Biorthonormal Eigenbasis of a Markovian Master Equation for the Quantum Brownian Motion

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Introduction

Quantum harmonic oscillator interacting with thermal field.

Markovian master equation:

- Effect of environment on the oscillator.
- Used in quantum optics, quantum information, decoherence, etc.

Outline:

1. Non-hermitian eigenvalue problem–biorthogonal eigenbasis in space coordinate, properties of solution.
2. Application to simple systems.

System

Oscillator in thermal reservoir ($\hbar = c = 1$)

$$H = \omega_0 a^\dagger a + \sum_k \omega_k a_k^\dagger a_k + \lambda \sum_k V_k (a^\dagger a_k + a a_k^\dagger)$$

Weak coupling, trace over the field

$$i \frac{\partial}{\partial \tau} \hat{\rho}(\tau) = C \hat{\rho}(\tau)$$

Collision operator $\quad C = C_0 + C_d$

Free oscillator

$$C_0 \hat{\rho} = \bar{\omega}_0 [a^\dagger a, \hat{\rho}\,]$$

Non-hermitian

$$\begin{aligned} C_d \hat{\rho} = {} & \tfrac{i}{2}(b + \tfrac{1}{2})(2a\hat{\rho}a^\dagger - a^\dagger a \hat{\rho} - \hat{\rho} a^\dagger a) \\ & + \tfrac{i}{2}(b - \tfrac{1}{2})(2a^\dagger \hat{\rho} a - a a^\dagger \hat{\rho} - \hat{\rho} a a^\dagger) \end{aligned}$$

Dimensionless variables

$$\tau \equiv \gamma t, \qquad \bar{\omega}_0 \equiv \omega_0/\gamma, \qquad x = \sqrt{M\omega_0}\, q, \qquad b = \tfrac{1}{2} + \bar{n} = \tfrac{1}{2}\coth\left(\tfrac{1}{2}\omega_0 \beta\right)$$

Non-Hermitian eigenvalue problem

Existing solution:

- Classical system for anharmonic solids (Prigogine 1962).
- Initial value problem (Agarwal 1973), damping basis (Briegel et.al. 1993), finite level system Kraus operator (Nakazato et.al. 2006), numerical solution with Gaussian ansatz.

Non-Hermitian eigenvalue problem in coordinate space.

Right eigenvector	Left eigenvector	Biorthonormality
$C\|f_\nu\rangle\!\rangle = z_\nu \|f_\nu\rangle\!\rangle$	$\langle\!\langle g_\nu\|C = \langle\!\langle g_\nu\|z_\nu$	$\langle\!\langle g_\mu\|f_\nu\rangle\!\rangle = \delta_{\mu\nu}$
	$\Leftrightarrow C^\dagger\|g_\nu\rangle\!\rangle = z_\nu^*\|g_\nu\rangle\!\rangle$	$\sum_\nu \|f_\nu\rangle\!\rangle\langle\!\langle g_\nu\| = I$

Time evolution $$|\rho(t)\rangle\!\rangle = \sum_\nu e^{-iz_\nu\tau}|f_\nu\rangle\!\rangle\langle\!\langle g_\nu|\rho(0)\rangle\!\rangle$$

Separable coordinates

Bring the equation into separable form.

(1) Center coordinates

$$Q \equiv (x+\tilde{x})/2$$
$$r \equiv x - \tilde{x}$$

$$C = \frac{\bar{\omega}_0}{2}\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \tilde{x}^2} + x^2 - \tilde{x}^2\right)$$
$$+ i\frac{1}{4}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \tilde{x}}\right)(x+\tilde{x}) - (x-\tilde{x})\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \tilde{x}}\right)\right]$$
$$+ i\frac{b}{2}\left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \tilde{x}}\right)^2 - (x-\tilde{x})^2\right]$$

(2) Wigner representation

$$\rho^W(Q,P)$$
$$\equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dr\, e^{-iPr}\rho(Q,r)$$

$$C = -i\bar{\omega}_0\left(P\frac{\partial}{\partial Q} - Q\frac{\partial}{\partial P}\right)$$
$$+ i\frac{1}{2}\left(\frac{\partial}{\partial Q}Q + \frac{\partial}{\partial P}P\right) + i\frac{b}{2}\left(\frac{\partial^2}{\partial Q^2} + \frac{\partial^2}{\partial P^2}\right)$$

