Canonical partition functions and Lee-Yang zeros in QCD

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collaboration with

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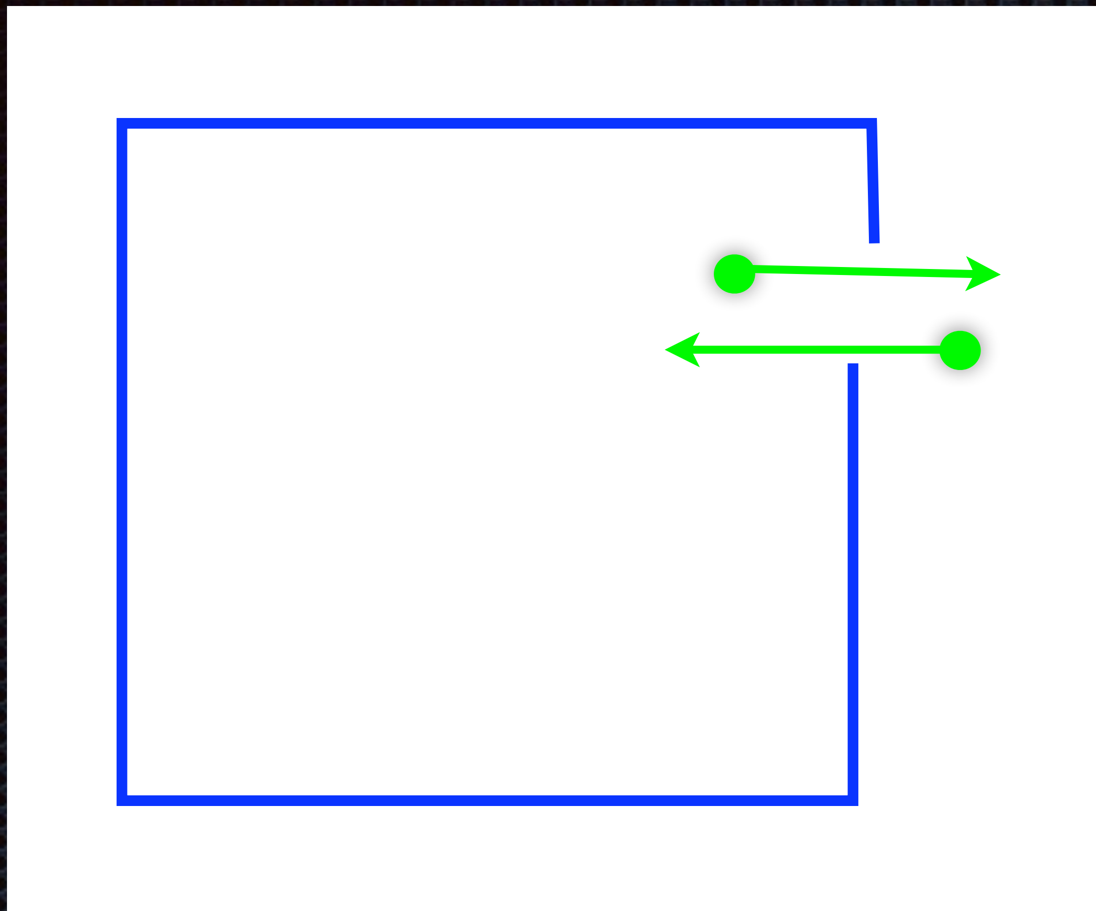
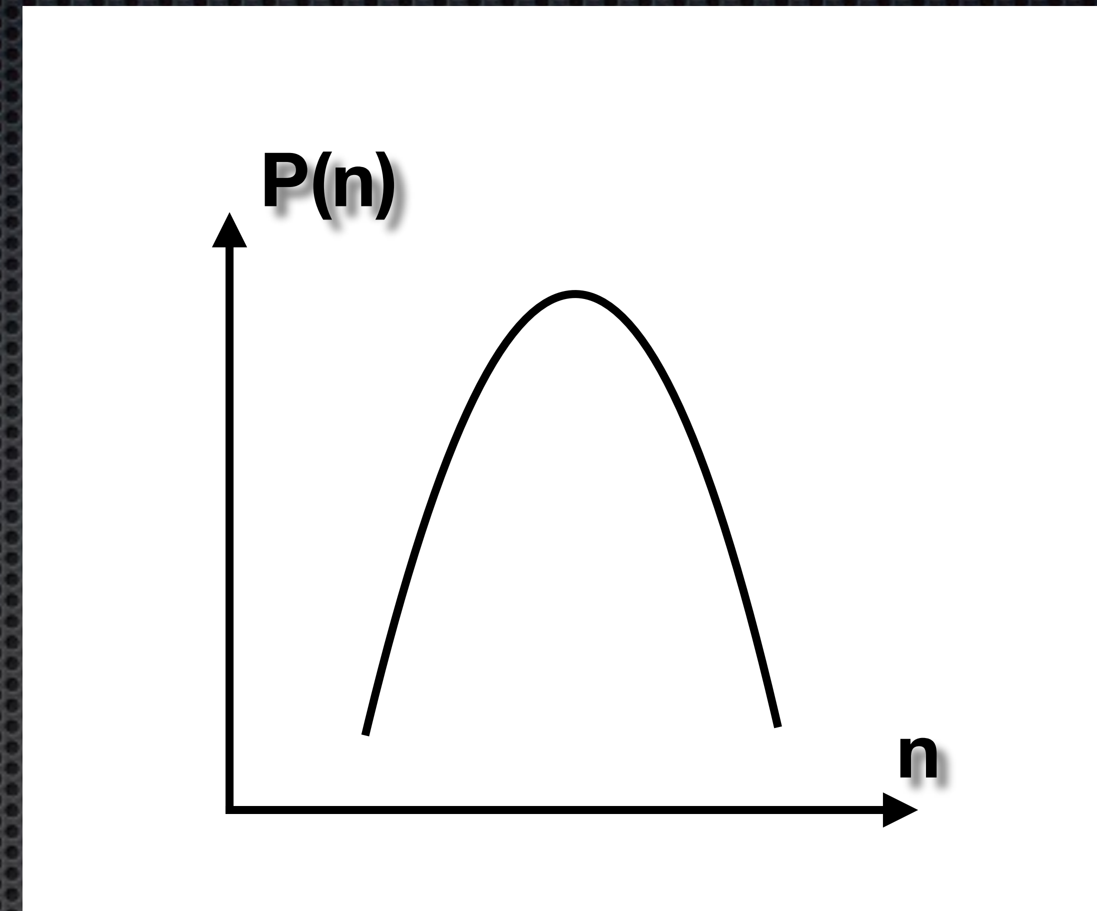
S. M. Nishigaki(Shimane)

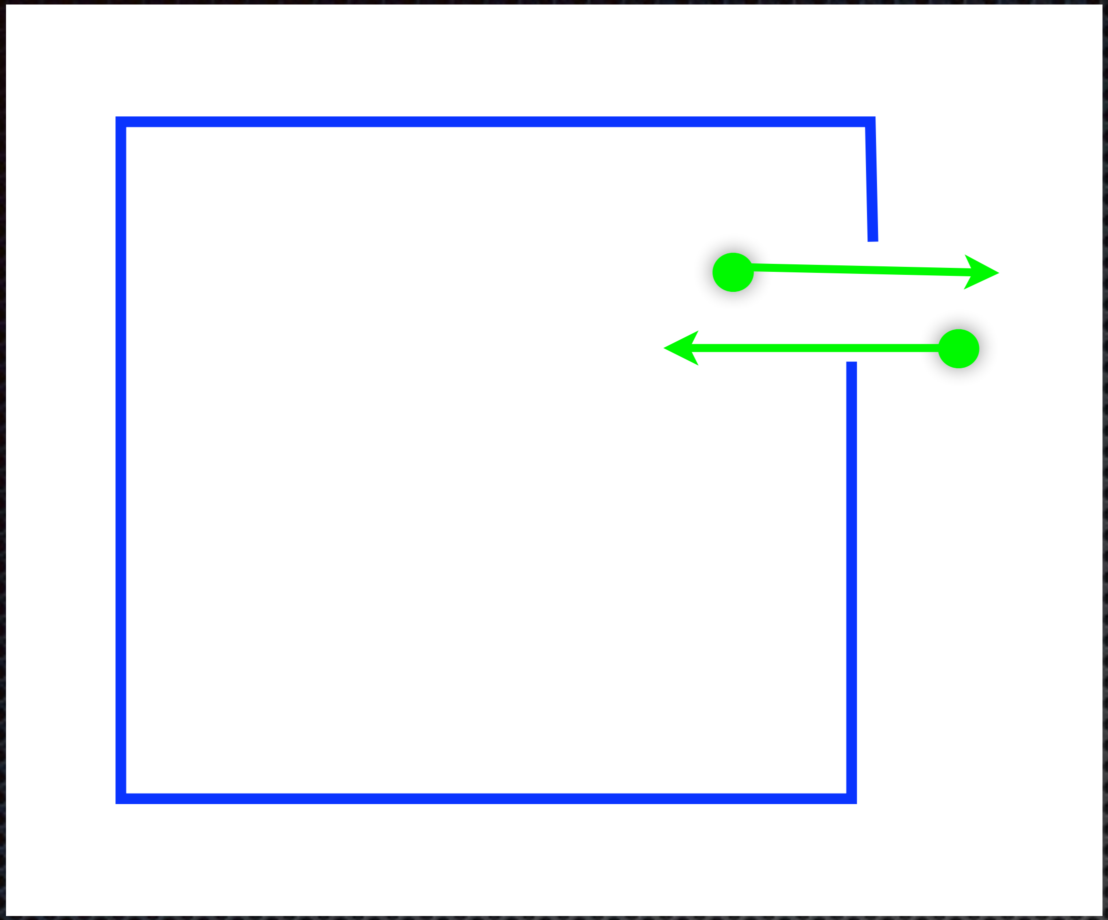
KN, S. Motoki, Y. Nakagawa, A. Nakamura, T. Saito [PTEP01A103(2012)]
A. Nakamura, KN [arXiv:1305.0760]
KN, K. Kashiwa, A. Nakamura, S. M. Nishigaki arXiv:1410.0783

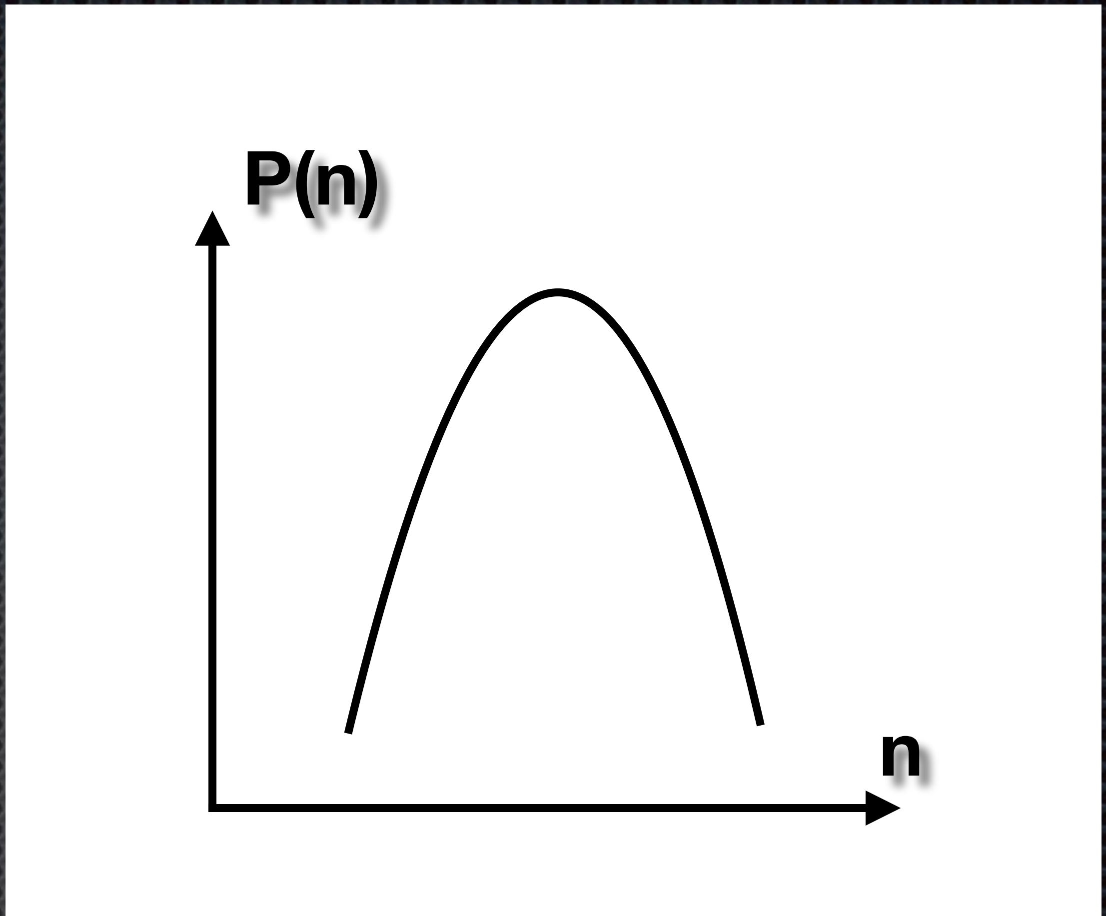




A grand canonical partition function

=> n-particle state with a probability P(n)





A grand canonical partition function

=> n-particle state with a probability P(n)

What is the shape of this probability in QCD ?

Contents

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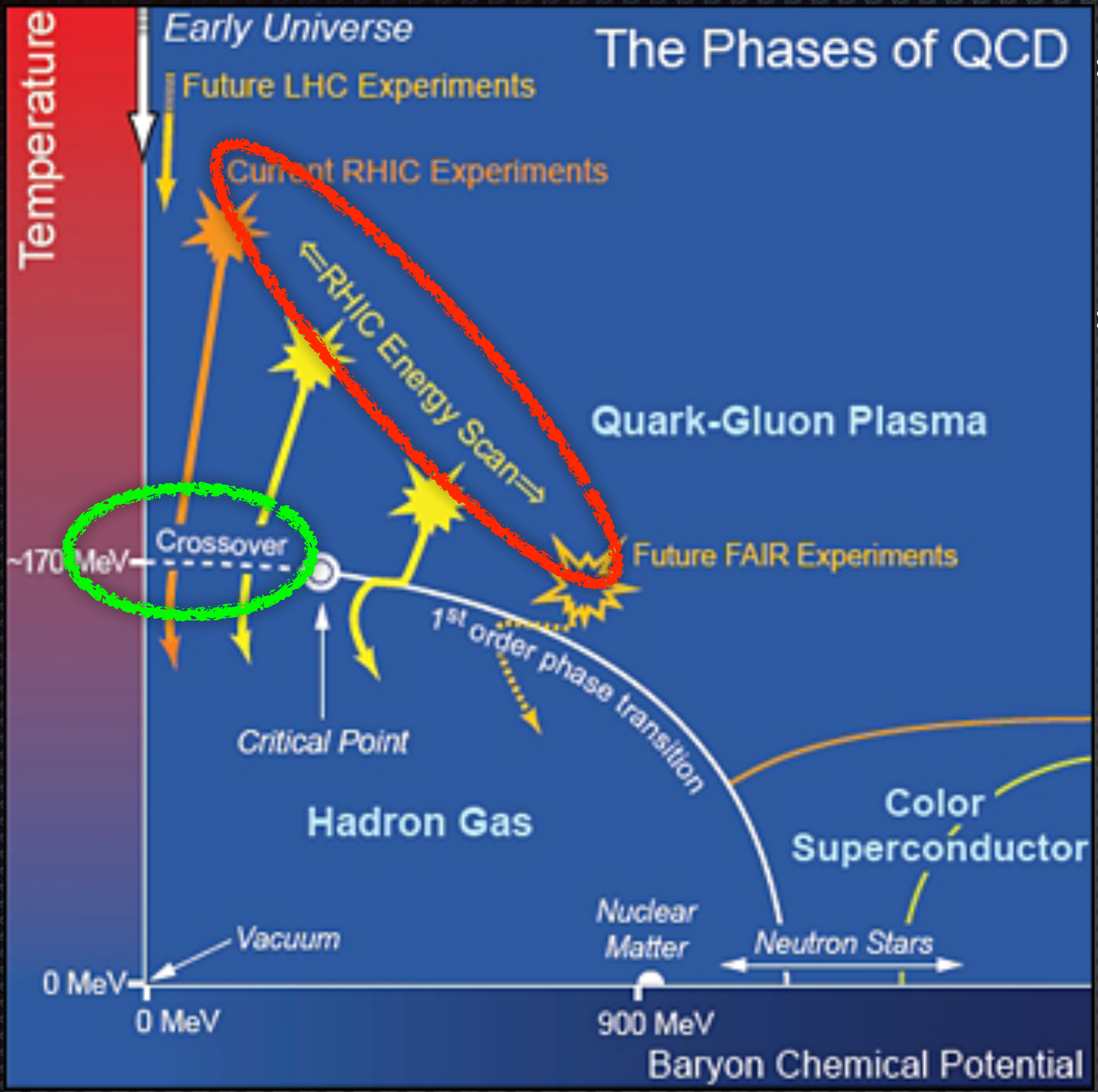
Introduction

Today, I would like to focus on

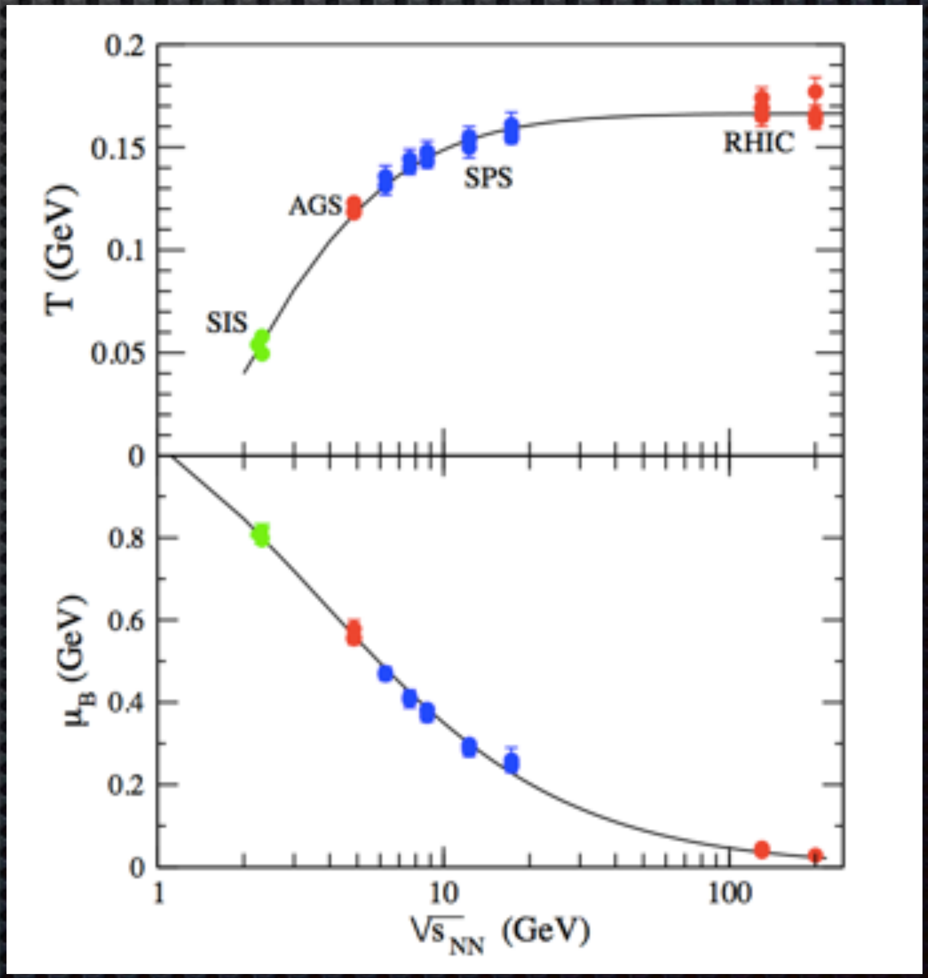


- **Beam energy scan experiment : exp. data at finite density**
- **Developments of techniques for finite density LQCD**

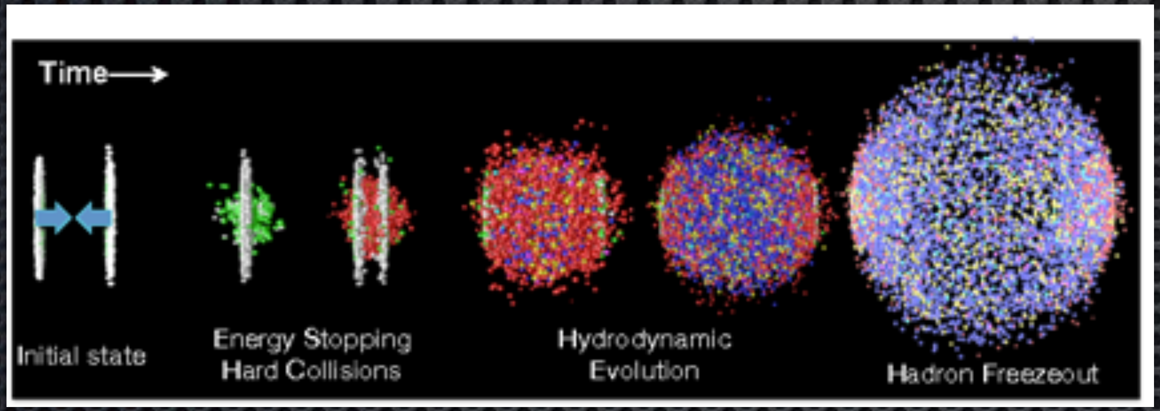
BES experiments

Investigate phase diagram using HIC with different beam energy

- Number of hadrons measured in heavy ion collision
  - success of thermal statistical models [e.g.Andronic, et., al, (2005)]
  - provide information at freeze out point (μ_B and T)



Cleymans et. al. PRC73, 034905('06)

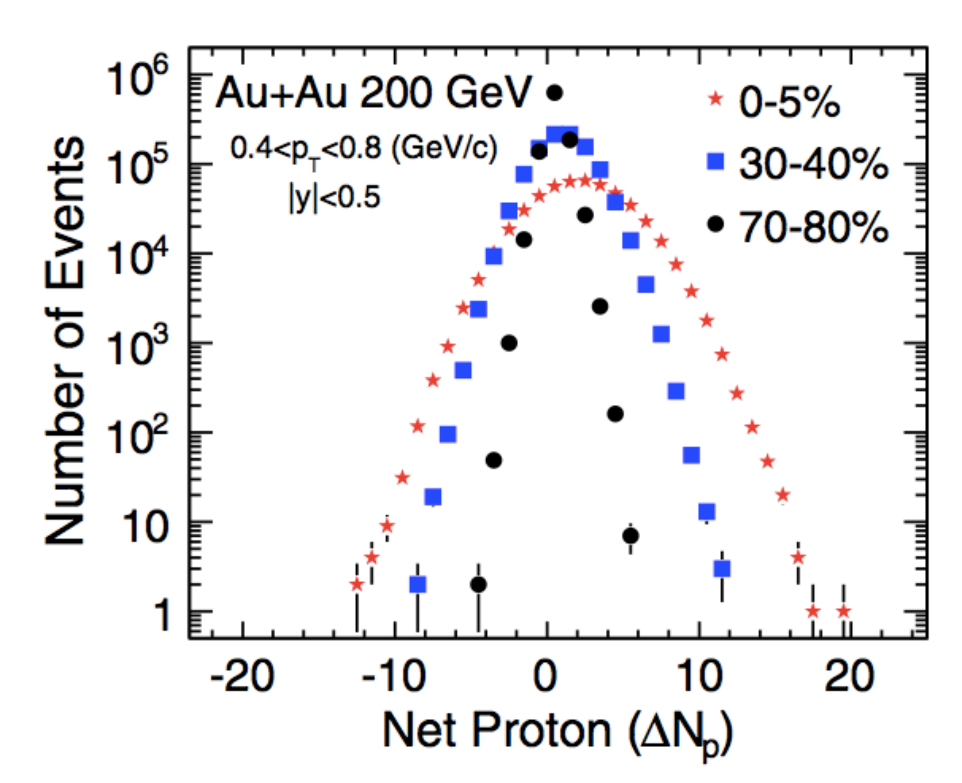


Nayak, Pramana, 79, 719('12)

BES experiments

Grand canonical ?

- Due to the experimental setup, hadrons can be observed for a limited region
  - seeing a part of fireball
  - analogous to grand canonical system



Agaarwal et.al. PRL105, 022302('10), arXiv:1004.4959.

Canonical approach

a method to study the probability distribution

$$Z(\mu) = \mathrm{tr} e^{-\beta(\hat{H}-\mu\hat{N})}$$

$$= \sum_{n=-N}^{N} Z_n e^{n\mu/T}$$

= probability for an n-particle state

$$Z_n = \langle n | e^{-\beta \hat{H}} | n \rangle$$

(references for canonical approach)

Barbour, Davies, Sabeur, PLB215, 567(1988) 2^4, Barbour, Bell NPB372, 385(1992)., Barbour et. al., arXiv:hep-lat/9705042

A. Hasenfratz, D. Toussaint, NPB371, 539('92) 2^4

de Forcrand, Kratochvila NPB Proc. Suppl. 153, 62 (2006), Kratochvila, de Forcrand, 0509143, PoS Lat2005.

Ejiri, PRD78, 074507(2008) 16^3x4

Li, Meng, Alexandru, Liu, 0810.2349, PoS Lat(2008) , Li, 1002.4459, PoS, Lat(2009) , Li, Alexandru, Liu, Meng 1005.4158, Phys.Rev. D82 (2010) 054502 , Li, Alexandru, Liu, PRD84, 071503, arXiv: 1103.3045

Canonical approach

It can be applied to both theory and experiment.

$$Z(\mu) = \mathrm{tr} e^{-\beta(\hat{H}-\mu\hat{N})}$$

$$= \sum_{n=-N}^{N} Z_n e^{n\mu/T}$$

$$Z_n = \langle n | e^{-\beta\hat{H}} | n \rangle$$

probability to observe n-particle state

calculable in LQCD at μ=0

- There may be an opportunity to compare theory with exp.
- Practically, there are controversy
  - difficulty to measure neutron
  - non-equlibrium

What we can learn from distribution

Shape of the distribution ~ signal for CEP

- Higher order moments of the distribution of conserved charges are sensitive to the correlation length

$$\sigma^2 = \langle(\delta N)^2\rangle, S = \langle(\delta N)^3\rangle/\sigma^3, \kappa = \langle(\delta N)^4\rangle/\sigma^4 - 3$$

  - S : skewness (asymmetry)
  - kappa : kurtosis : sharpeness

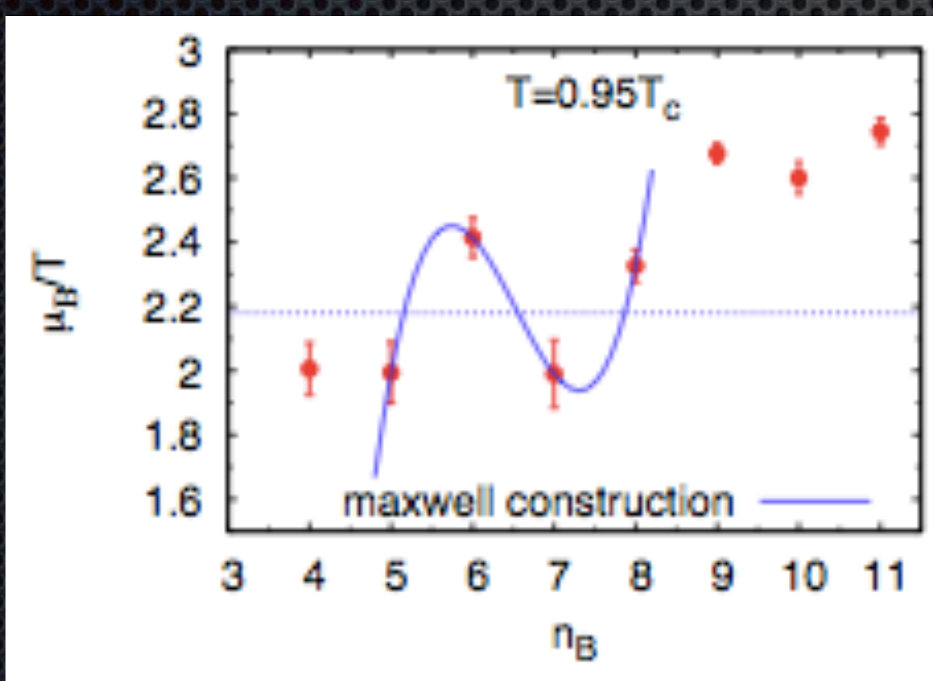
Hatta, Stephanov, PRL91, 102003(2003), Stephanov, PRL102, 032301(2009), Asakawa, Ejiri, Kitazawa, PRL103, 262301 (2009) , Stephanov PRL 107, 052301 (2011), etc

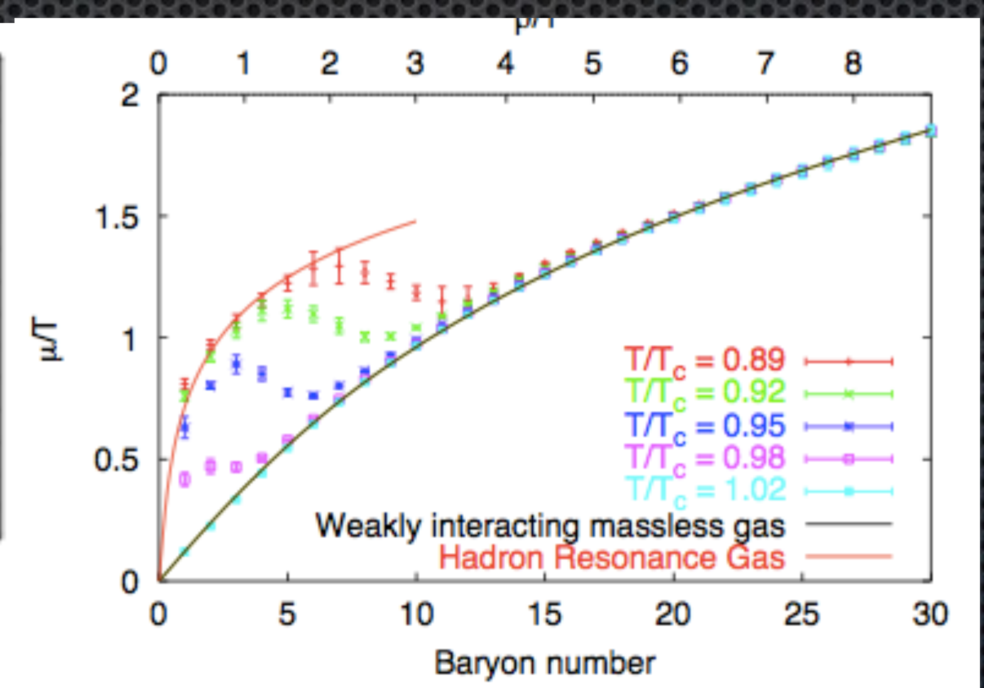
What we can learn from the distribution

First order phase transition from Maxwell construction

- Chemical potential ~ an energy to add one particle

$$\mu \equiv F(n+1) - F(n), \quad (Z_n = e^{-F(n)/T})$$
$$= -T(\ln Z_{n+1} - \ln Z_n)$$





(Left) A.Li, PoS Lat09,
(Right) de Forcrand & Kratochvila, NPB Proc. Suppl. 153, 62 (2006),

What we can learn from the distribution

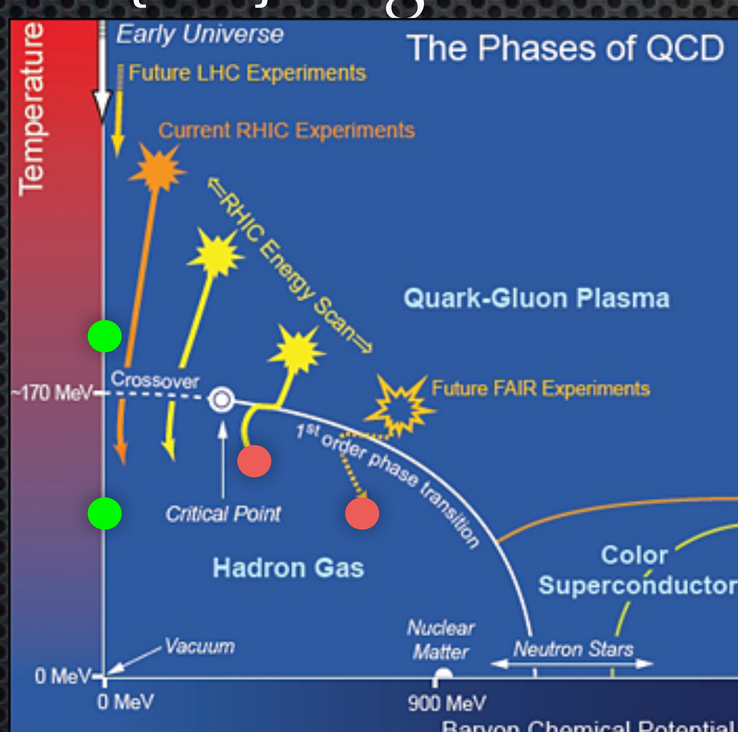
Canonical approach extends data at a given μ to wide range.

