Spinning Black Hole Binary Spacetime via Asymptotic Matching

Hiroyuki Nakano (A05)

Department of Physics, Kyoto University

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Intro.: Binary black holes (BBHs) and Motivation


- Want GRMHD simulations for long time evolution.
- Let’s consider an analytic spacetime!
We use **various approximations** in the analytical treatment.

<table>
<thead>
<tr>
<th>Zone</th>
<th>Region of Validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>IZ1</td>
<td>$0 &lt; r_1 \ll r_{12}$</td>
</tr>
<tr>
<td>IZ2</td>
<td>$0 &lt; r_2 \ll r_{12}$</td>
</tr>
<tr>
<td>NZ</td>
<td>$m_A \ll r_A \ll \lambda$</td>
</tr>
<tr>
<td>FZ</td>
<td>$r_{12} \ll r &lt; \infty$</td>
</tr>
<tr>
<td>IZ-NZ BZ</td>
<td>$m_A \ll r_A \ll r_{12}$</td>
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<td>NZ-FZ BZ</td>
<td>$r_{12} \ll r \ll \lambda$</td>
</tr>
</tbody>
</table>

$m_A$: Ath BH’s mass  
$r_{12}$: Orbital separation  
$r_A$: Distance from BH with $m_A$  
$r$: Distance from the COM  
$\lambda$: GW wavelength
Intro.: IZ/NZ/FZ metrics

- Inner zone (IZ)
  - Black hole perturbation
- Near zone (NZ)
  - Post-Newtonian ($1/c$)
- Far zone (FZ)
  - Post-Minkowskian ($G$)
- Buffer zone (BZ)
  - Smooth transition

The original work was done for the initial data (Non-spinning case).

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Non-precessing, spinning black hole binaries for long time evolution

For short time evolution (a previous work):

— Gallouin et al., CQG 29, 235013 (2012).

We do not discuss the details of the FZ metric, because there is no restriction of time evolution

- in the metric itself.
- in the matching between the NZ and FZ metrics.

The FZ metric is derived in harmonic coordinates.

Near Zone (cont’d)

PN approach

\( n \)th PN order: Correction of \( \mathcal{O}(v^{2n}/c^{2n}) \) for the leading order

NZ metric

\[
g_{\mu\nu}^{\text{NZ}} = \eta_{\mu\nu} + h_{\mu\nu}^{\text{NZ}},
\]

\( \eta_{\mu\nu} \): Minkowski metric, \( h_{\mu\nu}^{\text{NZ}} \): PN metric perturbation

\[
g_{00}^{\text{NZ}} + 1 = \frac{2m_1}{r_1} + \frac{m_1}{r_1} \left[ 4v_1^2 - (n_1 \cdot v_1)^2 \right] - 2 \frac{m_2}{r_1^2}
\]

\[
- m_1m_2 \left[ \frac{2}{r_1r_2} + \frac{r_1}{2r_12} - \frac{r_1^2}{2r_2r_12} + \frac{5}{2r_1r_2} \right]
\]

\[
+ \frac{4m_1m_2}{3r_12} (n_{12} \cdot v_{12}) + \frac{4}{r_1^2} \epsilon_{ijk} v_1^i s_1^j n_1^k + (1 \leftrightarrow 2) + \mathcal{O}(v^6),
\]

\[
g_{0i}^{\text{NZ}} = - \frac{4m_1}{r_1} v_1^i - \frac{2}{r_1^2} \epsilon_{ijk} s_1^i n_1^k + (1 \leftrightarrow 2) + \mathcal{O}(v^5),
\]

\[
g_{ij}^{\text{NZ}} - \delta_{ij} = \frac{2m_1}{r_1} \delta_{ij} + (1 \leftrightarrow 2) + \mathcal{O}(v^4),
\]
Near Zone (cont’d)

\( m_A \): Mass, \( s^i_A \): Spin angular momentum, 
\( y^i_A \): Location, \( v^i_A \): Velocity

Notation: 
\( r_{12} = |y_1 - y_2|, \quad n_{12} = (y_1 - y_2)/r_{12}, \)
\( v_{12} = v_1 - v_2, \quad r_A = |x - y_A|, \quad n^i_A = (x^i - y^i_A)/r_A. \)

In harmonic coordinates


Quasi-circular orbits

Ignoring the radial decay, the orbit can be considered as
\( r_{12} = r_{12} \{\cos \omega t, \sin \omega t, 0\} \).

Orbital angular frequency:
\[
\omega = \sqrt{\frac{m}{r_{12}^3}} \left[ 1 + \frac{m}{2r_{12}} \left( \frac{m_1 m_2}{m^2} - 3 \right) + \mathcal{O}(v^4) \right].
\]
Black hole perturbation

Kerr solution (mass: \( M \), Kerr parameter: \( A \)) + perturbation \( h_{\mu\nu}^{IZ} \)

\[
g^{IZ}_{\mu\nu} = g^{Kerr}_{\mu\nu} + h_{\mu\nu}^{IZ}.
\]


Boyer-Lindquist (BL) coordinates \( \{t_{BL}, r_{BL}, \theta_{BL}, \phi_{BL}\} \)?

We want to use a similar coordinates to the NZ one, and want to have the property of horizon-penetrating.

Harmonic coordinates \((\Box x^{\mu} = 0) + \) horizon-penetrating

\( \rightarrow \) Cook-Scheel harmonic coordinates \( \{T, X, Y, Z\} \)

\[
T = t_{BL} + \frac{r_+^2 + A^2}{r_+ - r_-} \ln \left| \frac{r_{BL} - r_+}{r_{BL} - r_-} \right| , \quad Z = (r_{BL} - M) \cos \theta_{BL},
\]

\[
X + i Y = (r_{BL} - M + i A)e^{i \phi_{IK}} \sin \theta_{BL} ; \quad \phi_{IK} = \phi_{BL} + \frac{A}{r_+ - r_-} \ln \left| \frac{r_{BL} - r_+}{r_{BL} - r_-} \right| .
\]

—– Cook and Scheel, PRD 56, 4775 (1997).
Strategy to calculate $h^\text{IZ}_{\mu\nu}$

- Find an appropriate solution for the Newman-Penrose scalar

$$\psi_0 = \sum_{\ell, m} R_\ell m(r) z_\ell m(t) Y^\ell m(\theta, \phi),$$

which satisfies the Teukolsky equation.


- We need to prepare a so-called Hertz potential $\Psi$ which is calculated from the Newman-Penrose scalar $\psi_0$.

- Then, the vacuum perturbation on the Kerr background is obtained from $\Psi$ via the Chrzanowski procedure as

$$h^\text{IZ}_{\mu\nu} = \hat{h}_{\mu\nu} [\Psi],$$

where $\hat{h}_{\mu\nu} [\cdot]$ is a differential operator.

—– Chrzanowski, PRD 11 2042 (1975).
Asymptotic IZ-NZ matching

\[
g^{NZ}_{\mu\nu} = \frac{\partial x^\alpha_{IZ}}{\partial x^\mu_{NZ}} \frac{\partial x^\beta_{IZ}}{\partial x^\nu_{NZ}} g_{\alpha\beta}^{IZ}
\]

We carry out \textbf{asymptotic matching} order by order with respect to \((m_2/r_{12})^{1/2}\).

- The IZ metric is described by the harmonic, horizon-penetrating coordinates \(X^\alpha\) and the parameters \(\Lambda^\alpha = (M, A, z_R, m, z_I, m)\).

- The NZ metric is written with the parameters \(\lambda^\alpha = (m_1, m_2, r_{12}, s_1^i, s_2^i)\) in the harmonic PN coordinates \(x^\alpha\).

In our previous work, we discussed asymptotic matching for \(t - t_0 \ll r_{12}\).

--- Gallouin \textit{et al.}, CQG 29, 235013 (2012).

For long time evolution of BBH systems, we remove the assumption, \(t \ll r_{12}\) where we set \(t_0 = 0\) here.
Explicit expressions for the coordinate transformation

Using

- the orbital phase evolution $\omega t = \phi = \phi(t)$
- the evolution of the orbital separation $r_{12} = r_{12}(t)$

in the PN approach ($\tilde{x}^\alpha = \{t, \tilde{x}, \tilde{y}, z\}$, $\tilde{r}_1 = \sqrt{\tilde{x}^i \tilde{x}_i}$),

\[
T = t - \sqrt{\frac{m_2}{r_{12}}} \sqrt{\frac{m_2}{m}} \tilde{y}_C + \frac{m_2}{r_{12}} \left( \frac{1}{3} \frac{\tilde{r}_1^3 - 3 \tilde{x}_C^2 \tilde{r}_1}{r_{12}^2} - \left(1 + \frac{1}{2} \frac{m_2}{m}\right) t \right)
\]

\[
= t - \sqrt{\frac{m_2}{r_{12}}} \sqrt{\frac{m_2}{m}} \tilde{y}_C + \frac{m_2}{r_{12}} \left( \frac{1}{3} \frac{\tilde{r}_1^3 - 3 \tilde{x}_C^2 \tilde{r}_1}{r_{12}^2} \right) + \frac{5}{384} \frac{(2m + m_2)(r_{12}^3 - r_{12}(0)^3)}{m^2 m_1},
\]

\[
X = \tilde{x} + \frac{m_2}{r_{12}} \left[ - \frac{\tilde{r}_1^2 \tilde{x}_C \cos \phi - \tilde{x}_C^2 \tilde{x}}{r_{12}^2} + \tilde{x} \left(1 - \frac{\tilde{x}_C}{r_{12}}\right) + \frac{1}{2} \frac{\tilde{r}_1^2 \cos \phi}{r_{12}} - \frac{1}{2} \frac{m_2 \tilde{y}_C \sin \phi}{m} \right],
\]

\[
Y = \tilde{y} + \frac{m_2}{r_{12}} \left[ - \frac{\tilde{r}_1^2 \tilde{x}_C \sin \phi - \tilde{x}_C^2 \tilde{y}}{r_{12}^2} + \tilde{y} \left(1 - \frac{\tilde{x}_C}{r_{12}}\right) + \frac{1}{2} \frac{\tilde{r}_1^2 \sin \phi}{r_{12}} + \frac{1}{2} \frac{m_2 \tilde{y}_C \cos \phi}{m} \right],
\]

\[
Z = z + \frac{m_2}{r_{12}} \left( \frac{\tilde{x}_C^2 z}{r_{12}^2} + z \left(1 - \frac{\tilde{x}_C}{r_{12}}\right) \right).
\]
We need to combine the IZ, NZ and FZ metrics smoothly.

**Transition functions:**

\[ f(r, r_0, w, q, s) := \begin{cases} 
0, & r \leq r_0, \\
\frac{1}{2} \{1 + \tanh[(s/\pi)\chi(r, r_0, w) - q^2/\chi(r, r_0, w)]\}, & r_0 < r < r_0 + w, \\
1, & r \geq r_0 + w, 
\end{cases} \]

\[ \chi(r, r_0, w) := \tan[\pi(r - r_0)/2w]. \]

**Full global metric:**

\[ g_{\alpha\beta} = \{1 - f_{\text{far}}(r)\}\{f_{\text{near}}(x)[f_{\text{inner},1}(r_1)g^{(3)}_{\alpha\beta} + \{1 - f_{\text{inner},1}(r_1)\}g^{(1)}_{\alpha\beta}] \\
+ [1 - f_{\text{near}}(x)][f_{\text{inner},2}(r_2)g^{(3)}_{\alpha\beta} + \{1 - f_{\text{inner},2}(r_2)\}g^{(2)}_{\alpha\beta}] \} + f_{\text{far}}(r)g^{(4)}_{\alpha\beta}. \]
Numerical study: Non-precessing, spinning case

— Gallouin et al., CQG 29, 235013 (2012).

— But, for short time evolution.

- The deviation from $R = 0$ means the error of approximation.

$R$ calculated from the global metric for $m_1 = m_2 = m/2$. Spin: $(\chi_1, \chi_2) = (0.9, 0.9)$, Orbital separation: $b = 10m$ and $20m$. 
Whether the contribution of $R \neq 0$ is small?

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - (g_{\alpha[\gamma R_{\delta}\beta} - g_{\beta[\gamma R_{\delta}\alpha}) + \frac{1}{3} R g_{\alpha[\gamma g_{\delta}\beta}$$

$$R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} + 2R_{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2.$$
Numerical study: Non-spinning case

--- Mundim et al., PRD 89, 084008 (2014).

\( m_1 = m_2 = M/2 \)

**Left:** 1st and 2nd order matching.

**Right:** \(|R|\) vs. the orbital separations \( r_{12} \).
Discussion

On the asymptotically matched spacetime for BBH,

- GRMHD simulations
- Ray-tracing
- Visualization

Next step:

- What is for precessing BBHs?
  complicated motion, spin directions, matching ...