(3) “Action-angle" variables

$$\bar{J} \equiv (P^2+Q^2)/2b$$
$$\tan\alpha \equiv Q/P$$

$$\boxed{C = -i\bar{\omega}_0\frac{\partial}{\partial \alpha} + i\left(\frac{\partial}{\partial \bar{J}}\bar{J}\frac{\partial}{\partial \bar{J}} + \frac{\partial}{\partial \bar{J}}\bar{J} + \frac{1}{4\bar{J}}\frac{\partial^2}{\partial \alpha^2}\right)}$$

Right eigenfunctions

Right eigenvalue problem

$$C F_{mn}^{\pm} = z_{mn}^{\pm} F_{mn}^{\pm}$$

Right eigenfunction

$$F_{mn}^{\pm}(\bar{J}, \alpha) = N\, e^{\pm i n \alpha}\, \bar{J}^{\frac{n}{2}}\, L_m^n(\bar{J}) e^{-\bar{J}}, \qquad m \geq n \geq 0$$

Eigenvalue

$$z_{mn}^{\pm} = \pm\, n \bar{\omega}_0 - i(m - n/2)$$

Equilibrium function

$$F_{\text{eq}} = F_{00}(\bar{J}, \alpha) = \frac{1}{\sqrt{2\pi b}}\, e^{-\bar{J}}$$

Left eigenfunctions

Left eigenvalue problem $C^\dagger G^{\pm *}_{mn} = z^{\pm *}_{mn} G^{\pm *}_{mn}$

Introducing weight $\langle\!\langle A|B\rangle\!\rangle_{\mathsf{w}} \equiv \int_0^\infty d\bar{J} \int_0^{2\pi} d\alpha\, e^{\bar{J}} A^*(\bar{J},\alpha) B(\bar{J},\alpha), \qquad e^{\bar{J}} \sim 1/F_{\mathsf{eq}}$

Adjoint operator $C^\dagger_{\mathsf{w}} = -i\bar{\omega}_0 \frac{\partial}{\partial\alpha} - i\left(\frac{\partial}{\partial\bar{J}}\bar{J}\frac{\partial}{\partial\bar{J}} + \frac{\partial}{\partial\bar{J}}\bar{J} + \frac{1}{4\bar{J}}\frac{\partial^2}{\partial\alpha^2}\right)$

$$\Rightarrow \quad \boxed{(G_{\mathsf{w}})^{\pm}_{mn} = F^{\pm}_{mn}}$$

Orthogonal $\int_0^\infty dJ\, e^{J/b} \int_0^{2\pi} d\alpha\, F^{\sigma *}_{mn}\, F^{\sigma'}_{m'n'} = \delta_{mm'}\delta_{nn'}\delta_{\sigma\sigma'}$

Complete $\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{\pm} F^{\pm *}_{mn}(\bar{J},\alpha) F^{\pm}_{mn}(\bar{J}',\alpha') = e^{-\bar{J}'}\delta(\alpha-\alpha')\delta(J-J')$

Lyapunov function
$$\langle\!\langle \rho(\tau)|\rho(\tau)\rangle\!\rangle_{\mathsf{w}} = \sum_{m,\,n,\,\pm} \langle\!\langle \rho(0)|e^{iC^\dagger_{\mathsf{w}}\tau}|f^{\pm}_{mn}\rangle\!\rangle_{\mathsf{w}} \langle\!\langle f^{\pm}_{mn}|e^{iC^\dagger_{\mathsf{w}}\tau}|\rho(0)\rangle\!\rangle_{\mathsf{w}}$$
$$= \sum_{m,\,n,\,\pm} e^{-(2m-n)\tau} |\langle\!\langle f^{\pm}_{mn}|\rho(0)\rangle\!\rangle_{\mathsf{w}}|^2$$

Eigenfunctions in position coordinates

Weighted scalar product $$\langle\langle g_{\mathrm{w}}|f\rangle\rangle_{\mathrm{w}} = \int_{-\infty}^{\infty} dPdQ\, e^{(P^2+Q^2)/2} G_{\mathrm{w}}(P,Q)F(P,Q)$$

Fourier transform $$f(Q,r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iPr} F(P,Q), \qquad g_{\mathrm{w}}(Q,r) = \cdots$$

Measure $$dPdQ\, e^{(P^2+Q^2)/2} \quad \longrightarrow \quad ?$$

Un-weighted scalar product $$\begin{aligned}\langle\langle g|f\rangle\rangle &= \int_{-\infty}^{\infty} dPdQ\, G(P,Q)F(P,Q) \\ &= \int_{-\infty}^{\infty} drdQ\, g^*(Q,r)f(Q,r)\end{aligned}$$

Left eigenvalue problem $$C^\dagger G^{\pm *}_{mn} = z^{\pm *}_{mn} G^{\pm *}_{mn}$$

Adjoint operator $$C^\dagger = -i\bar{\omega}_0 \frac{\partial}{\partial\alpha} - i\left(\frac{\partial}{\partial \bar{J}}\bar{J}\frac{\partial}{\partial \bar{J}} - \bar{J}\frac{\partial}{\partial \bar{J}} + \frac{1}{4\bar{J}}\frac{\partial^2}{\partial\alpha^2}\right)$$

Left eigenfunction: $$\boxed{G^{\pm}_{mn} = e^{\bar{J}} F^{\pm}_{mn}} \neq F^{\pm}_{mn}$$

Eigenfunctions in space coordinates

Right eigenfunction

$$f_{mn}^{\pm} = \frac{e^{-Q^2/2b - br^2/2}}{\sqrt{2\pi b}} \sum_{\mu=0}^{m-n} \sum_{\nu=0}^{\mu} \sum_{\sigma=0}^{n} c_{mn}^{\pm\mu\nu\sigma} \left(\frac{Q}{\sqrt{2b}}\right)^{2(\mu-\nu)+n-\sigma} H_{2\nu+\sigma}\left(\sqrt{b/2}\, r\right)$$

Left eigenfunction

$$g_{mn}^{\pm} = \sum_{\mu=0}^{m-n} \sum_{\nu=0}^{\mu} \sum_{\sigma=0}^{n} d_{mn}^{\pm\mu\nu\sigma} \left(\frac{Q}{\sqrt{2b}}\right)^{2(\mu-\nu)+n-\sigma} \left(\frac{\partial}{\partial r}\right)^{2\nu+\sigma} \delta\left(\sqrt{b/2}\, r\right)$$

Biorthonormal basis

$$\int_{-\infty}^{\infty} dQdr\, g_{\mu}^{*} f_{\nu} = \delta_{\mu\nu}, \qquad \sum_{\nu} g_{\nu}^{*}(Q,r) f_{\nu}(Q',r') = \delta(Q-Q')\delta(r-r')$$

Density functions are expanded in $|f_\nu\rangle\!\rangle$

$$|\Psi\rangle\!\rangle = \sum_{\nu} |f_\nu\rangle\!\rangle \langle\!\langle g_\nu|\Psi\rangle\!\rangle$$

$\langle\!\langle g_\nu|$ as dual vectors serve as projector

$g_{00} = \delta(r)$ projects out the probability component (diagonal component $x = \tilde{x}$).

Symmetry of eigenfunctions

Eigenvalue independent of b

$$C(b)|f_\nu(b)\rangle\rangle = z_\nu |f_\nu(b)\rangle\rangle$$

Unitary operator

$$C(b') = UC(b)U^\dagger, \qquad\qquad |f_\nu(b')\rangle\rangle = U|f_\nu(b)\rangle\rangle$$

Symmetry in Liouville space — symmetry in Hamiltonian H gives symmetry in Liouville operator $L = -i(H \times 1 - 1 \times H)$, not vice versa.

$$f^{\pm}_{mn} \sim \exp\left[-\left(\frac{Q}{\sqrt{2b}}\right)^2 - \frac{(\sqrt{2b}\,r)^2}{4}\right]\left(\frac{Q}{\sqrt{2b}}\right)^\mu H_\nu\left(\frac{\sqrt{2b}\,r}{4}\right), \qquad \begin{array}{l} Q = (x+\tilde{x})/2 \\ r = x - \tilde{x} \end{array}$$