Accessible values of μ are limited both in theory and experiments

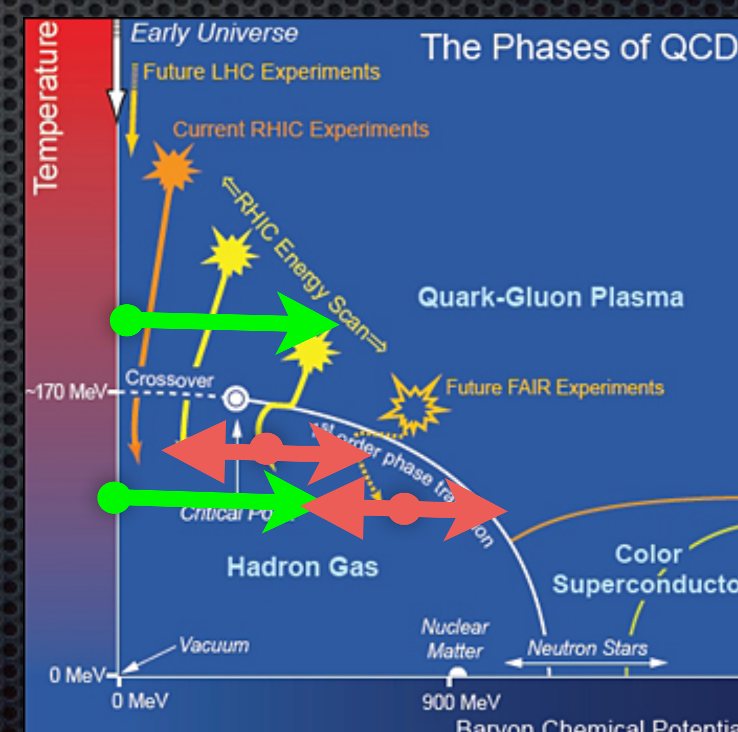
- lattice QCD(Monte Carlo) is possible at μ=0
- HIC data are obtained at chem. freeze-out

$$Z(\mu) = \sum Z_n e^{n\mu/T}$$

{Zn} is given



Z(μ) for any μ

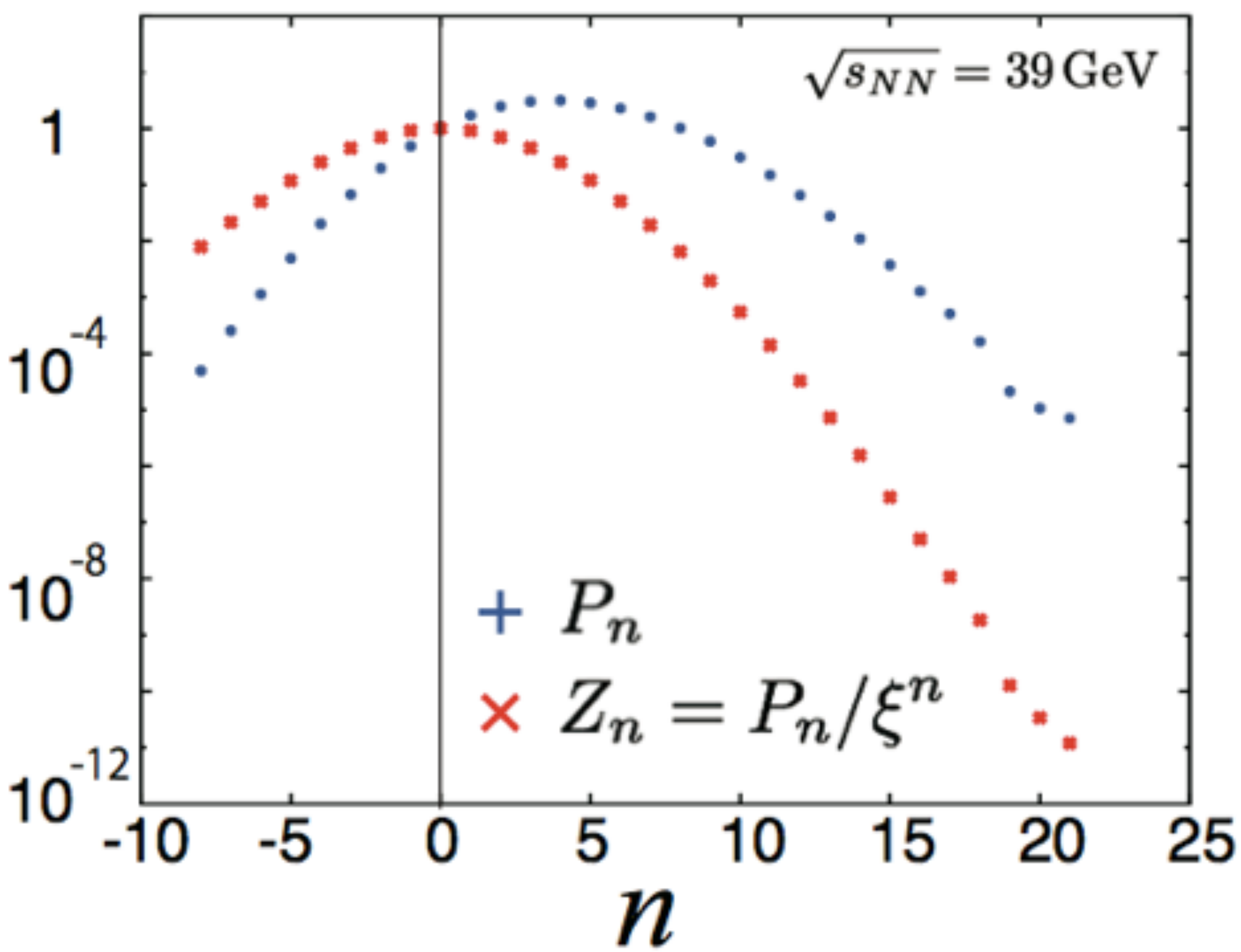


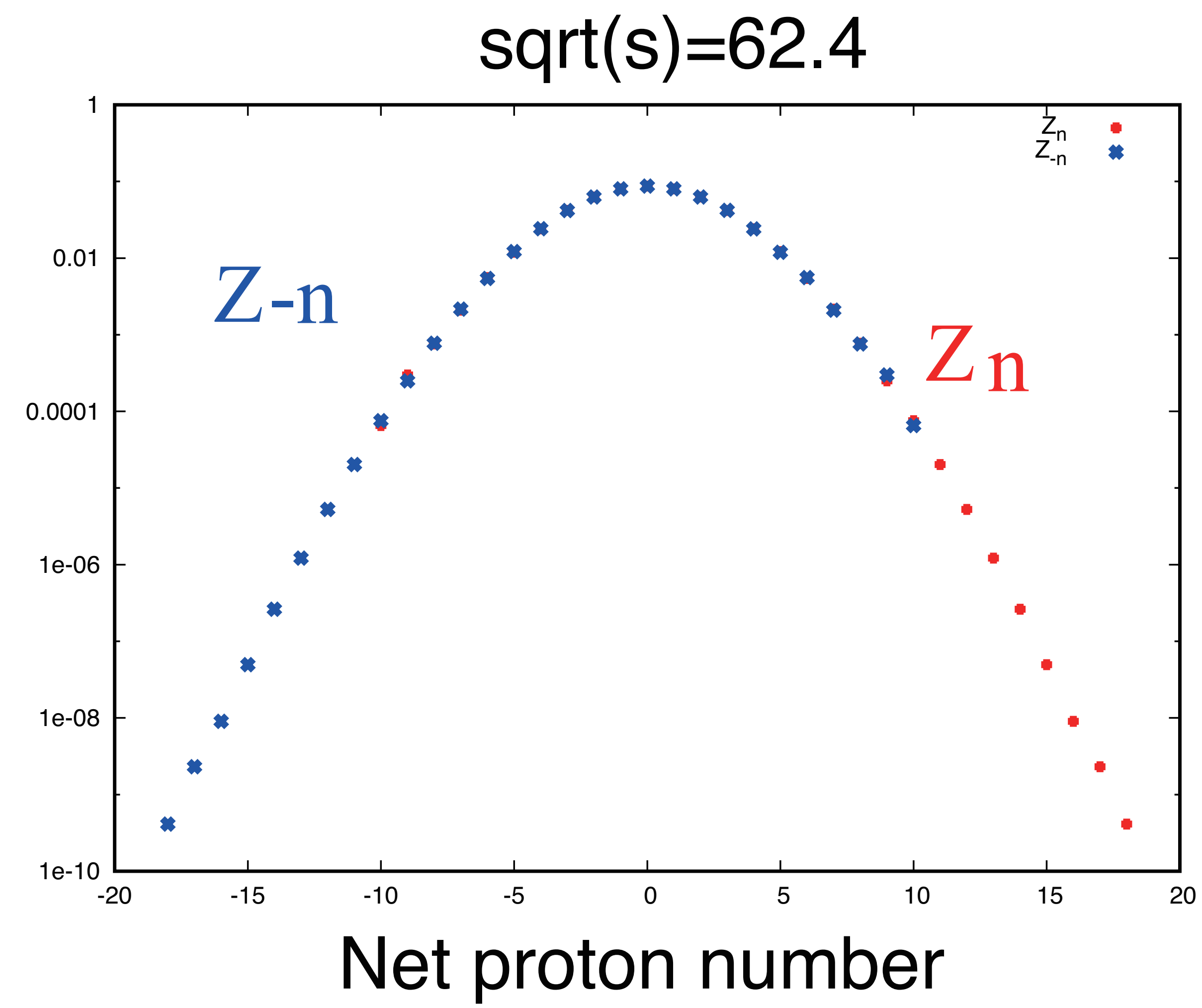
What can we learn from the distribution

Canonical approach extends data at a given μ to wide range.

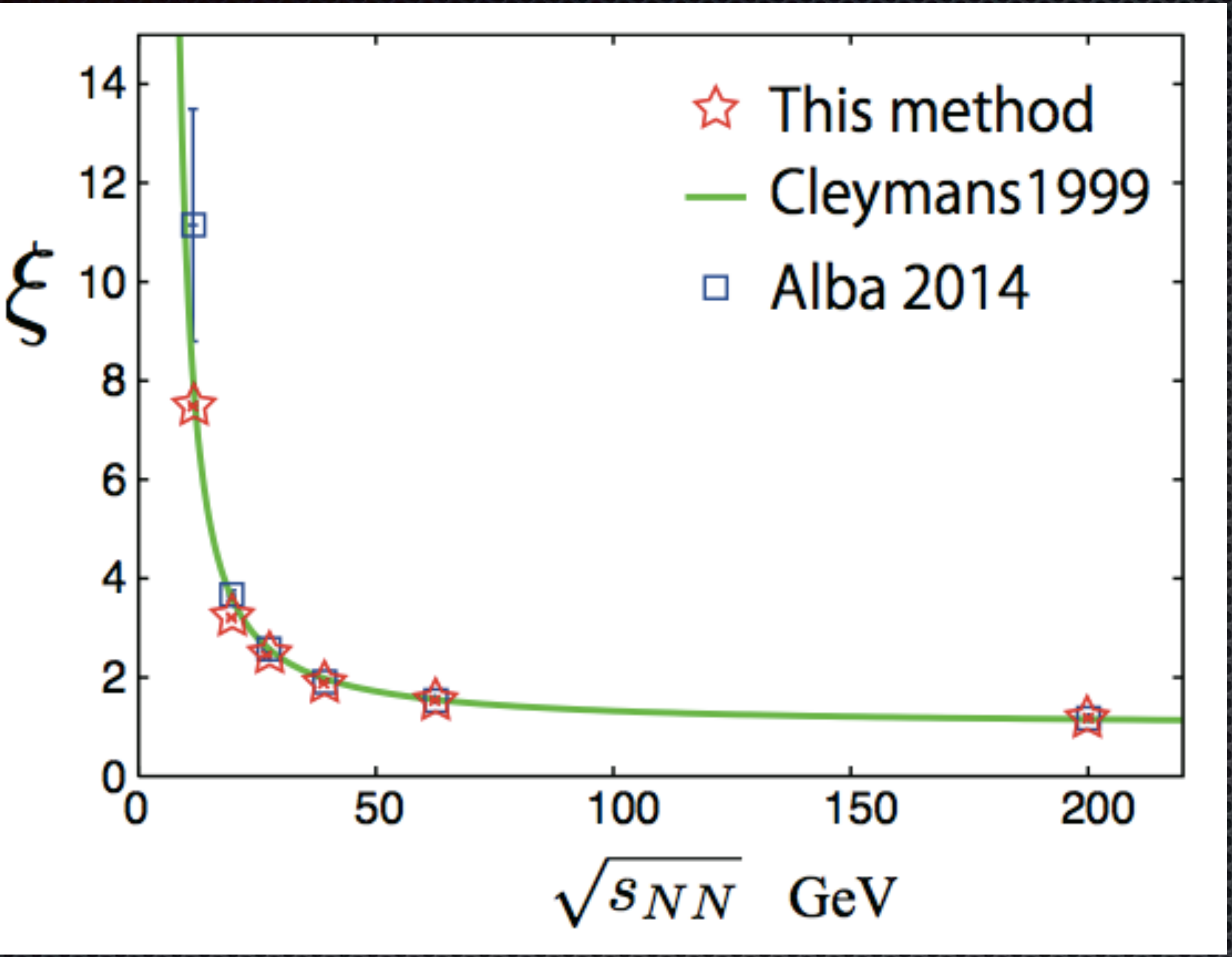
$$P_n \propto Z_n e^{n\mu/T}$$

Application to experimental data of proton number distribution [Nakamura, Nagata(2013)]



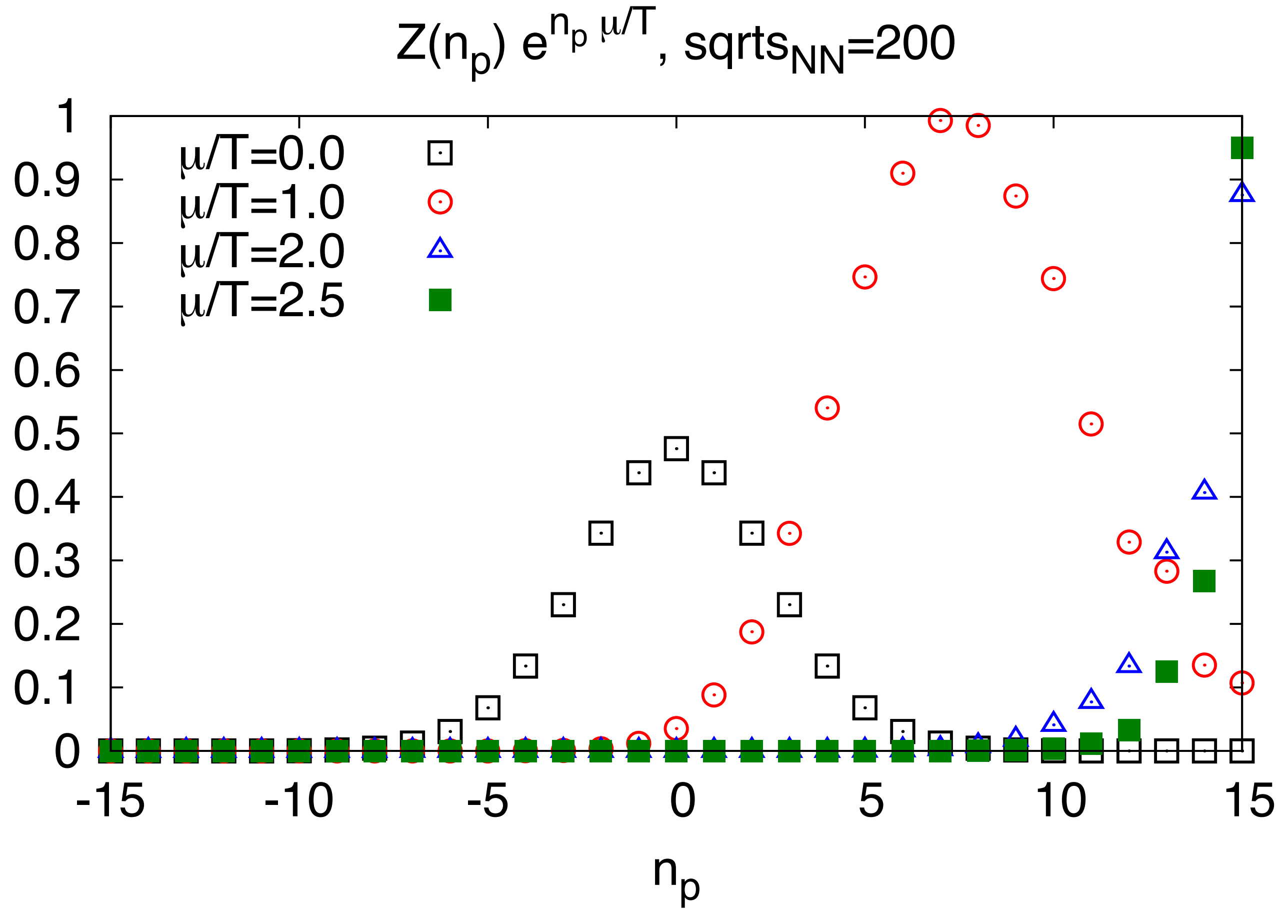


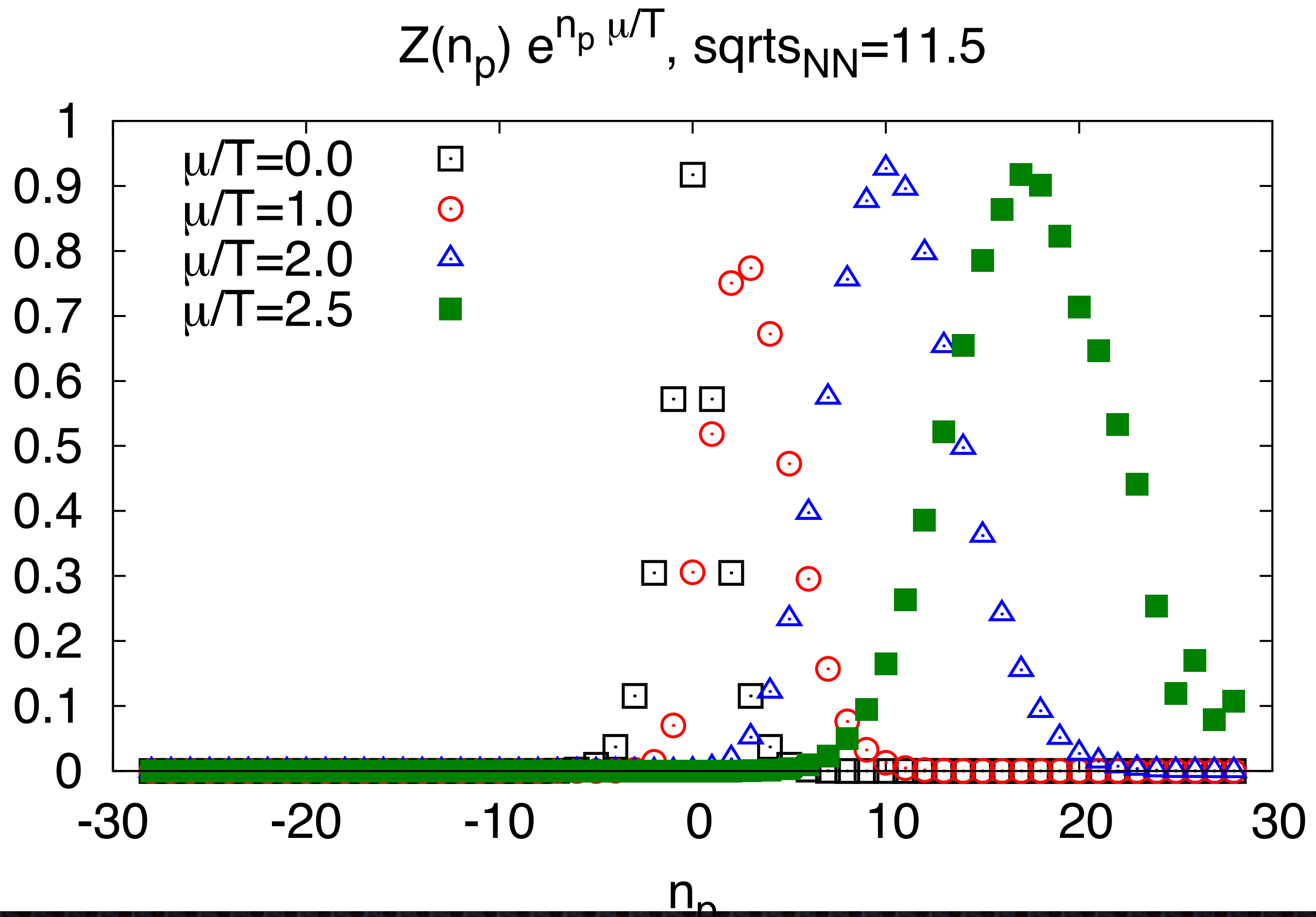
Extraction of Zn and μ/T



$$\xi = \exp(\mu/T)$$

- μ/T from the CP invariance vs thermal statistical


$Z(n_p)\, e^{n_p \mu/T}$, $sqrts_{NN}=200$
μ/T=0.0
μ/T=1.0
μ/T=2.0
μ/T=2.5
n_p


$Z(n_p)\, e^{n_p \mu/T}$, $sqrts_{NN}=11.5$
μ/T=0.0
μ/T=1.0
μ/T=2.0
μ/T=2.5
n_p

Studies of Zn

- Higher order moments can be a signal
- However, it is unclear if freeze out points in experiments hit CEP
- The data would contain information of finite density QCD even if the CEP is achieved.

It is important to study the shape of the distribution theoretically.

- find properties sensitive to the shape
- clarify its physical meaning

Lattice QCD simulations

Lattice simulations

Canonical approach : extends data at a given μ to wide range

How do we obtain Zn ? : reduction formula + reweighing

$$Z(\mu) = \int \mathcal{D}U (\det \Delta(\mu))^{N_f} e^{-S_g}$$

- Reduction formula

$$\det \Delta = \xi^{-N_{\rm red}/2} C_0(\{U\}) \prod (\xi + \lambda_n(\{U\})), \quad \xi = e^{-\mu/T},$$
$$= C_0 \sum_{n=-N_{\rm red}/2}^{N_{\rm red}/2} c_n \xi^n$$

[Gibbs ('86). Hasenfratz, Toussaint('92). Adams('03, '04), Borici('04). KN&AN('10), Alexandru &Wenger('10)]

- We use a reweighing in μ

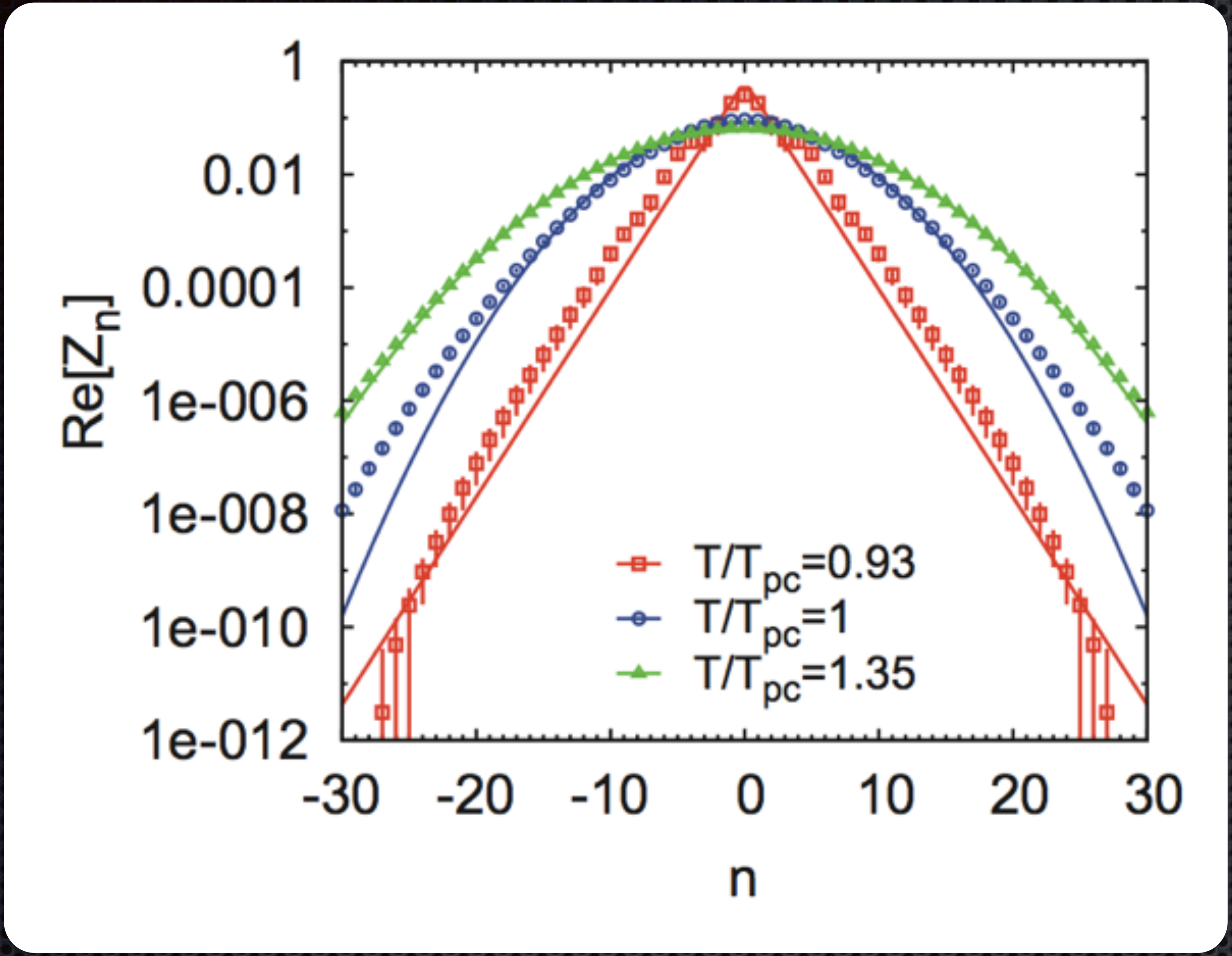
[Barbour, et. al. ('91).]

$$Z(\mu) = \sum Z_n e^{n\mu/T}, \quad Z_n \propto \left\langle \frac{C_0^{N_f} d_n}{(\det \Delta(0))^{N_f}} \right\rangle$$

Lattice simulations

- gauge configurations are generated at $\mu = 0$ and use reweighing
- volume : 8^3x4, 10^3x4
- mass : mps/mv~0.8
- action : clover-improved Wilson fermion +renormalization improved gauge
- # of statistics : 400 (20 trajectory-intervals, 3000 therm.)

