zero distribution of theta function

\[
\frac{\mu}{T} = \frac{(2l + 1)\pi i}{3} - \frac{3(2k + 1)}{4VT^3c_2}
\]

Zeros approaches to the RW transition point as 1/V. Spacing of zeros is a prediction.
How to achieve LY zeros?

- Calculation of Zn: truncation is inevitable

\[
Z(\mu) = \sum_{n=-\infty}^{\infty} Z_n e^{n\mu/T}
\]

\[
\rightarrow \sum_{n=-n_0}^{n_0} Z_n e^{n\mu/T}
\]

- Cauchy integral + recursive division + multi-precision arithmetic

\[
Z(\mu) = \sum Z_n e^{n\mu/T} \rightarrow \prod \left(1 - \frac{\xi}{\xi_i}\right)
\]
Validity of SPA

\[ Z(\mu) = C \sum e^{-9n^2/(4VT^3c_2) + 3n\mu/T} = C\vartheta(z, \tau) \]
Lee-Yang zeros - 20 years
RW periodicity & phase transition

\[ \beta = 1.80 \]

- \( \mu_I = 0.00 \)  
- \( \mu_I = 0.16 \)  
- \( \mu_I = \pi/12 \)  
- \( \mu_I = 0.28 \)

\[ \beta = 1.95 \]

- \( \mu_I = 0.00 \)  
- \( \mu_I = 0.16 \)  
- \( \mu_I = \pi/12 \)  
- \( \mu_I = 0.28 \)
Taylor coefficients of free energy

\[-\frac{f}{T^4} = \sum_{n=0}^{\infty} c_{2n} (\mu/T)^{2n}\]
Lee-Yang zeros : from CPF to Phase transition

- Lee-Yang zeros [Lee & Yang 1952]

\[
Z(\mu) = \sum Z_n e^{n\mu/T} \propto \prod (1 - \xi/\xi_i)
\]

- \(Z(\mu) = 0\) is an origin of a thermodynamic non-analyticity
Convergence on the negative real axis

- Infinite sum of higher order terms is bounded on the negative real axis.

\[ |Z(\mu) - \sum_{|n|<N} Z_n \xi^n| = | \sum_{|n|\geq N} Z_n \xi^n| < |Z_N \xi^N| \]

- c.f. Leibnitz’s test for an alternating series

\[ 1/2 - 1/3 + 1/4 - 1/5 + 1/6 + \cdots < 1/2 \]