Hyperbolic rotation symmetry

$$\begin{pmatrix} x' \\ \tilde{x}' \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix},$$

Rotation angle

$$\exp(-\theta) = \sqrt{2b}$$

Wave packet

Wave packet

$$\Psi(Q,r) = \frac{1}{\sqrt{2\pi b_i}}\, e^{-Q^2/2b_i - b_i r^2/2} = \sum_{m=0}^{\infty} \left[e^{-\tau}\,(1-\chi)\right]^m f_{m0}(b_f)$$

$\chi = b_i/b_f\,,$ $b_f =$ temperature of reservoir.

Ψ a density function

- Diagonal component, probability $x = \tilde{x}$ or $r = 0$.
- Off-diagonal component, quantum correlation, consider $Q = 0$.

Decoherence process: hotter environment, $0 < \chi_d < 1$

Coherence process: cooler environment, $\chi_c > 1$

- Can be understood as decay of non-equilibrium modes.

Change in quantum coherence length

Measure of quantum correlation.

Quantum coherence length

$$l = (\Delta r)_{Q=0}$$
$$= \sqrt{\langle r^2 \rangle_{Q=0} - \langle r \rangle^2_{Q=0}}$$

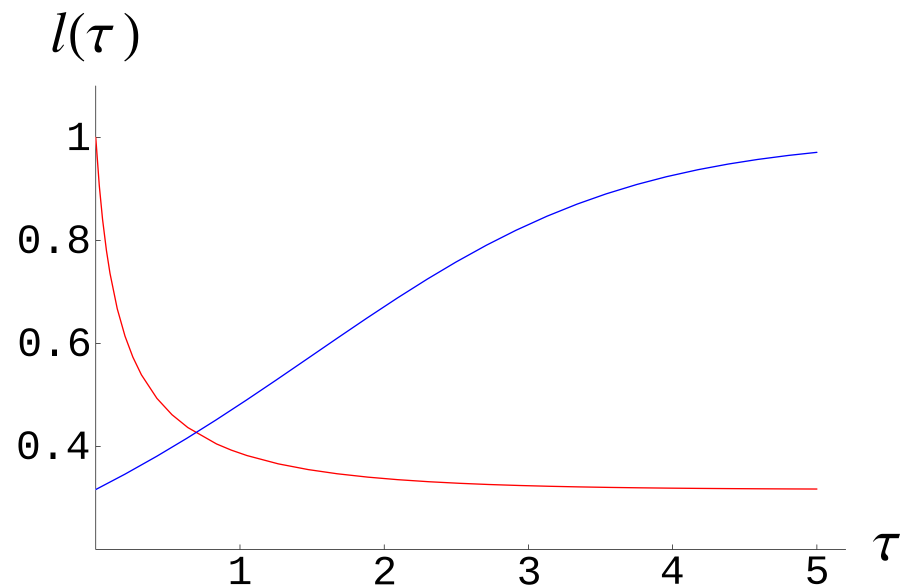
$$\langle r^n \rangle_{Q=0} \equiv \frac{\int_{-\infty}^{\infty} r^n \Psi(0,r|0)\,dr}{\int_{-\infty}^{\infty} \Psi(0,r|0)\,dr}$$

Time evolution

$$l(\tau) = \frac{1}{\sqrt{b_f[1 + (\chi - 1)e^{-\tau}]}}$$

High temperature

$$b_f \to k_B T/\omega_0, \qquad l \to 0$$



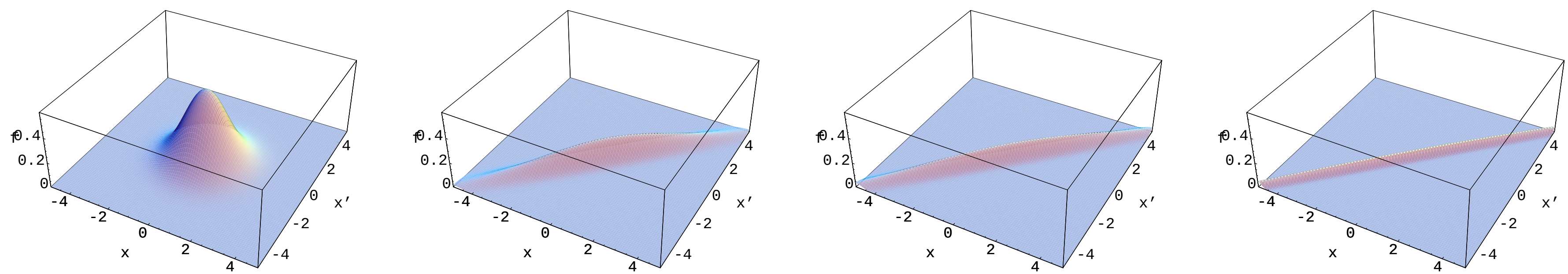
Decoherence: $\chi_d = 0.1, \quad b_i = 1, \quad b_f = 10.$

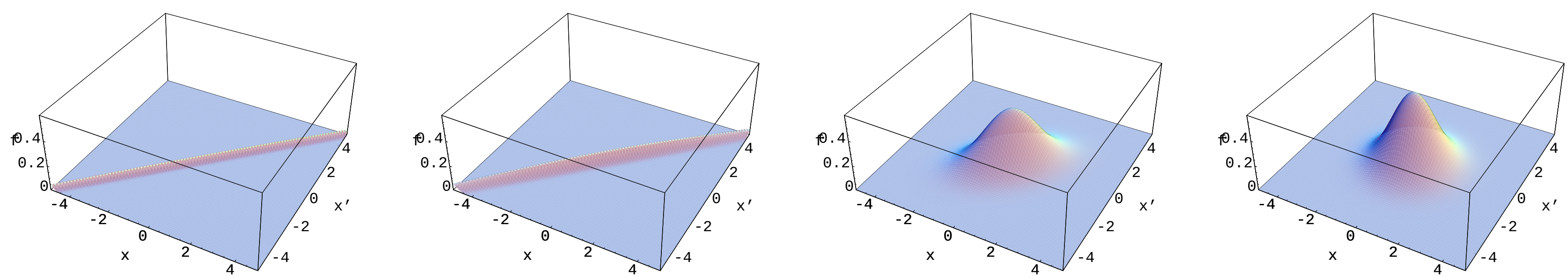
Coherence: $\chi_c = 10, \quad b_i = 10, \quad b_f = 1.$

Time asymmetry process

Decoherence: $\chi_d = 0.01$, $b_i = 0.5$, $b_f = 50$

$\tau = 0$ $\tau = 0.1$ $\tau = 0.2$ $\tau = 1$





$\tau = 0$ $\tau = 1$ $\tau = 5$ $\tau = 10$

Coherence: $\chi_d = 100$, $b_i = 50$, $b_f = 0.5$

Estimate time scale

Find functional dependence of time scales that qualitatively described the processes.

Total change in l

$$\Delta l = |l(\infty) - l(0)| \\ = \text{distance between solid line}$$

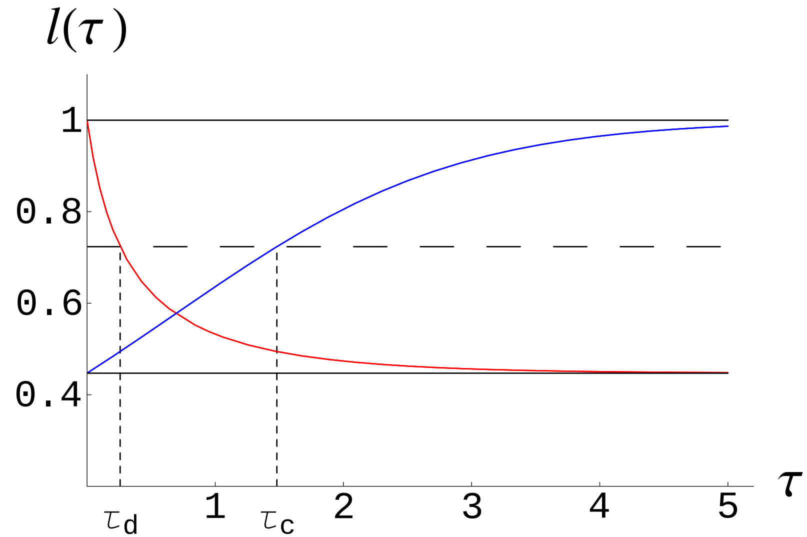
Time to reach

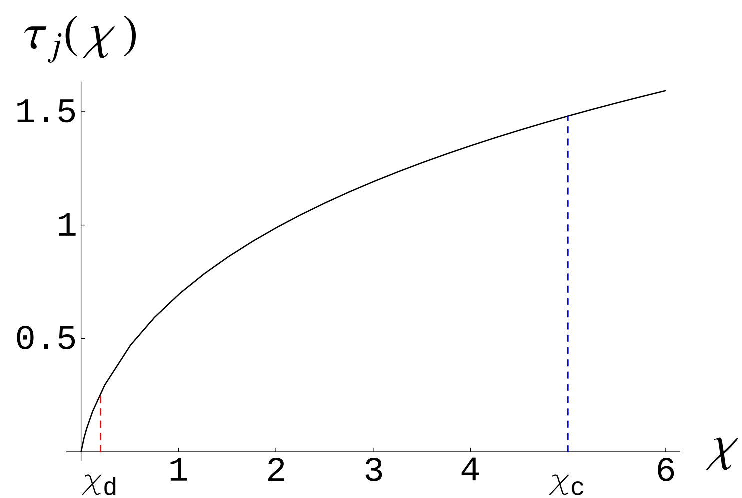
$$l(\tau_j) = l(0) \pm \tfrac{1}{2}\Delta l \\ = \tfrac{1}{2}[l(0) + l(\infty)]$$

Estimate

$$\tau_j = \ln|1-\chi| - \ln\left|1 - \frac{4\chi}{(1+\sqrt{\chi})^2}\right|$$

$$0 < \chi_d < 1 < \chi_c$$





$\tau_d \approx 0.26$, $\chi_d = 0.2$, $b_i = 1$, $b_f = 5$.
$\tau_c \approx 1.48$, $\chi_c = 5$, $b_i = 5$, $b_f = 1$.

High and low temperature limit

Estimate

$$\tau_j = \ln|1-\chi| - \ln\left|1 - \frac{4\chi}{(1+\sqrt{\chi})^2}\right|, \qquad 0 < \chi_d < 1 < \chi_c$$

Low T limit

$$\tau_c \sim \ln(\chi_c), \qquad \chi_c \gg 1$$

- Consistent with $\quad \Psi \sim \sum_m [e^{-\tau}(1-\chi)]^m f_{m0}(b_f)$

High T limit

$$\tau_d \sim \chi_d, \qquad \chi_d \ll 1$$

$$\Rightarrow \qquad t_d \sim \frac{\omega_0}{\gamma k_B T} \frac{1}{(\Delta r)^2_{Q=0}}$$

- Using the fact

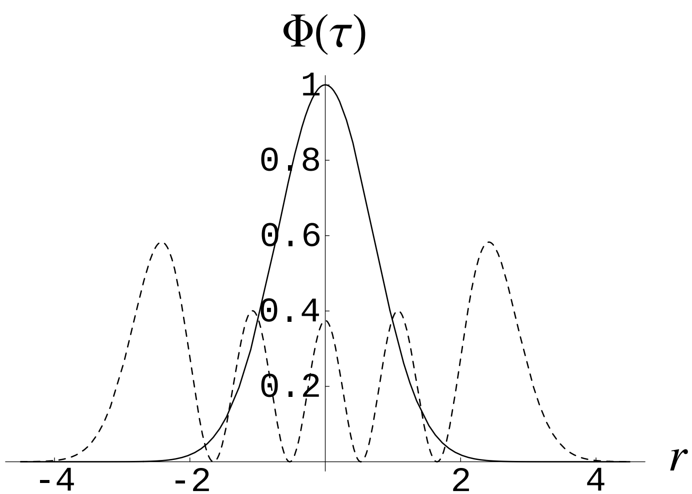
$(\Delta r)^2_{Q=0} = 1/b_i, \qquad b_f \approx k_B T/\omega_0$ and $\chi_d = b_i/b_f$

Disappearance of interference pattern

Disappearance of off-diagonal component as decay of non-equilibrium modes.

Example: 4th excited state of simple harmonic oscillator $\Phi \equiv |4\rangle\langle 4|$

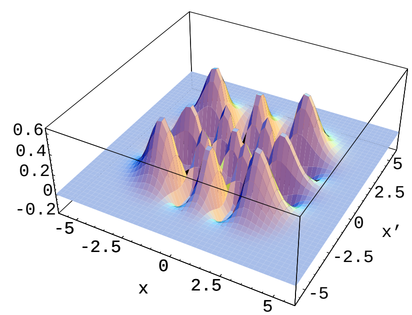
$$\Phi(\tau) = f_{00}(\tfrac{1}{2}) - 8e^{-\tau} f_{10}(\tfrac{1}{2}) + 24e^{-2\tau} f_{20}(\tfrac{1}{2}) - 32e^{-3\tau} f_{30}(\tfrac{1}{2}) + 16e^{-4\tau} f_{40}(\tfrac{1}{2})$$



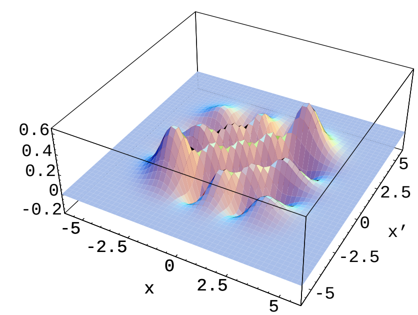
$\ldots$ $\tau = 0$

—- $\tau \to \infty$

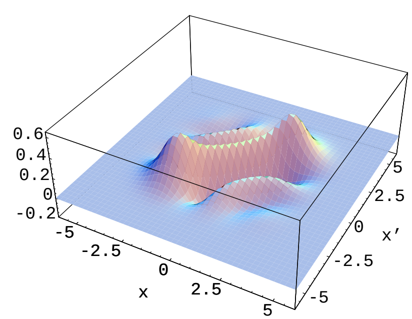
$\tau = 0$



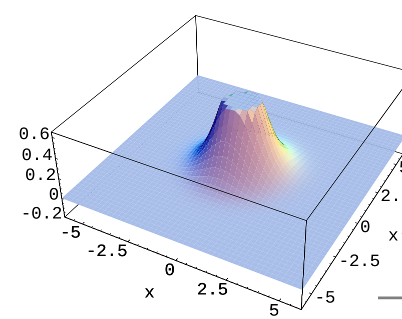
$\tau = 0.02$



$\tau = 0.5$



$\tau = 5$



Summary

Solve the non-hermitian eigenvalue problem of the collision operator for a quantum Brownian particle.

Obtain biorthogonal basis. In coordinate space, the left eigenfunctions are represented by distribution.

The solution exhibits hyperbolic rotation symmetry with respect to temperature.

Time evolution of quantum processes can be analysed in terms of the decay of non-equilibrium modes.

Some expressions

Caldeira-Leggett equation $\quad C_{\text{cl}}(Q,r) = -i\overline{\omega}_{\text{cl}}\left(P\frac{\partial}{\partial Q} - Q\frac{\partial}{\partial P}\right) + \frac{i}{2}\left(\frac{\partial}{\partial P}P + b_{\text{cl}}\frac{\partial^2}{\partial P^2}\right)$

Hu-Paz-Zhang equation $C_{\text{vir}}(Q,r) = -i\overline{\omega}_{\text{vir}}\left(P\frac{\partial}{\partial Q} - Q\frac{\partial}{\partial P}\right) + \frac{i}{2}\left(\frac{\partial}{\partial P}P + b\frac{\partial^2}{\partial P^2} + d\frac{\partial^2}{\partial P\partial Q}\right)$

$$
\begin{aligned}
N &= \sqrt{\frac{(m-n)!}{2\pi b(m!)^3}} \\
c_{mn}^{\pm\mu\nu\sigma} &= (\pm 1)^{n+\sigma}\frac{(-1)^{\mu+\nu}}{i^n 2^{2\nu+\sigma}\mu!}\sqrt{\frac{(m-n)!}{m!}}\begin{pmatrix} m \\ n+\mu \end{pmatrix}\begin{pmatrix} \mu \\ \nu \end{pmatrix}\begin{pmatrix} n \\ \sigma \end{pmatrix} \\
d_{mn}^{\pm\mu\nu\sigma} &= (\sqrt{2/b})^{2\nu+\sigma-1} c_{mn}^{\pm\mu\nu\sigma} \\
L_m^n(y) &= \sum_{\mu=0}^{m-n}(-1)^{\mu+n}\frac{m!}{\mu!}\begin{pmatrix} m \\ n+\mu \end{pmatrix} y^\mu\,, \quad m \geq n
\end{aligned}
$$

Some authors define associated Laguerre polynomials $L_m^{\prime n}$ differently. They are related to L_m^n by $L_m^n = (-1)^n L_{m-n}^{\prime n}$.

A few examples of $f^{\pm}_{mn}(Q,r,b)$

A few examples of $f^{\pm}_{mn}(Q,r,b)$ and their decay rate $\Gamma_{\text{decay}} = m - n/2$, with $\widetilde{Q} = Q/\sqrt{2b}$, $\widetilde{r} = \sqrt{2b}\, r$.

$f^{\pm}_{mn}$	Expansion in polynomials	Decay rate
$f^{\pm}_{00}$	$\frac{1}{\sqrt{2\pi b}}\, e^{-\widetilde{Q}^2 - \widetilde{r}^2/4}$	0
$f^{\pm}_{10}/f^{\pm}_{00}$	$\frac{1}{2} - \widetilde{Q}^2 + \frac{1}{4}\widetilde{r}^2$	1
$f^{\pm}_{11}/f^{\pm}_{00}$	$-i[\pm\widetilde{Q} + \frac{1}{2}\widetilde{r}]$	0.5
$f^{\pm}_{20}/f^{\pm}_{00}$	$\frac{1}{8}[3 + 4\widetilde{Q}^4 - 4\widetilde{Q}^2(3 + \frac{1}{2}\widetilde{r}^2) + \widetilde{r}^2(1 + \frac{1}{4}\widetilde{r}^2)]$	2
$f^{\pm}_{21}/f^{\pm}_{00}$	$\frac{-i}{2\sqrt{2}}[\mp 2\widetilde{Q}^3 - \widetilde{Q}^2\widetilde{r} \pm \widetilde{Q}(3 + \frac{1}{2}\widetilde{r}^2) + \frac{1}{2}\widetilde{r}(1 + \frac{1}{2}\widetilde{r}^2)]$	1.5
$f^{\pm}_{22}/f^{\pm}_{00}$	$\frac{1}{2\sqrt{2}}(1 - 2\widetilde{Q}^2 \mp 2\widetilde{Q}\widetilde{r} - \frac{1}{2}\widetilde{r}^2)$	1
$f^{\pm}_{30}/f^{\pm}_{00}$	$\frac{1}{48}[15 - 8\widetilde{Q}^6 + 12\widetilde{Q}^4(5 + \frac{1}{2}\widetilde{r}^2)$	3
	$-6\widetilde{Q}^2(15 + 3\widetilde{r}^2 + \frac{1}{4}\widetilde{r}^4) + \frac{1}{2}\widetilde{r}^2(9 + \frac{3}{2}\widetilde{r}^2 + \frac{1}{4}\widetilde{r}^4)]$	
$f^{\pm}_{40}/f^{\pm}_{00}$	$\frac{1}{384}[105 + 16\widetilde{Q}^8 - 32\widetilde{Q}^6(7 + \frac{1}{2}\widetilde{r}^2) + 24\widetilde{Q}^4(35 + 5\widetilde{r}^2 + \frac{1}{4}\widetilde{r}^4)$	4
	$-8\widetilde{Q}^2(105 + \frac{45}{2}\widetilde{r}^2 + \frac{9}{4}\widetilde{r}^4 + \frac{1}{8}\widetilde{r}^6) + 2\widetilde{r}^2(15 + \frac{9}{4}\widetilde{r}^2 + \frac{1}{4}\widetilde{r}^4 + \frac{1}{32}\widetilde{r}^6)]$	