Result - Zn



Lines :

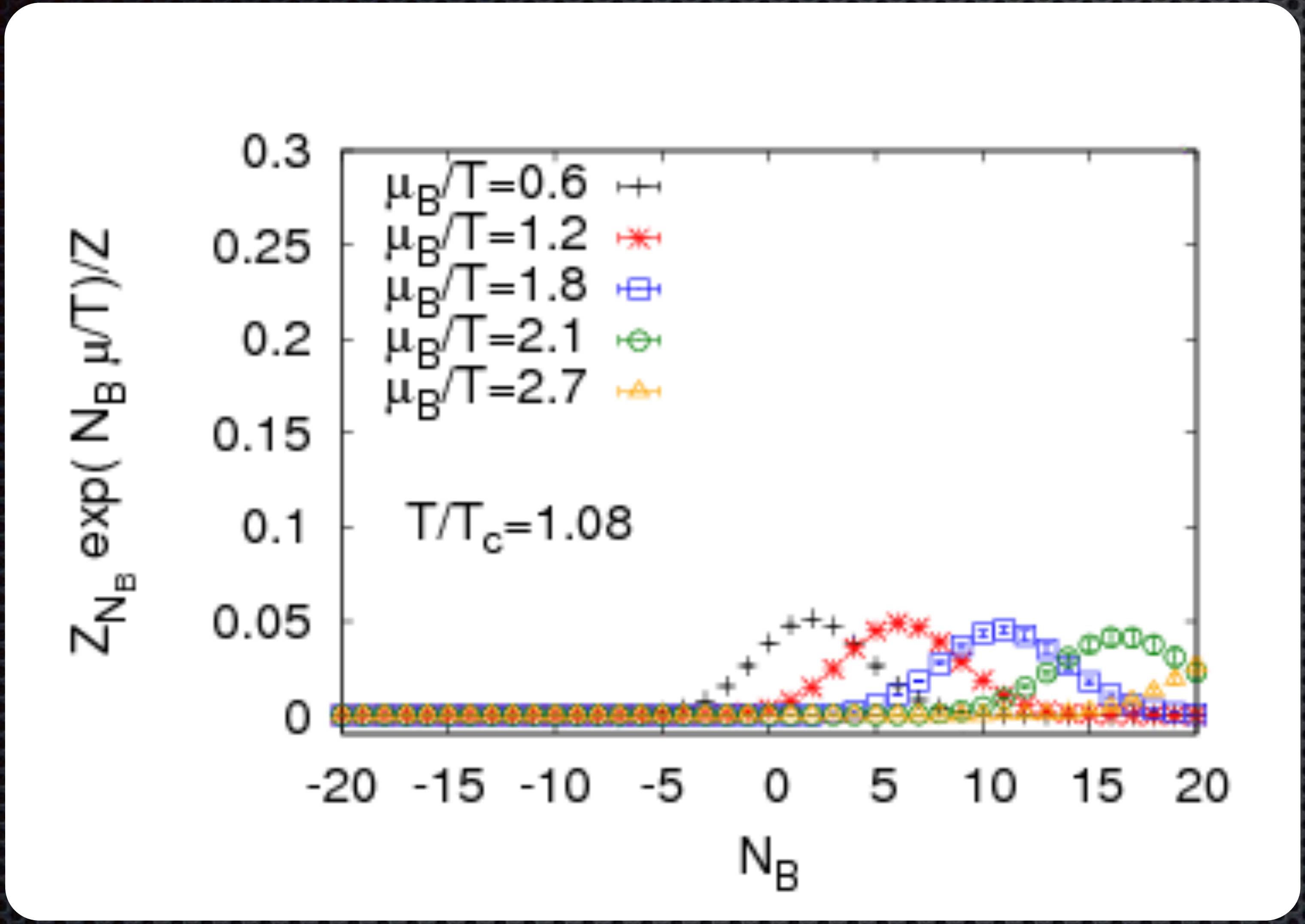
Gaussian for T/Tpc = 1.35 and 1,
exp(- a|n|) for T/Tpc = 0.93.

[KN, S. Motoki, Y. Nakagawa, A. Nakamura. T. saito. PTEP(2012).]

- volume : 8^3x4, 10^3x4, mass : mps/mv~0.8
- gauge configurations are generated at μ = 0 and use reweighing
- action : clover-improved Wilson fermion +renormalization improved gauge
- # of statistics : 400 (20 trajectory-intervals, 3000 therm.)

Result - Zn exp(μ/T) at high T

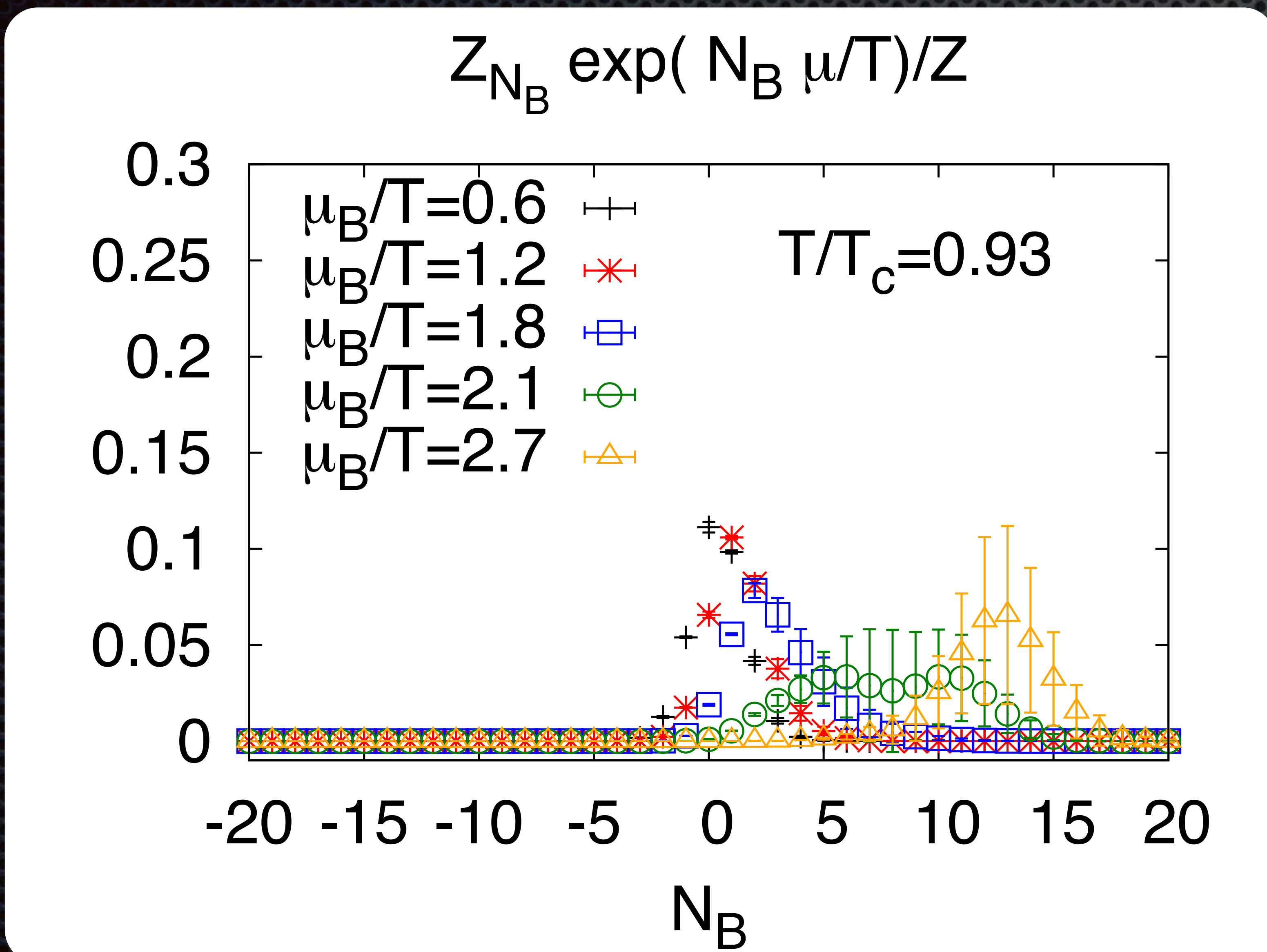
If Zn is the Gaussian, then the baryon number distribution is also Gaussian.



$$P_n \propto Z_n e^{n\mu/T}$$

Result - Zn exp(μ/T) at low T

Increasing μ at low T, a non-trivial shape change has been observed



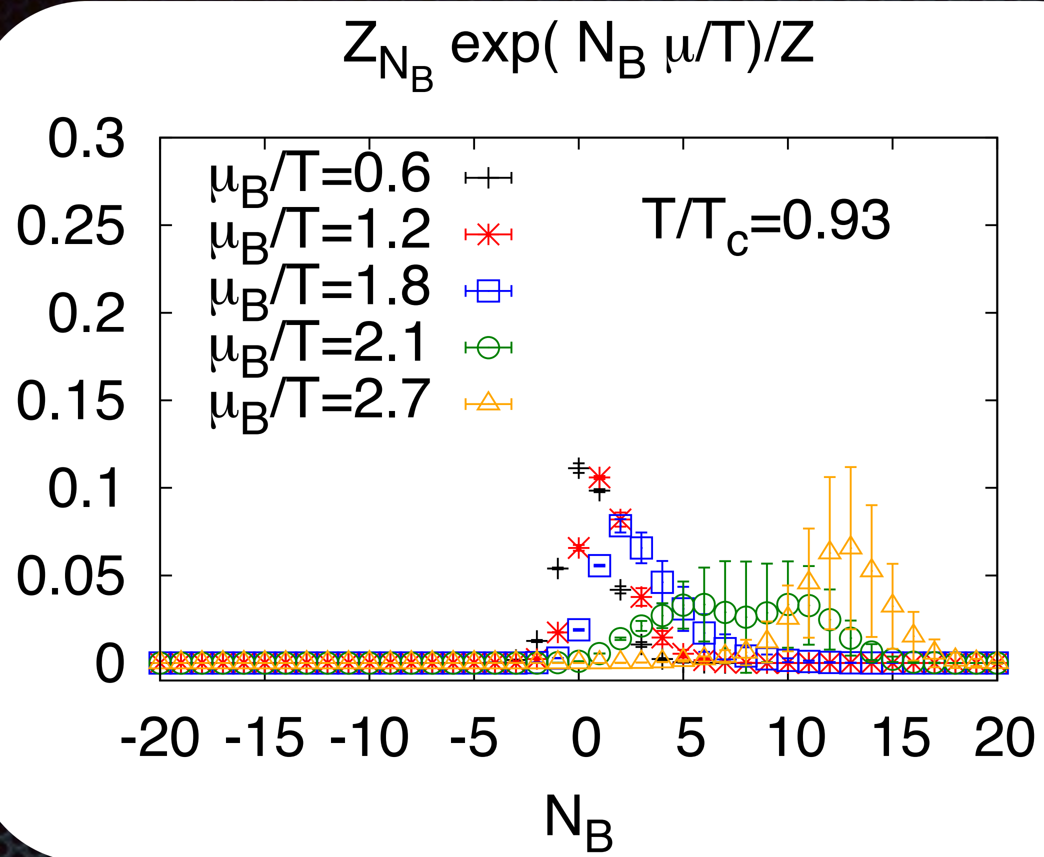
T/Tc = 0.93

μ_B/T = 1.8 : right tail

μ_B/T = 2.1 : flat

μ_B/T = 2.7 : left tail

Baryon number distribution & fluctuations

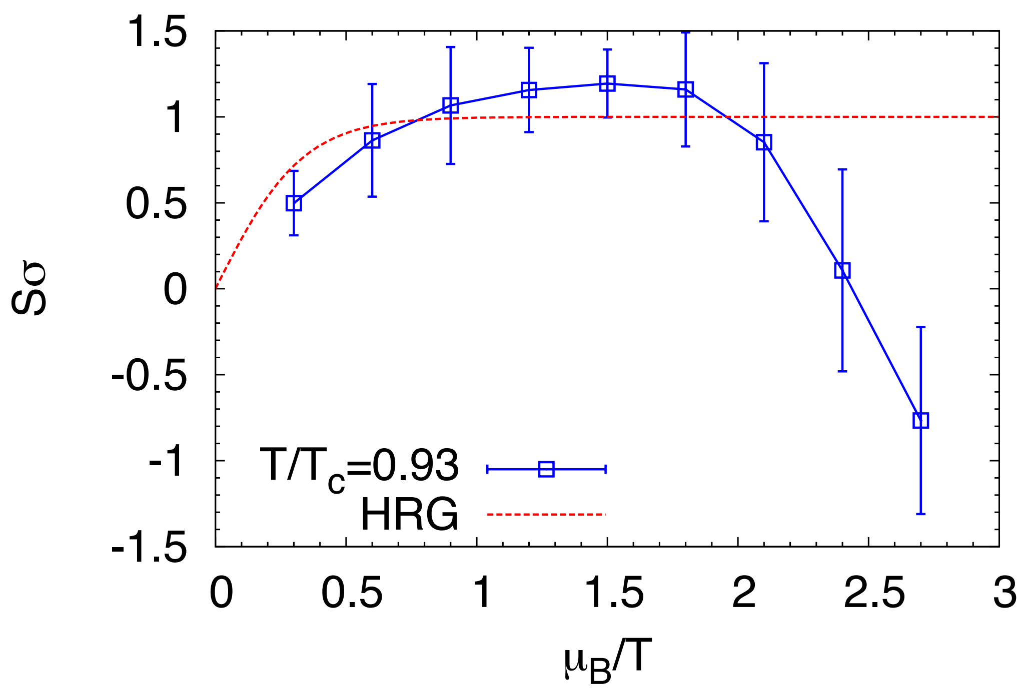


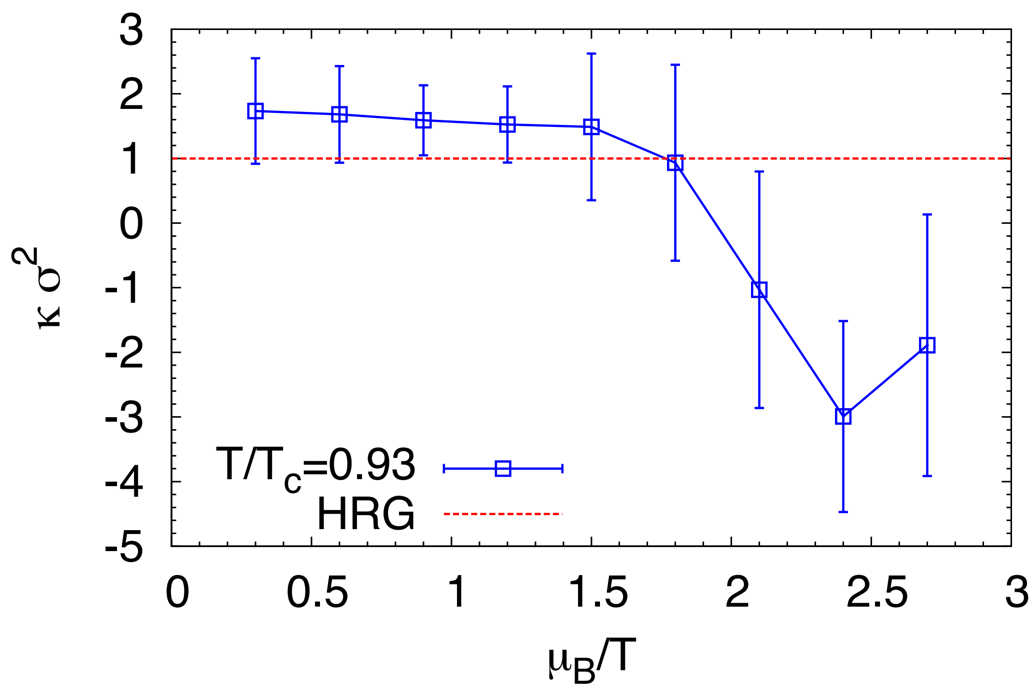
$$\langle(\delta N)^m\rangle = \sum_{n_B}(\delta N)^m Z_{n_B} e^{n_B\mu_B/T}/Z_{GC}$$

$$\sigma^2 = \langle(\delta N)^2\rangle$$

$$S = \frac{\langle(\delta N)^3\rangle}{\sigma^3}$$

$$\kappa = \frac{\langle(\delta N)^4\rangle}{\sigma^4} - 3$$





Lee-Yang zeros : from CPF to Phase transition

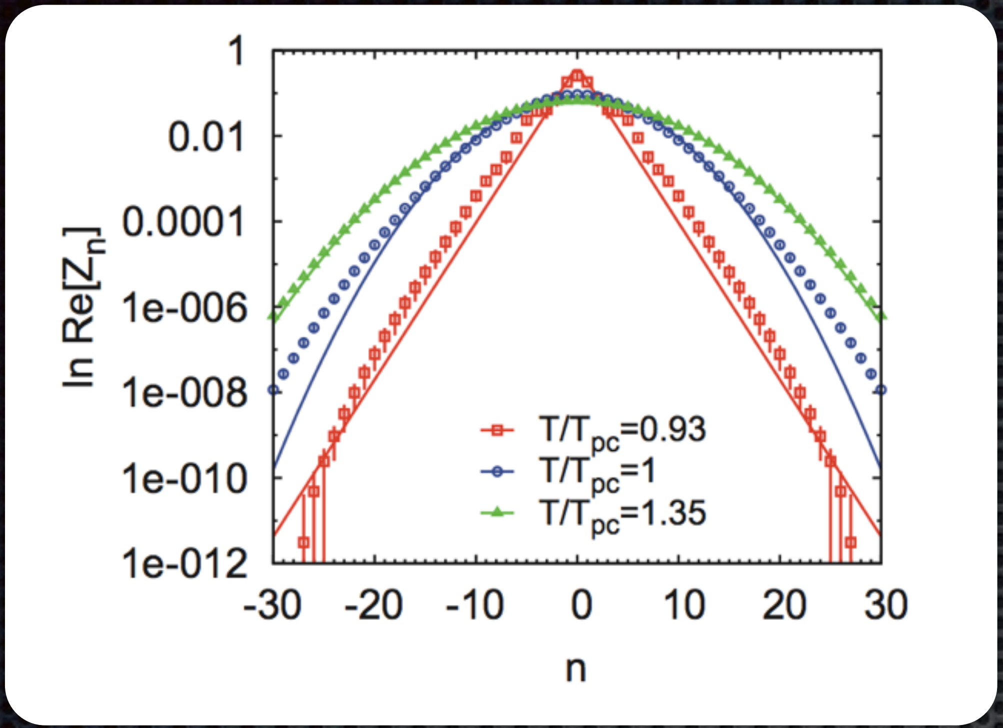
Lee-Yang zero Theorem :

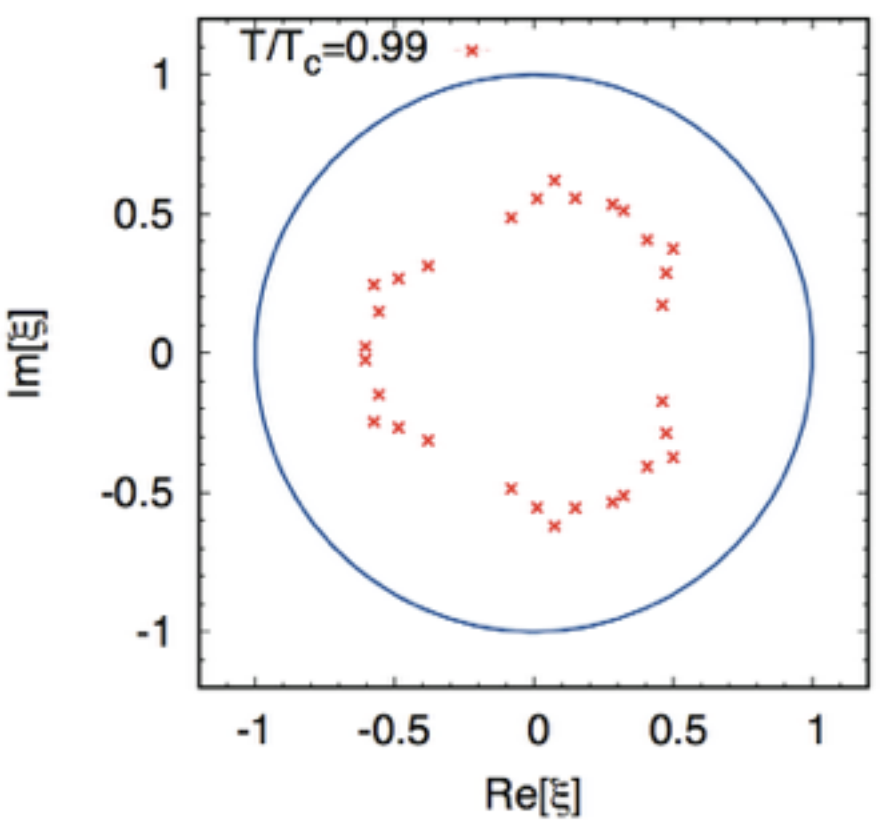
zeros of the partition function control the analyticity of the free energy [Lee & Yang 1952]

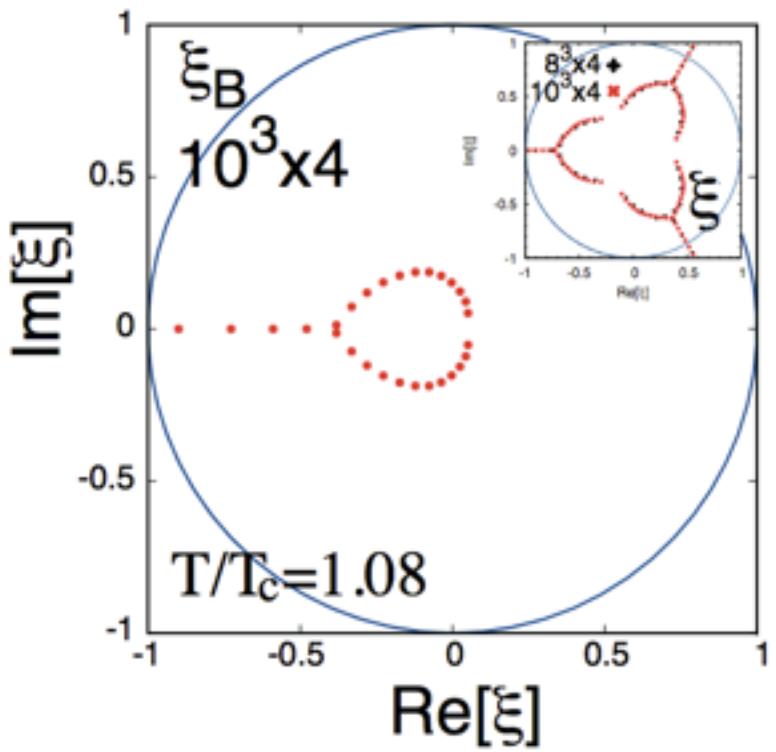
$$
\begin{aligned}
Z(\mu) &= \sum Z_n e^{n\mu/T} \\
&\propto \prod (1 - \xi/\xi_i)
\end{aligned}
$$

Result-Lee-Yang zeros

- [Nakamura, Nagata(2013)]







Result-Lee-Yang zeros

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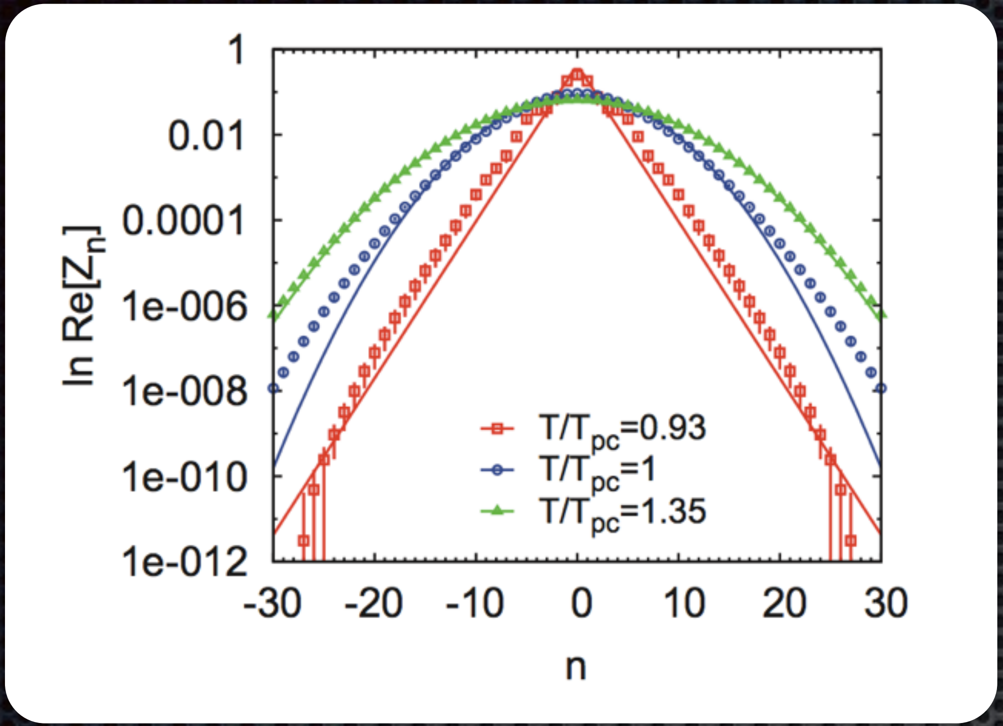
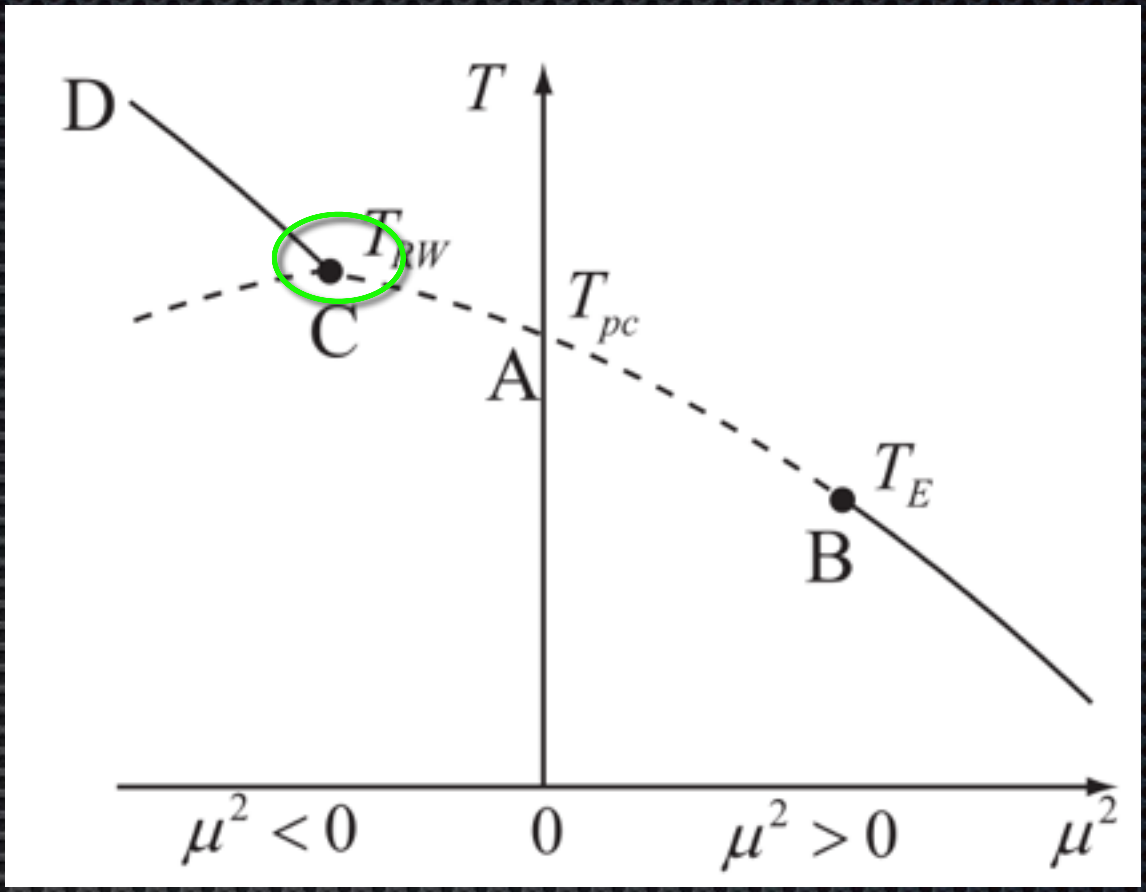
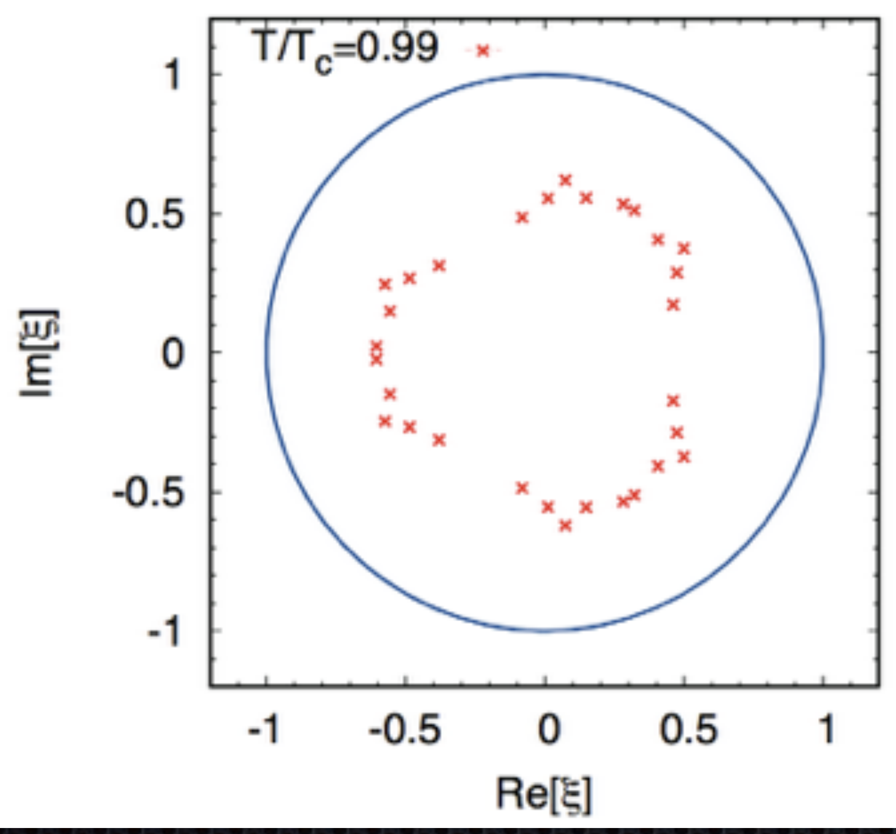
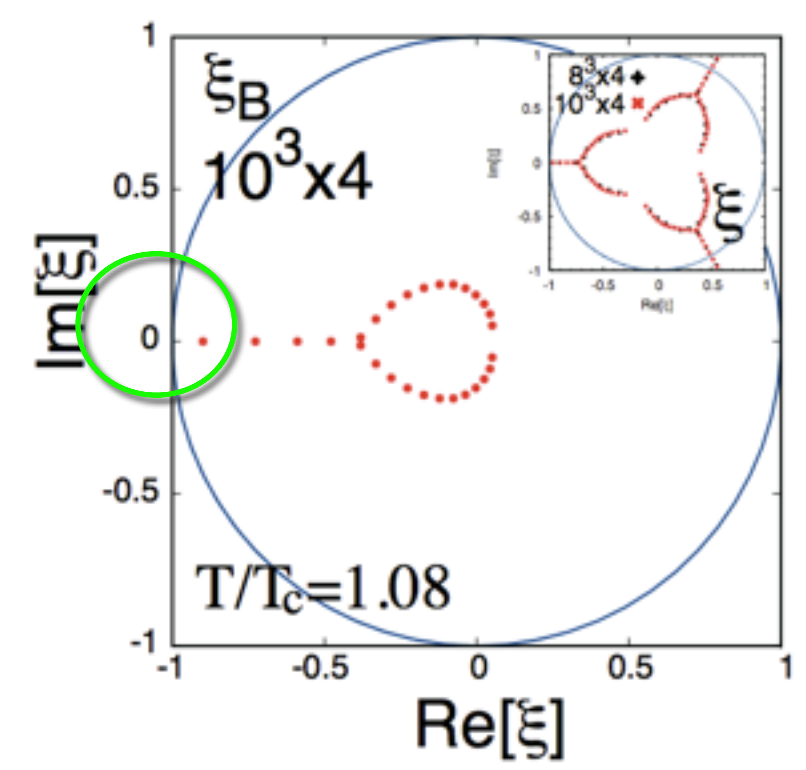


