Symanzik improvement of Yang-Mills gradient flow observables

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Outline

- The Yang-Mills gradient flow equation & its properties
- Some current applications
- The gradient flow equation on the lattice
- Large cutoff effects and their tree-level anatomy
- 4+1 dimensional formulation & Symanzik improvement
- Classical a-expansion of observables & the flow equation
- Conclusions & Outlook

The Yang-Mills gradient flow equation

Starting point: Yang-Mills theory in 4-dimensions:

$$S_g[A] = -\frac{1}{2g_0^2}\int \mathrm{d}^4x \ \mathrm{tr}\left\{F_{\mu\nu}(x)F_{\mu\nu}(x)\right\}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Add extra (flow) time coordinate t and define the gauge field $B_\mu(t,x)$

$$\begin{aligned} G_{\mu\nu}(t,x) &= \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)] \\ \partial_t B_\mu(t,x) &= D_\nu G_{\nu\mu}(t,x) \quad \left(= -\frac{\delta S_g[B]}{\delta B_\mu(t,x)}\right), \qquad B_\mu(0,x) = A_\mu(x) \end{aligned}$$

The linearized gradient flow equation reduces to the heat equation (use gauge freedom to diagonalize RHS):

$$\partial_t B_\mu(t,x) = -\partial_\nu\partial_\nu B_\mu(t,x), \qquad B_\mu(t,x) = (4\pi t)^{-2}\int \mathrm{d}^4y \, \mathrm{e}^{-\frac{(x-y)^2}{4t}} A_\mu(y)$$

$\Rightarrow$ The gauge field $B_\mu(t,x)$ is smoothed over a range with radius $r(t) = \sqrt{8t}$ (2σ range of the Gaussian smoothing function).

Properties of the Yang-Mills gradient flow

- Correlation functions of (gauge invariant) observables at $t > 0$ are renormalized

$$\langle O[B] \rangle = \frac{1}{Z} \int D[A] O[B] \exp\left(-S_g[A]\right)$$

once the coupling g_0 is renormalized as usual!

- Local gauge invariant composite fields at $t > 0$ such as

$$E(t,x) = -\tfrac{1}{2}\,\mathrm{tr}\left\{G_{\mu\nu}(x,t) G_{\mu\nu}(x,t)\right\}$$

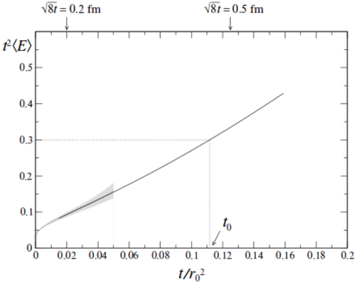
are renormalized; no mixing with other fields of same or lower dimensions!

- Established to all orders in perturbation theory [Lüscher & Weisz '2012];
- Explicit one-loop calculation (infinite volume, dimensional regularization) [Lüscher 2010]:

$$\langle E(t,x) \rangle = \frac{3 g^2_{\overline{\mathrm{MS}}}(\mu)}{16\pi^2 t^2} \left(1 + \frac{1.0978 + 0.0075 \times N_f}{4\pi} g^2_{\overline{\mathrm{MS}}}(\mu) + \mathrm{O}(g^4)\right), \quad \mu = \frac{1}{\sqrt{8t}}$$

$\Rightarrow$ $E(t,x)$ is, for $t > 0$, a renormalized field; unlike $E(0,x)$ which has a quartic and a logarithmic divergence!

- Convenient (implicit) definition of reference scale, e.g. t_0 [Lüscher 2010] easy to measure, small statistical fluctuations, mild quark mass dependence also in χPT [Bär & Golterman 2013]



$$t^2\langle E(t,x)\rangle\big|_{t=t_0} = 0.3$$

- Non-perturbative definition of a renormalized "gradient flow coupling" at scale $\mu = 1/\sqrt{8t}$:

$$g^2_{\rm GF}(\mu) \stackrel{\rm def}{=} \frac{16\pi^2}{3} t^2\langle E(t,x)\rangle$$

