U(1) axial anomaly with chiral fermion at finite temperature

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Osaka University (Ph.D student)
→ ??? (Looking for a post)

(Related talk by G.Cossu today 17:00- )

Reference:
Full paper is in preparation:
PURPOSE:
We would like to judge the $U(1)_A$ symmetry is restored above the QCD critical temperature or not
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$U(1)_A$ sym. may be restored
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KEYWORD : Chiral symmetry
1. Introduction
Sym. of $N_f=2$ QCD ($m=0$)

\[ \mathcal{L} : SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A \]

\[ \int d^4x \bar{\psi}(D\psi) \psi \quad \psi(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix} \]

At zero temperature…

SU(2) chiral symmetry is spontaneously broken

$U(1)_A \text{ Sym. is violated by the anomaly}$
Anomaly = Sym. Violation by Quantum Eff.

Nf=2 massless QCD has chiral symmetries

\[ Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left[ - \int d^4x \, \bar{\psi}(i\slashed{D})\psi \right] \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix} \]
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Nf=2 massless QCD has chiral symmetries

\[ Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[ - \int d^4x \ \bar{\psi} (\mathcal{D}) \psi \right] \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix} \]

\[ \left\{ \begin{array}{c} \psi \rightarrow \psi' = e^{i\gamma_5 \theta} \psi \\ \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} e^{i\gamma_5 \theta} \end{array} \right\} \leftarrow U(1)_A \text{Trf(Chiral trf)} \\
\text{Rotate all R/L handed quark opposite dir.} \\
\gamma_5 \mathcal{D} = -\mathcal{D} \gamma_5 \]

Anomaly = Sym. Violation by Quantum Eff.

Nf=2 massless QCD has chiral symmetries

\[
Z = \int D A_\mu D \bar{\psi} D \psi \exp \left[ - \int d^4 x \bar{\psi}(\not{D})\psi \right] \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix}
\]

\[
\begin{align*}
\psi & \rightarrow \psi' = e^{i\gamma_5 \theta} \psi \\
\bar{\psi} & \rightarrow \bar{\psi}' = \bar{\psi} e^{i\gamma_5 \theta}
\end{align*}
\]

\[ \leftrightarrow \text{U}(1)_A \text{Trf(Chiral trf)} \]

Rotate all R/L handed quark opposite dir.

\[ \gamma_5 \not{D} = - \not{D} \gamma_5 \]

Action is inv., Measure is not invariant.

U(1)$_A$ Anomaly, Chiral anomaly

Anomaly = Sym. Violation by Quantum Eff.

Nf=2 massless QCD has chiral symmetries

\[ Z = \int D A_\mu D \bar{\psi} D \psi \exp \left[ - \int d^4 x \, \bar{\psi} (\not{D}) \psi \right] \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix} \]

\[ \psi \rightarrow \psi' = e^{i \gamma_5 \theta} \psi \]
\[ \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} e^{i \gamma_5 \theta} \]
\[ \text{←U(1)}_A \text{Trf}(\text{Chiral trf}) \]

Action is inv., Measure is not invariant. \( \text{U(1)}_A \) Anomaly, Chiral anomaly

\[ D \bar{\psi} D \psi = D \bar{\psi}' D \psi' e^\Gamma, \quad \Gamma \sim i \int d^4 x [\text{tr} \epsilon^{\mu \nu \alpha \beta} F_{\mu \nu} F_{\alpha \beta}] \]

Chiral symmetry breaking in QCD \((N_f=2, \ m_{ud}=0)\)

\[
T = 0 \quad SU(2)_L \times SU(2)_R \times U(1)_V \times \underbrace{U(1)_A}_{\text{Anomaly}}
\]

\[
\rightarrow \quad SU(2)_V \times U(1)_V \quad \text{Residual symmetry}
\]
Chiral symmetry breaking in QCD ($N_f=2$, $m_{ud}=0$)

\[ T = 0 \]

\[
\frac{SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A}{SSB} \rightarrow SU(2)_V \times U(1)_V \quad \text{Residual symmetry}
\]

\[ T > T_c \]

\[
SU(2)_V \rightarrow SU(2)_L \times SU(2)_R \quad \text{Restored}
\]

\[
U(1)_A \rightarrow ??
\]
Chiral symmetry breaking in QCD ($N_f=2$, $m_{ud}=0$)

\[ T = 0 \]

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\frac{SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A}{\text{SSB}} \rightarrow SU(2)_V \times U(1)_V \quad \text{Residual symmetry}
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\[ T > T_c \]

\[
SU(2)_V \rightarrow SU(2)_L \times SU(2)_R \quad \text{Restored}
\]

\[
U(1)_A \rightarrow ??
\]

What happens on the anomaly above the $T_c$?
U(1)$_A$ may be restored...

