

Position space formulation for fermion on honeycomb lattice

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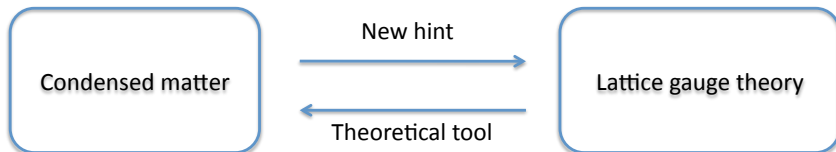
Talk at “Heavy hadrons and exotics”, March 6, 2015

In collaboration with

- [Masaki Hirotsu](#) (Osaka), T.O. and [Eigo Shintani](#) (Mainz)
Nucl. Phys. B885 (2014) 61, arXiv:1303.2886(hep-lat),
- [Aya Kagimura](#) (Osaka), Hirotsu, T.O. and Shintani work in progress.

Dirac fermion in condensed matter system: A new laboratory for lattice gauge theory

- **Graphene**
 - Topological insulator
- } Electrons hopping
on the atomic lattice
massless Dirac fermions at low energy



We study graphene as a first step.

Graphene system looks similar to staggered fermion.

Fermion hopping on hypercubic lattice

→ massless Dirac fermion with flavors

Two approaches in staggered fermion

1. Momentum space approach

Susskind '77, Sharatchandra et al.'81, Doel et al.'83, Golterman-Smit'84

2. Position space formulation

Kluberg-Stern et al. '83

Split the lattice sites into “space” and “internal” degrees of freedom.

Exact $U(1)$ chiral symmetry is manifest.

	Staggered fermion	Graphene
Momentum space	○	○
Position space	○	?
U(1) symmetry	○	?

This approach is absent in graphene system.

Goal of our work

- Find hidden exact $U(1)$ “flavor-chiral” symmetry in the Hamiltonian for graphene.
- To do so, we apply the position space formalism, which was first used for staggered fermion.

2. Tight-binding model

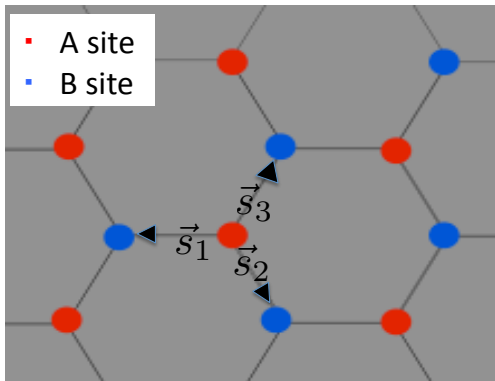
Outline

- ✓ 1. Introduction
2. Tight-binding Model
3. Dirac point and Discrete Symmetries
4. Position space formulation
5. Hidden exact symmetry
6. Summary

$$H = t \sum_{\vec{x}} \sum_{i=1}^3 [a^\dagger(\vec{x})b(\vec{x} + \vec{s}_i) + h.c.]$$

$$+ t' \sum_x \sum_{j=1}^6 [a^\dagger(\vec{x})a(\vec{x} + \vec{b}_j) + b^\dagger(\vec{x} + \vec{s}_1)b(\vec{x} + \vec{s}_1 + \vec{b}_j)]$$

$t = 2.8$ eV, $t' = 0.1$ eV a, a^\dagger : ann., cr. operator at site A
 b, b^\dagger : ann., cr. operator at site B



Nearest-neighbor bond

$$\begin{cases} \vec{s}_1 = a_0(-1, 0), \\ \vec{s}_2 = \frac{a_0}{2}(-1, \sqrt{3}) \\ \vec{s}_3 = \frac{a_0}{2}(1, \sqrt{3}) \end{cases}$$

Fundamental lattice vector

$$\begin{cases} \vec{b}_1 = \vec{s}_2 - \vec{s}_3, & \vec{b}_2 = \vec{s}_2 - \vec{s}_1, \\ \vec{b}_3 = \vec{s}_3 - \vec{s}_1, & \vec{b}_4 = \vec{s}_3 - \vec{s}_2, \\ \vec{b}_5 = \vec{s}_1 - \vec{s}_2, & \vec{b}_6 = \vec{s}_1 - \vec{s}_3 \end{cases}$$