- Coupling at scale $\mu_0 = 1/\sqrt{8t_0}$:

$$g^2_{\rm GF}(\mu_0) = \frac{16\pi^2}{3} \times 0.3 = 15.8 \quad \Rightarrow \quad \alpha_{\rm GF}(\mu_0) = \frac{15.8}{4\pi} = 1.257$$

Applications of the gradient flow II

- Consider $\langle E(t,x)\rangle$ in a finite box of dimension L^4, fix the ratio $c = \sqrt{8t}/L$ and define

$$\bar{g}^2_{\mathrm{GF}}(L) = \mathcal{N}(c)^{-1} t^2 \langle E(t,x)\rangle, \qquad \lim_{c\to 0} \mathcal{N}(c) = \frac{3}{16\pi^2}$$

- defines family of renormalized couplings, with parameter c.
  (typical range from 0.2 to 0.5; $c > 0.5$ implies that $2r(t) > L$, i.e. "smearing around the universe")
- The normalization constant is calculable in lowest order perturbation theory; depends on b.c's for the gauge field; periodic in spatial directions, and in the time direction
  - periodic b.c.'s [Fodor et al. 2012]
  - SF (Dirichlet) b.c.'s [Fritzsch & Ramos 2012]
  - twisted periodic b.c.'s [Ramos 2013]
  - open-SF (Neumann-Dirichlet) b.c.'s [Lüscher 2013]
- QCD, α_s determination: advantage of gradient flow coupling at low energies, but loses to SF coupling at high energies

$\Rightarrow$ pursue mixed approach [Fritsch et al. (ALPHA coll.), 2014]

Applications of the gradient flow III

- Small flow time expansion & operator renormalization [Lüscher 2013]
- Definition of renormalized energy-momentum tensor [Suzuki 2013ff; Del Debbio et al.,2013; Patella et al, 2014]
- enormalized energy-momentum tensor & SU(3) thermodynamics [Asakawa et al. (FlowQCD collaboration), 2014 (cf. Hatsuda on 4 March)]
- Extension to fermions possible [Lüscher 2013]
- Use flow quantities to check for autocorrelations in Monte-Carlo simulations; significant coupling to slow modes [Lüscher & Schaefer,2012]
- Assess quality of lattice actions, new improvement conditions,...
- ...

The lattice community has only just begun to explore the possibilities! Expect much more to come;
<u>However</u>: Improvements and/or combination with other techniques may be required;
<u>Here</u>: systematic reduction of $O(a^2)$ cutoff effects.

The gradient flow on the lattice

- consider generic SU(3) lattice action with 4-link and 6-link Wilson loops (normalization: $c_0 + 8c_1 + 16c_2 + 8c_3 = 1$):

$$S[U; c_i^{(a)}] = \frac{1}{g_0^2} \sum_x \mathrm{tr}\left(1 - c_0^{(a)} \Box - c_1^{(a)} \Box\!\Box - c_2^{(a)} \boxminus - c_3^{(a)} \boxplus \right)$$

  Expectation values defined by integral over U_μ.

- Gradient flow equation, choose the gradient of a lattice action:

$$\partial_t V_\mu(t,x) = -g_0^2 \partial_{x,\mu} S[V; c_i^{(f)}] V_\mu(t,x), \qquad V_\mu(t=0,x) = U_\mu(x)$$

  $\Rightarrow c_i^{(a)} \neq c_i^{(f)}$ in general!

- Observables: we focus on $E(t,x)$; two options:
  1. define a lattice version of $G_{\mu\nu}(t,x)$ (e.g. clover leaf of plaquettes in $\mu - \nu$ plane), then form

$$E^{\mathrm{cl}}(t,x) = -\tfrac{1}{2}\,\mathrm{tr}\{G^{\mathrm{cl}}_{\mu\nu}(t,x) G^{\mathrm{cl}}_{\mu\nu}(t,x)\}$$

  2. $E(t,x)$ is an action density; choose a lattice action density such that

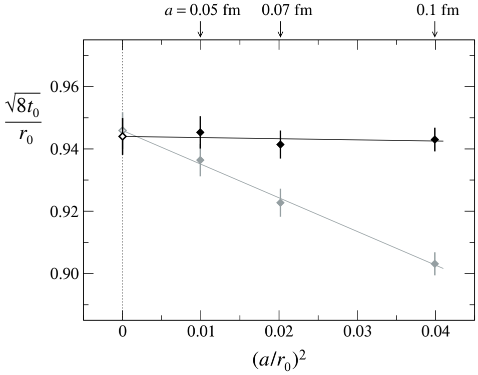
$$a^4 \sum_x E(t,x) = g_0^2 S[V; c_i^{(o)}]$$

  $\Rightarrow$ yet another set of $c_i^{(o)}$!

Popular choices for action parameters & terminology:

- Wilson action, Wilson flow, plaquette observable:
  $c_0 = 1, c_1 = c_2 = c_3 = 0;$
- Tree-level Lüscher-Weisz action, Symanzik flow:
  $c_0 = 5/3, c_1 = -1/12, c_2 = c_3 = 0;$

Wilson action & flow;
t_0 from clover (upper points) and plaquette observable
[Lüscher 2010]



$O(a^2)$ cutoff effects are surprisingly large & and even larger in QCD!

Anatomy of tree-level $O(a^2)$ effects in $\langle E(t,x)\rangle$ I

$$
\begin{aligned}
t^2\langle E(t,x)\rangle &= g^2\int_{-\pi/a}^{\pi/a} \mathrm{d}^4 p\,\mathrm{tr}\left[K^{(\mathrm{o})}_{\mu\nu}(p,0)\bar{D}_{\mu\nu}(p,\lambda,\alpha)\right],\\
\bar{D}_{\mu\nu}(p,\lambda,\alpha) &= (\mathrm{e}^{-tK^{(\mathrm{f})}(p,\alpha)})_{\mu\rho}\left(K^{(\mathrm{a})}(p,\lambda)^{-1}\right)_{\rho\sigma}(\mathrm{e}^{-tK^{(\mathrm{f})}(p,\alpha)})_{\sigma\nu},
\end{aligned}
$$

- λ, α: gauge fixing parameters for the action and flow equation, respectively.
- Observable, gradient flow and action characterized by kernels $K_{\mu\nu}(p)$ of "free lattice actions":

$$
\begin{aligned}
S^{(\mathrm{a,o,f})} &= \tfrac{1}{2}\int_{-\pi/a}^{\pi/a} \mathrm{d}^4 p\, A^b_\mu(-p)K^{(\mathrm{a,o,f})}_{\mu\nu}(p,\lambda)A^b_\nu(p) + \mathrm{O}(A^3),\\
K^{(\mathrm{a,o,f})}_{\mu\nu}(p,\lambda) &= K^{\mathrm{cont}}_{\mu\nu}(p,\lambda) + a^2 R^{(\mathrm{a,o,f})}_{\mu\nu}(p,\lambda) + \mathrm{O}(a^4)\\
K^{\mathrm{cont}}_{\mu\nu}(p,\lambda) &= p^2\delta_{\mu\nu} + (\lambda-1)p_\mu p_\nu
\end{aligned}
$$

Extend momentum integrals to infinity, then evaluate traces:

$$\langle E(t,x)\rangle = \frac{3g^2}{16\pi^2 t^2}\left\{1+\frac{a^2}{t}\left[\left(d_1^{(\mathrm{o})}-d_1^{(\mathrm{a})}\right)J_{4,-2}+\left(d_2^{(\mathrm{o})}-d_2^{(\mathrm{a})}\right)J_{2,0} - 2d_1^{(\mathrm{f})}J_{4,0}-2d_2^{(\mathrm{f})}J_{2,2}\right]+\mathrm{O}(a^4)\right\}+\mathrm{O}(g^4)$$

where

$$J_{n,m}=\frac{t^{(m+n)/2}\int_{-\infty}^{\infty}\mathrm{d}^4p\,\mathrm{e}^{-2tp^2}\,p^n p^m}{\int_{-\infty}^{\infty}\mathrm{d}^4p\,\mathrm{e}^{-2tp^2}},\qquad p^n\Big|_{n=2,4,\ldots}=\sum_\mu p_\mu^n,\qquad p^{-n}=1/p^n.$$

All momentum integrals can be evaluated:

$$J_{2,0}=1,\quad J_{2,2}=3/2,\quad J_{4,0}=3/4,\quad J_{4,-2}=1/2$$

Hence:

$$\langle E(t,x)\rangle=\frac{3g^2}{16\pi^2t^2}\left\{1+\frac{a^2}{t}\underbrace{\left(d^{(\mathrm{o})}-d^{(\mathrm{a})}-3d^{(\mathrm{f})}\right)}_{=d^{total}}+\mathrm{O}(a^4)\right\}.$$

Anatomy of tree-level $O(a^2)$ effects in $\langle E(t,x)\rangle$ III

For each d-coeffcient we may choose Wilson-plaquette, Lüscher-Weisz or the clover kernel or combinations thereof:

$$d^{(\mathrm{a,o,f})} = \begin{cases} -\frac{3}{72}, & & \text{plaquette (pl)}, \\ \frac{1}{72}, & \left(= -\frac{1}{24} - \frac{2}{3}c_1\right) & \text{Lüscher-Weisz (lw)}, \\ -\frac{15}{72}, & & \text{clover (cl)}. \end{cases}$$

Popular combinations, $d^{\mathrm{total}} = d^{(\mathrm{o})} - d^{(\mathrm{a})} - 3d^{(\mathrm{f})}$:

- Clover observable, Wilson action & flow:

$$d^{\mathrm{total}} = (-15+3+9)/72 = -3/72$$

- Wilson observable & action & flow:

$$d^{\mathrm{total}} = (-3+3+9)/72 = 9/72$$

- Clover observable, Lüscher-Weisz action & flow:

$$d^{\mathrm{total}} = (-15-1-3)/72 = -19/72$$

- Lüscher-Weisz observable & action & flow:

$$d^{\mathrm{total}} = (1-1-3)/72 = -3/72$$

N.B.: not improved!

Qualitative understanding the $O(a^2)$ effects in t_0^{plaq} vs. t_0^{cl}

- Strategy: keep the standard definition of t_0 fixed (clover definition) and look at cutoff effects in t_0 from the Wilson/plaquette definition.
- Define $r(t) = \sqrt{8t}$ and the coupling $\bar{g}(r)$ in the GF scheme at scale $r(t_0)$

$$t^2 \langle E_{\mathrm{cl}}(t,x)\rangle|_{t=t_0} = \frac{3}{16\pi^2}\bar{g}^2(r(t_0)) = 0.3 \quad \Rightarrow \bar{g}^2(r(t_0)) = 15.8$$

- While this relation is (by definition) exact we otherwise have

$$t^2 \langle E_{\mathrm{plaq}}(t,x)\rangle|_{t=t_0^{\mathrm{plaq}}} = \frac{3}{16\pi^2}\bar{g}^2(r(t_0^{\mathrm{plaq}}))\left(1 + \Delta d\frac{a^2}{t_0} + \mathrm{O}(a^4)\right) = 0.3$$

  Here,

$$\Delta d = d^{\mathrm{pl}} - d^{\mathrm{cl}} = \frac{-3+15}{72} = \frac{12}{72} = \frac{1}{6}$$

- Use 1-loop or 2-loop evolution of the GF coupling to obtain the $O(a^2)$ shift in t_0^{plaq} w.r.t. the reference (clover) definition.

Integration of the RG equation

Recall the definition of the β-function:

$$r\frac{\partial g(r)}{\partial r} = -\beta(g) = b_0 g^3 + b_1 g^5 + \ldots$$

with the universal coefficients ($N = 3$)

$$b_0 = \left(11 - \tfrac{2}{3}N_\mathrm{f}\right)/(4\pi)^2, \qquad b_1 = \left(102 - \tfrac{28}{3}N_\mathrm{f}\right)/(4\pi)^4$$

Writing $r(t_0^{\mathrm{plaq}}) = r(t_0) + \Delta r$ with Δr small (an $\mathrm{O}(a^2)$ effect):

$$\bar{g}^2(r(t_0) + \Delta r) = \frac{\bar{g}^2(r(t_0))}{1 + \Delta d \dfrac{a^2}{t_0}}$$