Two supporting evidences

1. Finite temperature = Theories on $L^3 \times (1/T)$
   $T=\infty \leftrightarrow$ Theory in $D=3$ (No anomaly)
   Anomaly disappears at infinite temperature
   (Anomaly could be disappeared at finite temperature?)

2. Cohen’s argument (1996, Next page)
   SU(2) chiral symmetry restoration may be related to $U(1)$_A restoration
Cohen: SU(2) & U(1)\textsubscript{A} may be restored

Fact: T>T\textsubscript{c}, when m→0, chiral symmetry is restored \iff \langle \bar{\psi}\psi \rangle = 0
Cohen: SU(2) & U(1)$_A$ may be restored (1996)

Fact: $T > T_c$, when $m \rightarrow 0$, chiral symmetry is restored $\iff \langle \overline{\psi} \psi \rangle = 0$

Cohen:
If we assume $\langle \overline{\psi} \psi \rangle = 0$, gap in the Dirac spectrum, it may lead $\chi_{U(1)}^A = 0$

Order parameter of SU(2) chiral sym.
Order parameter of U(1)$_A$ Sym. (Later)
※Additional condition is needed
Cohen: SU(2) & U(1)A may be restored
(1996)

Fact: T>Tc, when m→0, chiral symmetry is restored ⇔ ⟨\bar{\psi}\psi⟩ = 0

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If we assume ⟨\bar{\psi}\psi⟩ = 0, gap in the Dirac spectrum, it may lead

Order parameter
of SU(2) chiral sym.

Order parameter of
U(1)A Sym. (Later)

※Additional condition is needed

Spectral rep:

⟨\bar{\psi}\psi⟩ = \lim_{m→0} \int_0^∞ d\lambda \, \rho(\lambda) \frac{2m}{\lambda^2 + m^2}

lim_{m→0} \chi_{U(1)A} = \lim_{m→0} \int_0^∞ d\lambda \, \rho(\lambda) \frac{4m^2}{(\lambda^2 + m^2)^2}

Banks-Casher rel.
Cohen: SU(2) & U(1)\textsubscript{A} may be restored

Fact: T > T\textsubscript{c}, when m \to 0, chiral symmetry is restored \iff \langle \bar{\psi}\psi \rangle = 0

Cohen:

If we assume \langle \bar{\psi}\psi \rangle = 0, gap in the Dirac spectrum, it may lead \chi_{U(1)\textsubscript{A}} = 0

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\]

\[
\rightarrow \text{Let’s check this by Lattice QCD!}
\]
Previous studies (DW type) are controversial!

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<tr>
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What makes such difference?
Fermion, Volumes or Topology?
By the way

If $U(1)_A$ symmetry is restored?

Ref: http://personal.kent.edu/~mstrick6/
By the way

If $U(1)_A$ symmetry is restored?

We are interested in this point

Ref: http://personal.kent.edu/~mstrick6/
By the way

If $U(1)_A$ symmetry is restored?

![Diagram showing the crossover and target points](image-url)
By the way

If $U(1)_A$ symmetry is restored?

The order of the transition may be changed:
$2\text{nd} \rightarrow 1\text{st}$ (Pisarski & Wilczek 1983)
By the way

If U(1)A symmetry is restored?

The order of the transition may be changed:
2nd→1st (Pisarski&Wilczek 1983)
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1. Introduction

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5. Our setup

6. Our results

7. Summary
2. $U(1)_\text{A Sym.} \ & \ Correlators$
Sym. of QCD $\iff$ Degeneracy

\[ \langle \pi(x) \pi(0) \rangle \xrightleftharpoons{SU(2)_L \times SU(2)_R} \langle \sigma(x) \sigma(0) \rangle \]

\[ \langle \delta(x) \delta(0) \rangle \xrightleftharpoons{SU(2)_L \times SU(2)_R} \langle \eta(x) \eta(0) \rangle \]

\[
\begin{align*}
\pi(x) &= i \bar{\psi}(x) \gamma_5 \tau \psi(x) \\
\delta(x) &= \bar{\psi}(x) \tau \psi(x) \\
\sigma(x) &= \bar{\psi}(x) \psi(x) \\
\eta(x) &= i \psi(x) \gamma_5 \psi(x)
\end{align*}
\]

