Stress transmission in granular packs – why do conventional theories struggle?

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Jammed granular matter transmits stresses non-uniformly like no conventional solids, especially when it is on the verge of failure and immediately after yield. Jamming is caused by self-organization of granular matter under external loads, often giving rise to networks of force chains that support the loads non-uniformly. An ongoing debate in the literature concerns the correct way to model the static stress field in such media: good old theories, based on stress-strain relations, such as elasticity theory, or newcomer isostaticity theory. The two differ significantly. In particular, in 2D, isostaticity theory leads naturally to force chain solutions. More recently, it has been proposed that real granular materials are made of mixtures of regions, some behaving elastically and some isostatically. The theory to describe these systems has been named stato-elasticity [1,2].

In this talk, the first of a series, I present the rationale for moving away from displacement-based theories for purely isostatic materials. I discuss the concept of isostaticity and the conditions for marginal rigidity that this stress state is relevant to. I will then present the conceptual basis for the equations of isostaticity theory and, in particular, the need for stress-structure closure of the stress equations.

Speaker's related references
Stress transmission in granular packs –
Isostaticity theory and beyond

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In this, second talk of the series, I present a first-principles derivation of the stress-structure equation in 2D [1]. This equation closes the balance equations and together they form the field equations for stresses in purely isostatic materials. I then discuss the explicit solutions of these equations, both in uniform and non-uniform materials. I show that, while the solutions are perfect force chains in uniform materials, I then show that force chains is too simple a picture and that pure isostatic materials support more general solutions, where stresses along force chains attenuate, force chains branch, and stress 'leaks' away from main chains through secondary and tertiary chains into 'cones of influence' [2,3,4,5]. All these are direct solutions of the stress field equations with no need for additional modelling.

I then move on to discuss the rationale for going beyond isostaticity theory into a stato-elasticity theory - a theory that combines isostaticity with a defomation-based theory [2,3]. The basic idea is that marginal rigidity is a stress state that behaves as a critical point [6,7], therefore affecting close-by states that are nearly, but not exactly, isostatic. I claim that this picture applies to general non-isostatic granular materials. I will then outline briefly my ideas how to construct such a stato-elasticity theory.

Speaker's related references
An entropic formalism for the statistics of granular packs

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In this third lecture of a series on granular systems, I discuss a fundamental formalism for statistical analysis of these materials. The construction of such a formalism is an essential step toward derivation of equations of state and constitutive laws, in exactly the same way that statistical mechanics of thermal systems is essential for the derivation of the conventional equations of state.

First, I present a systematic method for characterization of the microstructure of granular packs in 2D and in 3D. This characterization has also the advantage that it quantifies the vague concept of 'structure' via a local structure tensor. Based on the structural characterization, I then construct an entropic formalism for granular packs and demonstrate how all structural properties and their distributions can be determined as expectation values over a certain 'partition function'. Several specific structural quantities are given as examples - quantities that can help determine large-scale permeability to flow through the granular packs, heat exchange, radiation heating, reactivity, catalysis, etc.

I then demonstrate how, with simple assumptions, such a formalism can describe phase separation in granular packs.

Speaker's related references
Rich dynamics of process-zone-limited fracture propagation in heterogeneous materials

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In the fourth lecture of the series I present a model for fractures propagating in disordered materials. Such fractures are often preceded by cracks that open in a so-called process zone. By association of these baby cracks into the main crack, and depending on their spatial and size distributions, the crack tip may at times be ahead of the stress field. In such situations, the dynamics of propagation can assume one of a number of propagation modes and a complex behaviour sets in.

A first-principles equation of motion has been developed for the tip of a line crack (not necessarily straight) in two dimensions, based on the Yoffe solution. The propagation kinetics are complemented with a generic relation between the tip velocity and the local stress, making it possible to solve exactly for the motion of the crack tip.

A detailed analysis of the solution is presented and it is found to give rise to rich dynamics. I first discuss the kinetics within an effective continuum approach. It is found that the mode of propagation depends only on one material-dependent parameter - the ratio between the arrest velocity in the constitutive velocity-stress relation and the local stress relaxation rate. This approach assumes absence of noise and gives three possible modes: (i) the crack tip may not propagate at all; (ii) the crack may grow at a constant speed; (iii) the crack may advance at a time-dependent velocity that oscillates periodically between a small (possibly zero) and a large value. The latter leads to arrest lines.

Going beyond effective medium, I include the effect of the statistics of the baby cracks in the process zone. This gives rise to spectacular dynamics. Depending on the particular statistics of the baby cracks, the propagation can assume one of further three modes: (i) intermittent - the crack velocity fluctuates erratically; (ii) quasi-periodic - the crack tip oscillates predominantly periodically, but with fluctuations in the periodicity; (iii) a continuous spectrum of noise-driven steady states. A state diagram of the propagation state in the parameter space of noise amplitude and frequency is proposed.

Speaker's related references
Stress transmission and incipient yield flow in dense granular materials

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Jammed granular materials transmit stresses non-uniformly unlike conventional solids, especially when it is on the verge of failure and immediately after yield. Jamming is caused by self-organization of granular matter under external loads, often giving rise to networks of force chains that support the loads non-uniformly. An ongoing debate in the literature concerns the correct way to model the static stress field in such media: good old elasticity theory or newcomer isostaticity theory. The two differ significantly and, in particular in 2D, isostaticity theory leads naturally to force chain solutions. More recently, it has been proposed that real granular materials are made of mixtures of regions, some behaving elastically and some isostatically. The theory to describe these systems has been named stato-elasticity [1,2].

In this talk, I first present the rationale for stato-elasticity theory. An important step towards the construction of this theory is a good understanding of stress transmission in the regions of pure isostatic states and new general solutions are presented for 2D isostatic regions. The solutions show that force chains is too simple a picture and that pure isostatic materials support generic solutions, where stresses along force chains attenuate, force chains branch, and stress 'leaks' away from main chains through secondary and tertiary chains into 'cones of influence' [3,4]. All these are direct solutions of the stress field equations with no need for additional modelling.

I will then show how the static stress equations are related directly to incipient yield flow and derive the equations that govern yield and creep rheology of dense granular matter [5,6]. These equations are general and also describe the seemingly unrelated kinetics of auxetic materials [7].

Speaker's related references

4. M. Gerritsen, G. Kreiss, R. Blumenfeld, Analysis of stresses in two-dimensional isostatic
5. R. C. Ball and R. Blumenfeld, From Plasticity to a renormalisation group, Phil. Trans. R. Soc. Lond. 360 , 731 (2003).