Expanding both sides and using that $\Delta r / r(t_0) = \sqrt{t_0^{\mathrm{plaq}}/t_0} - 1$, one obtains

$$\frac{t_0^{\mathrm{plaq}}}{t_0} = 1 - \frac{\Delta d}{\{-\beta(\bar{g})/g\}}\frac{a^2}{t_0} + \mathrm{O}(a^4), \qquad -\beta(g)/g|_{g^2=15.8, N_\mathrm{f}=0} = 1.10 + 1.02 + \mathrm{O}(g^6)$$

Nonetheless one obtains qualitatively correct results:

- The sign is correctly obtained: the plaquette definition comes from below!
- Switching on N_f decreases the β-function and increases the cutoff effect.

GF coupling in finite volume with twisted periodic b.c.'s

- fix the relation between flow time t and space-time volume L^4 by choosing a value for $c = \sqrt{8t}/L$.
- The trace algebra same as before, however: the numbers $J_{n,m}$ become linearly independent functions of c!

$\Rightarrow$ more conditions for improvement: each coefficient must vanish separately!

- Cannot be satisfied with LW/Symanzik type flow, need to be more general: include bent rectangles/chairs with coefficient c_2 & define the chair flow

$$c_0 = 1, \qquad c_1 = -1/12, \qquad c_2 = 1/24$$

$\Rightarrow$ complete tree-level $\mathrm{O}(a^2)$ improvement of $\langle E(t,x) \rangle$

- However, we then checked that the connected 2-point function

$$G(t,s,p) \stackrel{\text{def}}{=} a^4 \sum_x \mathrm{e}^{-ip(x-y)} \left[\langle E(t,x)E(s,y) \rangle - \langle E(t,x) \rangle \langle E(s,y) \rangle \right]$$

is not tree-level $\mathrm{O}(a^2)$ improved for $p \neq 0$!

$\Rightarrow$ need to be more systematic, i.e. apply Symanzik's procedure!

Symanzik's effective theory I

- Systematic expansion of lattice correlation functions in terms of a renormzalized effective continuum theory;
- Terms are organized by increasing canonical dimensions

$\Rightarrow$ reformulate as $4+1$ dimensional local theory [Lüscher & Weisz]:

$$S[V,L] = S_{\rm lat}[U] + a^4 \int_0^\infty \mathrm{d}t\, a^4 \sum_x \mathrm{tr}\Big(L_\mu(t,x) \times \Big\{ \left(\partial_t V_\mu(t,x)\right) V_\mu(t,x)^\dagger + \partial_{x,\mu}\left(g_0^2 S_{\rm lat}[V]\right)\Big\}\Big)$$

- Flow quantities as expectation values of local fields [in coordinates (t,x)]:

$$\langle O[V,L]\rangle = Z^{-1}\int D[V]D[L]\; O[V,L]\exp\left(-S[V,L]\right)$$

- $D[V]$ includes integration over $t=0$ gauge field $V_\mu(0,x)\equiv U_\mu(x)$
- Note: $V_\mu(t,x)$ is not constrained to be a solution to the flow equation;
- The gradient flow equation is enforced by the integration over L.
- For observables $O = O[U]$: definition reduces to usual 4-d expectation values!

Symanzik's effective theory, effective action and observables:

$$\begin{aligned} S_{\text{eff}}[V,L] &= S_0^{\text{cont}}[B,L] + a^2 S_{2,v}[B,L] + a^2 S_{2,b}[B,L] + O(a^4) \\ O_{\text{eff}} &= O_0 + a^2 O_2 + \mathrm{O}(a^4) \end{aligned}$$

$S_{2,v}$, $S_{2,b}$: 4+1 dimensional volume action and $t=0$ boundary action.
Expansion of lattice expectation values:

$$\langle O \rangle_{\text{latt}} = \langle O_0 \rangle_{\text{cont}} + a^2 \langle O_2 \rangle_{\text{cont}} - a^2 \langle O_0 S_{2,v} \rangle_{\text{cont}} - a^2 \langle O_0 S_{2,b} \rangle_{\text{cont}} + \mathrm{O}(a^4)$$

Possible terms appearing in $\mathrm{O}(a^2)$ contributions:

- A priori any terms with the correct dimension which share all symmetries of the lattice theory;

  $$S_{2,v}[L,B] = \int_0^\infty dt \int d^4x \sum_i T_i(t,x), \qquad S_{2,b}[L,B] = \int d^4x \sum_i O_i(t,x)|_{t=0}$$

  with gauge invariant O_i, T_i of dimension 8 and 6 respectively, polynomial in B_μ and L_μ and derivatives ($[L_\mu]=3$,$[B_\mu]=1$)
- Similarly, O_2 contains local terms of dimension $[O_0]+2$ sharing all the symmetries of the lattice observable.

Symanzik's effective theory III

BUT: This is a rather special $4+1$ dimensional theory!

- only tree diagrams are generated in the bulk [Lüscher & Weisz, 2012]

$\Rightarrow$ counterterms for bulk action and observables in the 4+1 dim. volume are determined by <u>classical</u> expansion!

- Need to expand classically the lattice bulk action & flow observables (s. below)

$\Rightarrow$ Full Symanzik analysis only required for $S_{2,b}$.
List of dimension 6 terms (reduced by use of flow equation):

$$O_1 = \mathrm{tr}\{J_{\mu\nu\rho}J_{\mu\nu\rho}\} \qquad O_4 = \mathrm{tr}\{L_\mu(0,x)J_{\nu\mu\nu}\}$$
$$O_2 = \mathrm{tr}\{J_{\mu\mu\rho}J_{\nu\nu\rho}\} \qquad O_5 = \mathrm{tr}\{L_\mu(0,x)L_\mu(0,x)\}$$
$$O_3 = \mathrm{tr}\{J_{\mu\mu\rho}J_{\mu\mu\rho}\}$$

where $J_{\mu\nu\rho} = D_\mu F_{\nu\rho}$

Symanzik's effective theory IV

- the counterterms $O_{1,2,3}$ are already fixed in the standard 4-d theory

$\Rightarrow$ cannot be modified!

- The only new counterterms are $O_{4,5}$
- Find by explicit calculation (tree-level) and directly from functional integral (general):

$$\int d^4x \, \langle O_4(0,x) O[B] \rangle \propto \int d^4x \, \langle O_5(0,x) O[B] \rangle$$

for observables $O[B]$ defined at $t > 0$.

$\Rightarrow$ O_5 is redundant

- Effect of O_4 insertion: change of boundary conditions at $t = 0$, possible implementation:

$$V_\mu(t=0,x) = U_\mu(x) \exp\left(c_4 g_0^2 \partial_{x,\mu} S_g[U] \right)$$

In the continuum this corresponds to the addition of a term

$$\propto a^2 D_\nu F_{\nu\mu}(x)$$

to $B_\mu(t=0,x)$

Smooth underlying continuum gauge field $B_\mu(x)$ (t-dependence suppressed), link variables are induced:

$$
\begin{aligned}
V_\mu(x) &= \mathcal{P}\exp\left\{a\int_0^1 d\lambda B_\mu\left(x+(1-\lambda)a\hat{\mu}\right)\right\} \\
&= \mathbb{1} + a\int_0^1 d\lambda\; B_\mu\left(x+(1-\lambda)a\hat{\mu}\right) \\
&\quad + a^2\int_0^1 d\lambda_1\int_0^{\lambda_1} d\lambda_2\; B_\mu\left(x+(1-\lambda_1)a\hat{\mu}\right) B_\mu\left(x+(1-\lambda_2)a\hat{\mu}\right) + \ldots \\
&= \mathbb{1} + aB_\mu(x) + a^2\tfrac{1}{2}\left(\partial_\mu B_\mu(x) + B_\mu^2(x)\right) \\
&\quad + \tfrac{1}{6}a^3\left(\partial_\mu^2 B_\mu(x) + 2B_\mu(x)\partial_\mu B_\mu(x) + \left(\partial_\mu B_\mu(x)\right)B_\mu(x) + B_\mu^3(x)\right) + \ldots
\end{aligned}
$$

<u>Note:</u> gauge transformations in the continuum and on the lattice are compatible:

$$
B_\mu(x) \to g(x)B_\mu(x)g(x)^{-1} + g(x)\partial_\mu g(x)^{-1} \quad \Leftrightarrow \quad V_\mu(x) \to g(x)V_\mu(x)g(x+a\hat{\mu})^{-1}
$$

Expansion of small Wilson loops, plaquettes etc.

Plaquette field:

$$P_{\mu\nu}(x) \quad = \quad V_\mu(x)V_\nu(x+a\hat{\mu})V_\mu(x+a\hat{\nu})^\dagger V_\nu(x)^\dagger$$

Using the gauge fixing tricks by Lüscher & Weisz '85 one obtains relatively quickly:

$$\begin{aligned} P_{\mu\nu}(x) &= \mathbb{1} + a^2 G_{\mu\nu}(x) + a^3 \tfrac{1}{2}(D_\mu + D_\nu) G_{\mu\nu}(x) \\ &\quad + a^4 \tfrac{1}{6} \left\{ \left(D_\mu^2 + \tfrac{3}{2} D_\nu D_\mu + D_\nu^2 \right) G_{\mu\nu}(x) + 3 G_{\mu\nu}^2(x) \right\} + \mathrm{O}(a^5) \\ D_\mu &= \partial_\mu + [B_\mu, \cdot], \quad [D_\mu, D_\nu] = [G_{\mu\nu}, \cdot] \end{aligned}$$

similar expansions for rectangles, chairs, etc.

- Obtain $\mathrm{O}(a^2)$ improved lattice expressions for $E(t,x)$ either from LW action density or

$$E_{\mathrm{lat}}(t,x) \stackrel{\mathrm{def}}{=} \frac{4}{3} E(t,x)|_{\mathrm{plaq}} - \frac{1}{3} E(t,x)|_{\mathrm{clover}} = E_{\mathrm{cont}}(t,x) + \mathrm{O}(a^4)$$

  however, up to total derivative terms (often irrelevant, but not always!).

Classical expansion of flow equation

- Lattice flow equation:

$$a^2\left(\partial_t V_\mu(t,x)\right) V_\mu(t,x)^\dagger = -\partial_{x,\mu}\left(g_0^2 S_{\mathrm{lat}}[V]\right), \qquad V_\mu(0,x) = U_\mu(x)$$

- The O(a^2) term for the LW flow has a simple structure

$$\partial_t B_\mu = D_\nu G_{\nu\mu} - \tfrac{1}{12} a^2 D_\mu^2 D_\nu G_{\nu\mu} + O(a^3)$$

  as expected: not O(a^2) improved
- This suggests a simple modification of the lattice flow equation (Zeuthen flow):

$$a^2\left(\partial_t V_\mu(t,x)\right) V_\mu(t,x)^\dagger = -\left(1 + \tfrac{1}{12} a^2 \nabla_\mu^* \nabla_\mu\right) \partial_{x,\mu}\left(g_0^2 S_{\mathrm{lat}}[V]\right)$$

- This removes <u>all</u> O(a^2) effects from the flow equation, corrections are in fact O(a^4).

Conclusions & Outlook

In principle, $O(a^2)$ effects can be removed from all gradient flow observables:

- use a non-perturbatively $O(a^2)$ improved 4-dim. pure gauge action
- use of classically improved flow observables, e.g. choose $E(t,x)$ as density of LW action;
- impose an $O(a^2)$ modified $t = 0$ boundary condition on $V_\mu(t,x)$;
- integrate the modified lattice flow equation, e.g. the Zeuthen flow;

In practice...

- 4-d gauge action and initial boundary conditions determined perturbatively

$\Rightarrow$ residual $O(a^2)$ artefacts even in pure gauge theory.

- Still important to completely eliminate the relatively large $O(a^2)$ effects from the gradient flow and observables.

Future:

- implementation of the Zeuthen flow in openQCD (done)
- application to the QCD running coupling (started)
- quenched/unquenched scaling tests for t_0 (planned)