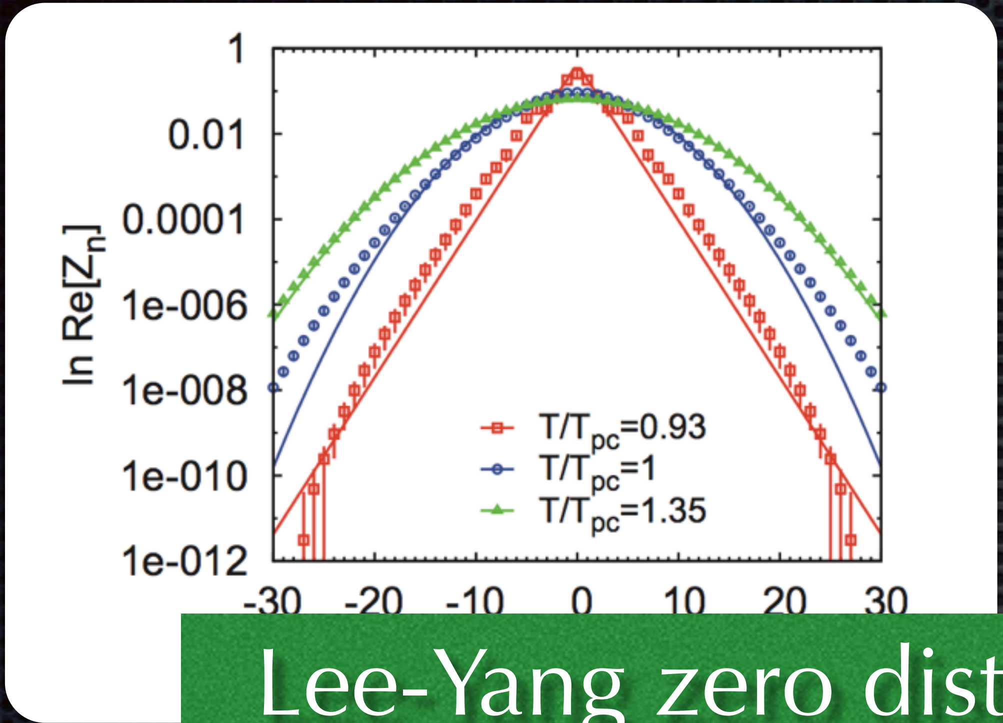


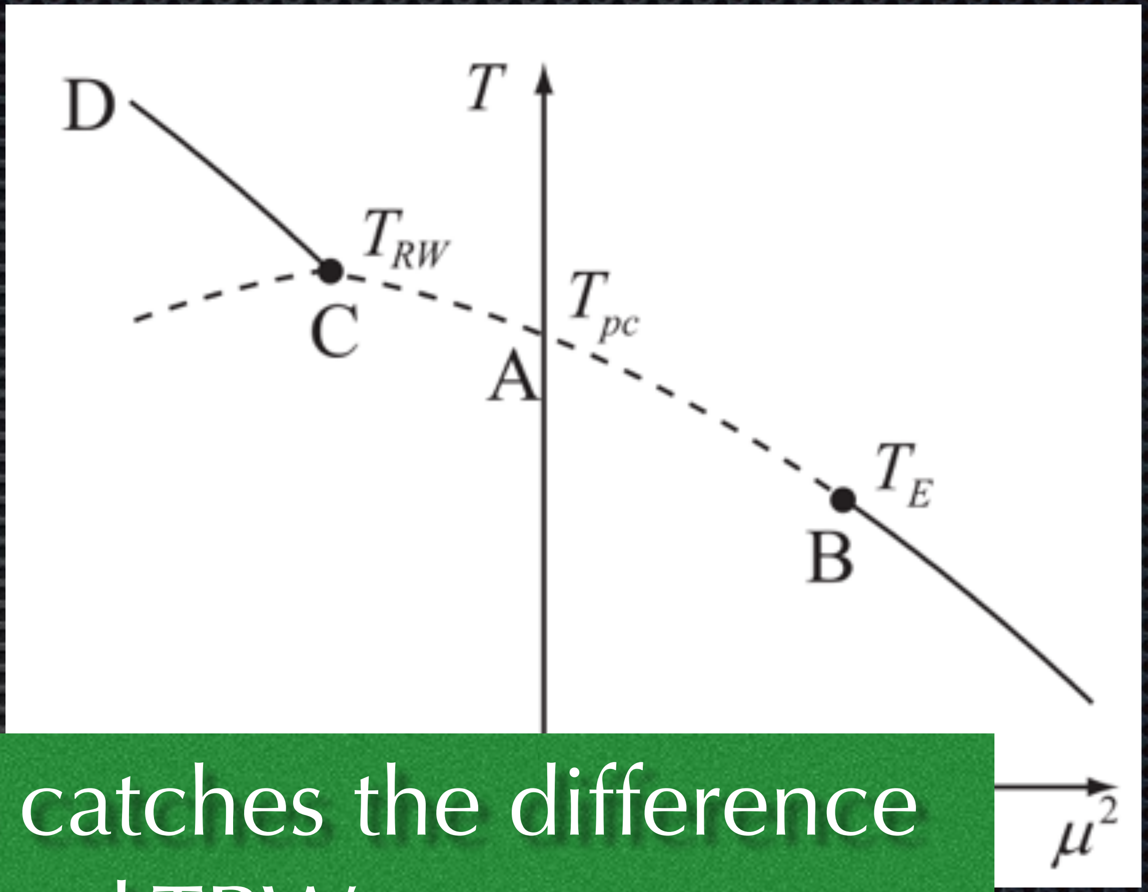




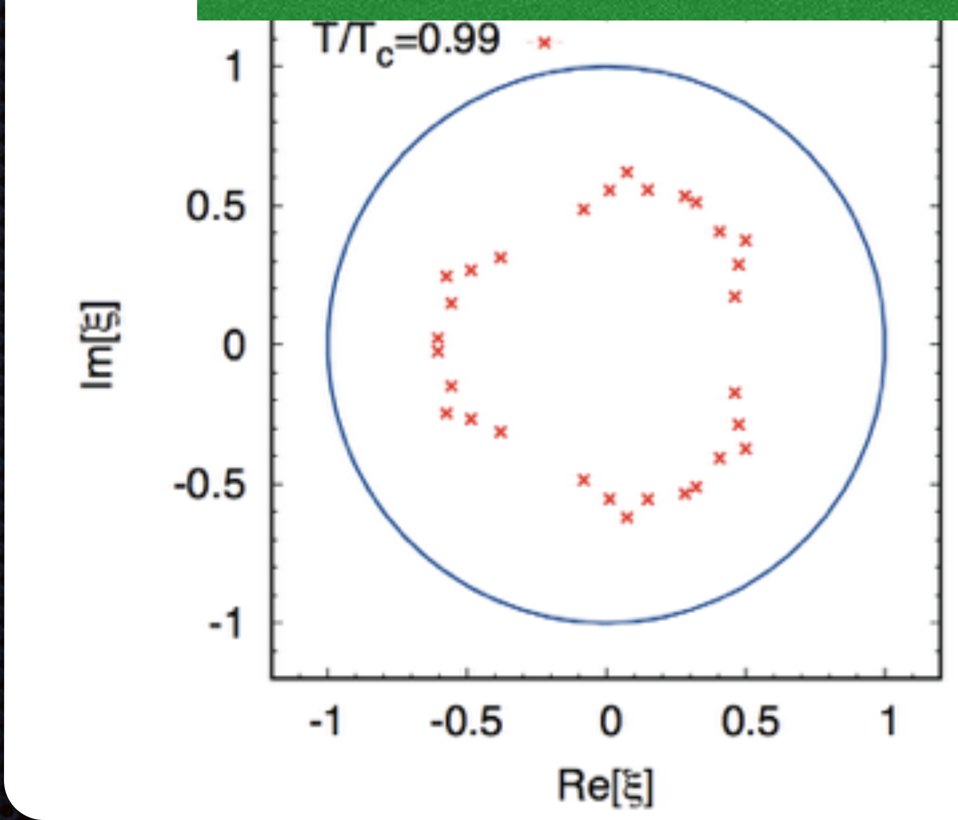
Result-Lee-Yang zeros

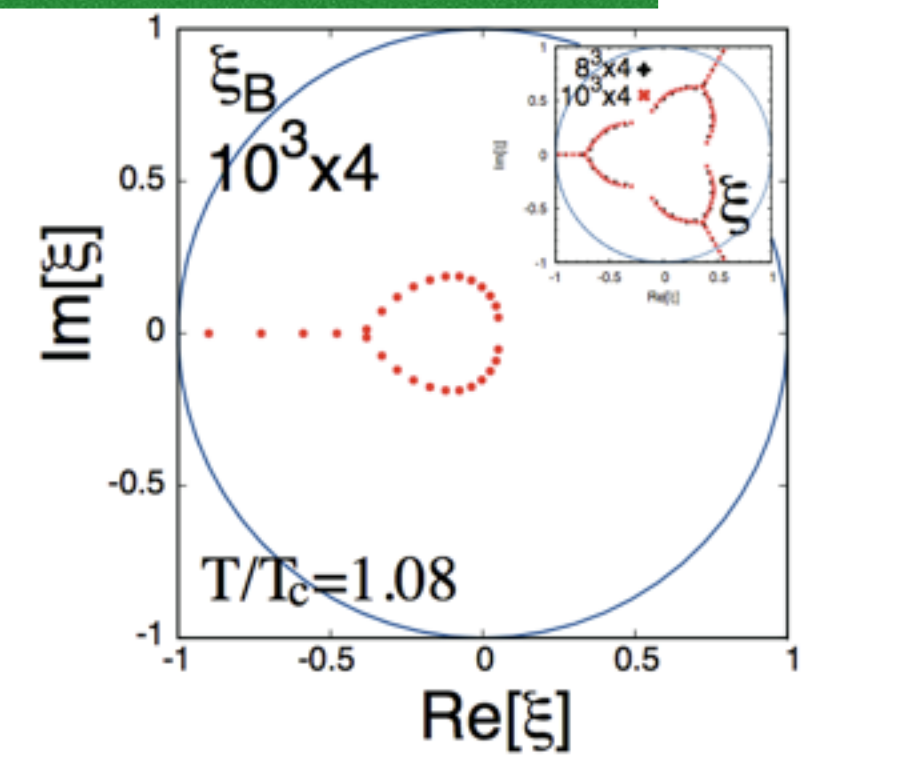
- [Nakamura, Nagata(2013)]





Lee-Yang zero distribution catches the difference between Tc and TRW





Result - 2

Questions and Subtleties in the calculation

- Lee-Yang zeros are sensitive to Zn; We need careful analysis
- There remains some questions
  - error bars in Zn : statistical stability
  - truncation of the polynomial : convergence

$$Z(\mu) = \sum_{n=-\infty}^{\infty} Z_n e^{n\mu/T} \to \sum_{n=-n_0}^{n_0} Z_n e^{n\mu/T} + (|n| > n_0)$$

We focus on high T region, and perform

- analytic calculation
- reanalysis of lattice data

Analytic calculation of Zn of high T QCD

Properties of high T QCD leads to Gaussian Zn

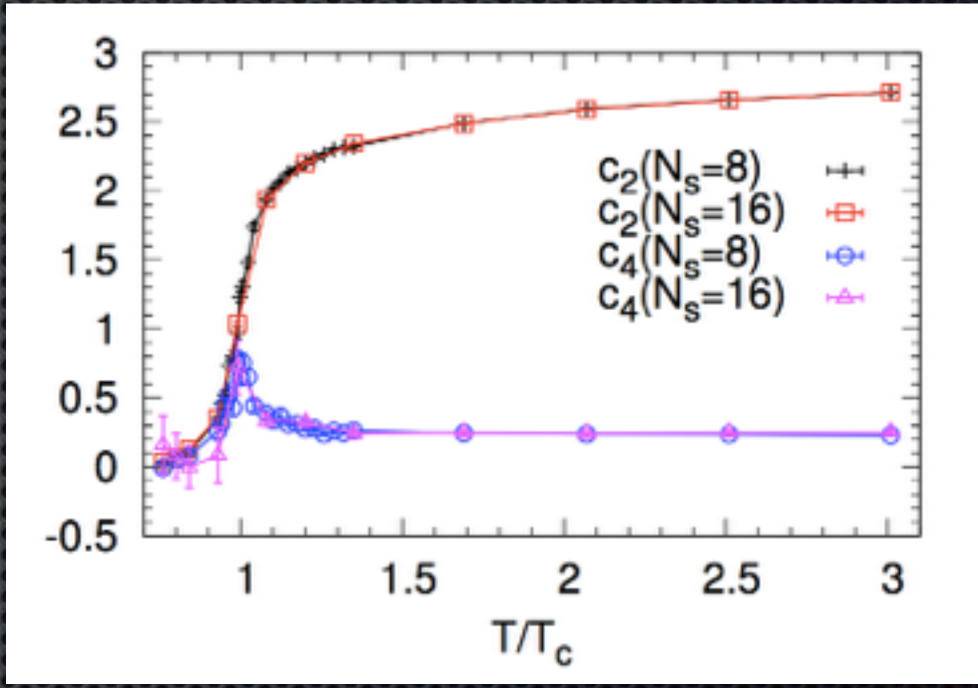
- Zn can be obtained from the F.T.

$$Z_n = \int Z(\theta) e^{in\theta} d\theta, \quad \mu/T = i\theta, (\theta \in \mathbb{R})$$

- Use properties of high T QCD

$$-\frac{f}{T^4} = c_0 + c_2(\mu/T)^2 + c_4(\mu/T)^4$$

$$Z\left(\frac{\mu_I}{T}\right) = Z\left(\frac{\mu_I}{T} + \frac{2\pi}{N_c}\right).$$



Nagata, Nakamura, JHEP(2012)

- also use the saddle point approximation

$$Z_n = Ce^{-n^2/(4T^3Vc_2)}, \quad (n \equiv 0 \mod 3)$$

Analytical calculation of LY zeros of high T QCD

Gaussian Zn + RW periodicity leads to Lee-Yang zeros as

$$\frac{\mu}{T} = \frac{(2l+1)\pi i}{3} - \frac{3(2k+1)}{4VT^3 c_2}$$

Analytical calculation of LY zeros of high T QCD

Gaussian Zn + RW periodicity leads to Lee-Yang zeros as

$$\frac{\mu}{T} = \frac{(2l+1)\pi i}{3} - \frac{3(2k+1)}{4VT^3 c_2}$$

High T QCD

$$Z_n = Ce^{-n^2/(4T^3Vc_2)}, \quad (n \equiv 0 \mod 3)$$

$$Z(\mu) = C \sum_{n_B=-\infty}^{\infty} e^{-9{n_B}^2/(4T^3Vc_2)+3n_B\mu/T}$$

Theta function

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{-\pi i n^2\tau + 2\pi i n z}$$

$$i\pi\tau = 9/(4T^3Vc_2),$$
$$2\pi i z = 3\mu/T$$

$$\vartheta(z,\tau) = 0 \leftrightarrow z = l + k\tau + \frac{1}{2} + \frac{\tau}{2}, (k, l \in \mathbb{Z})$$

Alternative calculation

Cancellation of two types of free energy allows Z=0

- Cancellation of free energies [Biskup et al('01)]

$$Z = e^{-Vf_I/T} + e^{-Vf_{II}/T}$$

$$\begin{cases} \mathrm{Re}[f_{\mathrm{I}} - f_{\mathrm{II}}] = 0 \\ \frac{V}{T}\mathrm{Im}[f_{\mathrm{I}} - f_{\mathrm{II}}] = (2k-1)\pi, (k \in \mathbb{Z}) \end{cases}$$

- Approximate solution for $c_2 >> c_4$

$$(\mu_R, \mu_I) = \left(\frac{3(2k-1)}{4VT^3 c_2}, -\pi/3\right)$$

  - It is also possible to solve it in the presence of c4
  - f_{I} and f_{III}, and f_{II} and f_{III}.

Analytical calculation of LY zeros of high T QCD

$$\frac{\mu}{T} = \underbrace{\frac{(2l+1)\pi i}{3}}_{\text{angular}} - \underbrace{\frac{3(2k+1)}{4VT^3 c_2}}_{\text{radial}}$$

$$\xi = \exp(-\mu/T)$$

Analytical calculation of LY zeros of high T QCD

$$\frac{\mu}{T} = \underbrace{\frac{(2l+1)\pi i}{3}}_{\text{angular}} - \underbrace{\frac{3(2k+1)}{4VT^3 c_2}}_{\text{radial}}$$

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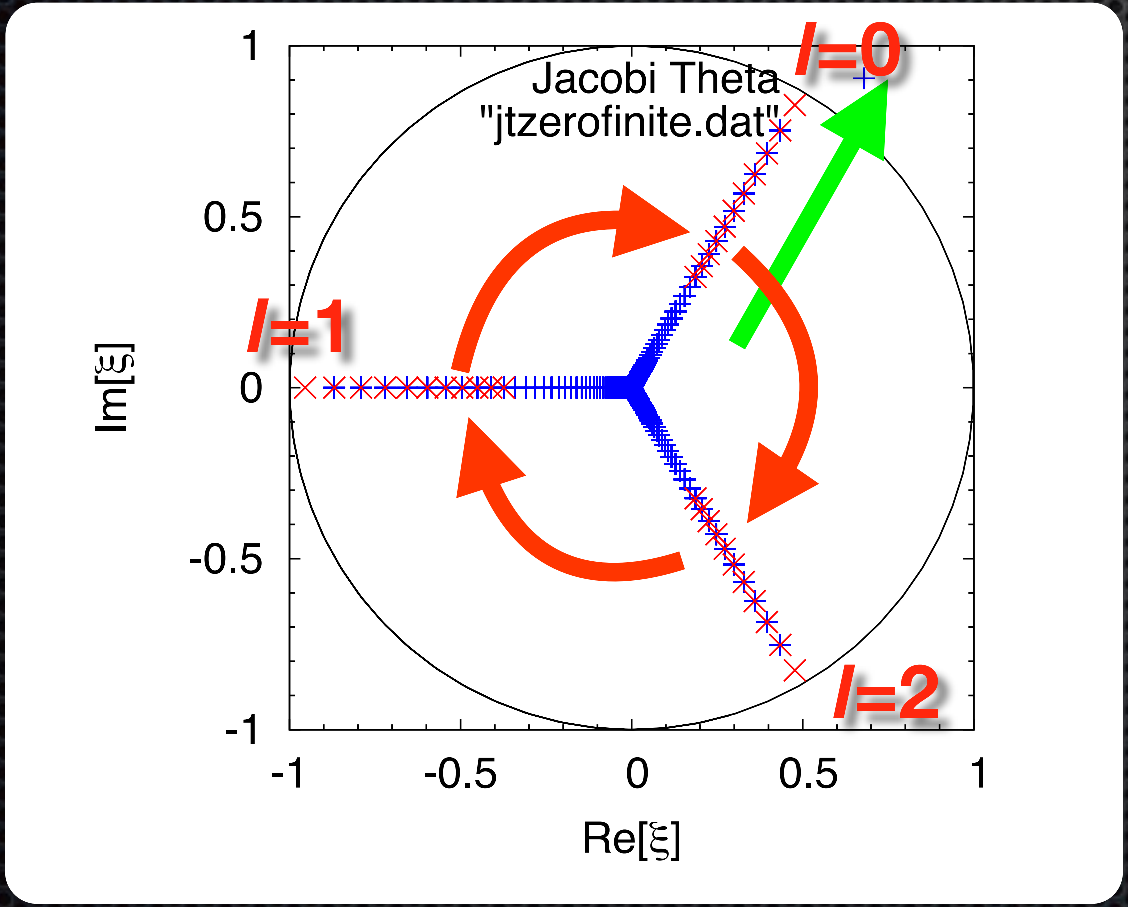
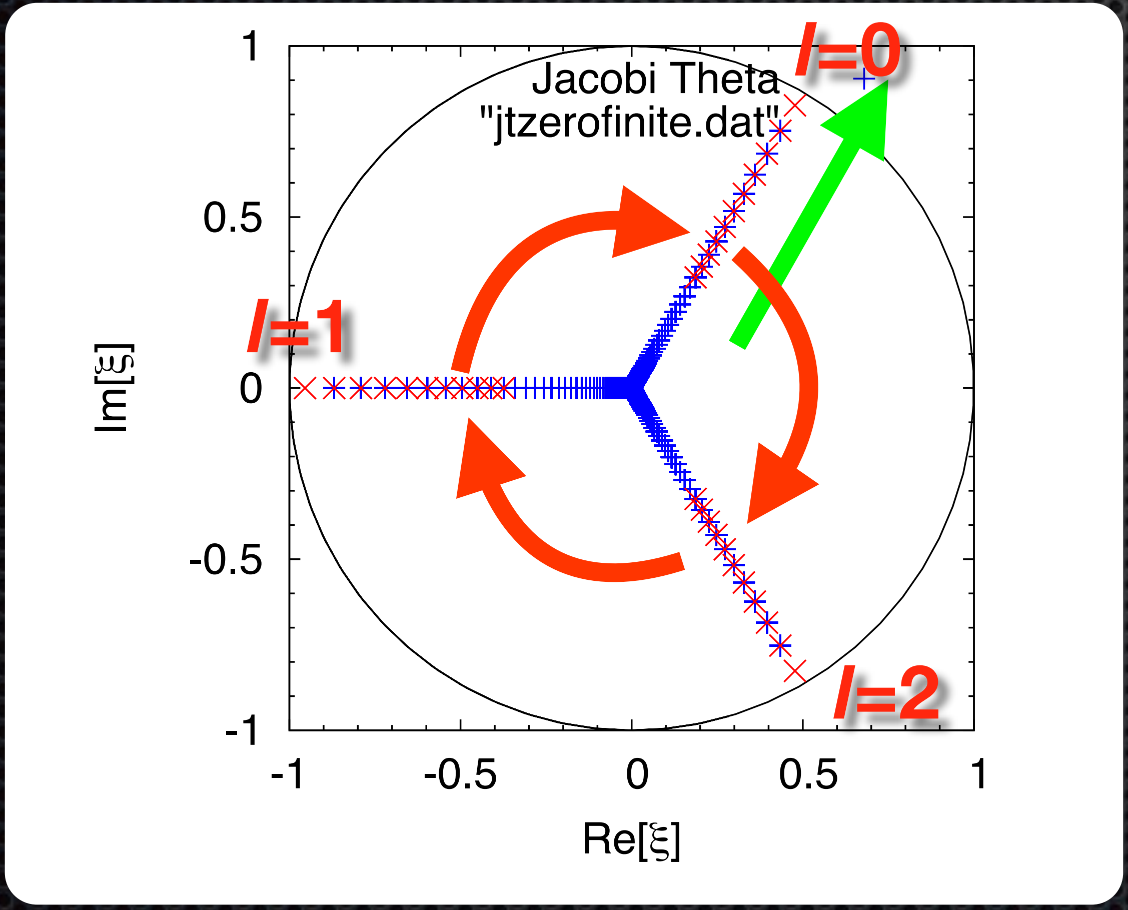
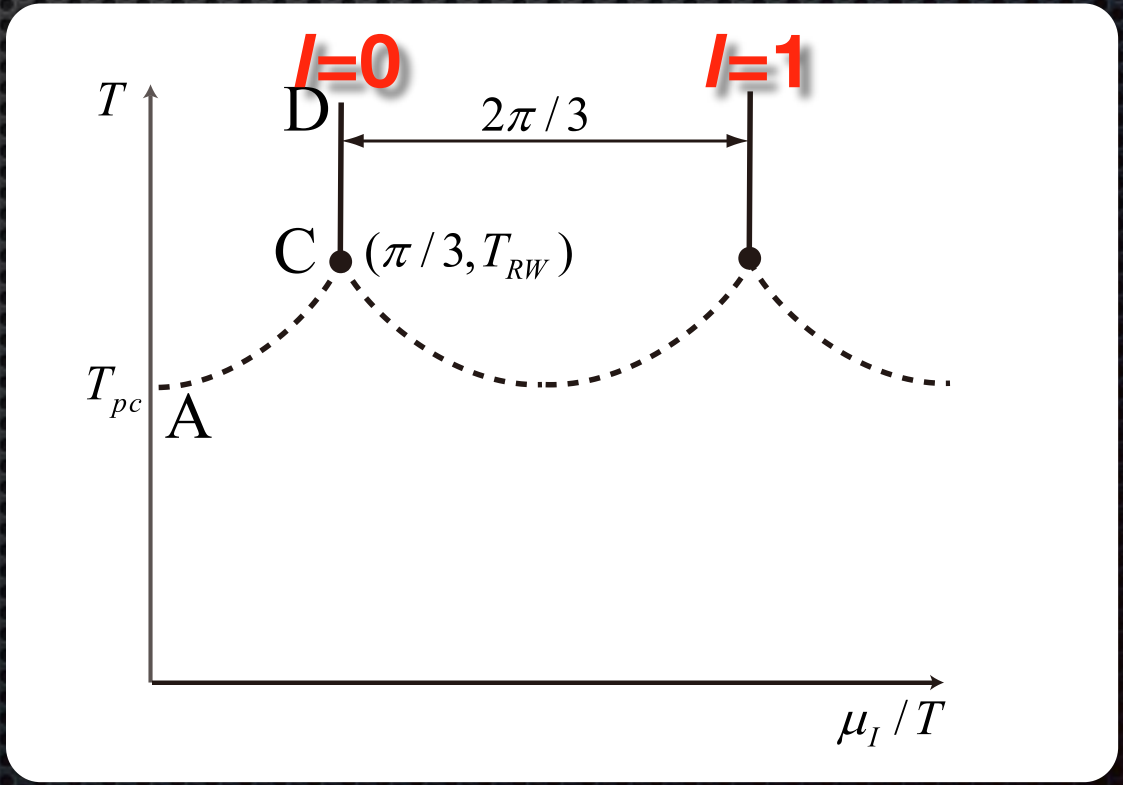


Analytical calculation of LY zeros of high T QCD

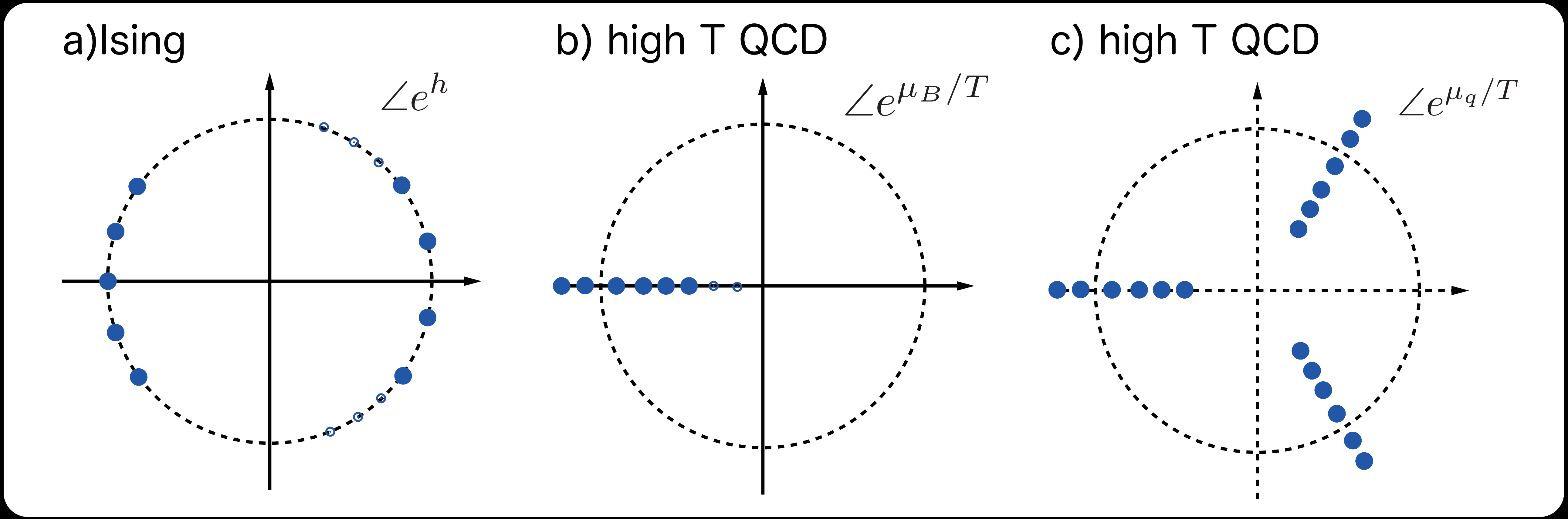
$$\frac{\mu}{T} = \underbrace{\frac{(2l+1)\pi i}{3}}_{\text{angular}} - \underbrace{\frac{3(2k+1)}{4VT^3 c_2}}_{\text{radial}}$$

$$\xi = \exp(-\mu/T)$$





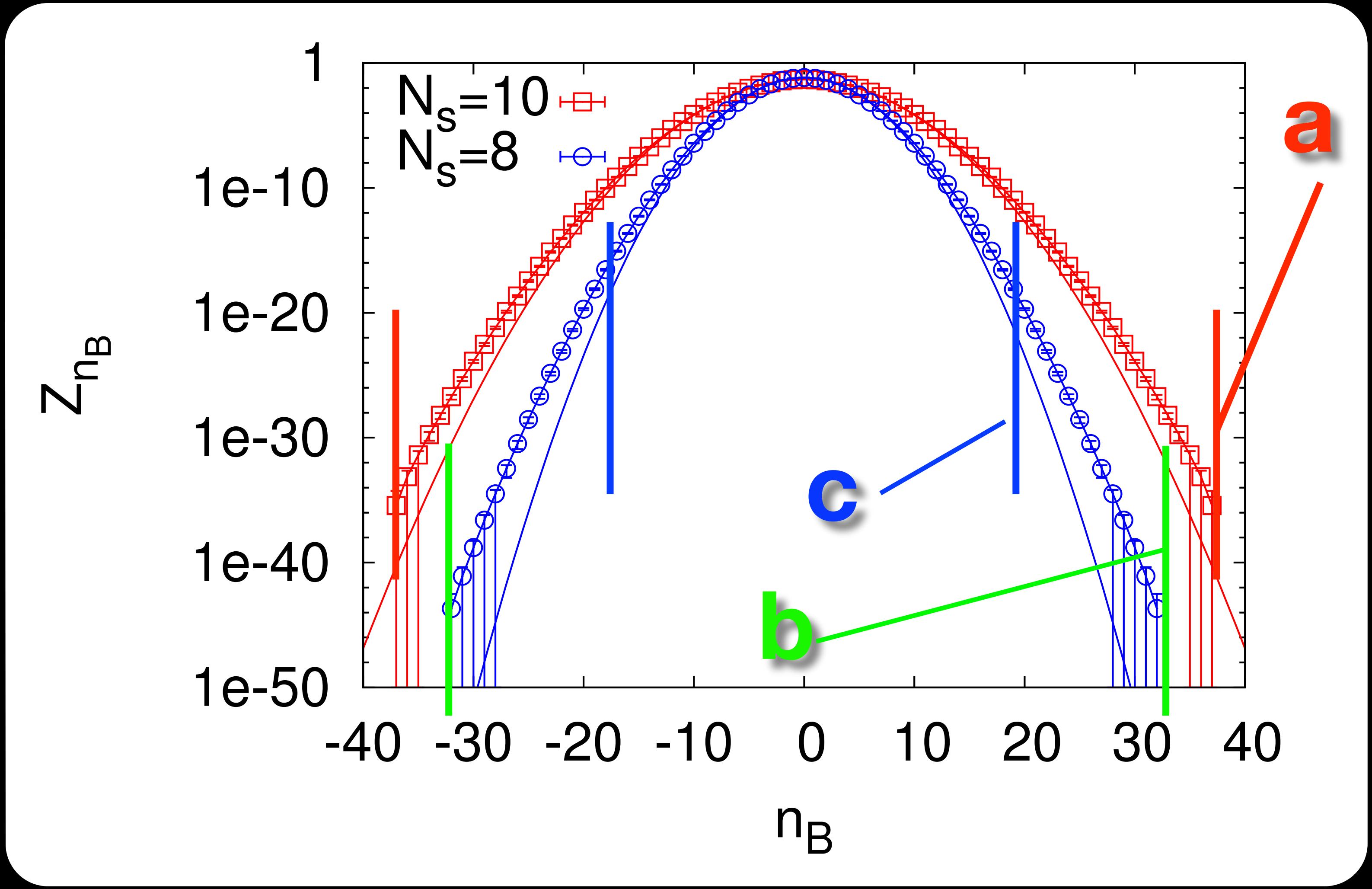
Lee-Yang zeros : Ising vs free fermion gas



Reanalysis of Lattice data & analytic result

- We reanalyze previous lattice data.
  - errors of Zn : bootstrap analysis (1000 BS samples.)
  - estimate the convergence
  - finite size scaling

$$|\mathrm{Re}[\xi]|^{VT^3} = \exp(-(3k+1)/(4c_2))$$



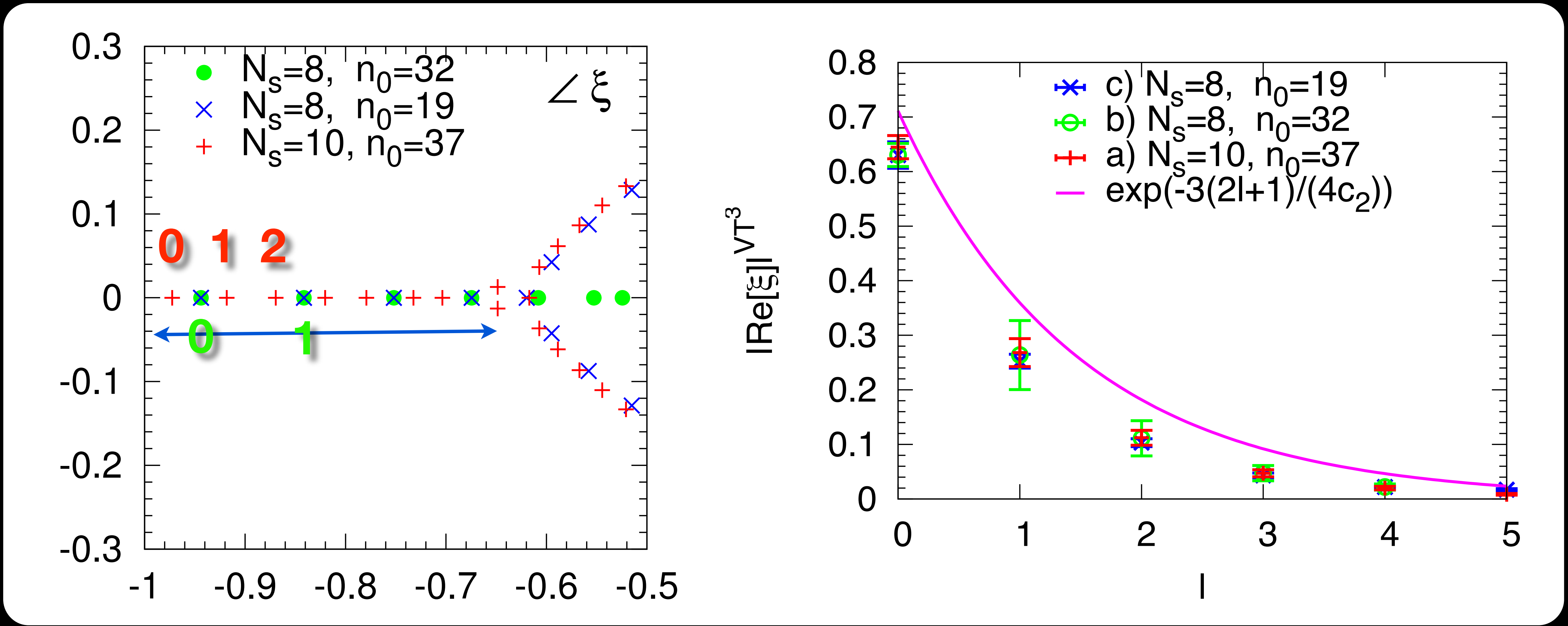
a) Ns=10 n0=37

b) Ns=8 n0=32

c) Ns=8 n0=19

Reanalysis of Lee-Yang zeros for high T QCD

Analytic and lattice calculation are consistent !



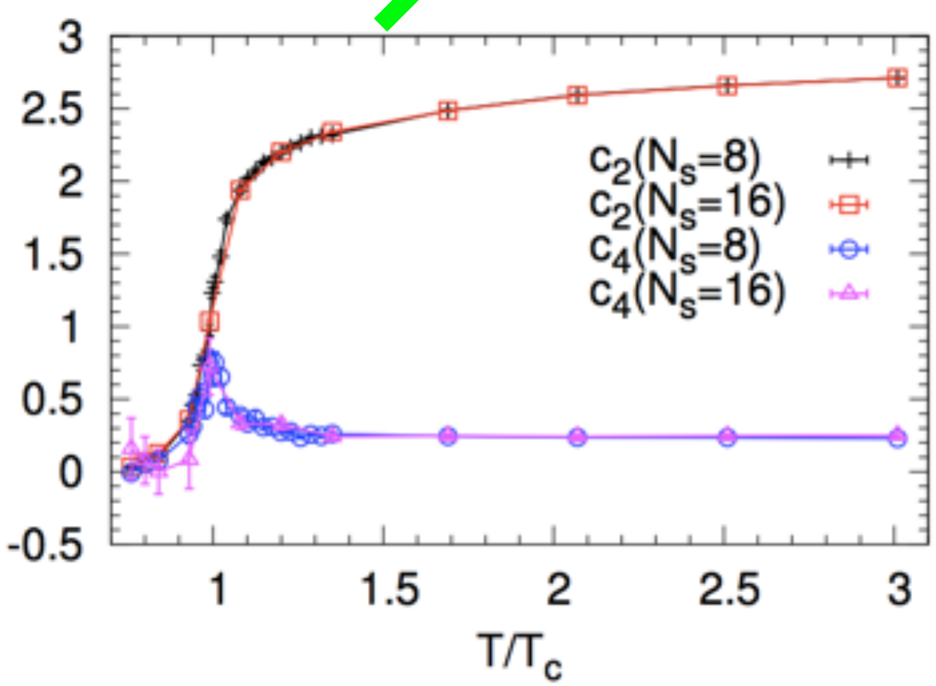
Solid Line : $|\mathrm{Re}[\xi]|^{VT^3} = \exp(-(3k+1)/(4c_2))$

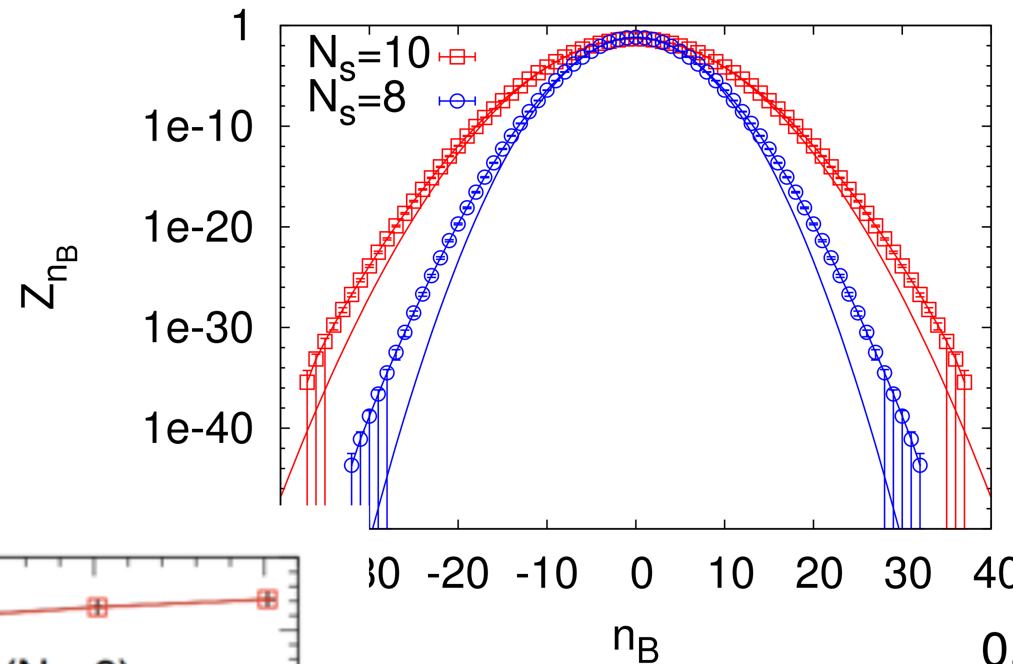
- Lee-Yang zeros near the unit circle are statistically stable
- b)-c) : convergence ok
- a)-c) : volume scaling consistent with the analytic one.
- underestimation : saddle point approximation

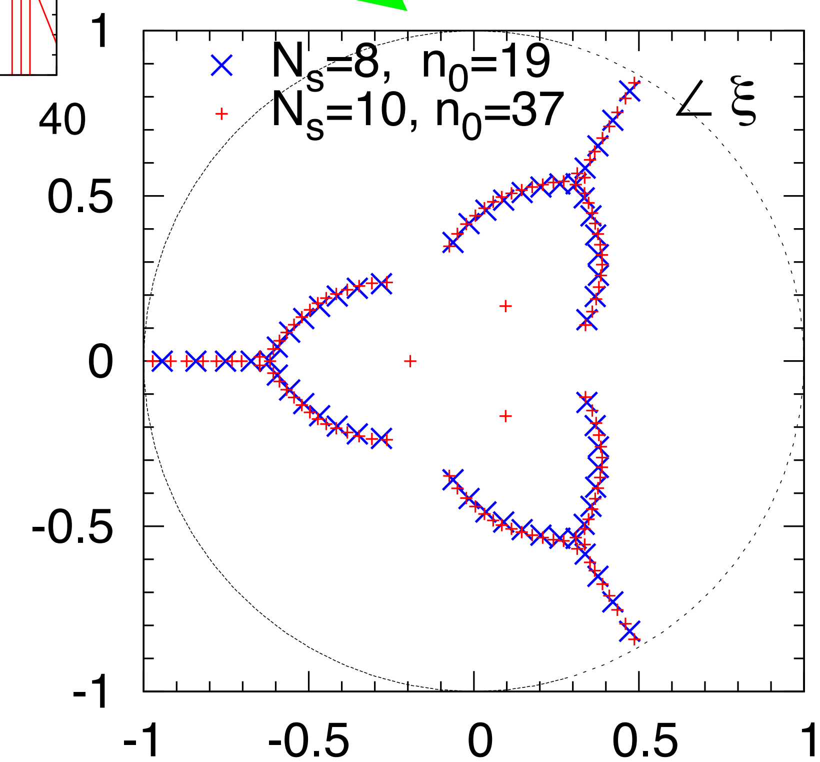
implication

c2-dominance
RW periodicity

Theta function





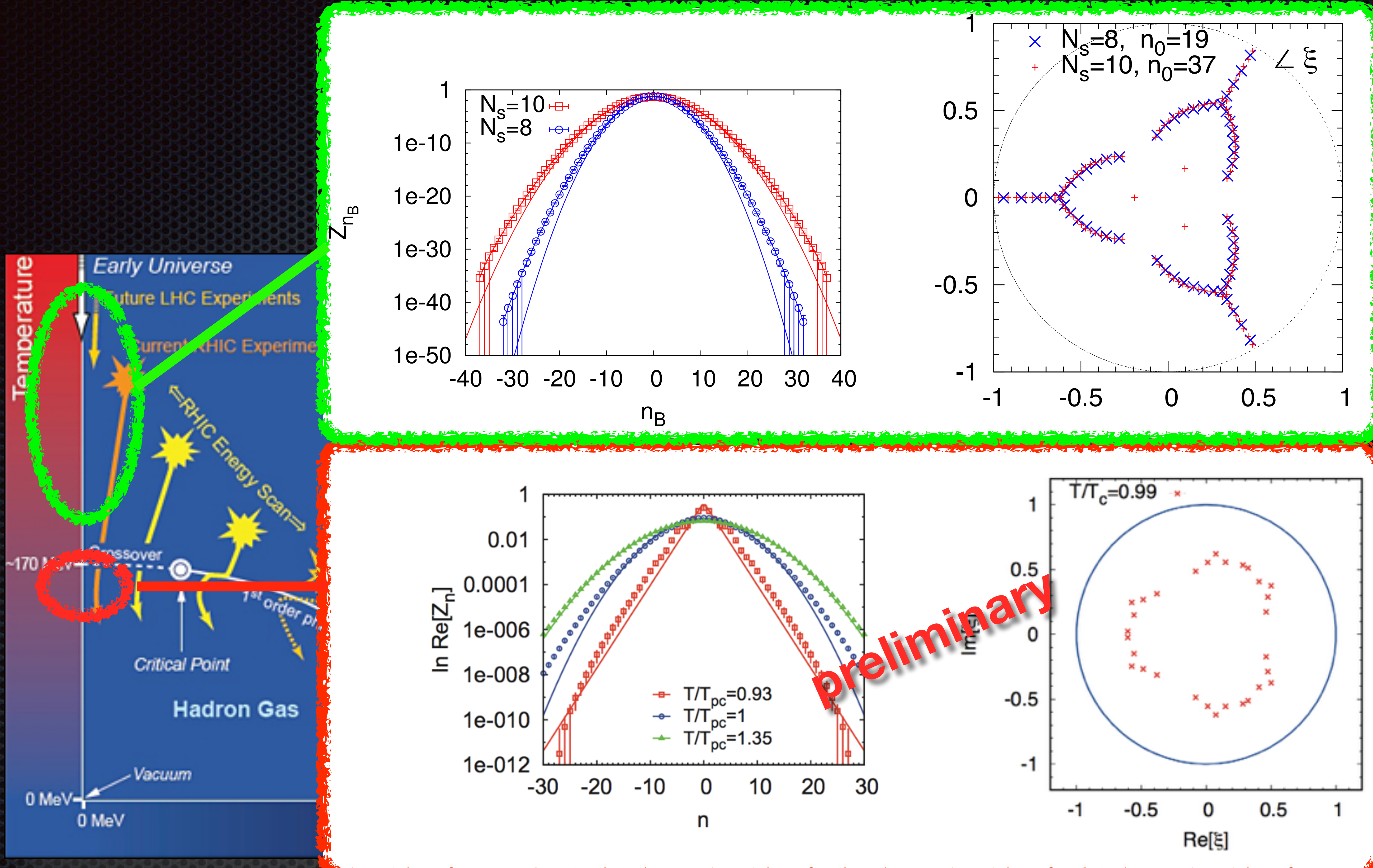


Completion of deconfinement

Gaussian behavior

Roberge-Weiss phase transition

Summary



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Summary

- Canonical approach
  - provides a way to extend data obtained at a certain μ/T to other values of μ/T.
  - can be applied not only to lattice QCD but also to experimental data.
- We showed that the net baryon number follows
  - Gaussian distribution at high temperatures
  - Gaussian shape is an indication of RW phase transition(high T)
- Lee-Yang zeros are sensitive to a shape of the canonical partition functions.

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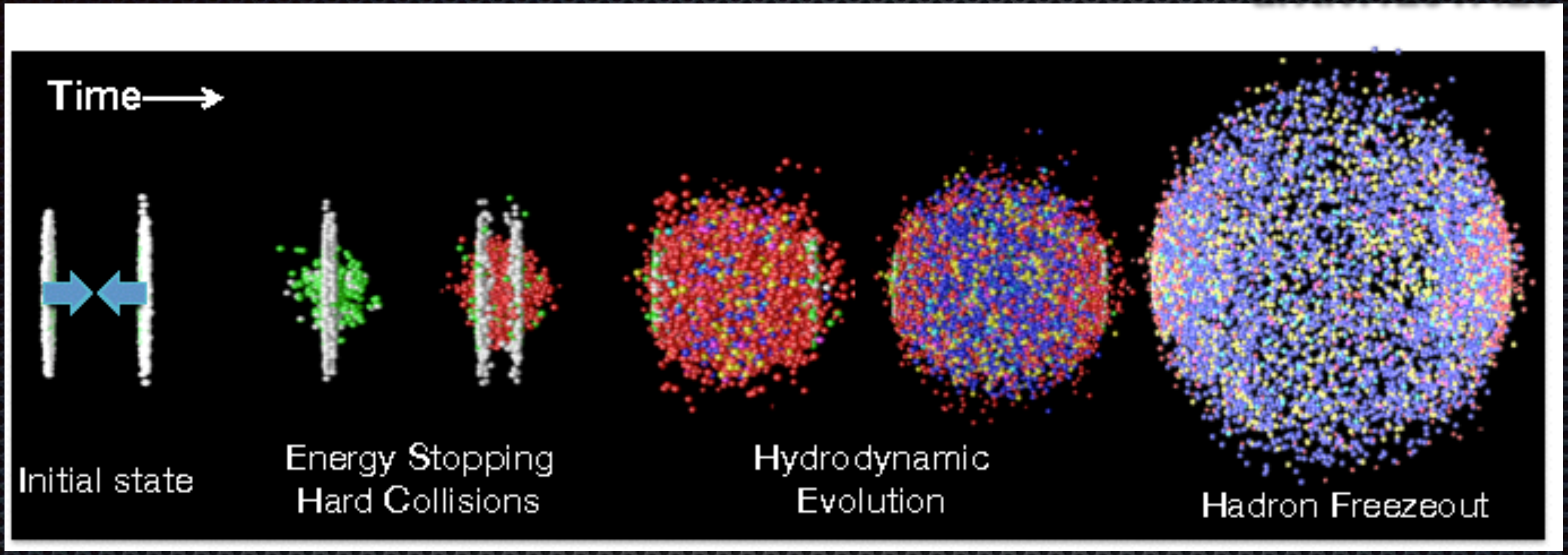
Applications to Beam energy scan experiment

Heavy ion collisions

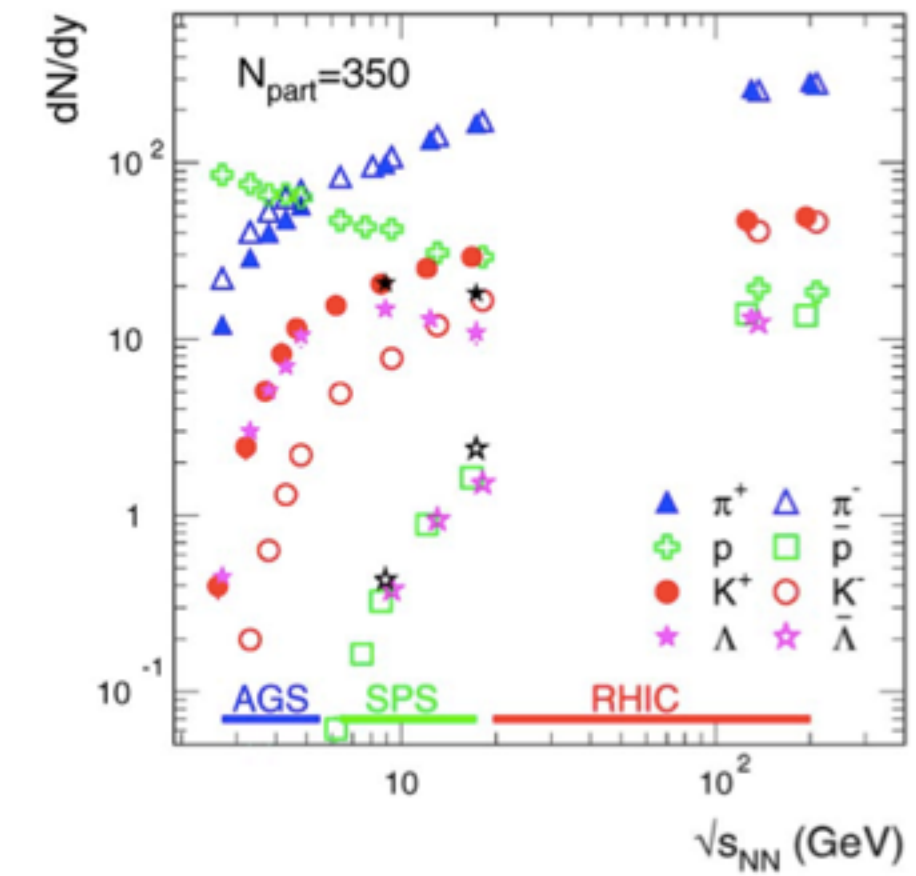
arXiv:1201.4264



- Number of hadrons
  - fixed at a time when inelastic process ceases
  - > chemical freeze-out

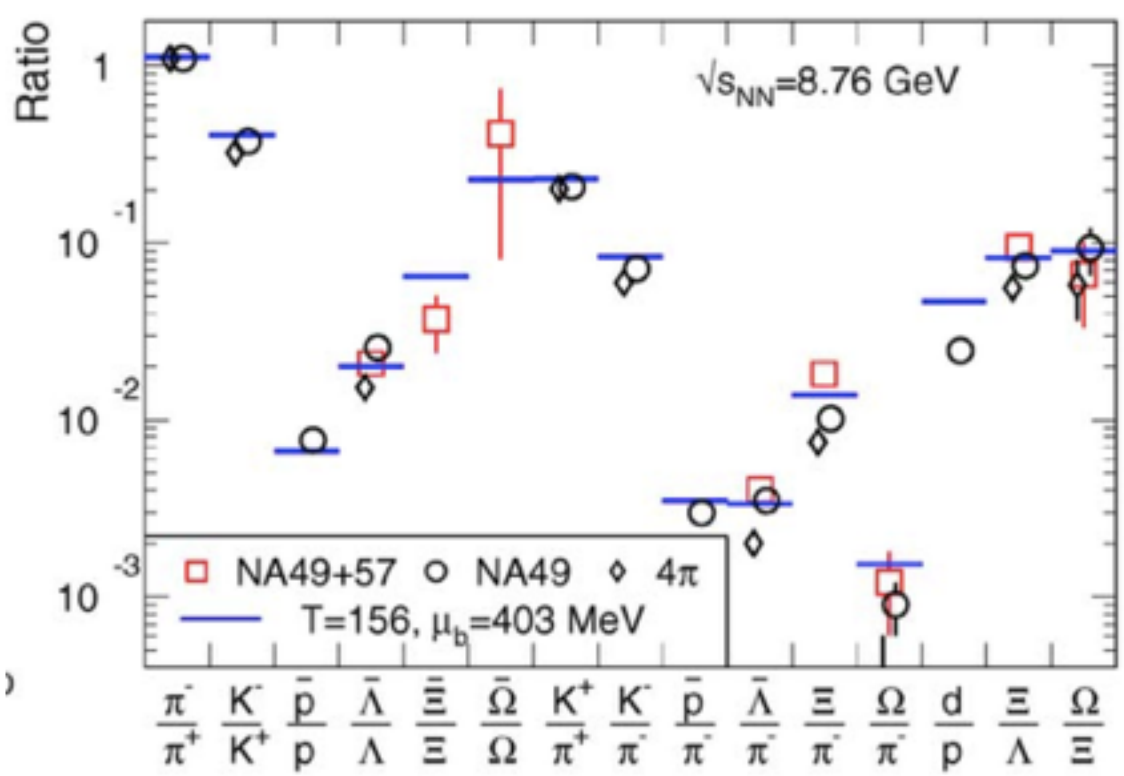
- Number of hadrons provide information at the freeze-out time

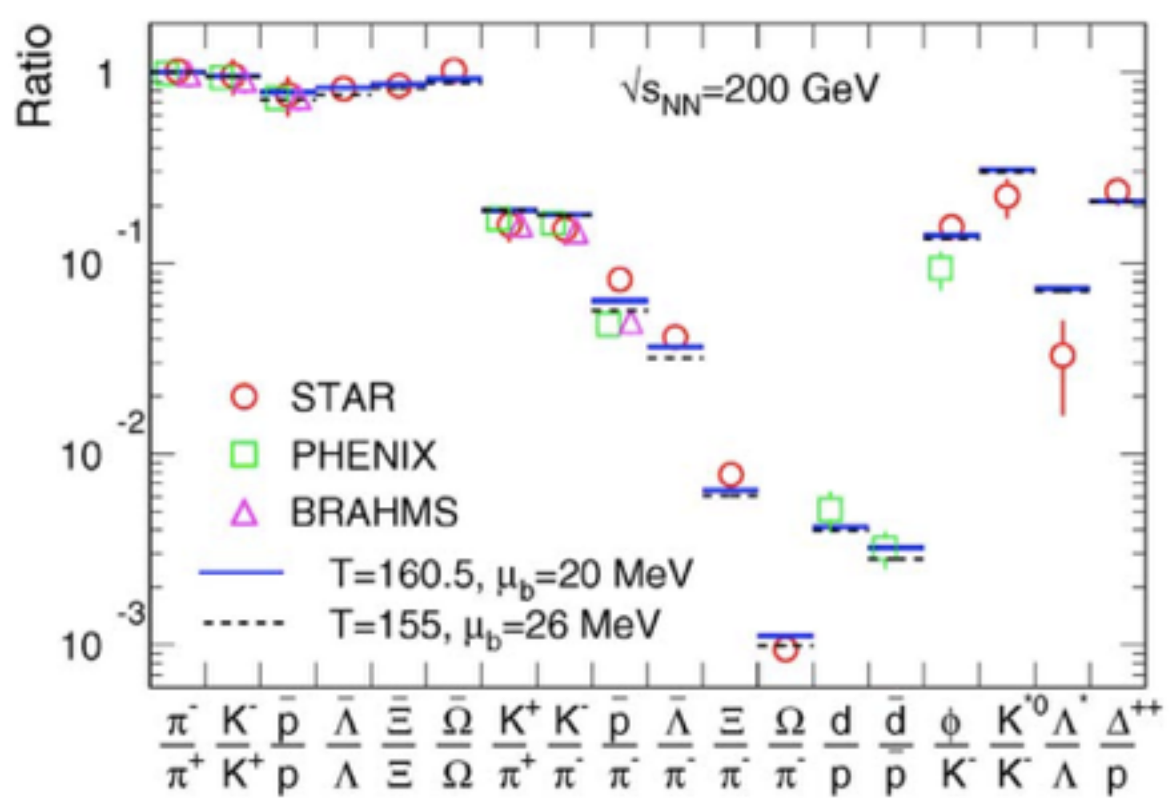
Hadron yields and thermal statistical model



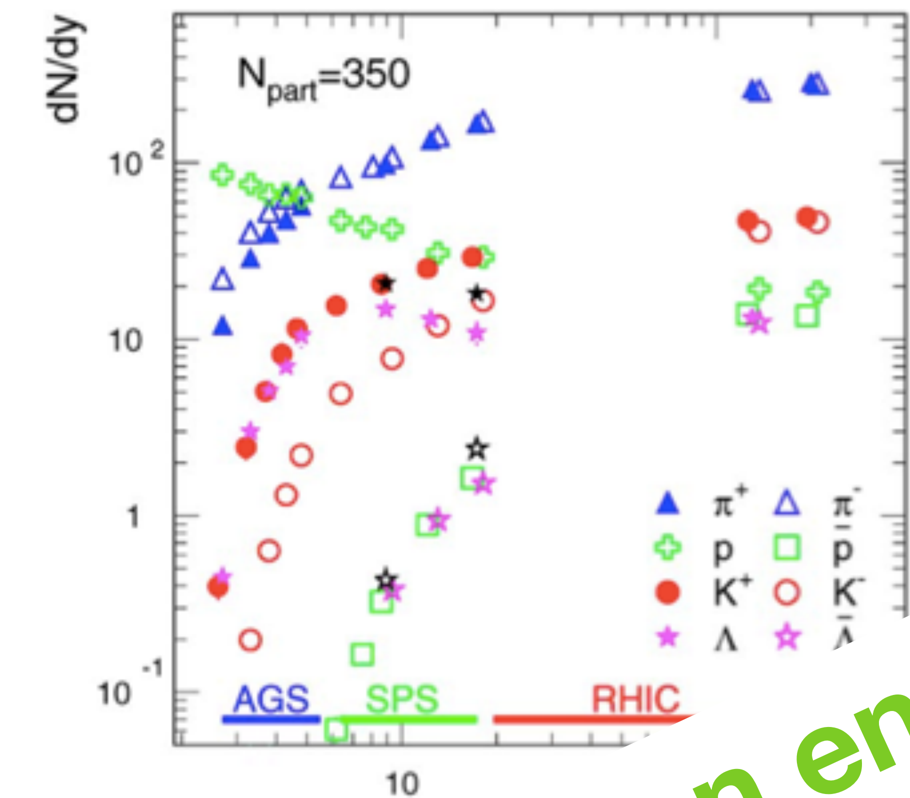
$$n_i = N_i/V = -\frac{T}{V}\frac{\partial \ln Z_i}{\partial \mu} = \frac{g_i}{2\pi^2}\int_0^\infty \frac{p^2\,dp}{\exp[(E_i-\mu_i)/T]\pm 1}$$

Andronic, et., al, (2005)





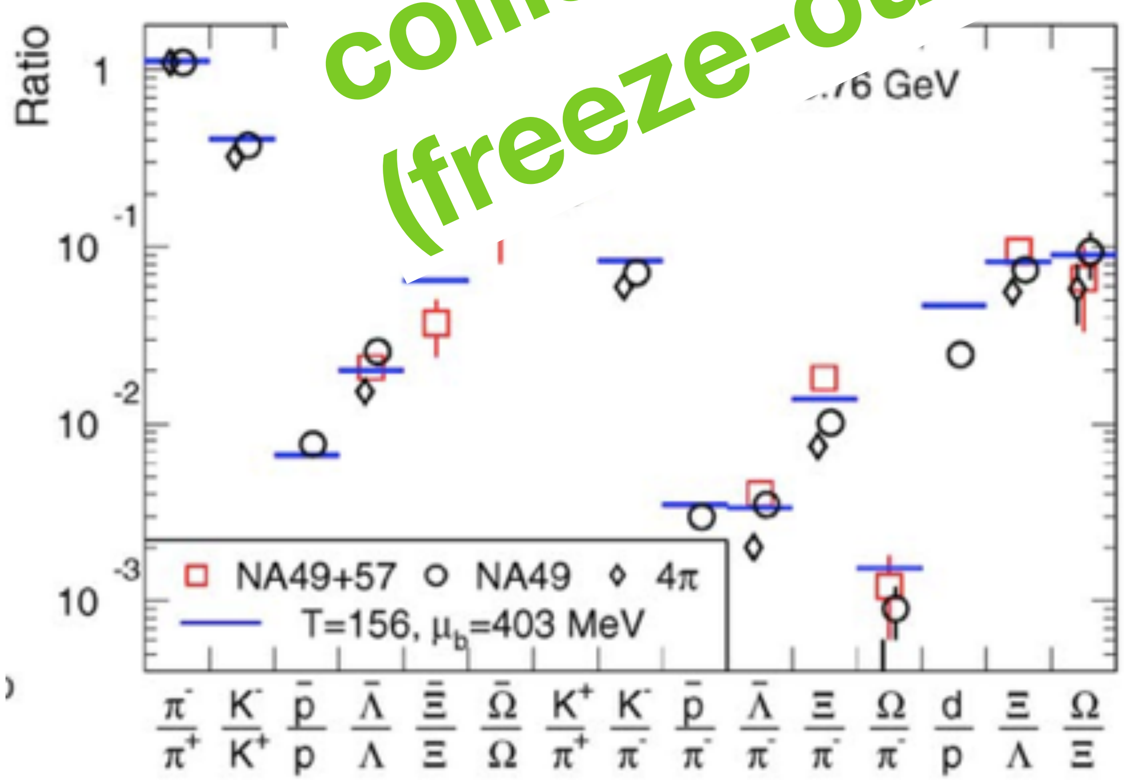
Hadron yields and thermal statistical model



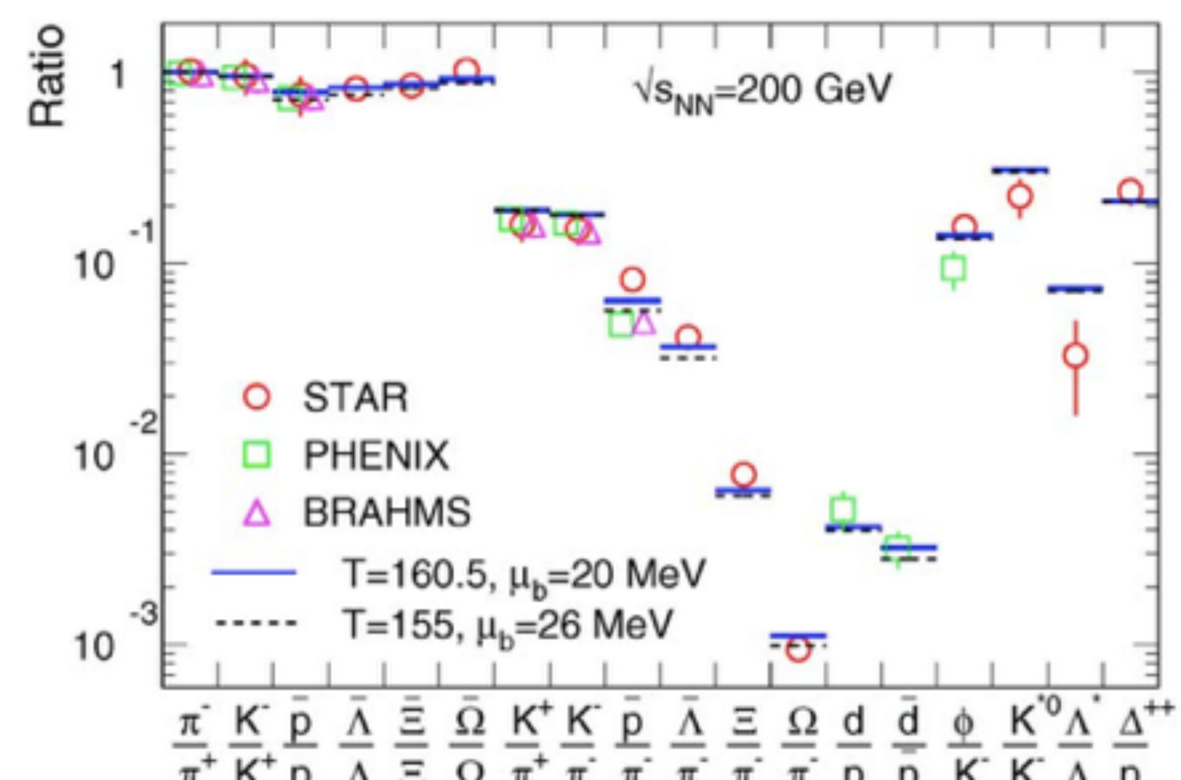
$$n_i = N_i/V = -\frac{T}{V}\frac{\partial \ln Z_i}{\partial \mu} = \frac{g_i}{2\pi^2}\int_0^\infty \frac{p^2\,dp}{\exp[(E_i-\mu_i)/T] \pm 1}$$

collision energy ~ μ, T

(freeze-out parameters)

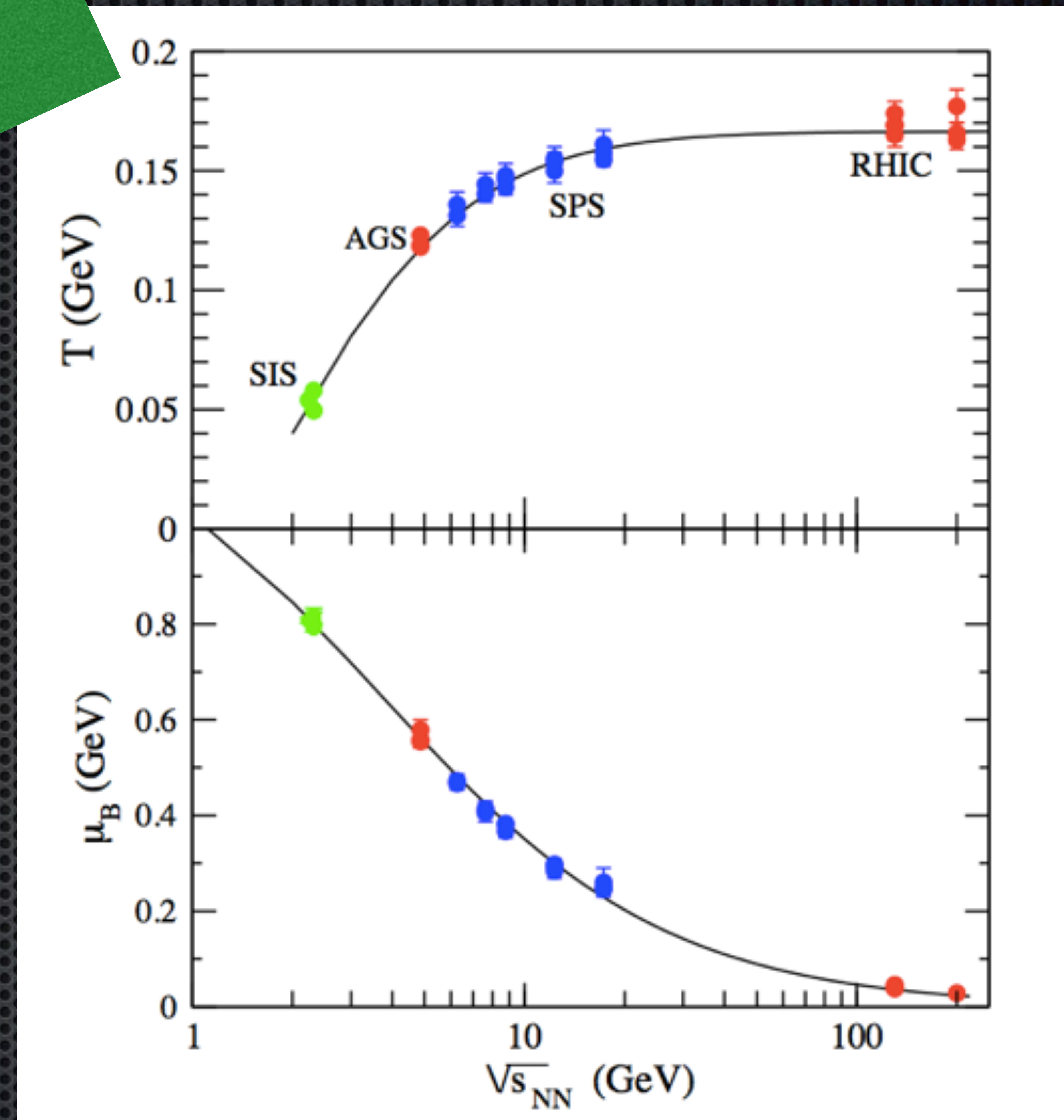


Andronic, et., al, (2005)



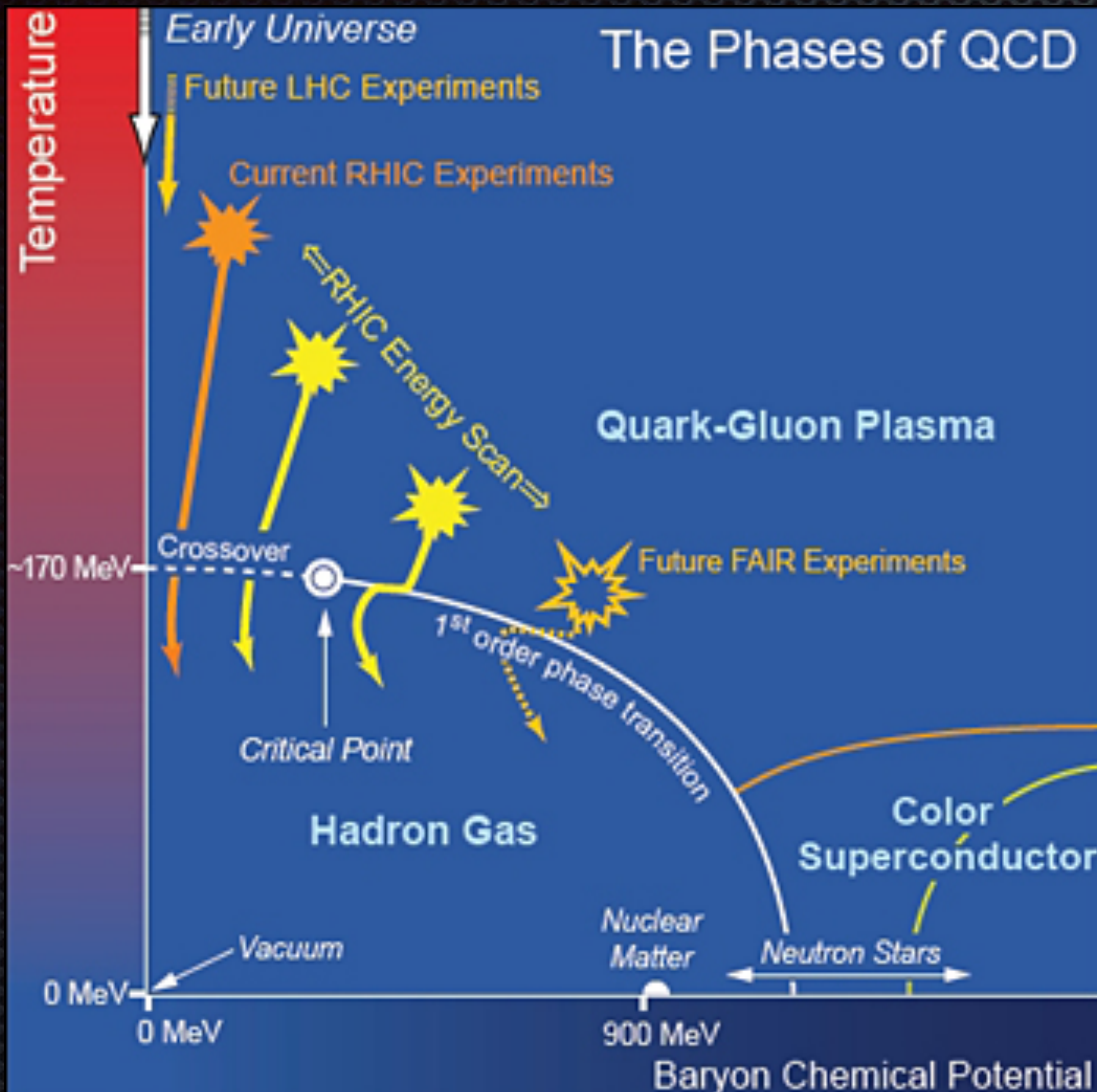
Energy dependence of freeze-out parameters

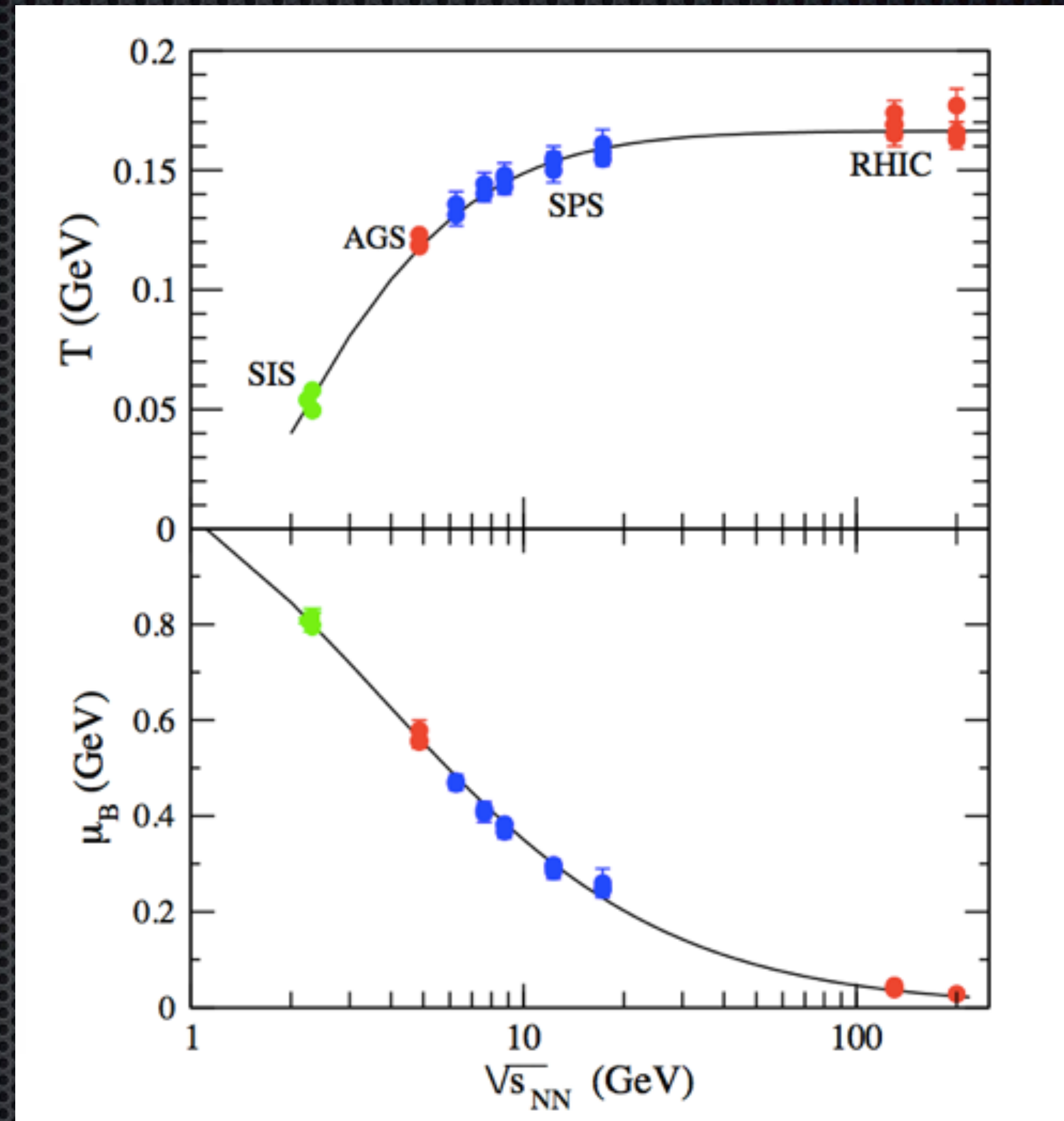
Freeze-out points are located on a line ~ freeze-out line



Cleymans et. al.

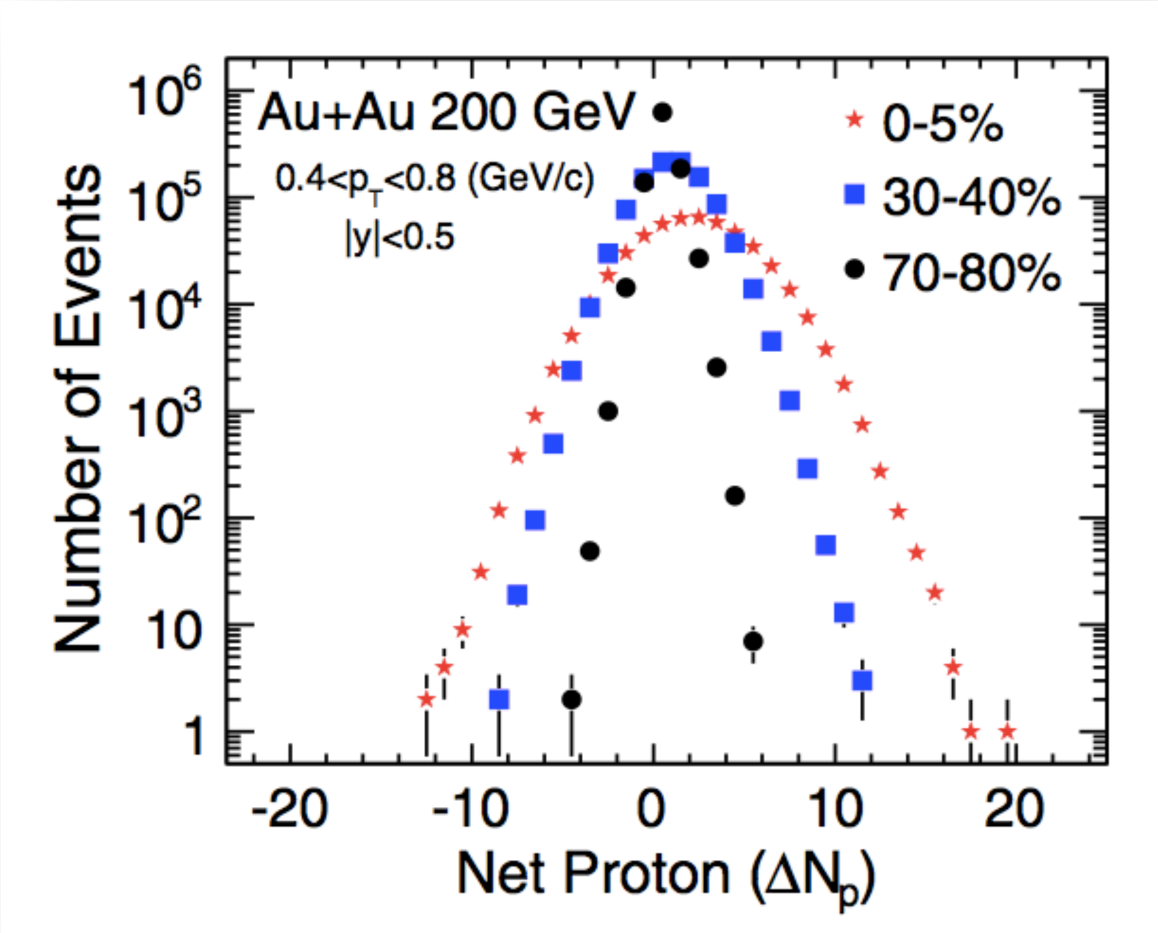
Energy dependence of freeze-out parameters





Cleymans et. al.

Fluctuation



Aggarwal et al. STAR,
PRL105, 022302('10)

Probability distribution for some hadrons are obtained. This is used to obtain event-by-event fluctuation at a freeze-out point.

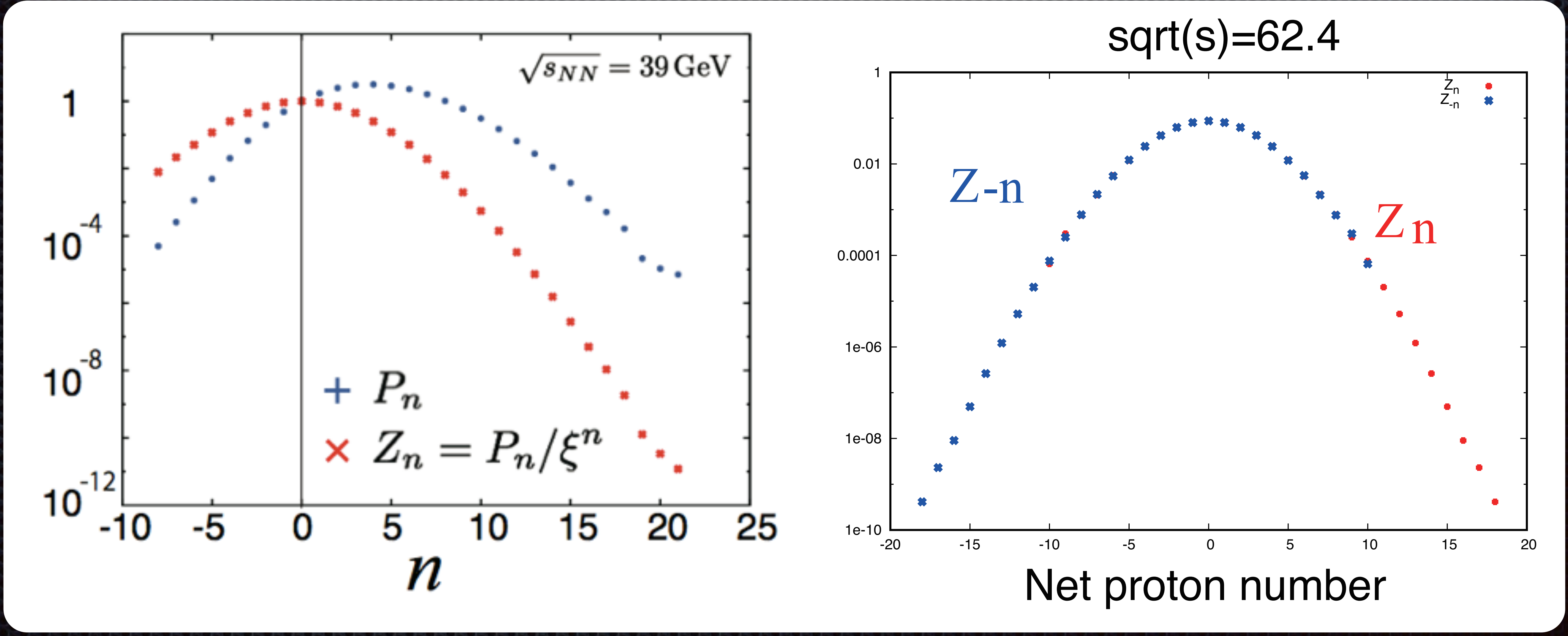
Extension of data to wide range of μ

The probability distribution of net baryon number

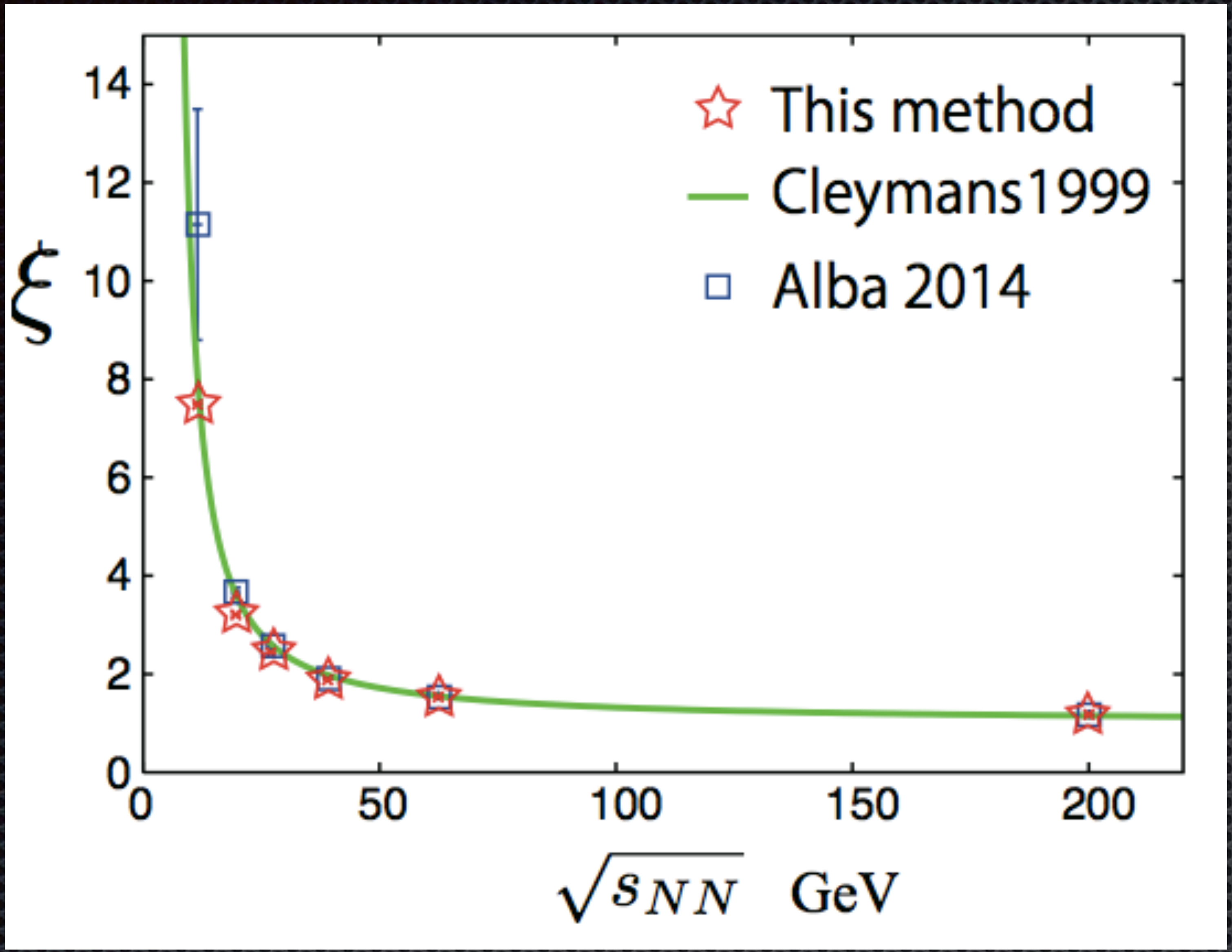
(here we use the proton number as an approximation)

$$P_n \propto Z_n e^{n\mu/T}$$

μ/T is determined from CP invariance : Z(n)= Z(-n)

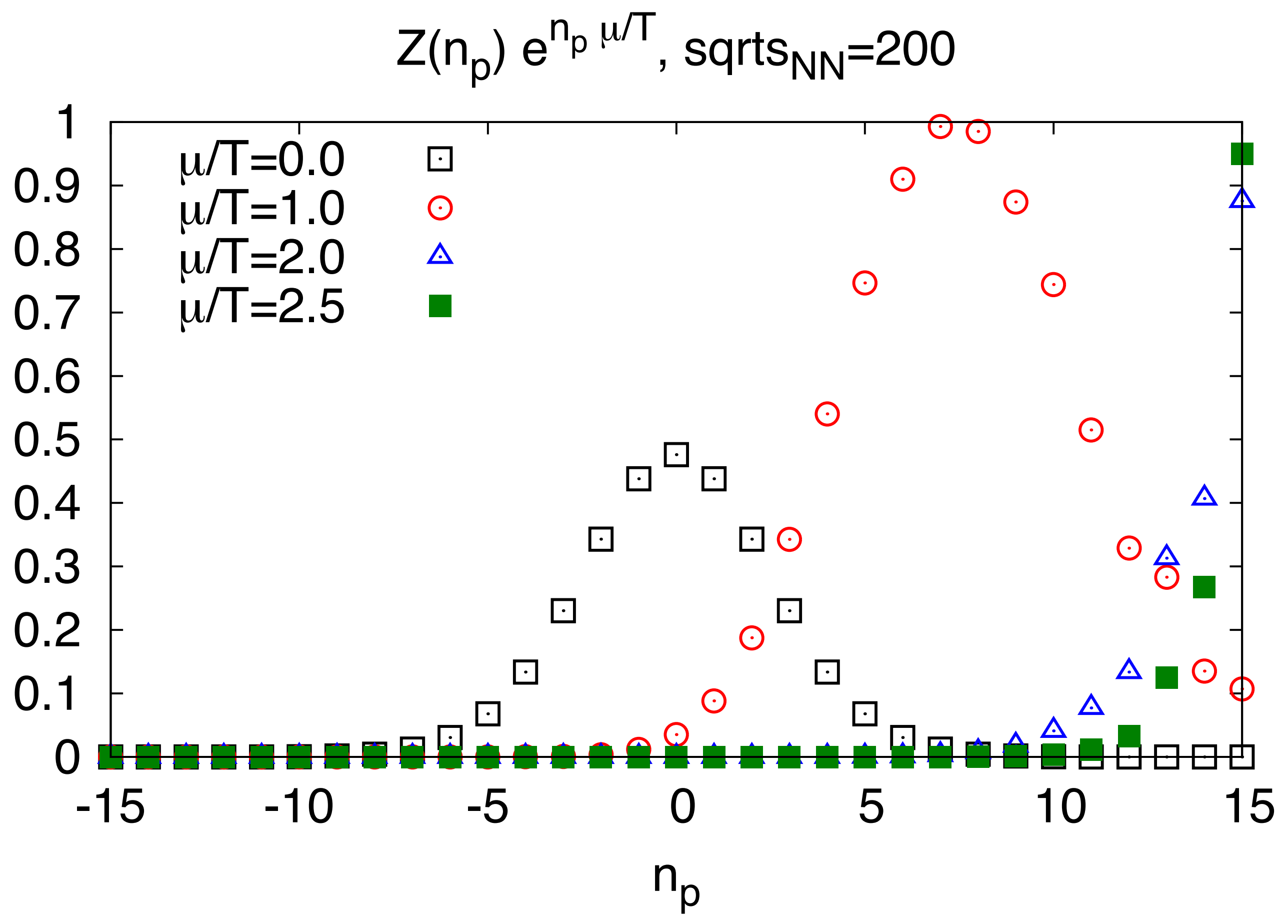


Extension of data to wide range of μ

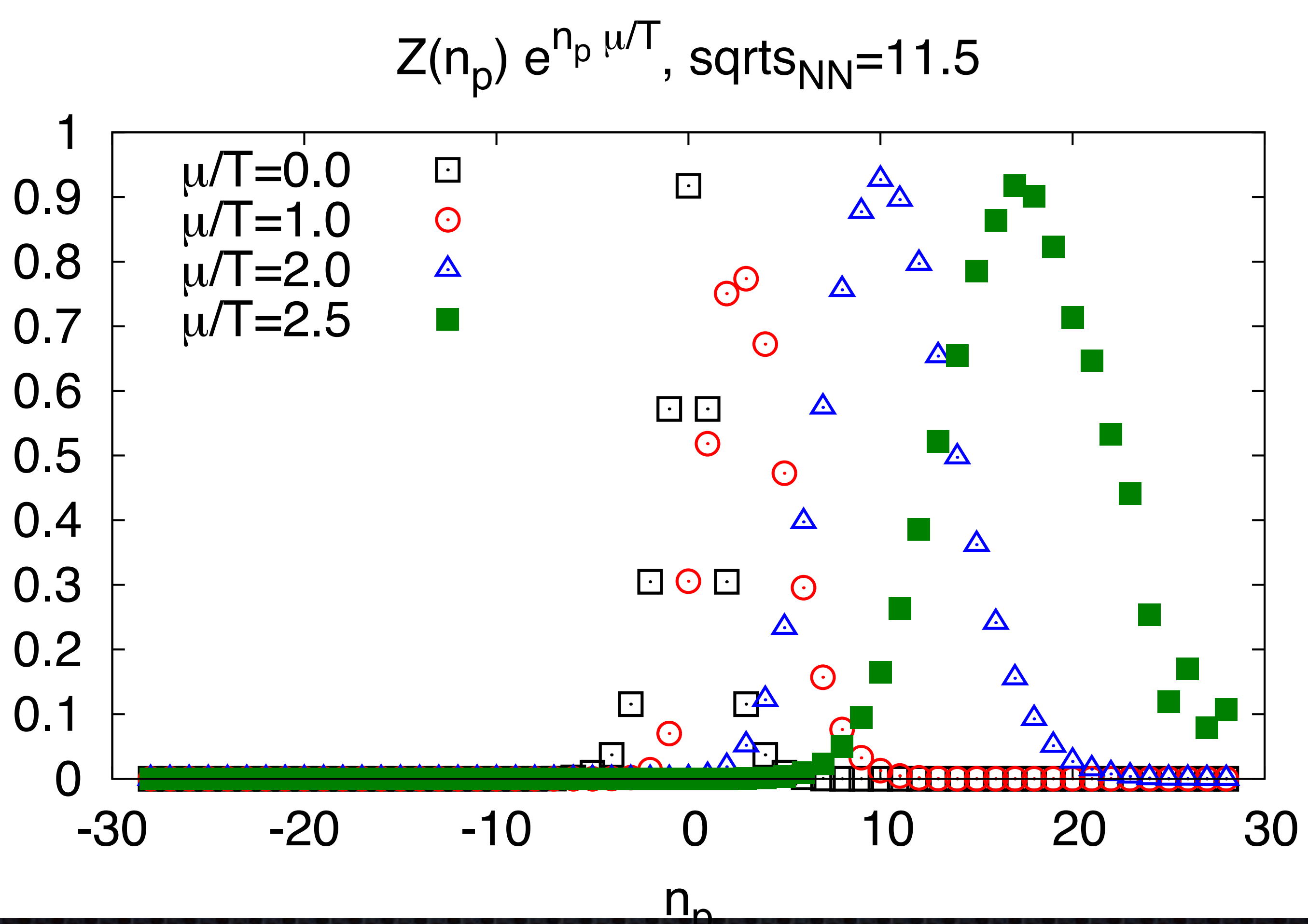


$$\xi = \exp(\mu/T)$$

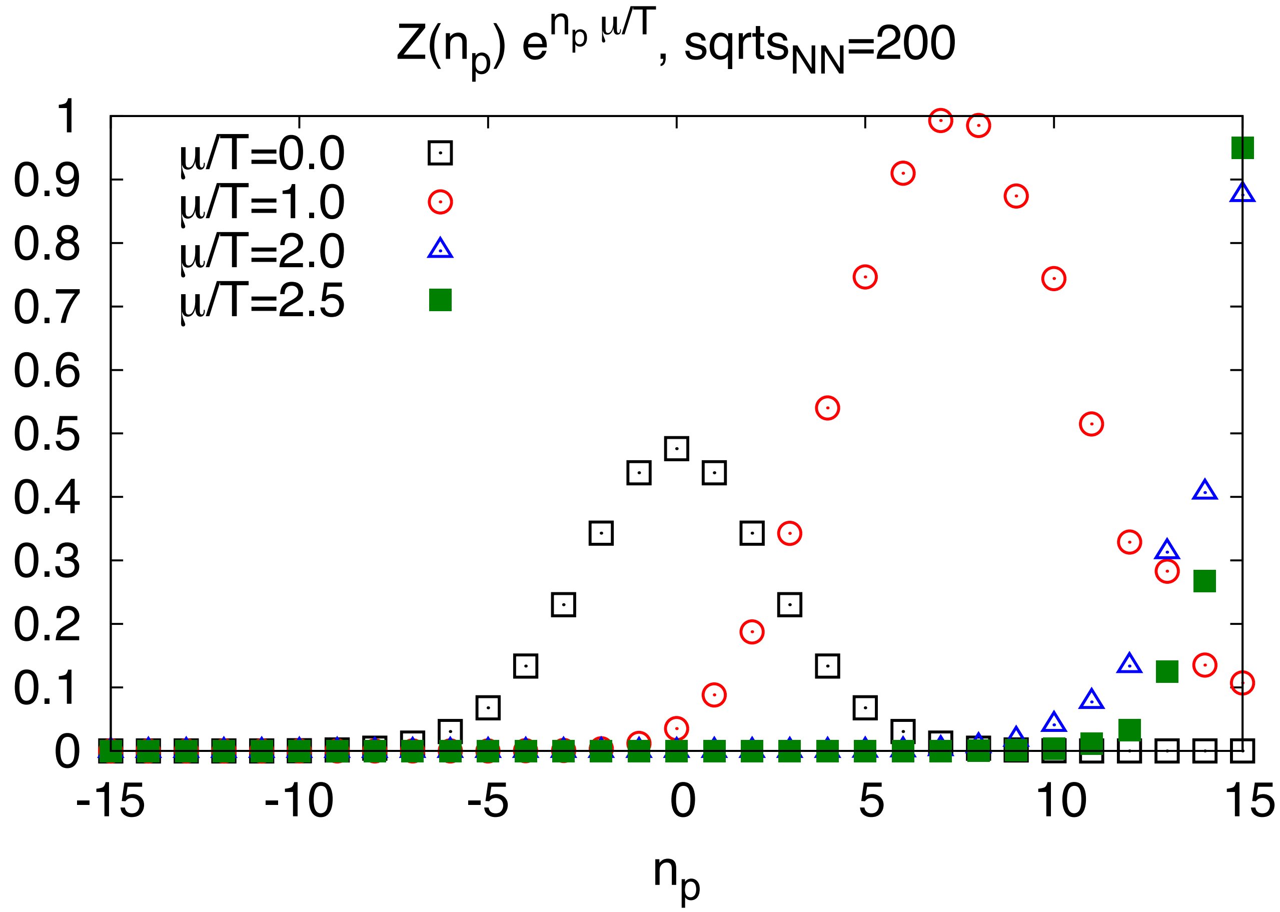
- μ/T obtained from CP invariance agree with those obtained from thermal statistical model for wide range of collision energies.

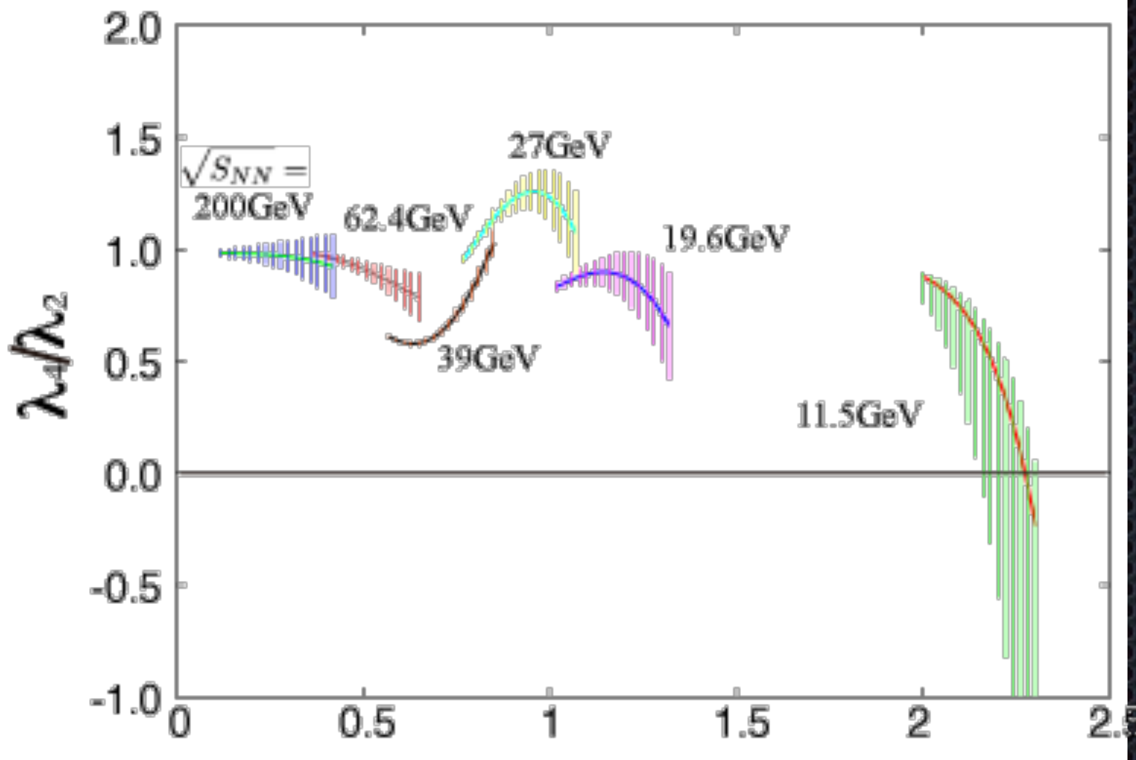


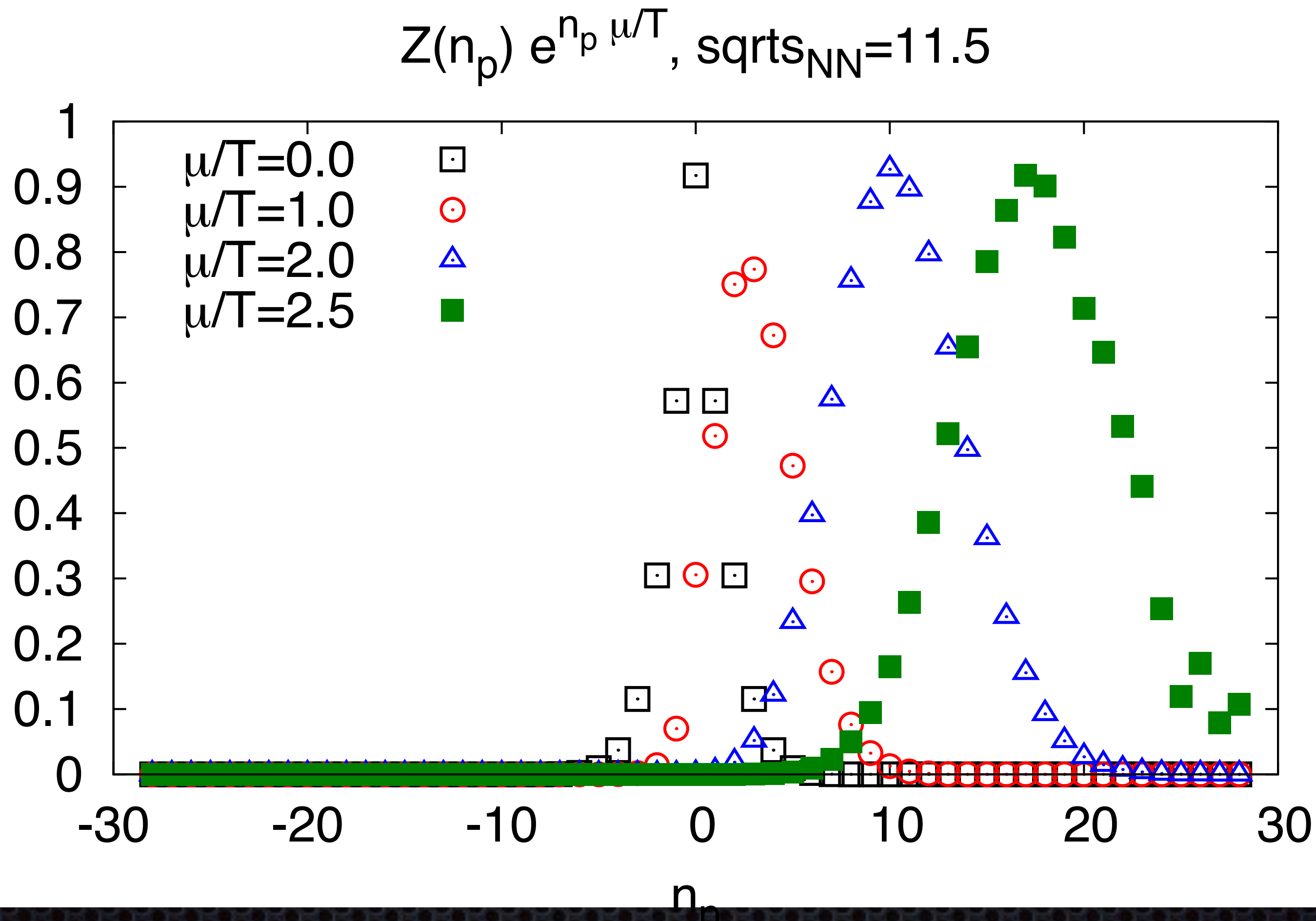
Z(n_p) e^{n_p μ/T}, sqrts_NN=200
μ/T=0.0
μ/T=1.0
μ/T=2.0
μ/T=2.5
n_p



Z(n_p) e^{n_p μ/T}, sqrts_NN=11.5
μ/T=0.0
μ/T=1.0
μ/T=2.0
μ/T=2.5
n_p



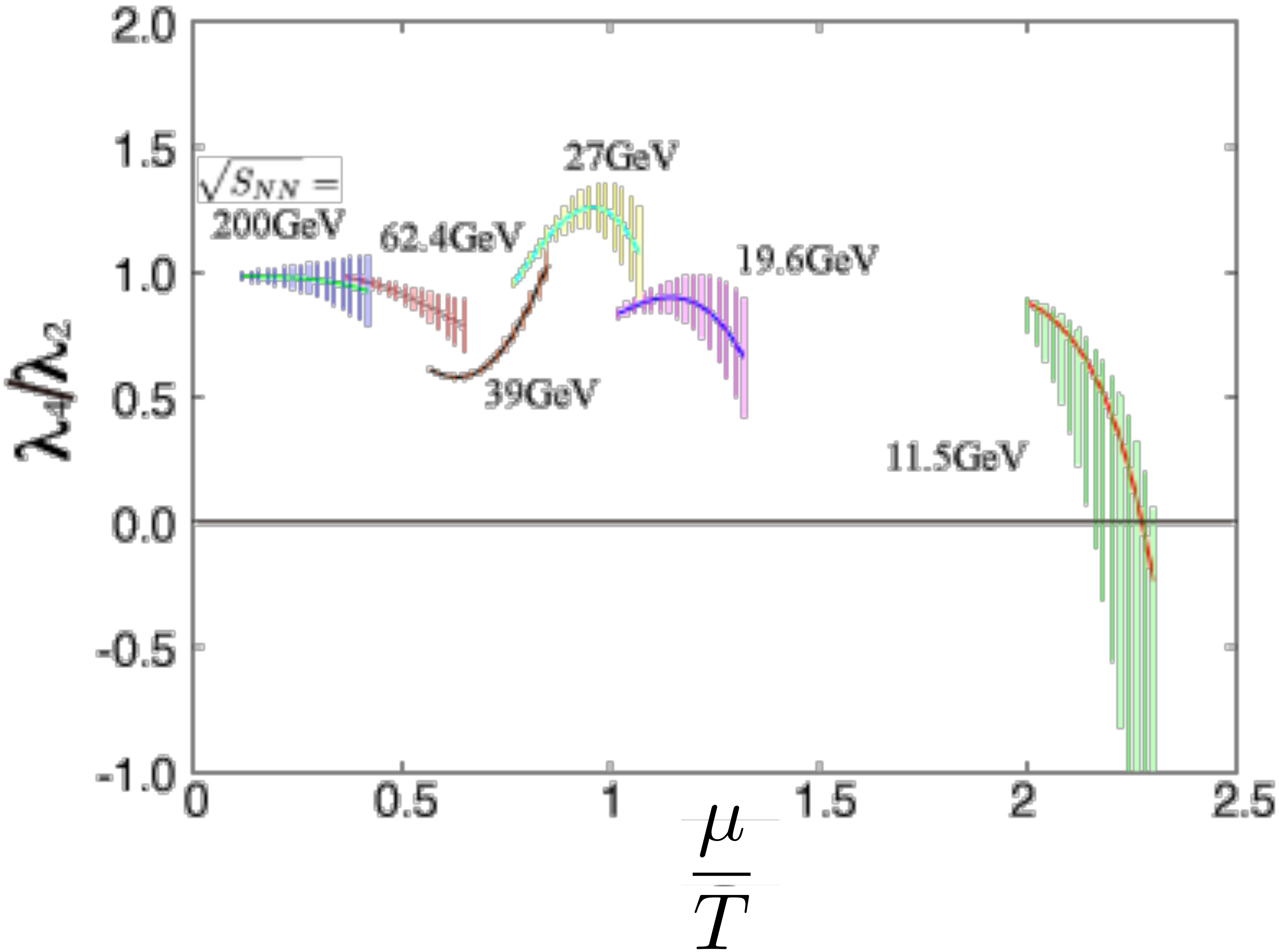




$$\lambda_n \propto (\partial/\partial\mu)^n \ln Z(\mu)$$

RHIC Data

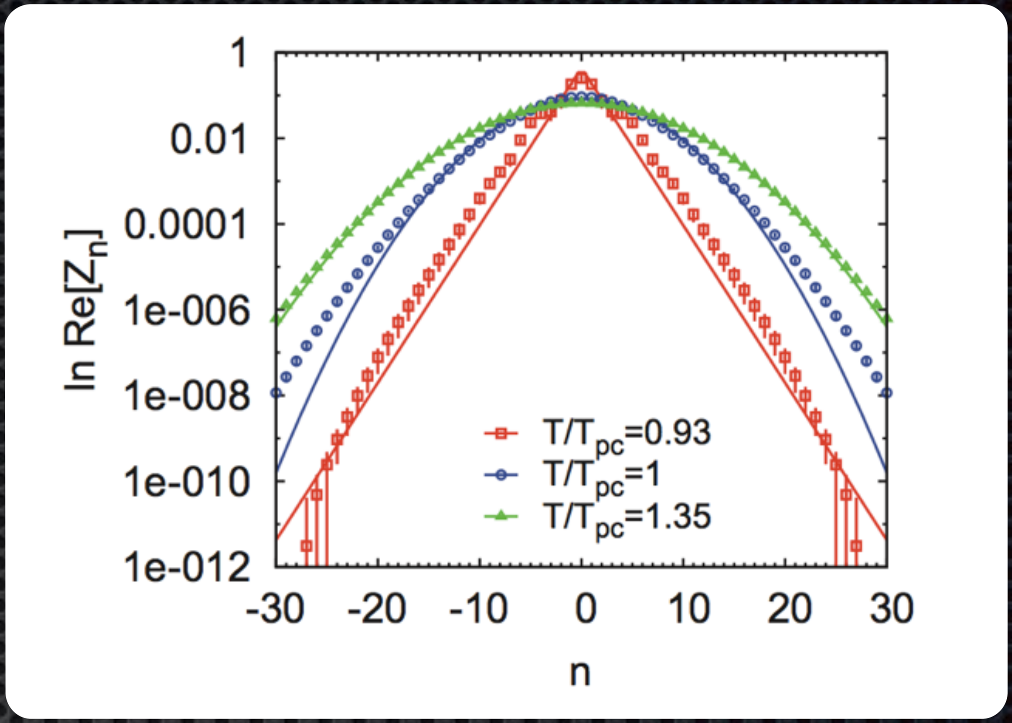
Kurtosis $\frac{\lambda_4}{\lambda_2}$ as a function of $\frac{\mu}{T}$



Buck up slides

How to achieve LY zeros ?

- **Calculation of Zn : truncation is inevitable**



$$Z(\mu) = \sum_{n=-\infty}^{\infty} Z_n e^{n\mu/T}$$

$$\rightarrow \sum_{n=-n_0}^{n_0} Z_n e^{n\mu/T}$$

Calculation of Lee-Yang zeros

Lee-Yang零点を２通りの方法で計算します。

- Zeros of the fugacity polynomial

$$Z(\mu) = \sum_n Z_n e^{n\mu/T}$$

- Cancellation of the free energy

$$Z = e^{-Vf_I/T} + e^{-Vf_{II}/T}$$

1. Cancellation of free-energy

Free energy is different in different RW phases

- RW phases are distinguished by the argument of Pol.
- The arg(Pol) is translated into A_4.

$$\mathcal{L}_4 = \bar{\psi}(\gamma_4(igA_4 + \mu))\psi$$

$$\mu \to \mu + igA_4 = \mu + i\omega T$$

- This modifies the free energy as

$$f_{\rm I} = -T^4(c_0 + c_2(\mu/T)^2 + c_4(\mu/T)^4),$$

$$f_{\rm II} = -T^4(c_0 + c_2(\mu'/T)^2 + c_4(\mu'/T)^4), \mu' = \mu + i\omega T$$

$$\mu, \mu' \in C$$

1. Cancellation of free-energy

Cancellation of two types of free energy allows Z=0

- Cancellation of free energies [Biskup et al('01)]

$$Z = e^{-Vf_I/T} + e^{-Vf_{II}/T}$$

$$\begin{cases} \mathrm{Re}[f_{\mathrm{I}} - f_{\mathrm{II}}] = 0 \\ \frac{V}{T}\mathrm{Im}[f_{\mathrm{I}} - f_{\mathrm{II}}] = (2k-1)\pi, (k \in \mathbb{Z}) \end{cases}$$

- Approximate solution for $c_2 >> c_4$

$$(\mu_R, \mu_I) = \left(\frac{3(2k-1)}{4VT^3 c_2}, -\pi/3 \right)$$

  - Possible to solve it with c4
  - f_I and f_{III}, and f_{II} and f_{III}.

2. Zeros of the fugacity polynomial

We show that the fugacity polynomial of high T QCD is well approximated by a well-known polynomial.

$$Z(\mu) = \sum_{n=-N}^{N} Z_n e^{n\mu/T}$$

- **We need Zn. : Fourier integral with the free energy as input (analytic)**
- **(We use the fugacity expansion in lattice simulation.)**

First, we derive Zn using the Fourier transformation.

$$Z(\mu) = \sum_{n=-N}^{N} Z_n e^{n\mu/T}$$

Zn from the Fourier transformation

$$Z_n = \int Z(\theta) e^{in\theta} d\theta, \quad \mu/T = i\theta, (\theta \in \mathbb{R})$$

$$f = -\frac{T}{V} \ln Z(\mu)$$

- We use the quartic form of f(μ) as input.
- f(μ) is obtained for real μ. On the other hand, the Fourier integral requires complex μ.

Next, we use the RW *periodicity*

$$Z_n = \int Z(\theta) e^{in\theta} d\theta, \quad \mu/T = i\theta, (\theta \in \mathbb{R})$$

We decompose the integral into three domains.

$$Z_n = \int_{-\pi/3}^{\pi/3} e^{-Vf(\theta)/T} e^{in\theta} \frac{d\theta}{2\pi} + \int_{\pi/3}^{\pi} e^{-Vf(\theta-2\pi/3)/T} e^{in\theta} \frac{d\theta}{2\pi}$$
$$+ \int_{\pi}^{5\pi/3} e^{-Vf(\theta-4\pi/3)/T} e^{in\theta} \frac{d\theta}{2\pi},$$

Shift of θ leads to

$$Z_n = \frac{3}{2\pi} \int_{-\pi/3}^{\pi/3} e^{-Vf(\theta)/T} e^{in\theta} d\theta, \; n \equiv 0 \mod 3$$

(This relation holds for any temperature, regardless of the RW phase transition)

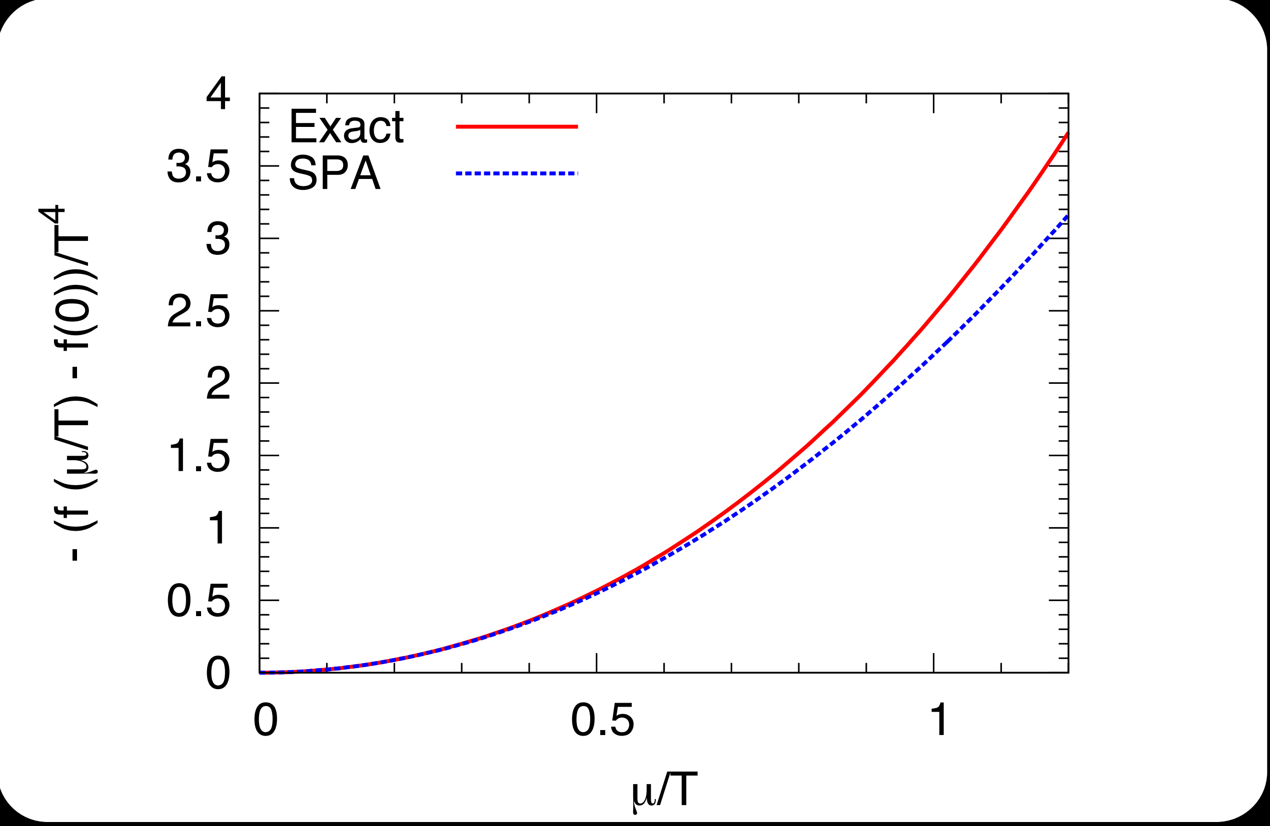
Now, we use the quartic expression of f(μ) and SPA

$$Z_n = \frac{3}{2\pi}\int_{-\pi/3}^{\pi/3} e^{VT^3(c_0 - c_2\theta^2 + c_4\theta^4)} e^{in\theta} d\theta, \quad n \equiv 0 \mod 3$$

At T/Tc > 1.1~1.2, c2/c4 ~ 10. We use the saddle point approximation.

$$Z_n = C e^{-n^2/(4T^3 V c_2)}, \quad (n \equiv 0 \mod 3)$$

- We assume the Gaussian Zn is valid for large n.
- SPA is valid for small Re[μ]



If Zn is Gaussian, then Z(μ) is a Jacobi-theta function.

$$Z(\mu) = C \sum_{n_B=-\infty}^{\infty} e^{-9{n_B}^2/(4T^3Vc_2)+3n_B\mu/T}$$

Theta function

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{-\pi i n^2\tau+2\pi i n z}$$

$$i\pi\tau = 9/(4T^3Vc_2),$$
$$2\pi i z = 3\mu/T$$

Zeros of theta function

$$\vartheta(z,\tau) = 0 \leftrightarrow z = l + k\tau + \frac{1}{2} + \frac{\tau}{2}, (k, l \in \mathbb{Z})$$

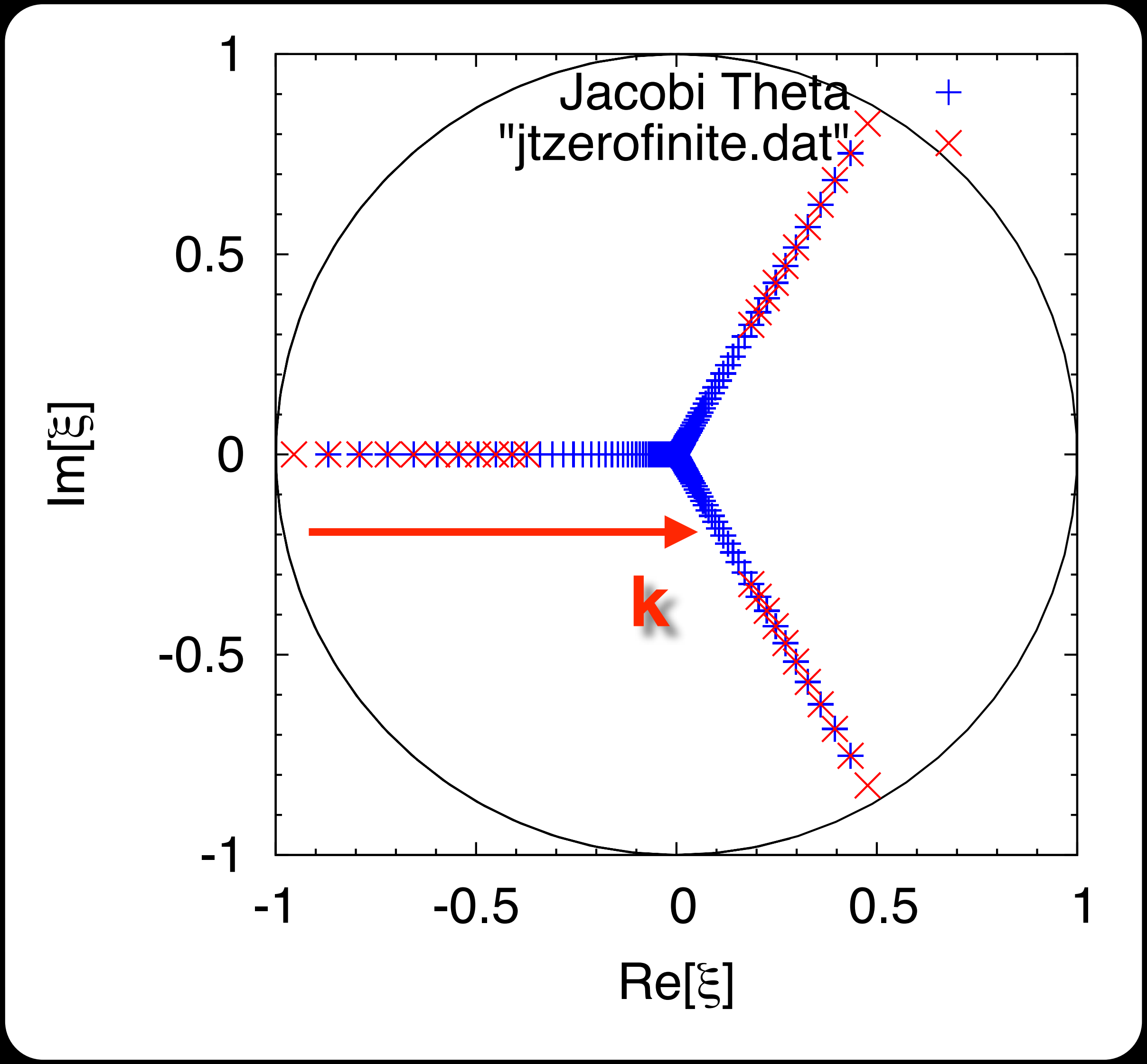
$$\frac{\mu}{T} = \frac{(2l+1)\pi i}{3} - \frac{3(2k+1)}{4VT^3c_2}$$

$$(\mu_R, \mu_I) = \left(\frac{3(2k-1)}{4VT^3c_2}, -\pi/3\right)$$

Method1

Lee-Yang zero distribution of theta function

$$\frac{\mu}{T} = \frac{(2l+1)\pi i}{3} - \frac{3(2k+1)}{4VT^3c_2}$$



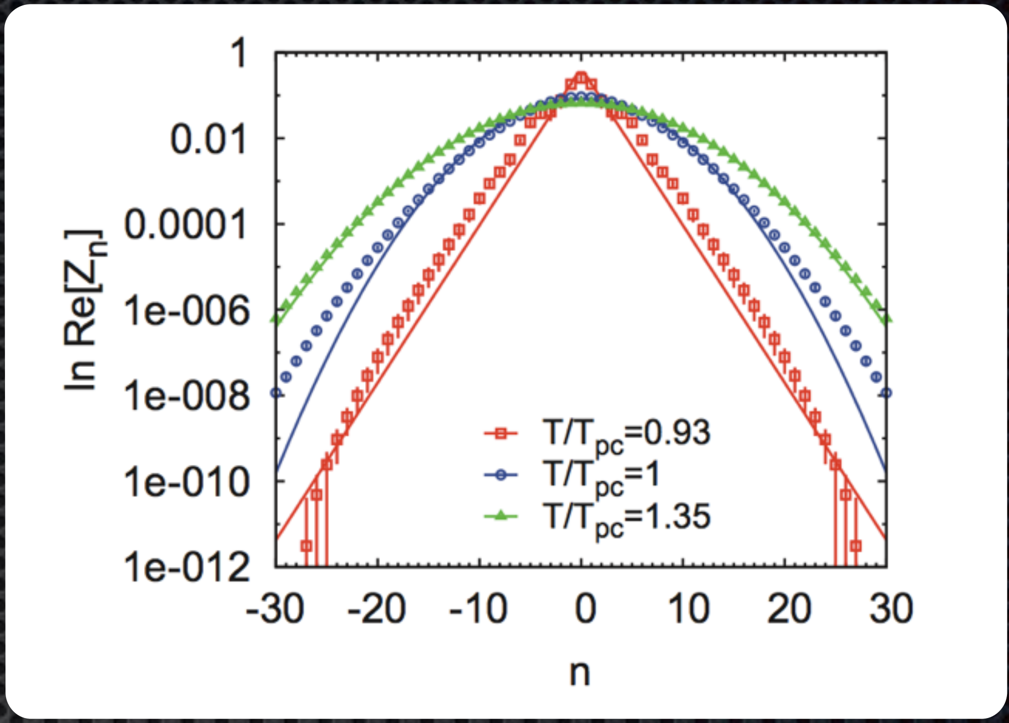
Zeros approaches to the RW transition point as 1/V.
Spacing of zeros is a prediction

How to achieve LY zeros ?

- **Calculation of Zn : truncation is inevitable**

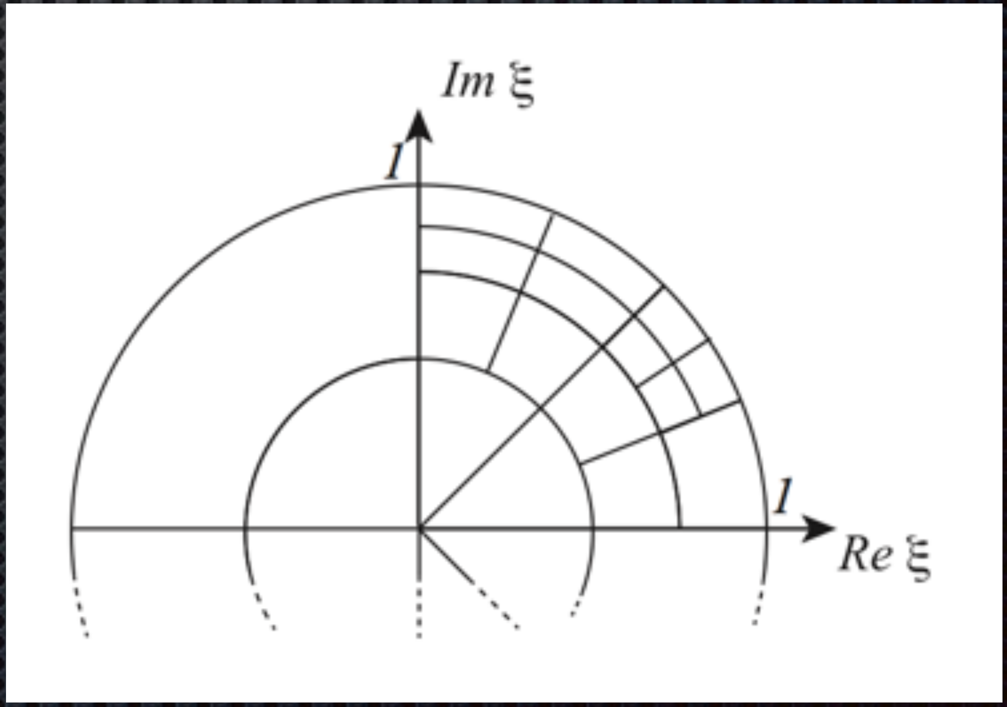
$$Z(\mu) = \sum_{n=-\infty}^{\infty} Z_n e^{n\mu/T}$$

$$\rightarrow \sum_{n=-n_0}^{n_0} Z_n e^{n\mu/T}$$



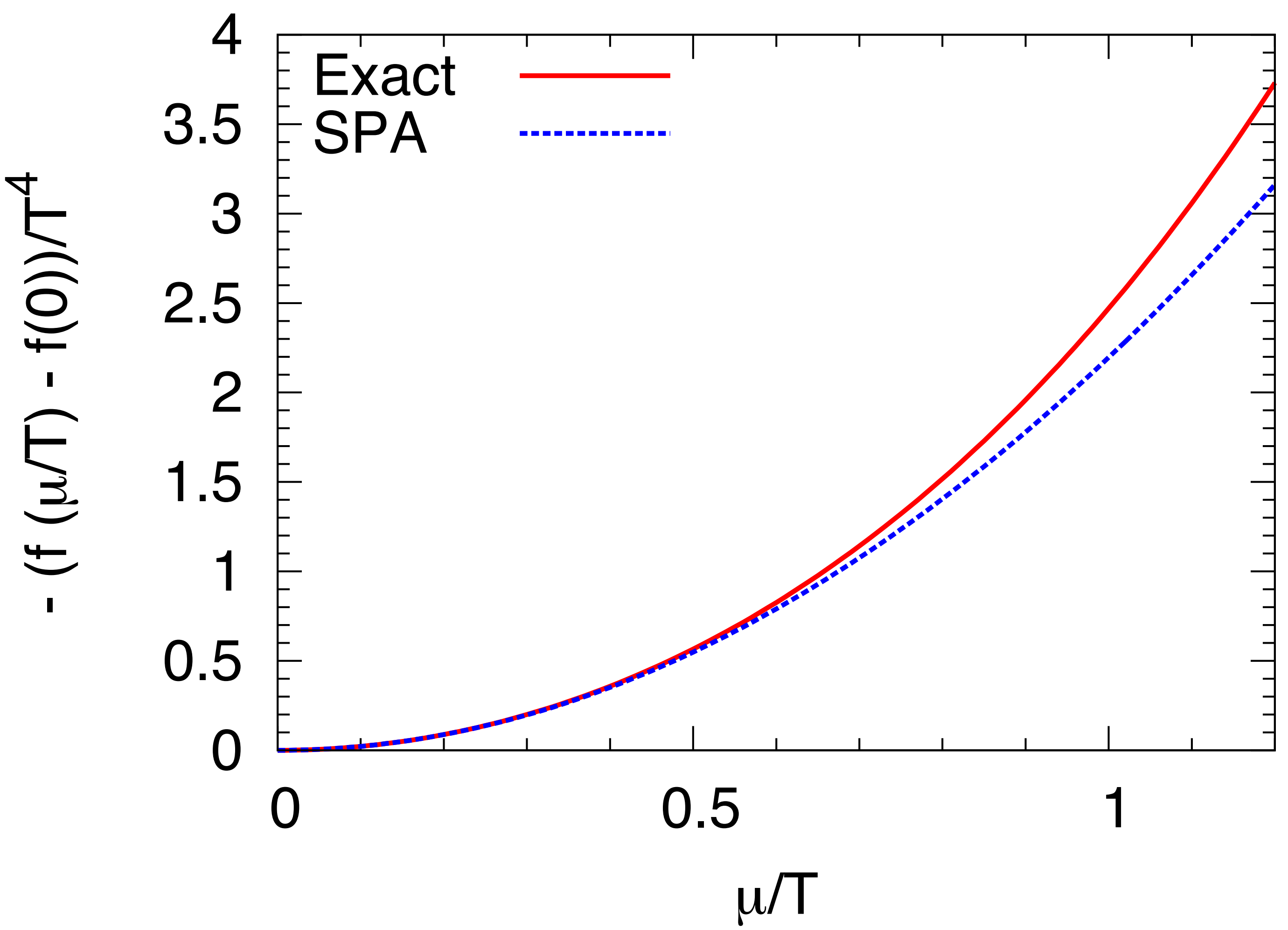
- **Cauchy integral + recursive division + multi-precision arithmetic**

$$Z(\mu) = \sum Z_n e^{n\mu/T} \rightarrow \prod (1 - \xi/\xi_i)$$

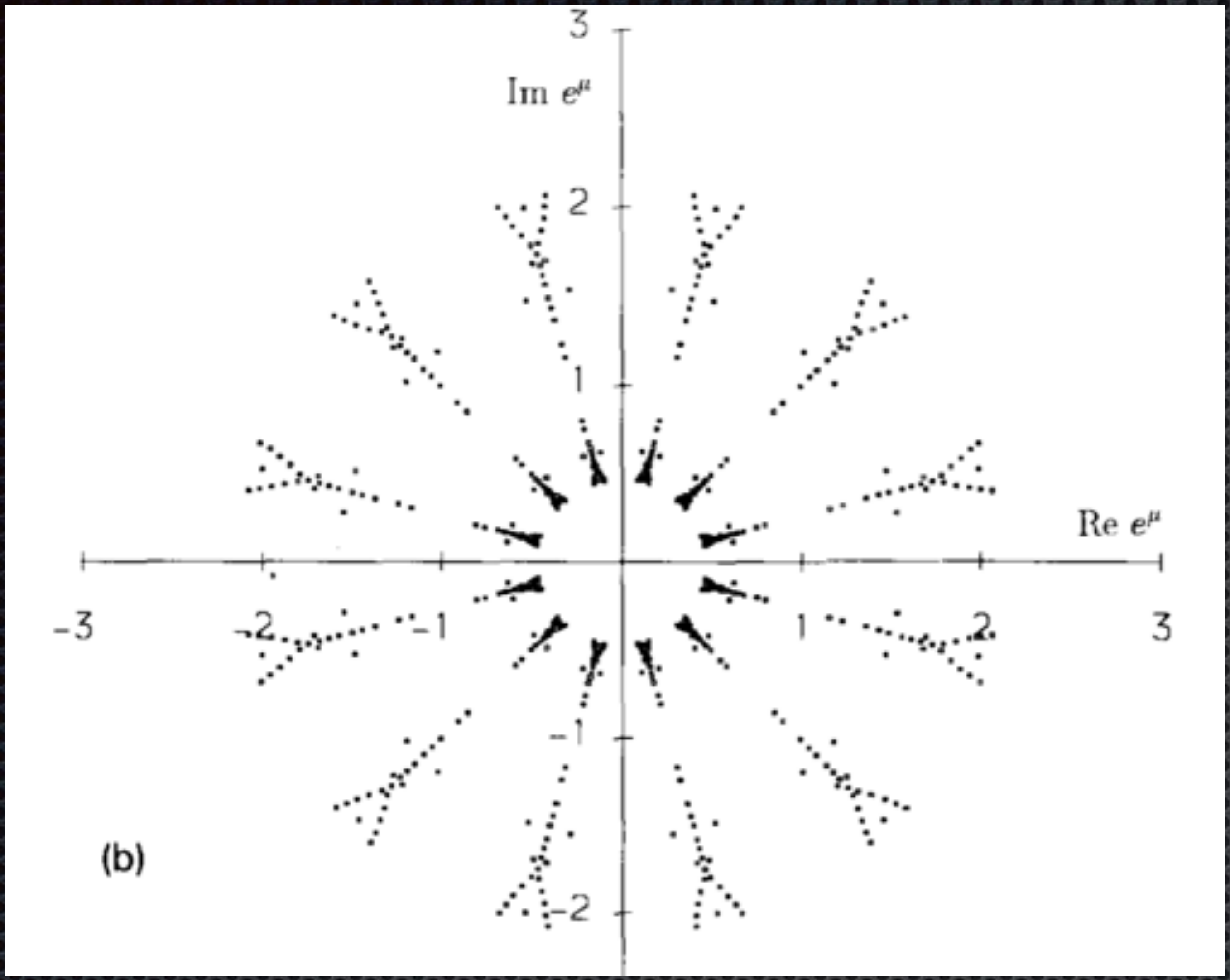


Validity of SPA

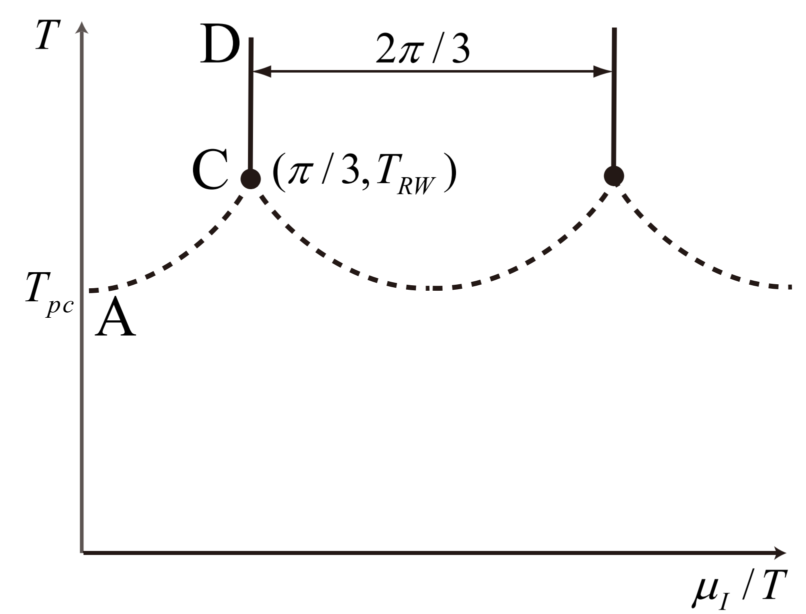
$$Z(\mu) = C \sum e^{-9n^2/(4VT^3c_2)+3n\mu/T}$$
$$= C\vartheta(z,\tau)$$

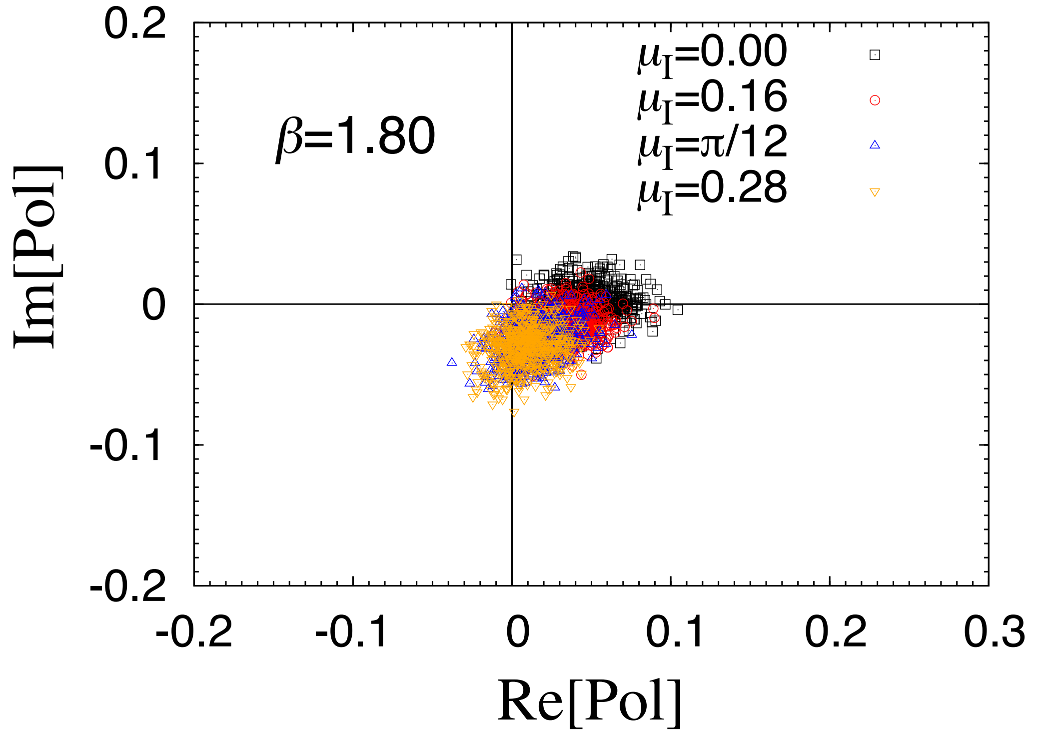


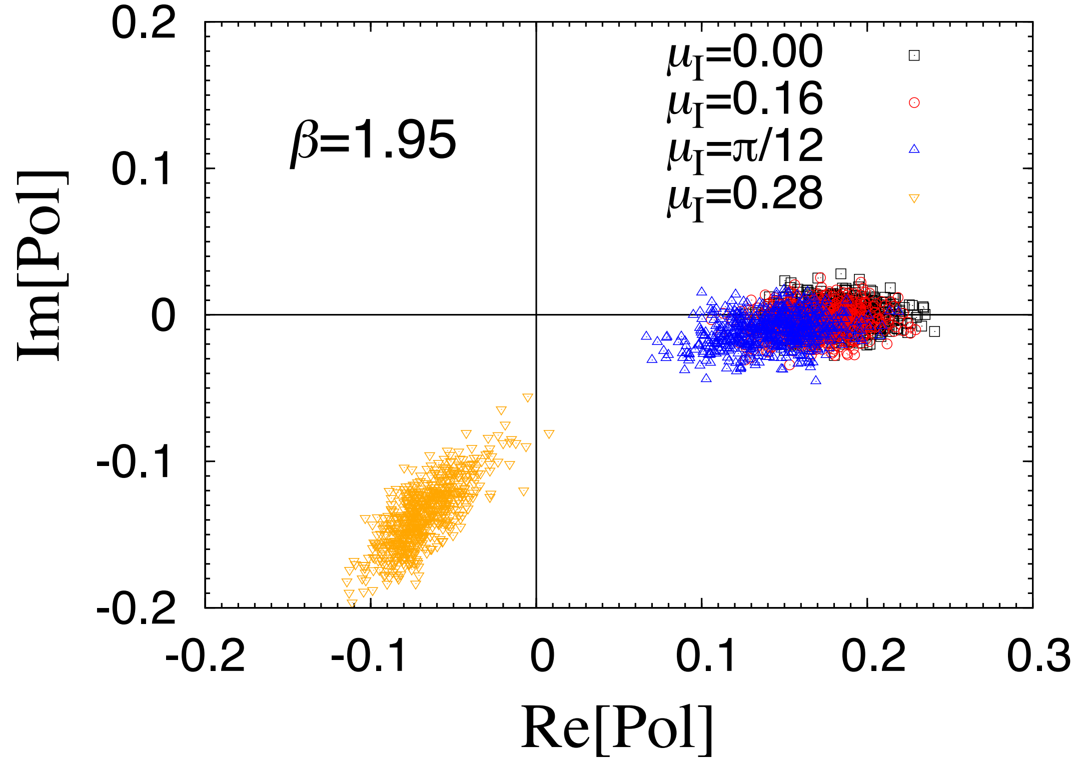
Lee-Yang zeros - 20 years



RW periodicity&phase transition

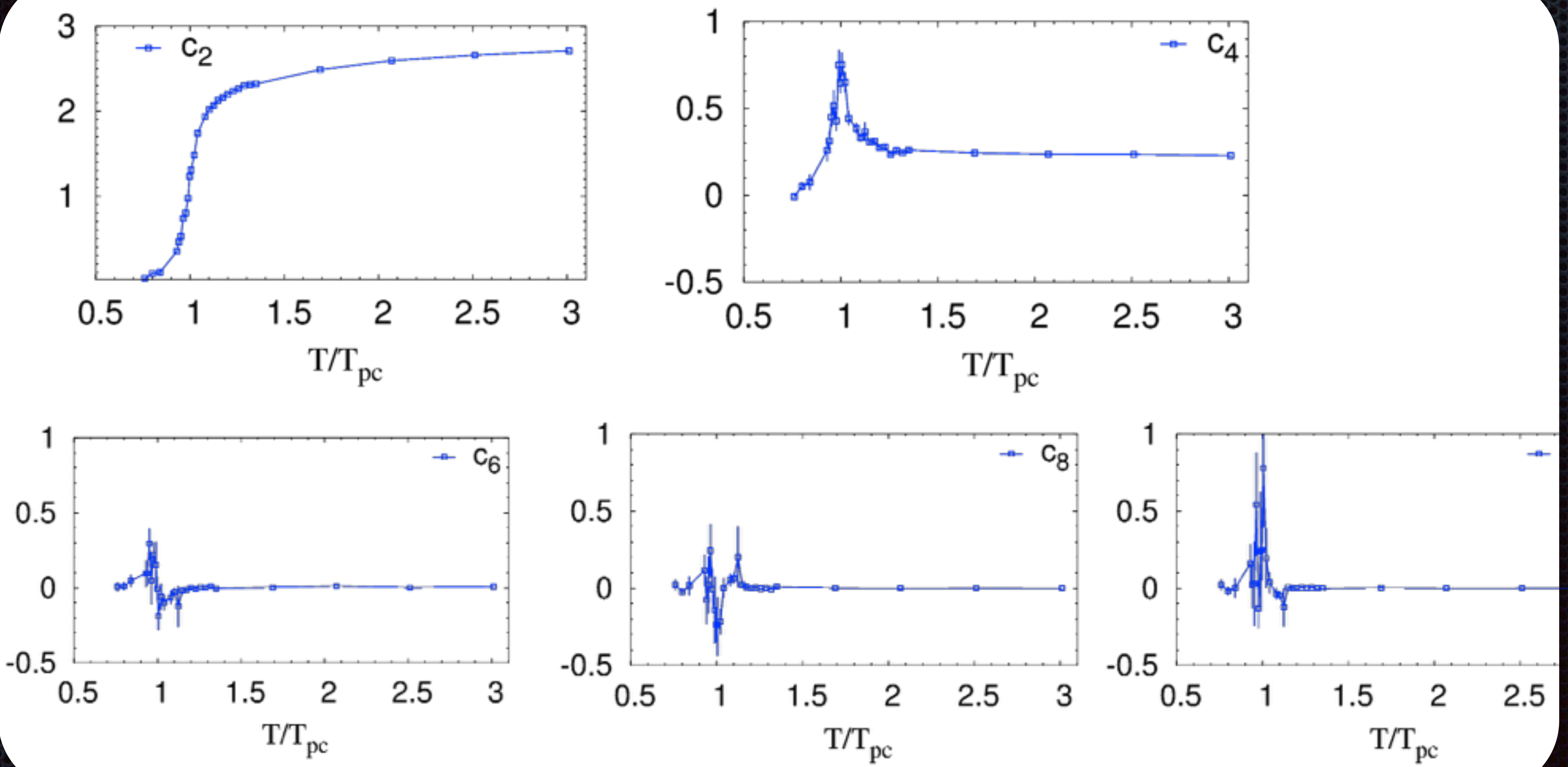






Taylor coefficients of free energy

$$-\frac{f}{T^4} = \sum_{n=0} c_{2n}(\mu/T)^{2n}$$

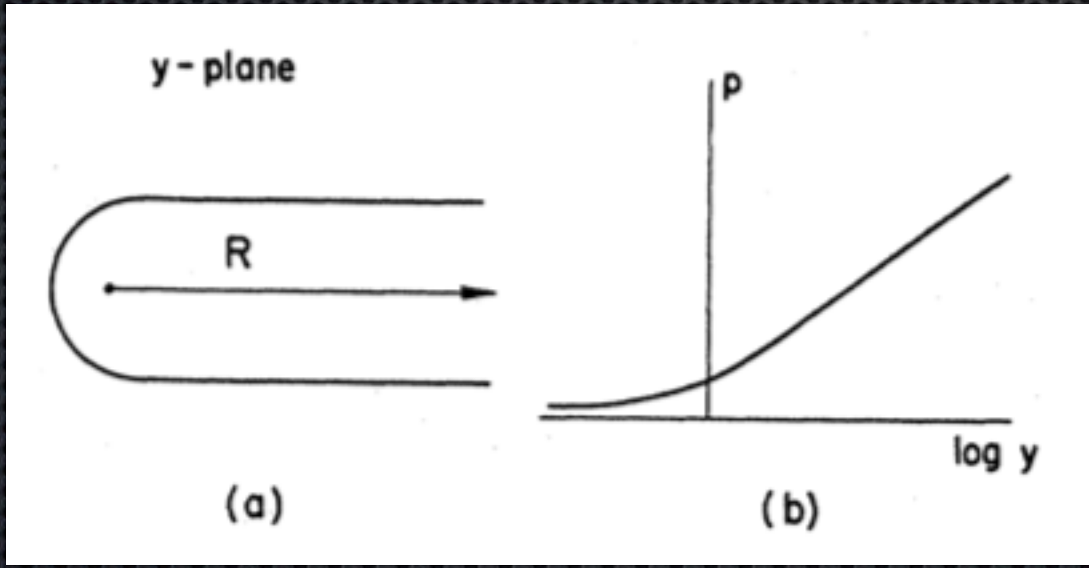


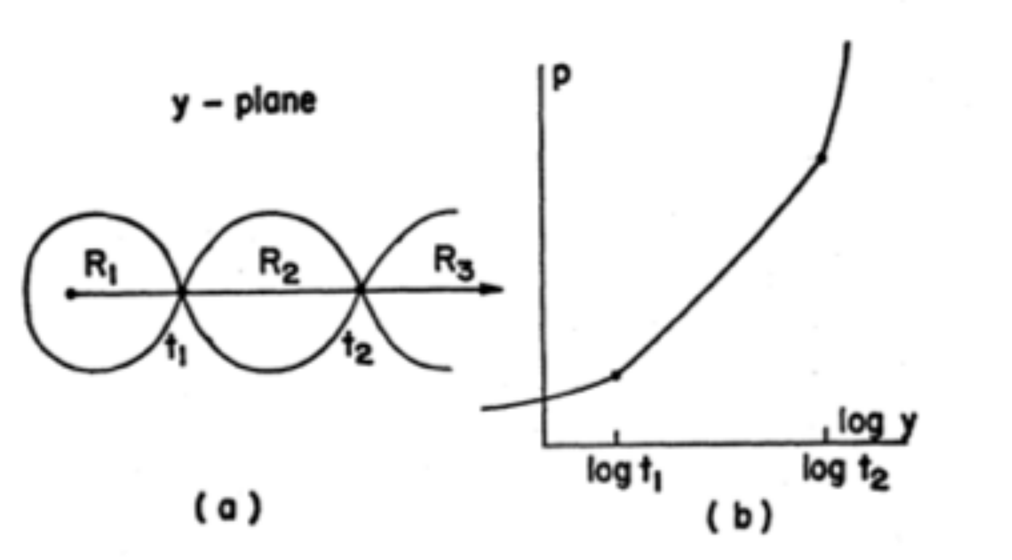
Lee-Yang zeros : from CPF to Phase transition

- **Lee-Yang zeros [Lee & Yang 1952]**

$$Z(\mu) = \sum Z_n e^{n\mu/T}$$
$$\propto \prod (1 - \xi/\xi_i)$$

- **Z(μ) = 0 is an origin of a thermodynamic non-analyticity**





Convergence on the negative real axis

- **Infinite sum of higher order terms is bounded on the negative real axis.**

$$|Z(\mu) - \sum_{|n|<N} Z_n \xi^n| = |\sum_{|n|\geq N} Z_n \xi^n|$$

$$< |Z_N \xi^N|$$

- c.f. Leibnitz's test for an alternating series

$$1/2 - 1/3 + 1/4 - 1/5 + 1/6 + \cdots < 1/2$$