Degeneracy of these channels

$\iff$ There are symmetries
Sym. of QCD $\iff$ Degeneracy

\[ \langle \pi(x)\pi(0) \rangle \xrightarrow{SU(2)_L \times SU(2)_R} \langle \sigma(x)\sigma(0) \rangle \]

\[ \langle \delta(x)\delta(0) \rangle \xrightarrow{SU(2)_L \times SU(2)_R} \langle \eta(x)\eta(0) \rangle \]

\[ \chi_{U(1)_A} \equiv \int d^4x \left[ \langle \pi(x)\pi(0) \rangle - \langle \delta(x)\delta(0) \rangle \right] \]

Order parameter of $U(1)_A$

If this quantity 0 at $V \rightarrow \infty$, $m \rightarrow 0$, $U(1)_A$ symmetry is restored.
Let’s check Cohen’s argument

\[ \langle \bar{\psi} \psi \rangle = \lim_{m \to 0} \int_0^\infty d\lambda \, \rho(\lambda) \frac{2m}{\lambda^2 + m^2} \]

Order parameter of SU(2) Chiral symmetry

\[ \lim_{m \to 0} \chi_{U(1)_A} = \lim_{m \to 0} \int_0^\infty d\lambda \, \rho(\lambda) \frac{4m^2}{(\lambda^2 + m^2)^2} \]

Order parameter U(1)A symmetry

What happens \( T > T_c \)

Checking U(1)\(_A\) Sym. restoration at \( T > T_c \) ....

• Directly measure \( \chi_{U(1)_A} \)
• Measure \( \rho(\lambda) \) → Has a gap?
3. Dirac spectrum

\[ \langle \bar{\psi} \psi \rangle = \lim_{m \to 0} \int_{0}^{\infty} d\lambda \, \rho(\lambda) \frac{2m}{\lambda^2 + m^2} \]
\[ \rho(\lambda) : \text{Dirac spectrum} \]

= Spectral density of the Dirac operator

\[
(\gamma_5 D) \psi_j = \lambda_j \psi_j = \gamma^\mu (\partial_\mu + A_\mu)
\]

\[ \rho(\lambda) : \text{Distribution of } \lambda \]

→ This reflects symmetry of quarks with the gauge field!
Argument by Cohen (1996)

If there is a gap in the Dirac spectrum (and can be ignored exact zero-modes)

\[ \lim_{m \to 0} \int d^4 x \left[ \langle \pi(x) \pi(0) \rangle - \langle \delta(x) \delta(0) \rangle \right] = \lim_{m \to 0} \int_0^\infty d\lambda \frac{4m^2 \rho(\lambda)}{(m^2 + \lambda^2)^2} = 0 \]

Cf : Aoki-Fukaya-Taniguchi (2012)
Argument by Cohen (1996)

If there is a gap in the Dirac spectrum (and can be ignored exact zero-modes)

\[ \lim_{m \to 0} \int d^4x [\langle \pi(x) \pi(0) \rangle - \langle \delta(x) \delta(0) \rangle] = \lim_{m \to 0} \int_0^\infty d\lambda \frac{4m^2 \rho(\lambda)}{(m^2 + \lambda^2)^2} = 0 \]

Cf: Aoki-Fukaya-Taniguchi (2012)

\( \lambda \sim 0 \) modes are important
4. Previous studies
### Previous studies (DW type) are controversial!

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What makes such difference?  
**Fermion, Volume or Topology?**
Chiral symmetry on the Lat. = Ginsparg-Wilson rel.
Chiral symmetry on the Lat. = Ginsparg-Wilson rel.

\[ S = \int d^4x \bar{\psi} \gamma_5 \psi \text{ is chiral symmetric } \iff \]

Cont. : \[ \gamma_5 \gamma_5 \gamma_5 \gamma_5 = 0 \]
Chiral symmetry on the Lat. = Ginsparg-Wilson rel.

\[ S = \int d^4x \bar{\psi} D \psi \] is chiral symmetric \(\iff\)

Cont. : \[ D \gamma_5 + \gamma_5 D = 0 \]

Lat. : \[ D \gamma_5 + \gamma_5 D = 2a D \gamma_5 D \]

Ginsparg-Wilson relation

\(a\): lattice spacing
Chiral sym. on the Lat.

