Precise determination of πN scattering from Roy-Steiner equations

J. Ruiz de Elvira

Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn

In collaboration with:
C. Ditsche, M. Hoferichter, B. Kubis, U.-G. Meißner.

[JHEP 1206 (2012) 043] and [JHEP 1206 (2012) 063]

+ work in progress

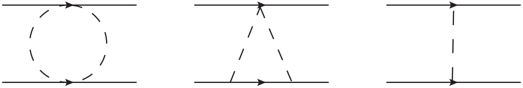
Hadron and Hadron Interactions in QCD, Kyoto, March 3, 2015

Outline

Contents

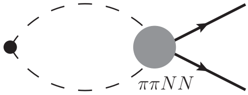
Motivation: Why πN scattering?

- **Low energies:** test chiral dynamics in the baryon sector
  $\Rightarrow$ low-energy theorems e.g. for the scattering lengths
- **Higher energies:** resonances, baryon spectrum
- **Input for NN scattering:** LECs c_i, πNN coupling



- **Crossed channel $\pi\pi \to \bar{N}N$:** nucleon form factors

$\Rightarrow$ probe the structure of the nucleon



- spectral functions of form factors
- vector form factors (P-waves)
- scalar form factors (S-waves)

The pion-nucleon σ-term

Scalar form factor of the nucleon:

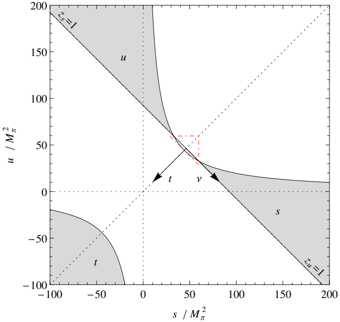
$$\sigma(t) = \langle N(p')|\hat{m}\left(\bar{u}u + \bar{d}d\right)|N(p)\rangle \qquad t = (p'-p)^2 \qquad \sigma_{\pi N} = \sigma(0)$$

- $\sigma_{\pi N}$ measures the **light-quark contribution** to the nucleon mass
- Unfortunately, no direct experimental access to it
- Linked to πN via the **Cheng-Dashen** theorem [Cheng, Dashen 1971]

$$\underbrace{F_\pi^2 \bar{D}^+(\nu=0, t=2M_\pi^2)}_{F_\pi^2\,(d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D} = \underbrace{\sigma(2M_\pi^2)}_{\sigma_{\pi N} + \Delta_\sigma} + \Delta_R$$

$|\Delta_R| \lesssim 2$ MeV [Bernard, Kaiser, Meißner 1996]

$\Delta_D - \Delta_\sigma = (-3.3 \pm 0.2)$ MeV [Gasser, Leutwyler, Sainio 1991]



- Phenomenological status:
  - "canonical value" $\sigma_{\pi N} \sim 45$ MeV, from **Cheng-Dashen** theorem based on KH80 input [Gasser, Leutwyler, Socher, Sainio 1988, Gasser, Leutwyler, Sainio 1991]
  - **GWU/SAID** PW analysis $\sigma_{\pi N} \sim (64 \pm 8)$ MeV [Pavan, Strakovsky, Workman, Arndt 2002]

Motivation: Why Roy-Steiner equations?

Roy(-Steiner) eqs. = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries**: analyticity, unitarity, crossing
- **Model independent** $\Rightarrow$ the actual parametrization of the data is irrelevant once it is used in the integral.
- Framework allows for systematic improvements (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with high precision:
  - $\pi\pi$-scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - πK-scattering: [Büttiker et al. (2004)]
  - $\gamma\gamma \to \pi\pi$ scattering: [Hoferichter et al. (2011)]

Contents

Warm up: Roy-equations for $\pi\pi$

- $\pi\pi \to \pi\pi \Rightarrow$ fully crossing symmetric in Mandelstam variables s, t, and $u = 4M_\pi - s - t$
- Start from twice-subtracted **fixed-t** DRs of the generic form

$$T^I(s,t) = c(t) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\left[\frac{s^2}{(s'-s)} - \frac{u^2}{(s'-u)}\right]\mathrm{Im}T^I(s',t)$$

- Subtraction functions c(t) are determined via crossing symmetry functions of the I=0,2 scattering lengths: a_0^0 and a_0^2
- PW-expansion of these DRs yields the **Roy-equations** [Roy (1971)]

$$t_J^I(s) = ST_J^I(s) + \sum_{J'=0}^{\infty}(2J'+1)\sum_{I'=0,1,2}\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s',s)\mathrm{Im}\, t_{J'}^{I'}(s')$$

- $K_{JJ'}^{II'}(s',s) \equiv$ kernels $\Rightarrow$ analytically known

Solving Roy-equations: flow information

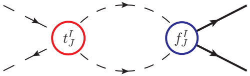
- **Roy-equations** rigorously valid for a finite energy range
  $\Rightarrow$ introduce a matching point s_m
- only partial waves with $J \leq J_{\text{max}}$ are solved
- assume isospin limit
- **Input**
  - High-energy region: $\text{Im}t_{IJ}(s)$ for $s \geq s_m$ and for all J
  - Higher partial waves: $\text{Im}t_{IJ}(s)$ for $J > J_{\text{max}}$ and for all s
- **Output**
  - Self-consistent solution for $\delta_{IJ}(s)$ for $J \leq J_{\text{max}}$ and $s_{\text{th}} \leq s \leq s_m$
  - Constraints on subtraction constants

Roy-Steiner equations for πN: difficulties

Key difficulties compared to $\pi\pi$ Roy-equations

- Crossing: coupling between $\pi N \to \pi N$ (s-channel) and $\pi\pi \to \bar{N}N$ (t -channel)
  $\Rightarrow$ hyperbolic dispersion relations [Hite, Steiner 1973], [Büttiker et al. 2004]
- Unitarity in t-channel, e.g. in single-channel approximation

$$\mathrm{Im} f_\pm^J(t) = \sigma_t^\pi f_\pm^J(t) t_J^I(t)^*$$



$\Rightarrow$ **Watson's theorem**: phase of $f_\pm^J(t)$ equals δ_{IJ} [Watson 1954]

solve with Muskhelishvili-Omnès techniques [Muskhelishvili 1953, Omnès 1958]

$\Rightarrow$ Omnès function: $\Omega_J^I(t) = \exp\left\{ \frac{t}{\pi} \int_{t_{th}}^{tm} dt' \frac{\delta_J^I(t)}{t'(t'-t)} \right\}$

- Large pseudo-physical region in t -channel
  $\Rightarrow$ $\bar{K}K$ intermediate states for s-wave in the region of the $f_0(980)$

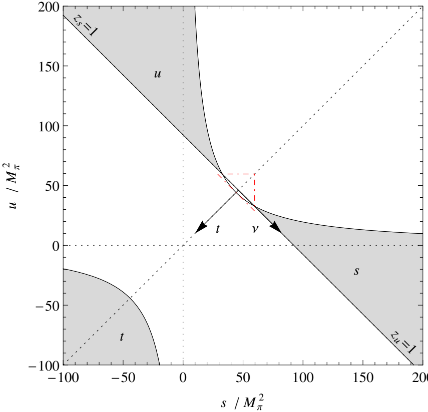
Contents

πN-scattering basics

$$\pi^a(q) + N(p) \to \pi^b(q') + N(p')$$

- **Isospin Structure**:
  $T^{ba} = \delta^{ba} T^+ + \epsilon^{ab} T^-$
- **Lorentz Structure**: $I \in \{+,-\}$
  $T^I = \bar{u}(p') \left(A^I + \frac{\not{q} + \not{q}'}{2} B^I \right) u(p)$
  $D^I = A^I + \nu B^I, \quad \nu = \frac{s-u}{4m}$
- **Isospin basis**: $I_s \in \{1/2, 3/2\}$
  $\{T^+, T^-\} \Leftrightarrow T^{1/2}, T^{3/2}$
- **PW projection**:
  s-channel pw: $f^I_{l\pm}$
  t-channel pw: $f^J_\pm$

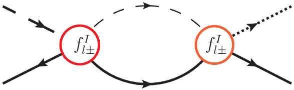
Bose symmetry $\Rightarrow$ even/odd $J \Leftrightarrow I = +/-$