Energy spectrum

Fourier transformation of H ($t'=0$ case) G. W. Semenoff, (1984)

$$H = t \sum_{\sigma} \int \frac{d^2k}{(2\pi)^2} (a^{\dagger}(k)b^{\dagger}(k)) \begin{pmatrix} 0 & \Phi(k) \\ \Phi^*(k) & 0 \end{pmatrix} \begin{pmatrix} a(k) \\ b(k) \end{pmatrix}$$

$$\Phi(k) = e^{i\vec{k}\cdot\vec{s}_1} + e^{i\vec{k}\cdot\vec{s}_2} + e^{i\vec{k}\cdot\vec{s}_3}$$

$$\text{Eigenvalue: } E = \pm|\Phi(k)|$$

One electron per site (= half-filled)

→ all negative energy states are filled(=Dirac sea) .

If there is a gap, the system is insulator .

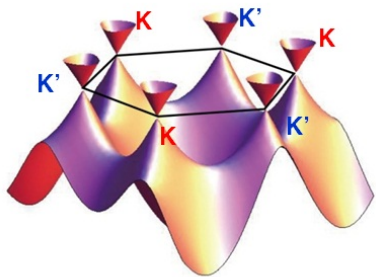
If no gap, the system can be metallic.



Dirac point : the lowest energy state of graphene

$$\Phi(k) = e^{i\vec{k}\cdot\vec{s}_1} + e^{i\vec{k}\cdot\vec{s}_2} + e^{i\vec{k}\cdot\vec{s}_3} = 0$$

Two Solutions:



$$K_1 = \left(0, \frac{4\pi}{3a_0}\right), K_2 = \left(\frac{2\pi}{\sqrt{3}a_0}, \frac{2\pi}{3a_0}\right),$$
$$\rightarrow \Phi(k) \propto 1 + e^{2\pi i/3} + e^{-2\pi i/3} = 0$$

LOW ENERGY APPROXIMATION

Leading order of expansion near K point

$$H_0 = v_F \int \frac{d^2k}{(2\pi)^2} \left[u^\dagger(k)(k_x\sigma_2 + k_y\sigma_1)u(k) - v^\dagger(k)(k_x\sigma_2 + k_y\sigma_1)v(k) \right]$$

$$u(k) = \begin{pmatrix} a(k - K_1) \\ b(k - K_1) \end{pmatrix}, \quad v(k) = \begin{pmatrix} a(k - K_2) \\ b(k - K_2) \end{pmatrix},$$

$$v_F = \frac{\sqrt{3}}{2} a_0 t \sim c/300$$

Emergence of massless Dirac fermion

- Continuous global symmetry of Heff

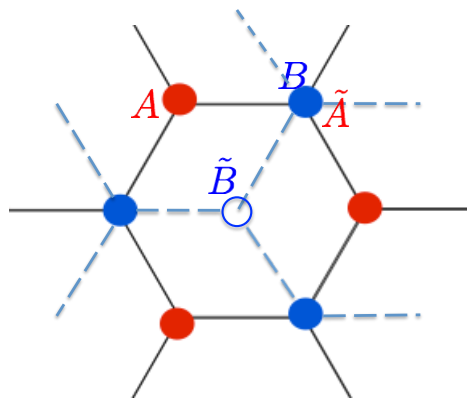
$$H_{\text{eff}} = v_F \int d^2x \psi^\dagger [(\tau_2 \otimes \sigma_1) \partial_1 + (\tau_2 \otimes \sigma_2)] \psi(x)$$

“Flavor/Chiral” symmetry

$$\delta\psi = i\Gamma\psi, \quad \delta\psi^\dagger = -i\psi^\dagger\Gamma \quad (\Gamma : \text{Hermitian matrix})$$

$1_{2 \times 2} \otimes 1_{2 \times 2}$	$\tau_1 \otimes \sigma_3$	$\tau_2 \otimes 1_{2 \times 2}$	$\tau_3 \otimes \sigma_3$
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AB stacked bilayer graphene



A B Upper layer
 \tilde{A} \tilde{B} Lower layer

B sites sit on top of \tilde{A}

Inter-layer hopping Hamiltonian

$$H_{\text{inter}} = \int \frac{d^2k}{(2\pi)^2} \left[(a^\dagger(k)b^\dagger(k)) \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} \tilde{a}(k) \\ \tilde{b}(k) \end{pmatrix} + h.c. \right]$$

Total Hamiltonian matrix for bi-layer graphene

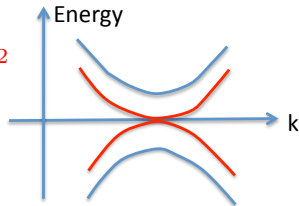
$$\begin{pmatrix} 0 & t\Phi(k) & 0 & 0 \\ t\Phi^\dagger(k) & 0 & \gamma & 0 \\ 0 & \gamma & 0 & t\Phi(k) \\ 0 & 0 & t\Phi^\dagger(k) & 0 \end{pmatrix}$$

Energy Eigenvalue E(k)

$$E(k) = \pm \left[\frac{\gamma^2}{2} + |t\Phi(k)|^2 \pm \sqrt{\left(\frac{\gamma^2}{2} + |t\Phi(k)|^2 \right)^2 - |t\Phi(k)|^4} \right]^{1/2}$$

Near Dirac point, $\Phi(k)$ behaves as $|t\Phi(k)| \sim v_F |\vec{k} - \vec{K}|$

$$E(k) \sim \pm \left[\gamma + \frac{v_F^2}{\gamma} (\vec{k} - \vec{K})^2 \right], \pm \frac{v_F^2}{\gamma} (\vec{k} - \vec{K})^2$$



Gapless mode with quadratic dispersion

3. Dirac point and discrete symmetry

What is known ?

Condensed matter theorist studied the Dirac point

- **Gapless** : P.R. Wallace 1947, J.W. McClure 1956
- **Existence of Dirac points by discrete symmetry:**
W.M. Lomer 1955, J.C. Slonczewski P.R. Weiss 1958
- **Local Stability:** C. Herring 1937, K. Asano & C. Hotta 2011

Existence of Dirac points

Discrete symmetries can predict the Dirac points

I : Inversion $A \leftrightarrow B$

C3: $2/3\pi$ rotation around a point A (or B)

Fixed points under transformation = K, K'

Invariance of Hamiltonian under I, C3

→ At K, K' there is no gap.

Local Stability against small de-tuning

Hatsugai-Fukui-Aoki 2006

Let us consider the free Hamiltonian on more general lattice with A-, B- sites.

$$H = \begin{pmatrix} R_0(k) + R_3(k) & R_1(k) - iR_2(k) \\ R_1(k) + iR_2(k) & R_0(k) - R_3(k) \end{pmatrix}$$

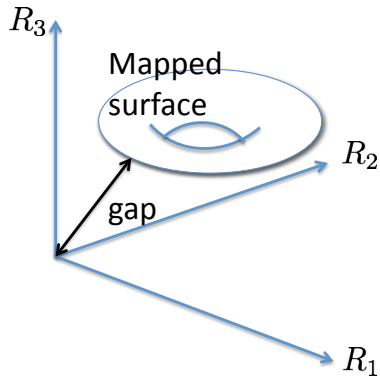
c.f. graphene $R_0(k) = R_3(k) = 0$, $R_1(k) - iR_2(k) = \Phi(k)$

Energy eigenvalue

$$E = R_0 \pm \sqrt{R_1^2 + R_2^2 + R_3^2}$$

Consider the map from the Brillouin zone (BZ) to \mathbb{R}^3

$$\vec{k} \rightarrow \vec{R}(\vec{k})$$



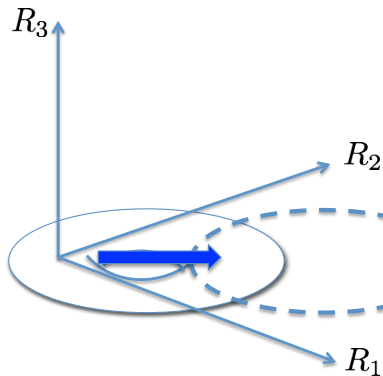
Energy gap is the minimum distance $\sqrt{R_1^2 + R_2^2 + R_3^2}$ between the mapped 2 dimensional surface and the origin.

**Fine-tuning is needed for zero gap.
One cannot expect the stability.**

However, when Hamiltonian has Z_2 symmetry

$$\{H, \sigma_3\} = 0$$

Then, $R_3(k) = 0$ is automatically satisfied.



- The map is restricted to x-y plane, which includes the origin.
- Zero gap can be achieved without fine-tuning.
- Zero gap occurs at 2 points.
- Any small deformation which keeps Z_2 symmetry preserves zero gap.

Existence of Dirac points are understood by discrete symmetries.

What about chiral flavor symmetries of the massless Dirac fermion?

Does it exist only in the continuum?

or even at finite lattice spacing?

(It may affect the property of the exciton excitation when there occurs a spontaneous symmetry breaking.)

4. Position space formulation

Graphene system looks similar to staggered fermion.

Two approaches in staggered fermion

1. Momentum space approach

Susskind '77, Sharatchandra et al.'81, C.v.d. Doel et al.'83, Golterman-Smit'84

2. Position space formulation ... Kluberg-Stern et al. '83

Split the lattice sites into “space” and “internal” degrees of freedom.
Exact chiral symmetry is manifest.

This approach is absent in graphene system.

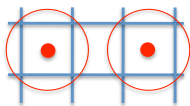
We try to construct similar formalism in graphene system.

c.f. Staggered fermion in 2dimension

Position space degree of freedom is relabeled into internal degrees of freedom.

$$S = \sum_X \sum_{\rho, \rho'} \sum_{\mu} \bar{\psi}(X) \left[(\Gamma_{\mu})_{\rho, \rho'} \nabla_{\mu} - \frac{i}{2} (\Lambda_{\mu})_{\rho, \rho'} \Delta_{\mu} \right] \psi_{\rho'}(X)$$

$$\chi(2X + \rho) \longrightarrow \psi_{\rho}(X) \quad \text{where}$$



$$\Gamma_1 = \tau_1 \otimes 1, \quad \Gamma_2 = \tau_3 \otimes \tau_1$$

$$\Lambda_1 = \tau_2 \otimes 1, \quad \Lambda_2 = \tau_3 \otimes \tau_2$$

$$\nabla_{\mu} \psi(X) \equiv \frac{1}{2} [\psi(X + \hat{\mu}) - \psi(X - \hat{\mu})]$$

$$\Delta_{\mu} \psi(X) \equiv [\psi(X + \hat{\mu}) + \psi(X - \hat{\mu}) - 2\psi(X)]$$

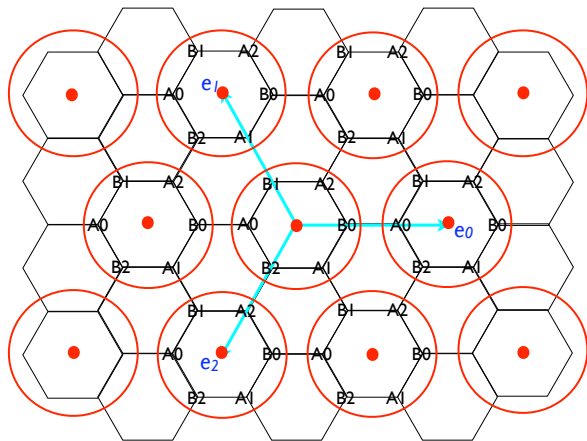
Exact chiral symmetry on the lattice $U(1)_A \subset SU(2)_A$

$$\delta\psi = \Gamma_5 \psi, \quad \delta\bar{\psi} = \bar{\psi} \Gamma_5$$

$$\text{where } \Gamma_5 \equiv \tau_3 \otimes \tau_3$$

Position space formulation for graphene

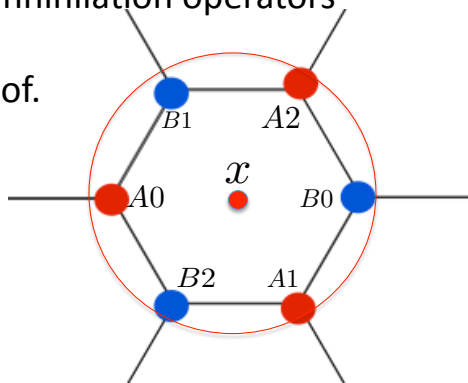
We relabel the honeycomb lattice \bullet as the fundamental lattice.
6 sites in hexagonal unit cell is now the internal degree.
Fundamental vectors $\vec{e}_{0,1,2}$ are those connecting \bullet .



$$\begin{cases} \vec{e}_0 = a(1, 0) \\ \vec{e}_1 = a\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ \vec{e}_2 = a\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \end{cases}$$


(a : lattice spacing)

- Central coordinate of hexagonal lattice is labeled by \vec{x}
- $I=A,B$ are the index for sublattices A,B
- ρ is the index for the 3 vertices (counter-clockwise)
- $\chi_{I\rho}^\dagger(x), \chi_{I,\rho}(x)$ are creation/annihilation operators
- Each lattice point has $6=2 \times 3$ dof.




The Hamiltonian involves 6x6 matrices.
 Corresponding to the indices I, ρ , we
 express 6x6 matrix using the tensor product
 of 2x2 matrix and 3x3 matrix.

$$A_{I\rho, I'\rho'} = \sum_{a=0}^3 (\tau^a)_{II'} \otimes (B^a)_{\rho\rho'}$$




6x6



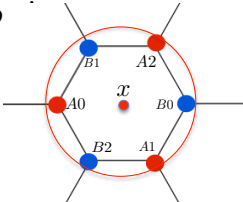
2x2

$I, I' = A, B$



3x3

$\rho, \rho' = 0, 1, 2$



- New formulation of tight-binding Hamiltonian

Mass term

$$\mathcal{H} = t \sum_{\vec{x}} \chi^\dagger(\vec{x}) \left[(\tau_1 \otimes M) \chi(\vec{x}) - i \sum_{\rho} (\tau_2 \otimes \Gamma_{\rho}) (\nabla_{\rho} \chi(\vec{x})) + \frac{1}{2} \sum_{\rho} (\tau_1 \otimes \Gamma_{\rho}) (\Delta_{\rho} \chi(\vec{x})) \right]$$

$H\chi$

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\nabla_{\rho} \chi(X) \equiv \frac{1}{2} [\chi(\vec{x} + \vec{e}_{\rho}) - \chi(\vec{x} - \vec{e}_{\rho})]$$

$$\Delta_{\rho} \chi(X) \equiv \chi(\vec{x} + \vec{e}_{\rho}) + \chi(\vec{x} - \vec{e}_{\rho}) - 2\chi(\vec{x})$$

Locality is manifest.

The mass term has 1 massive mode and 2 zero modes.

Democratic matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

B_0, B_1, B_2 can hop to
 A_0, A_1, A_2 with equal weight

Change of basis



M^{diag}

Massive mode

$$M^{\text{diag}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Massive mode can
 be integrated out

Zero modes

vertex	spin	flavor	integrating out	effective DOF
6	= 2	× 3	→	2 × 2

$$H_{\text{eff}} = v_F \int d^2x \psi^\dagger [(\tau_2 \otimes \sigma_1) \partial_1 + (\tau_2 \otimes \sigma_2)] \psi(x)$$

Comment: exact discrete symmetry

- Z3 symmetry: cyclic rotation

$$\rho: (0, 1, 2) \rightarrow (1, 2, 3)$$

- Parity

$$\vec{x} \rightarrow \vec{x}_P, A \leftrightarrow B$$


$$(\tau_1 \otimes 1)$$

- Possible global symmetry of Heff

$$H_{\text{eff}} = v_F \int d^2x \psi^\dagger [(\tau_2 \otimes \sigma_1) \partial_1 + (\tau_2 \otimes \sigma_2)] \psi(x)$$

$$\delta\psi = i\Gamma\psi, \quad \delta\psi^\dagger = -i\psi^\dagger\Gamma$$

4 possibilities for Γ (Γ : Hermitian matrix)

$1_{2 \times 2} \otimes 1_{2 \times 2}$	$\tau_1 \otimes \sigma_3$	$\tau_2 \otimes 1_{2 \times 2}$	$\tau_3 \otimes \sigma_3$
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“Flavor-Chiral” symmetry

Parity conserving mass term $m\tilde{\psi}^\dagger(\vec{k})(\tau_1 \otimes 1_{2 \times 2})\tilde{\psi}(\vec{k})$
 is prohibited by the last two symmetries

However, these could be violated by lattice artifacts.

5. Hidden exact symmetry

Hidden symmetry in graphene

We look for an exact symmetry as

$$\delta\chi(k) = \Gamma_5(k)\chi(k) \quad \lim_{k \rightarrow 0} \Gamma_5(k) = \Gamma_5^{\text{cont}}$$

Expanding $\Gamma_5(k)$ in powers of k , we look for solution to $[H(k), \Gamma_5(k)] = 0$ order by order.

- Series starting from $\tau_2 \otimes 1$ failed at 2nd order in k .
- Series starting from $\tau_3 \otimes \sigma_3$ survived at 3rd order in k
→ All order solution may exist for the latter?

To 3rd order in momentum expansion, only terms of the form $\tau_3 \otimes A$, $1 \otimes B$ (A,B: 3x3 matrix) appeared.

→ We take the following ansatz for the symmetry

$$\delta\chi(\vec{x}) = i\theta \left[\underbrace{3(\tau_3 \otimes X)\chi(\vec{x}) + \frac{1}{2} \sum_{\rho} (\tau_3 \otimes Y_{\rho})(\Delta_{\rho}\chi(\vec{x})) + \frac{1}{i} \sum_{\rho} (1 \otimes Z_{\rho})(\nabla_{\rho}\chi(\vec{x}))}_{\Gamma_5} \right]$$

Require that Hamiltonian is invariant under the symmetry.

$$[H, \Gamma_5] = 0$$



A set of algebraic equations for X, Y_{ρ}, Z_{ρ} and matrices M, Γ_{ρ} in Hamiltonian

Unique solution for X, Y, Z exists!

$$X = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix},$$

$$Y_0 = \begin{pmatrix} 0 & -i & i \\ i & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & -i \\ -i & i & 0 \end{pmatrix},$$

$$Z_0 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Z_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, Z_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Continuum limit (mass diagonal basis)

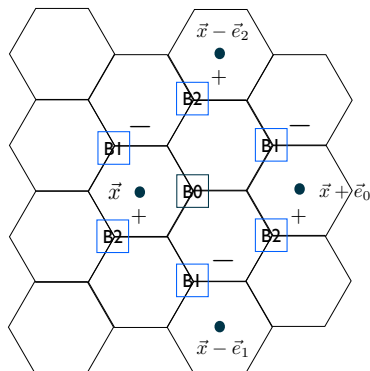
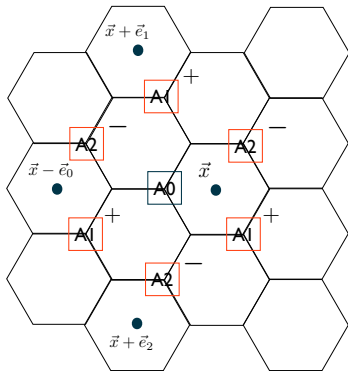
$$\delta\chi(x) \rightarrow \tau_3 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \tau_3 \otimes \sigma_3$$

Coincide with
“flavor-chiral sym.”

Symmetry in terms of conventional labeling

$$\delta a(\vec{x}) = \theta[a(\vec{x} + \vec{s}_2 - \vec{s}_3) - a(\vec{x} - \vec{s}_1 + \vec{s}_2) + a(\vec{x} + \vec{s}_3 - \vec{s}_1) - a(\vec{x} - \vec{s}_2 + \vec{s}_3) + a(\vec{x} + \vec{s}_1 - \vec{s}_2) - a(\vec{x} - \vec{s}_3 + \vec{s}_1)]$$

$$\delta b(\vec{x}) = \theta[b(\vec{x} + \vec{s}_2 - \vec{s}_3) - b(\vec{x} - \vec{s}_1 + \vec{s}_2) + b(\vec{x} + \vec{s}_3 - \vec{s}_1) - b(\vec{x} - \vec{s}_2 + \vec{s}_3) + b(\vec{x} + \vec{s}_1 - \vec{s}_2) - b(\vec{x} - \vec{s}_3 + \vec{s}_1)]$$



What about AB-stacked bi-layer graphene?

When interlayer interaction is absent, each layer has the symmetry :

$$\left\{ \begin{array}{l} \delta\chi = i\theta\Gamma_5\chi \\ \delta\tilde{\chi} = i\tilde{\theta}\Gamma_5\tilde{\chi} \end{array} \right. \quad \begin{array}{l} \chi : \text{electron in upper-layer} \\ \tilde{\chi} : \text{electron in lower-layer} \\ \theta, \tilde{\theta} : \text{parameters for "chiral" symmetry} \end{array}$$

One can show that the interlayer interaction Hamiltonian is invariant under transformation with $\theta = \tilde{\theta}$

Exact symmetry in bilayer graphene !

Gapless mode in bi-layer graphene is also protected by "flavor-chiral" symmetry.

Remark 1

What about the case including non-nearest neighbor hopping or higher?

→ No problem, “chiral-flavor” symmetry remains exact.

Due to the discrete rotational symmetry and parity, non-nearest neighbor hopping Hamiltonian is

expressed as $H_{\text{non nearest}} = P(H)$ (P: some polynomial)

$$[P(H), \Gamma_5] = 0, \quad \text{although } \{P(H), \sigma_3\} \neq 0$$

Only our exact symmetry can explain the gapless mode.

Similar arguments hold also for inter-layer hopping Hamiltonian.

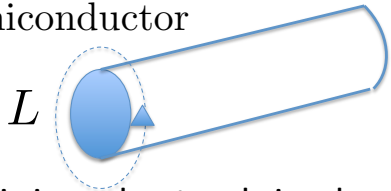
Remark 2

Example of physical interpretation:

(Non-chiral) carbon nanotube can be metallic or semiconductor depending on the radius L of the cylinder.

$\text{Mod}(L/a_0, 3) = 0$: metallic

$\text{Mod}(L/a_0, 3) = 1, 2$: semiconductor



In position space formalism, this is understood simply as periodic boundary condition for Dirac fermion.

6. Summary

We studied the position space formulation of graphene

Results

- ❖ Spin-flavor structure
- ❖ Manifest locality of the low energy Dirac theory
- ❖ Discovery of exact “flavor-chiral” symmetry on the lattice
- ❖ Alternative understanding of the existence of the Gapless mode is understood by “flavor-chiral” symmetry
- ❖ “flavor-chiral” symmetry remains for bilayer graphene
- ❖ The symmetry also holds with non-nearest hopping.

What is next?

- Symmetry with gauge interaction.
Whether it is exact at interacting level is yet to be seen.
- Nonperturbative lattice studies on graphene including gauge interaction.
 - Phase structure
 - Quantum Hall effect
 - Effect of impurities and boundaries
 - Atomic collapse

Our position space formalism will be helpful to give a clear physical interpretation of low energy excitation as massless Dirac particle.

Thank you for your attention.

Backup Slides

In order to determine X , Y_ρ , Z_ρ , we employ momentum representation of $\chi(\vec{x}), \chi^\dagger(\vec{x})$

$$\mathcal{H} = \int \frac{d^2k}{(2\pi)^2} \tilde{\chi}^\dagger(\vec{k}) \left[(\tau_1 \otimes \Lambda) + \sum_\rho e^{ik_\rho} (\tau_- \otimes \Gamma_\rho) + \sum_\rho e^{-ik_\rho} (\tau_+ \otimes \Gamma_\rho) \right] \tilde{\chi}(\vec{k})$$

with $\tau_\pm \equiv (\tau_1 \pm i\tau_2)/2$ and $\Lambda \equiv M - 1$, and for chiral transformation $\delta\tilde{\chi}(\vec{k}) = i\theta\tilde{\Gamma}_5(\vec{k})\tilde{\chi}(\vec{k})$ $\tilde{\Gamma}_5(\vec{k})$ is given as

$$\tilde{\Gamma}_5(\vec{k}) = (\tau_3 \otimes X) + \sum_\rho e^{ik_\rho} \gamma_\rho + \sum_\rho e^{-ik_\rho} \gamma_\rho^\dagger, \quad (2)$$

with

$$\gamma_\rho = \frac{\tau_3 + 1}{2} \otimes W_\rho^\dagger + \frac{\tau_3 - 1}{2} \otimes W_\rho. \quad (3)$$

W_ρ is defined as $W_\rho = \frac{1}{2}(Y_\rho + iZ_\rho)$.

Imposing $[\tilde{H}(\vec{k}), \tilde{\Gamma}_5(\vec{k})] = 0$, we obtain following equations;

$$\{\Lambda, X\} + \sum_{\rho} (\Gamma_{\rho} W_{\rho} + W_{\rho}^{\dagger} \Gamma_{\rho}) = 0 \quad (1)$$

$$\{\Gamma_{\rho}, X\} + \Lambda W_{\rho}^{\dagger} + W_{\rho} \Lambda = 0 \quad (2)$$

$$\Lambda W_{\rho} + W_{\rho}^{\dagger} \Lambda + \sum_{\sigma \neq \lambda (\sigma, \lambda \neq \rho)} (\Gamma_{\sigma} W_{\lambda}^{\dagger} + W_{\lambda} \Gamma_{\sigma}) = 0 \quad (3)$$

$$\Gamma_{\rho} W_{\rho}^{\dagger} + W_{\rho} \Gamma_{\rho} = 0 \quad (4)$$

$$\Gamma_{\rho} W_{\sigma} + W_{\sigma}^{\dagger} \Gamma_{\rho} = 0 \quad (\rho \neq \sigma). \quad (5)$$