Lat. : \[ D \gamma_5 + \gamma_5 D = 2aD \gamma_5 D \]

Ginsparg-Wilson relation

\( a \): lattice spacing

D (satisfies above) : 2 good things

1. Action has exact chiral symmetry

\[
\begin{align*}
\psi & \rightarrow \psi' = e^{i\gamma_5(1-aD)\theta} \psi \\
\bar{\psi} & \rightarrow \bar{\psi}' = \bar{\psi} e^{i\gamma_5 \theta}
\end{align*}
\]

\( \rightarrow \) The action has full SU(2) and U(1)\(_A\) symmetries

2. U(1)\(_A\) sym. is violated by the quantization as same as the continuum theory.
The overlap fermion satisfies the Ginsparg-Wilson relation exactly!

\[ D \gamma_5 + \gamma_5 D = 2aD \gamma_5 D \]

\[ D_{ov} = \frac{1 + m}{2} - \frac{1 - m}{2} \gamma_5 \text{sgn}(H_T) \]

H\(_T\) is a hermitian Dirac operator

○ Exact chiral symmetry on the lattice

× Bad for the numerical simulation because of the sign function (needs special care).
Previous result by JLQCD(2013)

Simulation with the overlap (Exactly chiral)
Volume : L=2 fm & fixing topology
Finite temperature simulation!
→ As a result, U(1) is restored above the Tc

Objections: Do finite volume affect near zero modes ?
, Does topology-fixing change the physics?

In this work, we change our set-up and Check U(1) restoration
By the way

**Ideal simulation?**

Ideal simulation

- Overlap action:
  Large volume,
  several volume

- Without topology
  fixing term
By the way

**Ideal simulation?**

### Ideal simulation

- Overlap action: Large volume, several volume
- Without topology fixing term

### Our set-up

- Overlap like action → reweighting to OV
- Large volume, several volume
- Without topology fixing term
5. Our set-up
Mobius Domain-wall fermion

= Better approximation of the overlap fermion

The overlap (Exactly chiral, used in JLQCD2013):

\[ D_{ov} = \frac{1 + m}{2} - \frac{1 - m}{2} \gamma_5 \text{sgn}(H_T) \]

\( H_T \) is a hermitian Dirac op.

Domain-wall fermion (used in RBC/LLNL)
1 digit precision chiral symmetry

\[ \tanh \left[ L_s \tanh^{-1}(H_T) \right] \]
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1 digit precision chiral symmetry

\[
\tanh \left[ L_s \tanh^{-1} (H_T) \right]
\]

\[
\tanh \left[ L_s \tanh^{-1} (2H_T) \right]
\]
Mobius Domain-wall fermion

= Better approximation of the overlap fermion

The overlap (Exactly chiral, used in JLQCD2013):

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Domain-wall fermion (used in RBC/LLNL)
1 digit precision chiral symmetry

\[ \tanh \left[ L_s \tanh^{-1}(H_T) \right] \]

Mobius Domain-wall fermion
(This work)
3 digit chiral symmetry

\[ \tanh \left[ L_s \tanh^{-1}(2H_T) \right] \]

→ Better approximation of the OV
(Still it violates the Ginsparg-Wilson rel.)
Lattice set up

Gauge action: tree level Symanzik
Fermion: Mobius DW (b=2, c=1, Scaled Shamir + Tanh)
with Stout smearing (3)

code: lrolro++ (G. Cossu et al.)

Resource: BG/Q (KEK)

<table>
<thead>
<tr>
<th>$L^3 \times L_t$</th>
<th>$\beta$</th>
<th>$m_{ud}$ (MeV)</th>
<th>$L_s$</th>
<th>$m_{res}$ (MeV)</th>
<th>Temp. (MeV)</th>
</tr>
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<tr>
<td>$16^3 \times 8$</td>
<td>4.07</td>
<td>30</td>
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Table 1: Our lattice set-up. Those with $m_{ud}^*$ are obtained by the stochastic reweighting of the Dirac operator determinant from the ensemble with the higher quark mass. Residual mass with ** is estimated by weighted average of $g_i$ with some threshold.

$m_{res}$: Scale of violation of Ginsparg-Wilson relation
6. Our results
9 Discussion

The eigenvalues by 100%, and may explain the fermion (star symbols). This result indicates that the low modes of the Möbius domain-wall mass by an weighted average of the Dirac operator.

Note that one can obtain the residual mass through Dirac operator through the normalization of Dirac operator.

We measure the violation of the Ginsparg-Wilson relation on each eigenmode of the Hermitian Dirac operators. The data for various lattices are presented. Figure 4: The eigenvalue histograms of the domain wall (left panels) and reweighted overlap (right) Dirac operators. The data for figure 5: The eigenvalue histograms of the domain wall (left panels) and reweighted overlap (right) Dirac operators. The data for (bottom) lattices are presented.

Figure 7: The violation of Ginsparg-Wilson relation (right) Dirac operators. The data for figure 4, 5, 6, 7 are presented.