πN-scattering basics: Unitarity relations

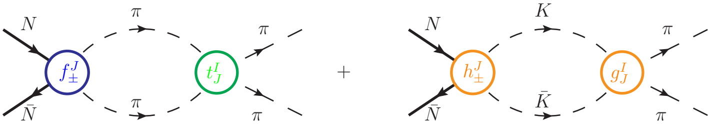
- **s-channel** unitarity relations $(I_s \in \{1/2, 3/2\})$:

$$\mathrm{Im}\, f_{l\pm}^{I_s}(W) = q\left|f_{l\pm}^{I_s}(W)\right|^2 \theta(W-W_+) + \frac{1-\left(\eta_{l\pm}^{I_s}(W)\right)^2}{4q}\,\theta(W-W_{\mathrm{inel}})$$



- **t-channel** unitarity relations: 2-body intermediate states: $\pi\pi + \bar{K}K + \cdots$

$$\mathrm{Im}\, f_{\pm}^{J}(t) = \sigma_t^{\pi}\left(t_J^{I_t}(t)\right)^* f_{\pm}^{J}(t)\,\theta(t-t_\pi) + 2c_J\sqrt{2}\,k_t^{2J}\sigma_t^{K}\left(g_J^{I_t}(t)\right)^* h_{\pm}^{J}(t)\,\theta(t-t_K)$$



- Only linear in $f_{\pm}^{J}(t) \Rightarrow$ less restrictive

Roy-Steiner equations for πN: HDR's

- **Hyperbolic DRs**: $(s-a)(u-a) = b = (s'-a)(u'-a)$ with $a, b \in \mathbb{R}$

$$A^+(s,t;a) = \frac{1}{\pi}\int_{s_+}^{\infty} ds' \left[\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right] \mathrm{Im}\, A^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \frac{\mathrm{Im}\, A^+(s',t')}{t'-t}$$

- Why **HDR**?
  - Combine all physical regions $\Rightarrow$ crucial for t-channel projection
  - Evade double-spectral regions $\Rightarrow$ the PW decompositions converge
  - Range of convergence can be maximized by tuning the free hyperbola parameter a
  - No kinematical cuts, manageable kernel functions
- Similar derivation to $\pi\pi$ **Roy equations**.
  - Expand imaginary parts in terms of s- and t-channel partial waves
  - Project onto s- and t-channel partial waves
  - Combine the resulting equations using s- and t-channel PW unitarity relations
- **Validity**: $W \in [W_+ = 1.08\ \mathrm{GeV}, 1.38\ \mathrm{GeV}]$, $\sqrt{t} \in [\sqrt{t_\pi} = 0.28\ \mathrm{GeV}, 2.00\ \mathrm{GeV}]$.
- Subtractions: subthreshold expansion around $\nu = t = 0$

$$\bar{A}^+(\nu,t) = \sum_{m,n=0}^{\infty} a^+_{mn}\nu^{2m}t^n$$

Contents

Solving t-channel: single channel

- Elastic-channel approximation: generic form of the integral equation

$$f(t) = \Delta(t) + (a+bt)(t-4m^2) + \frac{t^2(t-4m^2)}{\pi}\int_{t_\pi}^{\infty} dt' \frac{\operatorname{Im} f(t')}{t'(t'^2-4m^2)(t'-t)}$$

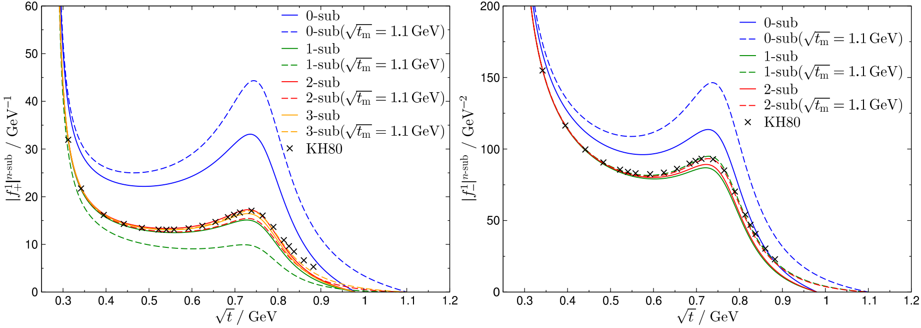
- $\Delta(t)$: Born terms, s-channel integrals, higher t -channel partial waves $\Rightarrow$ left-hand cut
- Introduce subtractions at $\nu = t = 0 \Rightarrow$ subthreshold parameters a, b
- Solution in terms of Omnès function:

$$f(t) = \Delta(t) + (t-4m^2)\Omega(t)(1-t\dot{\Omega}(0))a + t(t-4m^2)\Omega(t)b$$
$$-\Omega(t)\frac{t^2(t-4m^2)}{\pi}\left\{\int_{4M_\pi^2}^{t_m} dt' \frac{\Delta(t')\operatorname{Im}\Omega(t')^{-1}}{t'^2(t'-4m^2)(t'-t)} + \int_{t_m}^{\infty} dt' \frac{\Omega(t')^{-1}\operatorname{Im} f(t')}{t'(t'-4m^2)(t'-t)}\right\}$$

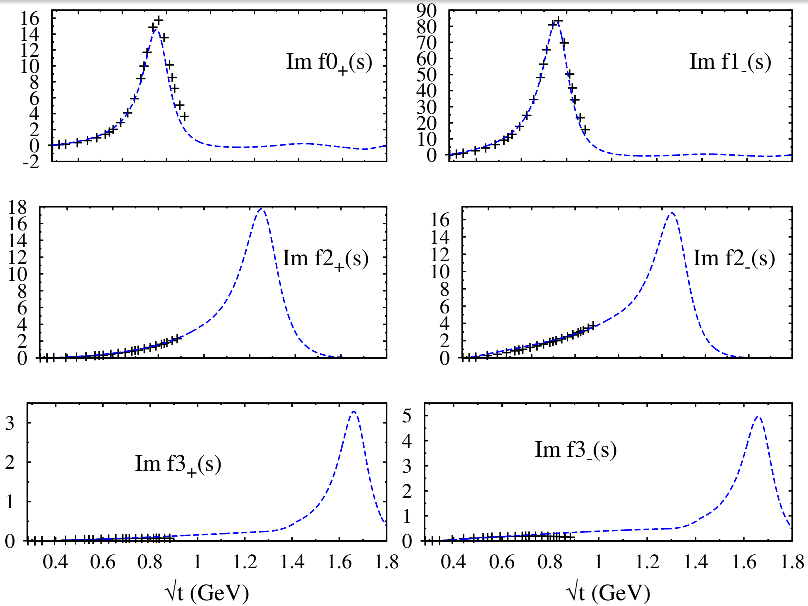
$$\Omega(t) = \exp\left\{\frac{t}{\pi}\int_{t_\pi}^{t_m}\frac{dt'}{t'}\frac{\delta(t')}{t'-t}\right\}$$

Solving t-channel: input and subtractions

- elastic channel approximation: $\sqrt{t_m} = 0.98 - 1.1$ GeV, for $t > t_m$ $\mathrm{Im} f^J_\pm(t) = 0$
- First step: check **consistency** with KH80 [Höhler 1983]
- Input needed:
  - $\pi\pi$ phase shifts: [Caprini, Colangelo, Leutwyler, (in preparation)]
  - πN phase shifts: SAID [Arndt et al. 2008], KH80
  - πN at high energies: Regge model [Huang et al. 2010]
  - πN parameters: KH80

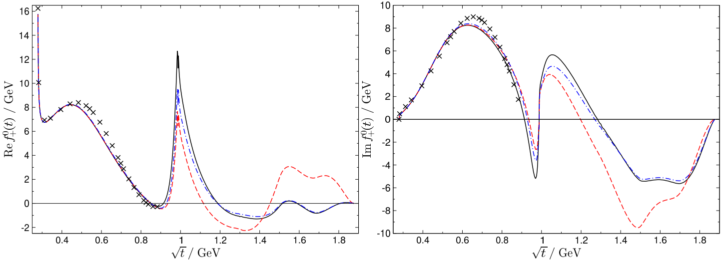


Solving t-channel: P, D and F waves up to $\bar{N}N$



Solving t-channel: S-wave results

- Important for the $\sigma_{\pi N}$ term
- $\bar{K}K$ channel important $\Rightarrow$ two-channel Muskhelishvili-Omnès problem
- also needed:
  - $K\bar{K}$ s-wave partial waves: [Büttiker. (2004)]
  - KN s-wave pw: SAID [Arndt et al. 2008], KH80
  - Hyperon couplings from [Jülich model 1989]
  - KN subthreshold parameters neglected



MO solutions in general consistent with KH80 results

Solving s-channel: flow of information

- General form of the s-channel integral equation

$$f^I_{l+}(W) = \Delta^I_{l+}(W) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty}\Big\{K^I_{ll'}(W,W')\,\mathrm{Im}\,f^I_{l'+}(W') + K^I_{ll'}(W,-W')\,\mathrm{Im}\,f^I_{(l'+1)-}(W')\Big\}$$

$\Rightarrow$ form of $\pi\pi$ Roy-Equations

- $\Delta^I_{l+}(W) \equiv$ t-channel contribution and pole term
- valid up to $W_m = 1.38$ GeV
- **Input**:
  - RS t-channel solutions
  - s-channel partial waves for $J > 1$ [SAID analysis]
  - s-channel partial waves for $W_m < W < 2.5$ GeV [SAID analysis]
  - high energy contribution for $W > 2.5$ GeV: Regge model [Huang et al. 2010]
- **Output**:
  - Self-consistent solution for S and P waves between $s_{\mathrm{th}} \leq s \leq s_m$
  - Constraints on subtraction constants $\Rightarrow$ subthreshold parameters

Solving s-channel: subtractions

- Existence and uniqueness of solutions [Gasser, Wanders 1999]
  $\Rightarrow$ no-cusp condition for each pw + 2 additional constraints are needed
- Take advantage of the precise data for pionic atoms [Gotta et al. 2005, 2010]
  $\Rightarrow$ Impose as a **constraint** scattering lengths from a combined analysis of pionic hydrogen and deuterium [Baru et al. 2011]

$$a_{0+}^{1/2} = (169.8 \pm 2.0)10^{-3} M_\pi^{-1} \quad a_{0+}^{3/2} = (-86.3 \pm 1.8)10^{-3} M_\pi^{-1}$$

$$\mathrm{Re}\, f_{l\pm}^I(s) = \mathbf{q}^{2l} \left(a_{l\pm}^I + b_{l\pm}^I \mathbf{q}^2 + \cdots \right)$$

10 subthreshold parameters are needed to match **d.o.f**
$\Rightarrow$ **three subtractions**

Solving s-channel: strategy

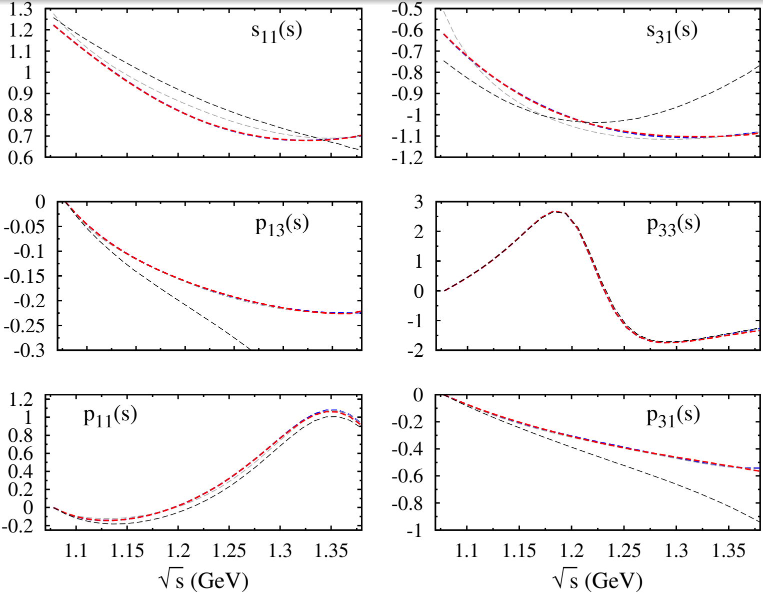
- Parametrize S and P waves up to $W < W_m$
  - Imposing a continuous and differentiable matching point
  - Using SAID partial waves as starting point
- Minimize difference between LHS and the RHS on a grid of points W_j

$$\chi^2 = \sum_{l,I_s,\pm} \sum_{j=1}^{N} \frac{\left(\mathrm{Re}\, f_{l\pm}^{I_s}(W_j) - F[f_{l\pm}^{I_s}](W_j)\right)^2}{\mathrm{Re}\, f_{l\pm}^{I_s}(W_j)}$$

$F[f_{l\pm}^{I_s}](W_j) \equiv$ right hand side of RS-equations

- Parametrization and subthreshold parameters are the fitting parameters

Solving s-channel: results



blue/red

⇕

LHS/RHS

after the fit

gray/black

⇕

LHS/RHS

before the fit

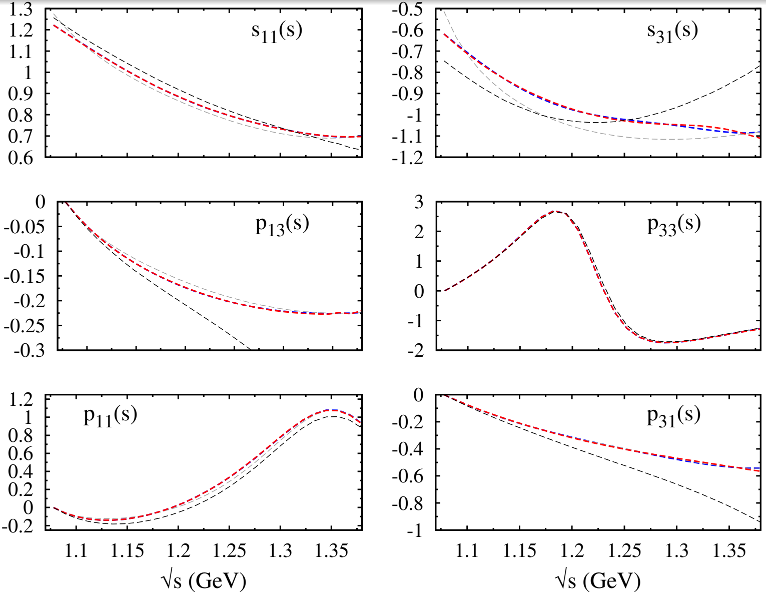
Notation: $L_{2I_s 2J}$

Solving the full RS system: strategy

- Full solution: self-consistent, iterative solution of the full RS system
  $\Rightarrow$ consistent set of **s**- and **t**-channel PWs & low-energy parameters
- However:
  - t-channel RS eqs. depend only weekly on s-channel PWs
  - resulting s-channel PW change little from **SAID**

A **full solution** can be achieved including in the s-channel RS eqs. the t-channel dependence on the **subthreshold parameters**

Results: s-channel PWs



blue/red

⇕

LHS/RHS

after the fit

gray/black

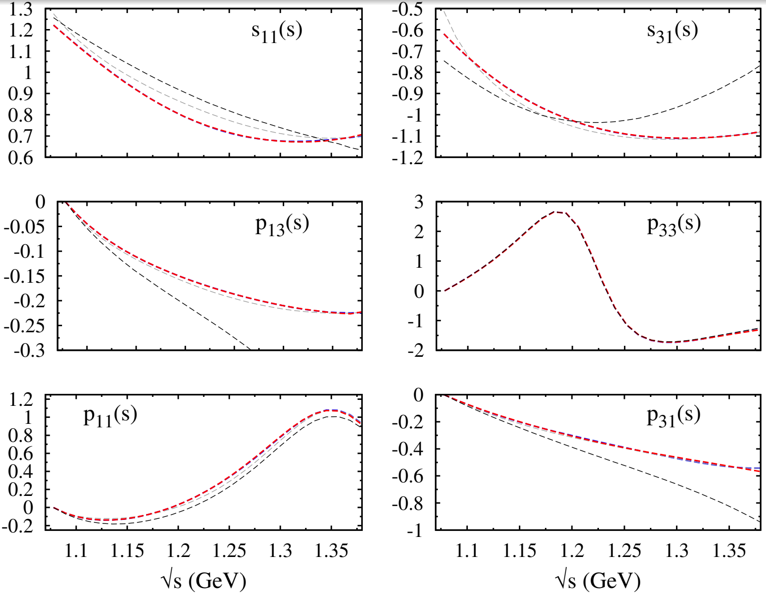
⇕

LHS/RHS

before the fit

Notation: $L_{2I_s 2J}$

Results: s-channel PWs



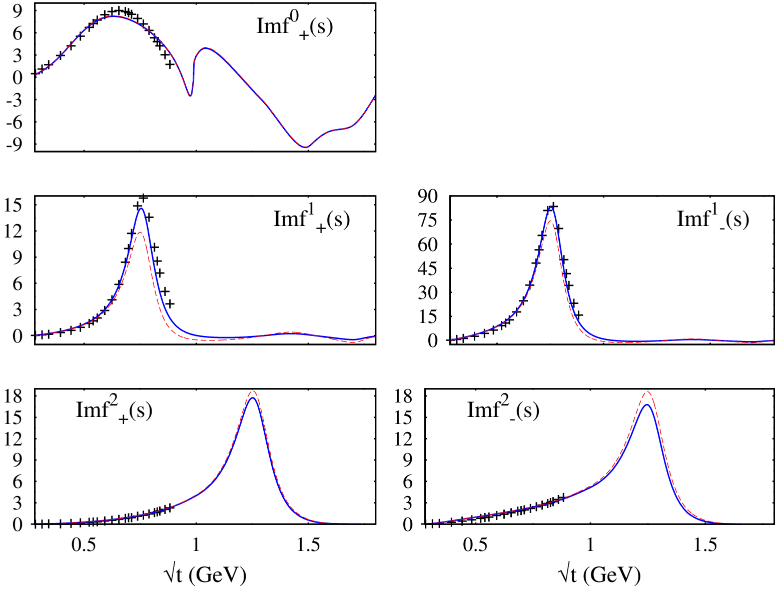
sizable $f_2(1275)$ effect

blue/**red** ⇕ **LHS**/**RHS** after the fit

gray/**black** ⇕ **LHS**/**RHS** before the fit

Notation: $L_{2I_s 2J}$

Results: t-channel PWs



blue ⇕ before the fit

red ⇕ after the fit

+ ⇕ KH80

Improvements with respect to KH80

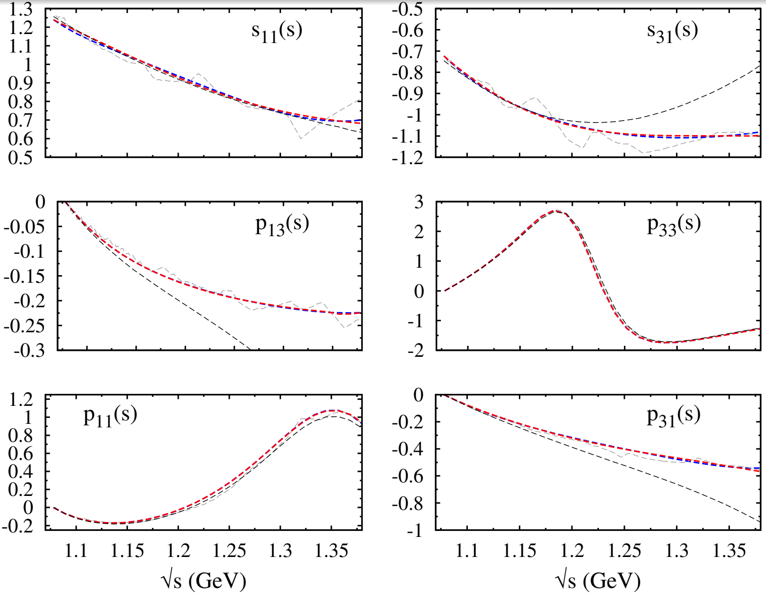
- Karlsruhe-Helsinki analysis **KH80** [Höhler et al. 1980]
  - comprehensive analyticity constraints based on fixed-t dispersion relations
  - old experimental data
- Here, an update of **KH80** results with modern input
  - HDR increase the range of validity of the equations
  - πN scattering length extracted from hadronic atoms $\Rightarrow$ essential for the $\sigma_{\pi N}$
  - Goldberger-Miyazawa-Oehme sum rule:

$$g^2_{\pi N}/4\pi = 13.7 \pm 0.2$$ [Baru et al. 2011]

compare: $$g^2_{\pi N}/4\pi = 14.28$$ [Höhler et al. 1983]

  - s-channel PWs from **SAID**
  - $f_2(1275)$ included $\Rightarrow$ sizable effect
- **KH80** is internally consistent $\Rightarrow$ RS reproduces **KH80** results with **KH80** input

Results: s-channel PWs with KH80 input

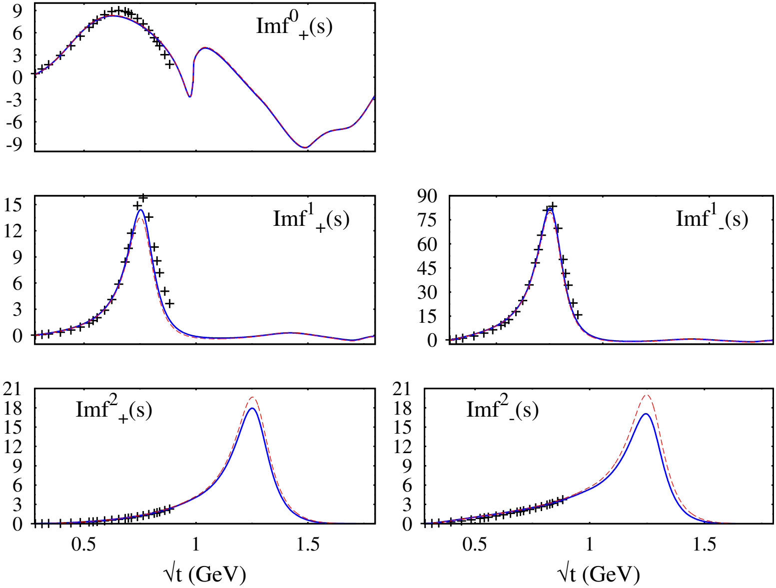


blue/red ⇕ **LHS/RHS** after the fit

gray/black ⇕ **LHS/RHS** before the fit

Notation: $L_{2I_s 2J}$

Results: t-channel PWs with KH80 input



blue

⇕

before the fit

red

⇕

after the fit

+

⇕

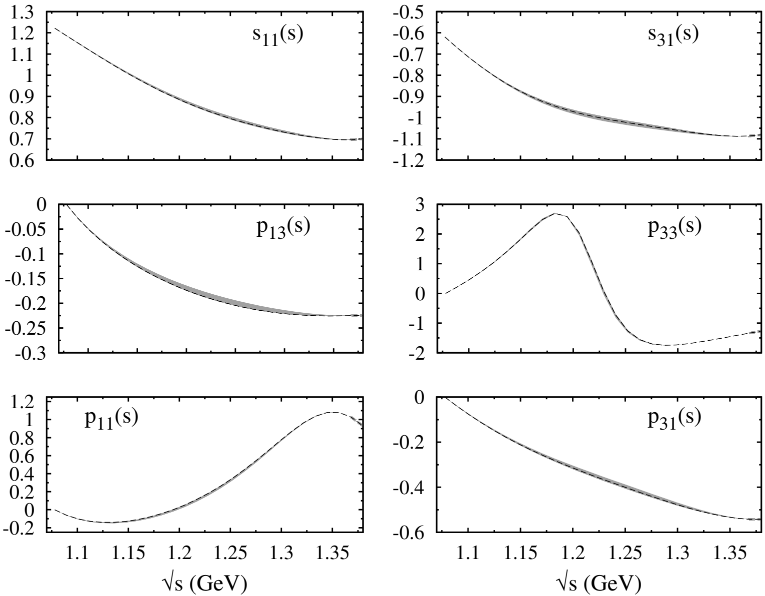
KH80

Solving the full RS system: uncertainties

- Statistical errors (at intermediate energies)
    - important correlations between subthreshold parameters
    - shallow fit minima

  $\Rightarrow$ Sum rules for subthreshold parameters become essential to reduce the errors

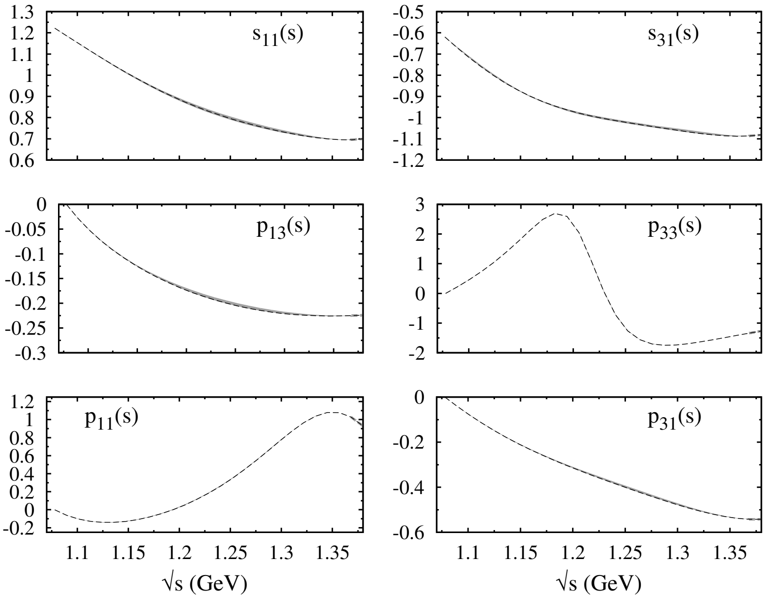
Uncertainties: statistical errors



Solving the full RS system: uncertainties

- Statistical errors (at intermediate energies)
  - important correlations between subthreshold parameters
  - shallow fit minima
  - $\Rightarrow$ Sum rules for subthreshold parameters become essential to reduce the errors
- Input variation (small)
  - small effect for considering s-channel KH80 input
  - very small effects from $L > 5$ s-channel PWs
  - small effect from the different S-wave extrapolation for $t > 1.3$ GeV
  - negligible effect of ρ' and ρ''
  - very significant effects of the D-waves ($f_2(1275)$)
  - F-waves shown to be negligible

Uncertainties: input variation

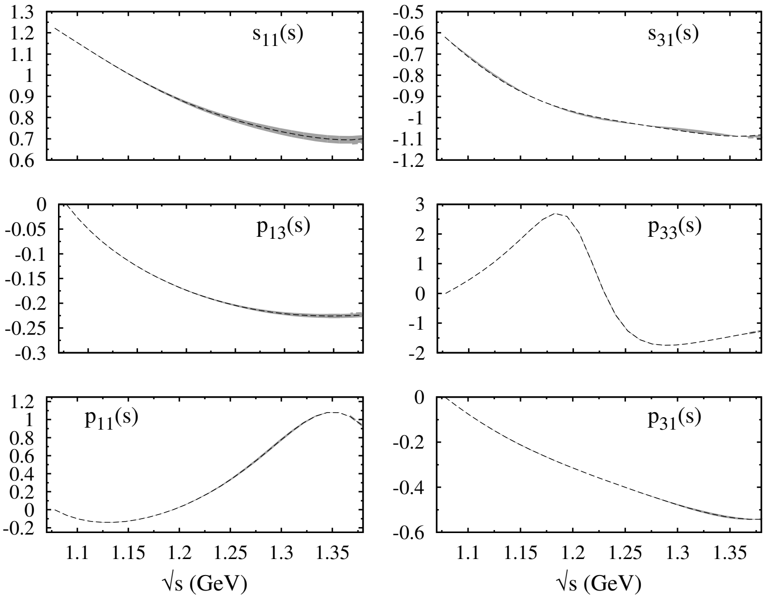


Solving the full RS system: uncertainties

- Statistical errors (at intermediate energies)
  - important correlations between subthreshold parameters
  - shallow fit minima

  $\Rightarrow$ Sum rules for subthreshold parameters become essential to reduce the errors
- Input variation (small)
  - small effect for considering s-channel KH80 input
  - very small effects from $L > 5$ s-channel PWs
  - small effect from the different S-wave extrapolation for $t > 1.3$ GeV
  - negligible effect of ρ' and ρ''
  - very significant effects of the D-waves ($f_2(1275)$)
  - F-waves shown to be negligible
- matching conditions (close to W_m)

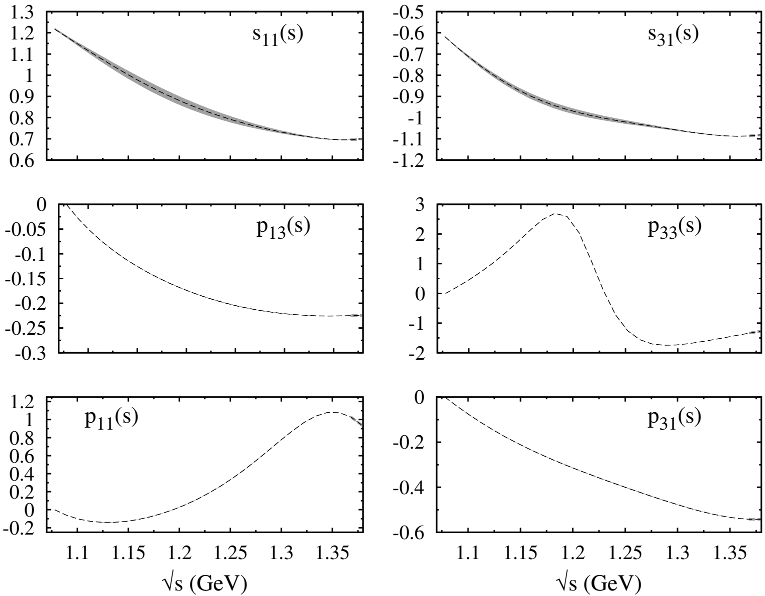
Uncertainties: matching conditions



Solving the full RS system: uncertainties

- Statistical errors (at intermediate energies)
  - important correlations between subthreshold parameters
  - shallow fit minima
  - $\Rightarrow$ Sum rules for subthreshold parameters become essential to reduce the errors
- Input variation (small)
  - small effect for considering s-channel KH80 input
  - very small effects from $L > 5$ s-channel PWs
  - small effect from the different S-wave extrapolation for $t > 1.3$ GeV
  - negligible effect of ρ' and ρ''
  - very significant effects of the D-waves ($f_2(1275)$)
  - F-waves shown to be negligible
- matching conditions (close to W_m)
- scattering length (SL) errors (on S-waves and subthreshold parameters)
  - very important for the $\sigma_{\pi\mathbf{N}}$

Uncertainties: scattering lengths



Solving the full RS system: $\sigma_{\pi N}$

Very preliminary results

$$\sigma_{\pi\mathbf{N}} = \Sigma_d + \Delta_D - \Delta_R - \Delta_\sigma$$

$$\Sigma_d = F_\pi^2 \left(d_{00}^+ + 2M_\pi^2 d_{01}^+ \right), \qquad \Delta_D - \Delta_R - \Delta_\sigma = -(1.8 \pm 2.2)\ \text{MeV}$$

$$\Sigma_d = 57.9 \pm 1.8\ \text{MeV}$$
$$\sigma_{\pi\mathbf{N}} = (56.1 \pm 2.8)\ \text{MeV} + (3 \pm 2.2)\ \text{MeV (IV)}$$

$\Rightarrow$with KH80 SL+(KH80 input): $\Sigma_d = 47.9$ MeV $\Rightarrow \sigma_{\pi\mathbf{N}} = 46$ MeV

to be compared with $\sigma_{\pi\mathbf{N}} = 45$ [Gasser, Leutwyler, Socher, Sainio 1988, Gasser, Leutwyler, Sainio 1991]

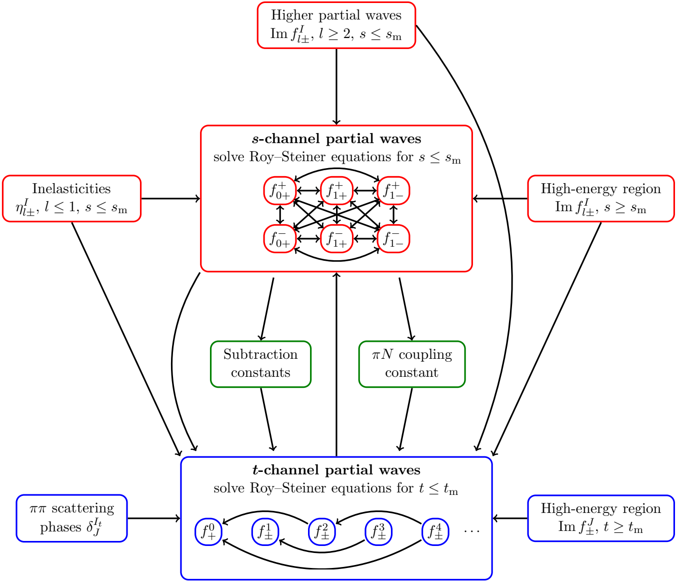
$\Rightarrow$KH80 is internally **consistent** but at odd with the modern SL determinations

Contents

Summary & Outlook

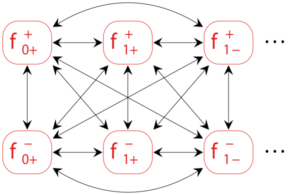
- **What has been done**:
  - Derived a closed system of Roy-Steiner equations (PWHDRs) for πN scattering
  - Constructed unitarity relations including $\bar{K}K$ intermediate states for the t-channel PWs
  - Optimized the range of convergence by tuning a for s- and t-channel each
  - Implemented subtractions at several orders
  - Solved the t-channel MO problem for a single- and two-channel approximation
    $\Rightarrow$ t-channel RS/MO machinery works
  - Numerical solution of the full system of RS eqs.
  - Precise determination of the $\sigma_{\pi N}$
  - Preliminary error analysis
- **What needs to be done**:
  - Error propagation still ongoing
  - Extraction of the Low Energy Constants
  - Possible improvements: higher PWs, more inelastic input, ...

Roy-Steiner equations for πN: flow of information

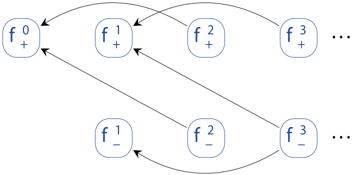


Solving Roy-Steiner equations for πN: Recoupling schemes

- s-channel subproblem:
  - Kernels are diagonal for $I \in \{+,-\}$, but unitarity relations are diagonal for $I_s \in \{1/2, 3/2\} \Rightarrow$ all partial-waves are interrelated
  - Once the t-channel PWs are known
    $\Rightarrow$ Structure similar to $\pi\pi$ Roy-equations



- t-channel subproblem:
  - Only higher PWs couple to lower ones
  - Only PWs with even or odd J are coupled
  - No contribution from f_+^J to f_-^{J+1}
    $\Rightarrow$ Leads to Muskhelishvili-Omnès problem



Spare slides

Roy-Steiner equations for πN: HDR's

- **Hyperbolic DRs**: $(s-a)(u-a) = b = (s'-a)(u'-a)$ with $a, b \in \mathbb{R}$

$$A^+(s,t;a) = \frac{1}{\pi}\int_{s_+}^{\infty} ds' \left[\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right] \mathrm{Im}\, A^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \, \frac{\mathrm{Im}\, A^+(s',t')}{t'-t}$$

$$B^+(s,t;a) = N^+(s,t) + \frac{1}{\pi}\int_{s_+}^{\infty} ds' \left[\frac{1}{s'-s} - \frac{1}{s'-u}\right] \mathrm{Im}\, B^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \, \frac{\nu}{\nu'} \frac{\mathrm{Im}\, B^+(s',t')}{t'-t}$$

$$N^+(s,t) = g^2\left(\frac{1}{m^2-s} - \frac{1}{m^2-u}\right)$$ similar for A^-, B^- and N^- [Hite/Steiner (1973)]

- Why **HDR**?
  - Combine all physical regions $\Rightarrow$ crucial for t-channel projection
  - Evade double-spectral regions $\Rightarrow$ the PW decompositions converge
  - Range of convergence can be maximized by tuning the free hyperbola parameter a
  - No kinematical cuts, manageable kernel functions

πN-scattering basics: partial waves

- **s-channel** projection:

$$f^I_{l\pm}(W) = \frac{1}{16\pi W}\Big\{(E+m)\big[A^I_l(s)+(W-m)B^I_l(s)\big]+(E-m)\big[-A^I_{l\pm1}(s)+(W+m)B^I_{l\pm1}(s)\big]\Big\}$$

$$X^I_l(s) = \int_{-1}^{1} \mathrm{d}z_s\, P_l(z_s) X^I(s,t)\Big|_{t=t(s,z_s)=-2q^2(1-z_s)} \quad \text{for } X\in\{A,B\} \text{ and } W=\sqrt{s}$$

- **McDowell symmetry**: $f^I_{l+}(W) = -f^I_{(l+1)-}(-W) \qquad \forall\, l\geq 0$
- **t-channel** projection:

$$f^J_+(t) = -\frac{1}{4\pi}\int_0^1 \mathrm{d}z_t\, P_J(z_t)\bigg\{\frac{p_t^2}{(p_tq_t)^J}A^I(s,t)\Big|_{s=s(t,z_t)} - \frac{m}{(p_tq_t)^{J-1}}z_t B^I(s,t)\Big|_{s=s(t,z_t)}\bigg\} \quad \forall\, J\geq 0$$

$$f^J_-(t) = \frac{1}{4\pi}\frac{\sqrt{J(J+1)}}{2J+1}\frac{1}{(p_tq_t)^{J-1}}\int_0^1 \mathrm{d}z_t\Big[P_{J-1}(z_t)-P_{J+1}(z_t)\Big]B^I(s,t)\Big|_{s=s(t,z_t)} \quad \forall\, J\geq 1$$

- **Bose symmetry** $\Rightarrow$ even/odd $J \Leftrightarrow I=+/-$

Roy-Steiner equations for πN: derivation

- Recipe to derive **Roy-Steiner** equations:
  - Expand imaginary parts in terms of s- and t-channel partial waves
  - Project onto s- and t-channel partial waves
  - Combine the resulting equations using s- and t-channel PW unitarity relations
- Similar structure to $\pi\pi$ **Roy equations**
- **Validity**: assuming Mandelstam analyticity
  - s-channel $\Rightarrow$ optimal for $a = -23.2M_\pi^2$

$$s \in \left[s_+ = (m + M_\pi)^2, 97.30\, M_\pi^2\right] \quad \Leftrightarrow \quad W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}]$$

  - t-channel $\Rightarrow$ optimal for $a = -2.71M_\pi^2$

$$t \in [t_\pi = 4M_\pi^2, 205.45\, M_\pi^2] \quad \Leftrightarrow \quad \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}]\ .$$

Roy-Steiner equations for πN: subtractions

- **Subtractions** are necessary to ensure the convergence of DR integrals
  $\Rightarrow$ asymptotic behavior
- Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior
- Parametrize high-energy information in (a priori unknown) subtraction constants
  $\Rightarrow$ matching to ChPT
- Subthreshold expansion around $\nu = t = 0$

$$\bar{A}^+(\nu, t) = \sum_{m,n=0}^{\infty} a^+_{mn} \nu^{2m} t^n \qquad \bar{A}^-(\nu, t) = \sum_{m,n=0}^{\infty} a^-_{mn} \nu^{2m+1} t^n$$

where

$$\bar{A}^+(s, t) = A^+(s, t) - \frac{g^2}{m} \qquad \bar{A}^-(s, t) = A^-(s, t) \ ,$$

similar expansion for $B^+(s, t)$ and $B^-(s, t)$

Roy-Steiner equations for πN: s-channel

s-channel RS equations

$$f^I_{l+}(W) = N^I_{l+}(W) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K^I_{ll'}(W,W')\,\mathrm{Im} f^I_{l'+}(W') + K^I_{ll'}(W,-W')\,\mathrm{Im} f^I_{(l'+1)-}(W') \right\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \sum_J \left\{ G_{lJ}(W,t')\,\mathrm{Im} f^J_+(t') + H_{lJ}(W,t')\,\mathrm{Im} f^J_-(t') \right\}$$

$$= -f^I_{(l+1)-}(-W) \qquad \forall\, l \geq 0\,, \quad \text{[Hite/Steiner (1973)]}$$

- $K^I_{ll'}(W,W')$, $G_{lJ}(W,t')$ and $H_{lJ}(W,t')$-Kernels: analytically known,
  e.g. $K^I_{ll'}(W,W') = \frac{\delta_{ll'}}{W'-W} + \ldots \quad \forall\, l,l' \geq 0\,,$
- **Validity**: assuming Mandelstam analyticity
  $\Rightarrow$ optimal for $a = -23.2 M_\pi^2$

$$s \in \left[s_+ = (m+M_\pi)^2, 97.30\, M_\pi^2\right] \quad \Leftrightarrow \quad W \in [W_+ = 1.08\text{ GeV}, 1.38\text{ GeV}]$$

Roy-Steiner equations for πN: t-channel

t-channel RS equations

$$f_+^J(t) = \tilde{N}_+^J(t) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l=0}^{\infty}\left\{\tilde{G}_{Jl}(t,W')\,\mathrm{Im} f_{l+}^I(W') + \tilde{G}_{Jl}(t,-W')\,\mathrm{Im} f_{(l+1)-}^I(W')\right\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \sum_{J'}\left\{\tilde{K}_{JJ'}^1(t,t')\,\mathrm{Im} f_+^{J'}(t') + \tilde{K}_{JJ'}^2(t,t')\,\mathrm{Im} f_-^{J'}(t')\right\} \qquad \forall\, J \geq 0\,,$$

$$f_-^J(t) = \tilde{N}_-^J(t) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l=0}^{\infty}\left\{\tilde{H}_{Jl}(t,W')\,\mathrm{Im} f_{l+}^I(W') + \tilde{H}_{Jl}(t,-W')\,\mathrm{Im} f_{(l+1)-}^I(W')\right\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \sum_{J'}\tilde{K}_{JJ'}^3(t,t')\,\mathrm{Im} f_-^{J'}(t') \qquad \forall\, J \geq 1\,,$$

- **Validity**: assuming Mandelstam analyticity
  $\Rightarrow$ optimal for $a = -2.71 M_\pi^2$

$$t \in [t_\pi = 4M_\pi^2, 205.45\, M_\pi^2] \quad \Leftrightarrow \quad \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}]\,.$$

RS-eqs for πN: Range of convergence

- Subthreshold expansion around $\nu = t = 0$

$$A^+(\nu,t) = \frac{g^2}{m} + d_{00}^+ + d_{01}^+ t + a_{10}^+ \nu^2 + \mathcal{O}(\nu^2 t, t^2)$$

$$A^-(\nu,t) = \nu a_{00}^- + a_{01}^- \nu t + a_{10}^- \nu^3 + \mathcal{O}(\nu^5, \nu t^2, \nu^3 t)$$

$$B^+(\nu,t) = g^2 \frac{4m\nu}{(m^2-s_0)^2} + \nu b_{00}^+ + \mathcal{O}(\nu^3, \nu t)\ ,$$

$$B^-(\nu,t) = g^2 \left[\frac{2}{m^2-s_0} - \frac{t}{(m^2-s_0)^2}\right] - \frac{g^2}{2m^2} + b_{00}^- + b_{01}^- t + +b_{10}^- \nu^2 + \mathcal{O}(\nu^2, \nu^2 t, t^2)$$

- pseudovector Born terms: $D^I = A^I + \nu B^I$

$$\bar{D}^+ = d_{00}^+ + d_{01}^+ t + d_{10}^+ \nu^2$$

$$d_{mn}^+ = a_{mn}^+ + b_{m-1,n}^+\ , \qquad d_{mn}^- = a_{mn}^- + b_{mn}^-\ .$$

- Sum rules for subthreshold parameters:

$$d_{00}^+ = -\frac{g^2}{m} + \frac{1}{\pi}\int_{s_+}^{\infty} \mathrm{d}s'\, h_0(s')\, \left[\mathrm{Im}\, A^+(s', z_s')\right]_{(0,0)} + \frac{1}{\pi}\int_{t_\pi}^{\infty} \frac{\mathrm{d}t'}{t'}\, \left[\mathrm{Im}\, A^+(t', z_t')\right]_{(0,0)}$$

$$h_0(s') = \frac{2}{s'-s_0} - \frac{1}{s'-a}$$

RS-eqs for πN: Range of convergence

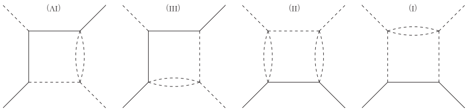
- Assumption: Mandelstam analyticity [Mandelstam (1958,1959)]

  $\Rightarrow$ T(s,t) can be written in terms double spectral densities: ρ_{st}, ρ_{su}, ρ_{ut}

  $$T(s,t) = \frac{1}{\pi^2}\iint ds'du' \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\iint dt'du' \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\iint ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)}$$

  integration ranges defined by the support of the double spectral densities ρ

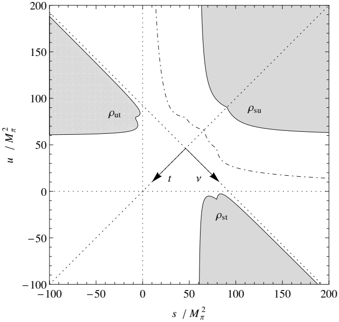
- Boundaries of ρ are given lowest lying intermediate states



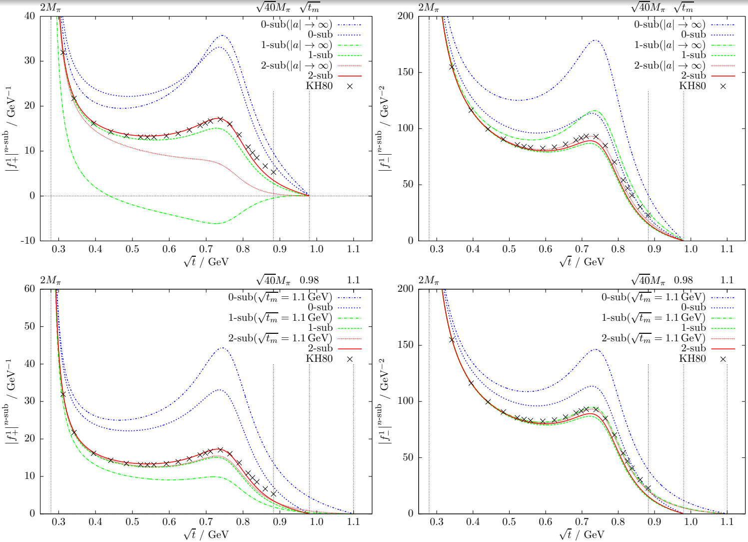
- They limit the range of validity of the HDRS:
  - Pw expansion converge

    $\Rightarrow z = \cos\theta \in$ Lehmann ellipses [Lehmann (1958)]
  - the hyperbolae $(s-a)(u-a) = b$ does not enter any double spectral region

    $\Rightarrow$ for a value of a, constraints on b yield ranges in s & t



Solving t-channel: P-wave results



MO solutions in general consistent with KH80 results

Solving t-channel: coupled channels

- Generic coupled-channel integral equation

$$\mathbf{f}(t) = \boldsymbol{\Delta}(t) + \frac{1}{\pi}\int_{t_\pi}^{t_\mathrm{m}} \mathrm{d}t' \, \frac{T^*(t')\Sigma(t')\mathbf{f}(t')}{t'-t} + \frac{1}{\pi}\int_{t_\mathrm{m}}^{\infty} \mathrm{d}t' \, \frac{\mathrm{Im}\,\mathbf{f}(t')}{t'-t}$$

- Formal solution as in the single-channel case (now with Omnès matrix $\Omega(t)$)
$\Rightarrow$ Two-channel Muskhelishvili-Omnès problem

$$\mathbf{f}(t) = \begin{pmatrix} f_+^0(t) \\ h_+^0(t) \end{pmatrix} \qquad \mathrm{Im}\,\Omega(t) = (T(t))^*\Sigma(t)\Omega(t)$$

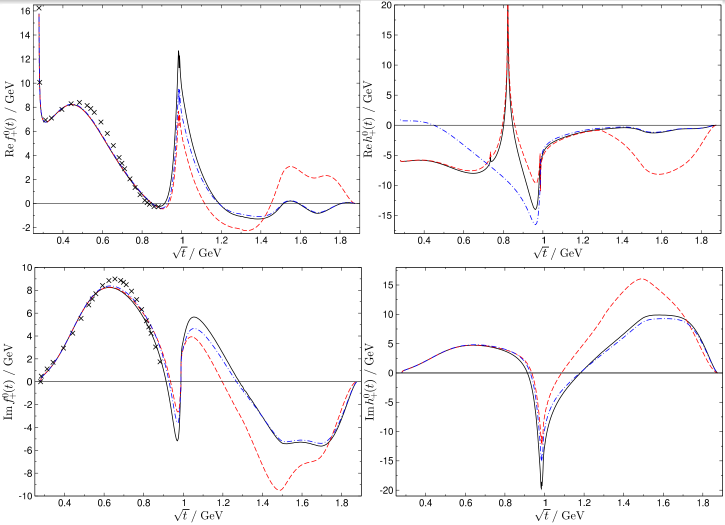
- Two linearly independent solutions Ω_1, Ω_2 [Muskhelishvili 1953]
- In general no analytical solution for the Omnès matrix but for its determinant [Moussallam 2000]

$$\det \Omega(t) = \exp\left\{ \frac{t}{\pi}\int_{t_\pi}^{t_\mathrm{m}} \mathrm{d}t' \, \frac{\psi(t')}{t'(t'-t)} \right\} .$$

Solving t-channel S-wave equations: input

- Input needed:
  - $\pi\pi$ s-wave partial waves: [Caprini, Colangelo, Leutwyler, (in preparation)]
  - $K\bar{K}$ s-wave partial waves: [Büttiker. (2004)]
  - πN and KN s-wave pw: SAID [Arndt et al. 2008], KH80
  - πN at high energies: Regge model [Huang et al. 2010]
  - πN parameters: KH80
  - Hyperon couplings from [Jülich model 1989]
  - KN subthreshold parameters neglected
- Two-channel approximation beaks down at $\sqrt{t_0} = 1.3$ GeV $\Rightarrow 4\pi$ channel
- From t_0 to $t = 2$ GeV, different approximations considered

Solving t-channel: S-wave results

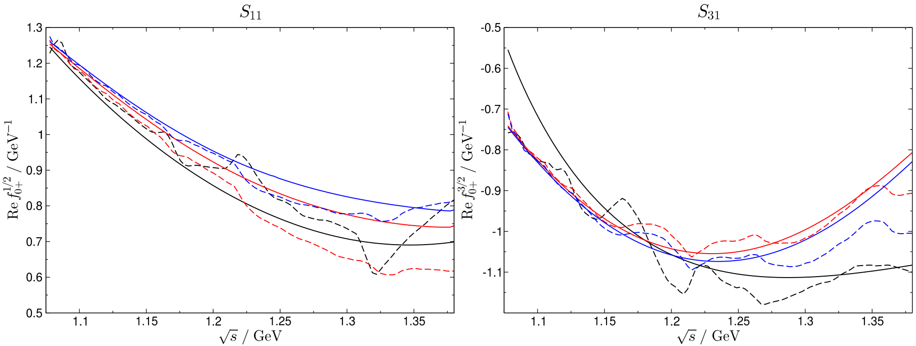


MO solutions in general consistent with KH80 results

Solving s-channel: consistency with KH80

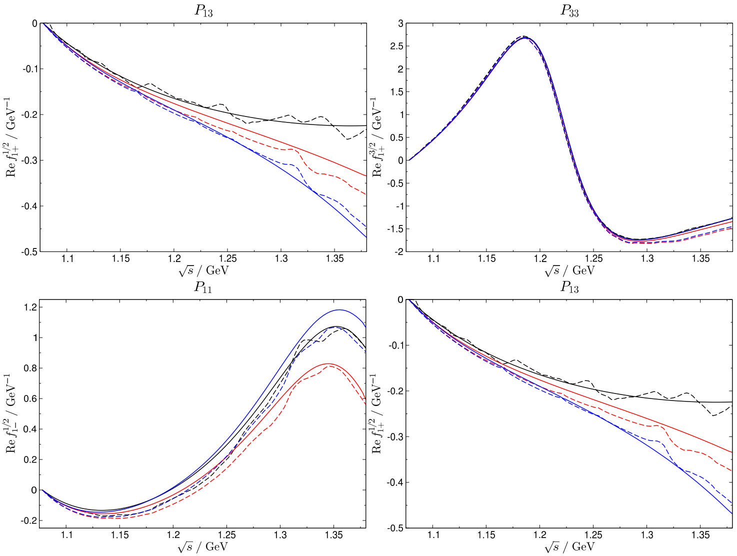
- Consistency with KH80
  - parametrize SAID S and P waves up to $W < W_m$
    Imposing a continuous and differentiable matching point
  - Compare between the input (LHS) and the output (RHS)

S-WAVES



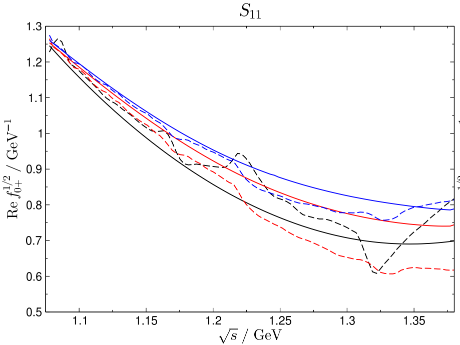
$\Rightarrow$ important discrepancies

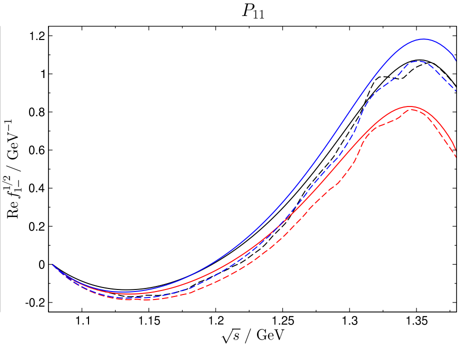
Solving s-channel: consistency with KH80. P-waves



Solving s-channel: consistency with KH80

- Consistency with KH80
  - parametrize SAID S and P waves up to $W < W_m$
    Imposing a continuous and differentiable matching point
  - Compare between the input (LHS) and the output (RHS)





$\Rightarrow$ important discrepancies

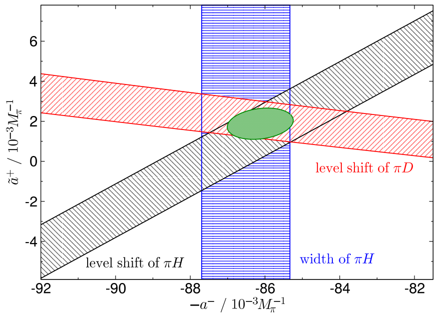
Solving s-channel: scattering length from hadronic atoms

$$\tilde{a}^{+} = a^{+} + \frac{1}{1+M_\pi/m_p}\left\{\frac{M_\pi^2 - m_{\pi^0}^2}{\pi F_\pi^2} c_1 - 2\alpha f_1\right\}$$

- $\pi H/\pi D$: bound state of π^- and p/d, spectrum sensitive to threshold πN amplitude
- Combined analysis of πH and πD:
  $a_0^+ \equiv a^+ = (7.5 \pm 3.1) \cdot 10^{-3} M_\pi^{-1}$
  $a_0^- \equiv a^- = (86.0 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$
- Large a^+ suggests a large $\sigma_{\pi N}$, but

$$\frac{a_{\pi^- p} + a_{\pi^+ p}}{2} = (-1.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$$

- Isospin breaking in $\sigma_{\pi N}$ could be important
- Revisit the Cheng-Dashen low-energy theorem



[Baru et al. 2011]