The violation is of course negligible for the overlap. The violation is of course negligible for the overlap.

→$U_A(1)$ looks violated?
3. Histogram for DW

No clear gap in the domain-wall spectrum in large volume system $\rightarrow U(1)_A$ looks violated.

Same result as the previous study by LLNL/RBC 2013
What was wrong with previous JLQCD?
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Possible causes
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previous JLQCD : Finite volume effect?
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**Same result as the previous study by LLNL/RBC 2013**

**What was wrong with previous JLQCD?**

**Possible causes**

- previous JLQCD: Finite volume effect?
- previous JLQCD: Topology fixing changes physics?
No clear gap in the domain-wall spectrum in large volume system \( \rightarrow \text{U(1)}_A \text{ looks violated.} \)

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Possible causes

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- previous JLQCD : Topology fixing changes physics?
- Ginsparg-Wilson violation in DW ?
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What was wrong with previous JLQCD?

Possible causes

- previous JLQCD: Finite volume effect?
- previous JLQCD: Topology fixing changes physics?
- Ginsparg-Wilson violation in DW?
Ginsparg-Wilson violation for each mode

\[ S = \int d^4 x \bar{\psi} D \psi \text{ is chiral symmetric } \Longleftrightarrow \]

Lattice: \[ D \gamma_5 + \gamma_5 D = 2aD \gamma_5 D \]

Eigen-function of \(D\) ↓

\[ g_i \propto \psi_i^\dagger \gamma_5 \left[D \gamma_5 + \gamma_5 D - 2aD \gamma_5 D\right] \psi_i \]

\(g_i = 0\) for the chiral fermion

What happens on the Mobius domain-wall?
Ginsparg-Wilson relation is violated even for improved domain-wall fermion

Near zero modes important for the issue...
Observation

Eigenmodes of improved domain-wall: Ginsparg-Wilson is violated
Observation

Eigenmodes of improved domain-wall: Ginsparg-Wilson is violated

Using reweighting technique, we switch the fermion determinant to the OV one (toward the ideal simulation...
Re-weighting tech. enables us to change another fermion determinant

\[ \langle \mathcal{O}\rangle_{\text{Overlap}} \propto \int D\bar{\psi}D\psi DA_\mu \mathcal{O} e^{-S_{\text{gauge}}} e^{-\bar{\psi}[D_{\text{OV}}] \psi} \]

\[ = \int DA_\mu \mathcal{O} e^{-S_{\text{gauge}}} \text{Det}[D_{\text{OV}}^2] \]

\[ = \int DA_\mu \mathcal{O} e^{-S_{\text{gauge}}} \text{Det}[D_{\text{OV}}^2] \frac{\text{Det}[D_{\text{DW}}^2]}{\text{Det}[D_{\text{DW}}^2]} \]

\[ = \int D\bar{\psi}D\psi DA_\mu \mathcal{O} R e^{-S_{\text{gauge}}} e^{-\bar{\psi}[D_{\text{DW}}] \psi} \]

\[ \propto \langle \mathcal{O} \mathcal{R} \rangle_{\text{Domain Wall}} \]

\[ R = \frac{\text{Det}[D_{\text{OV}}^2]}{\text{Det}[D_{\text{DW}}^2]} \]

Multiplying R and taking average, we obtain the result with the overlap determinant.
Reweighting suppress near zero-modes!

Spectral density for the overlap Dirac operator with different fermion determinant (OV or DW)

Red: Partially quenched = w/ det of DW
Blue: Reweighted = w/ det of OV

**DW/OV Dirac spectrum has different shape**

**Domain-wall**

L=16 Domain-wall Histogram(β=4.07) T=184 MeV

L=32 Domain-wall Histogram(β=4.07) T=184 MeV

**Overlap**

T~Tc

L=16 Overlap Histogram(β=4.07) T=184 MeV

L=32 Overlap Histogram(β=4.07) T=184 MeV

The overlap spectrums have gaps on the spectrum(λ ~20MeV)
7. Summary

1. 

1.1. Thanks to the reweighting, we can perform the simulation with the overlap in large volume without topology fixing

1.2. We find, the ov and DW spectrum have different shape

1.3. We find Ginsparg-Wilson violation for DW each eigenmodes

1.4. We find gaps in the spectrum for OV: T=180 MeV, T=200 MeV, 2 fm and 4 fm.

2. We are going to,

2.1. find a gap in the spectrum for finer lattice

2.2. do quantitative evaluation for the gap (Check vol. scaling)

2.3. check consistency with the measurement for $\chi^{U(1)}$

Reference: