Universal Mathematical Structure in Quasi 1D Fluid-Particle Systems

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ABSTRACT

We propose simple models of granular pipe flows and 2-lane traffic flow. The models produce the power spectrum in frequency $P(f) \sim f^{-4/3}$ analytically and numerically. The result agrees well with those observed in experiments and simulation of other models.

INTRODUCTION

Recently physics of granular materials has become a hot subject. Unlike usual solids, liquids or gases, granular materials are known to show fascinating dynamical behaviors [1].

Dynamical behaviors of grains flowing through a vertical pipe which can be regarded as a one dimensional realization of a fluidized bed is also a typical example of unusual features of granular motion [5–9]. Emergence of density waves (slugging) has been investigated by molecular dynamics [2], Stokesian dynamics simulation [3,4] and lattice-gas automata (LGA) simulations [5,6] and by the experiments using sand or glass beads in air [7,8] and metallic spheres in liquids [9]. In these days, we recognize that the emergence of density waves is closely related to the formation of traffic jams in a highway [10,11].

Theoretical analysis to understand jam formations has been focused for pure one dimensional cases. Although there are a variety of models such as optimal velocity models [12–14], and fluid models [15–17] to describe such phenomena, we believe that a universal mathematical structure exists which is independent of the choice of models. From the analysis for pure one dimensional models we support the validity of the idea on universality in traffic models [13,14,16,17]. Although these are interesting recent progress, we do not pursue the analysis for pure one dimensional models (see summary in Ref. [18]).

In realistic situations, however, e.g. highways have multi-lanes, and vehicles (particles) can pass slow vehicles (particles). When we include multi-lane effects in one dimensional models, separations between jam and non-jam phases become obscured. However, there is another universal law in quasi one-dimensional systems for dissipative discrete element flows, i.e. a power law in power spectrum of density auto-correlation function of vehicles or particles.

Power-law form of the power spectrum $P(f) \sim f^{-\alpha}$, where f is frequency, of density fluctuations was also found in both numerical simulations [5] and experiments [7,8]. Although their interpretations on the origin of the emergence of density waves are different, estimated values of the exponent α is close to each other $(1.3 < \alpha < 1.5)$.

In the previous experiment [7,8] which reported $\alpha \simeq 1.5$ the volume of air flow out of the bottom end of the pipe was not well controlled. Recently, Moriyama et al. [19–21] have presented better-controlled air flow out

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of the pipe and more accurate experimental results than the previous ones by increasing the number of trials [7,8]. Their estimation of the scaling exponent of the power spectrum $P(f) \sim f^{-\alpha}$ is $\alpha \simeq 1.33$. This result is identical to that by LGA [5], and is expected to be a universal law in quasi one dimensional dissipative flows such as traffic flows in a highway where $\alpha = 1.4$ was reported from the observation [11].

The purpose of this paper is to clarify the mechanism to appear $f^{-4/3}$ law in power spectra. We, thus at first, extend the one-dimensional model to a stochastic model supplemented by the white noise. We will demonstrate the simple model reproduces $\alpha = 4/3$ near the neutral curve of the linear stability analysis of uniform states. We also introduce a model of 2-lane traffic flows to confirm the universality of $f^{-4/3}$ law. We briefly explain the mechanism how to appear $f^{-4/3}$ law.

LANGEVIN MODEL

Let us introduce a one-dimensional model which reproduces $\alpha = 4/3$. This model is introduced by one of the authors [18,19,22]. The important mechanisms for particle dynamics are the drag between the air and particles, and the relaxation process to an optimal velocity which may be the sedimentation rate of particles. When the N particles are confined in a quasi-one dimensional container, the motion of particles may be described by the following nonlinear equation:

$$\ddot{r}_n + \zeta[\dot{r}_n - \tilde{W}(\{r_n\})] = TF(\{r_n\}) + f_n(t), \tag{1}$$

where r_n and ζ are the relative distance between the n-th and $n+1$-th particles, and the friction constant, respectively. The collision force $F(\{r_n\}) = \varphi'(r_{n+1}) + \varphi'(r_{n-1}) - 2\varphi'(r_n)$ comes from a soft core repulsive potential $\varphi(r_n)$. The parameter T represents the strength of repulsion. The optimal velocity $\tilde{W}(\{r_n\}) = U(\frac{r_{n+1}+r_n}{2}) - U(\frac{r_n+r_{n-1}}{2})$ is the linear combination of sedimentation rate $U(r)$ which is the nonlinear function of the local volume fraction [23] in general. The most crucial simplification of (1) is to assume that f_n is a Gaussian white noise with zero mean. It should be noticed that the drag ζ is irrelevant in systems where the bottom end of the pipe is fully open, because air in the pipe flows away together with particles. Thus, to observe density waves it is important to close the cock of the pipe.

Linearizing (1) around the uniform solution $\dot{r}_n = 0$ where $h = \bar{r}_n \equiv N^{-1}\sum_n r_n$, we obtain

$$\ddot{\tilde{r}}_k + \zeta[\dot{\tilde{r}}_k - iU'\tilde{r}_k \sin k] = 2T\varphi''(\cos k - 1)\tilde{r}_k + \tilde{f}_k(t), \tag{2}$$

where the argument of U' and φ'' is h. $\tilde{r}_k$ and $\tilde{f}_k(t)$ are respectively the Fourier transform of $\delta r_n = r_n - a$ and $f_n(t)$. Equation (2) has the solution $\tilde{r}_k(t) \propto \exp[\sigma_\pm t]$. $\mathrm{Re}[\sigma_+]$ represents the relevant eigenvalue of the linear problem, which becomes positive for $U'(h)^2\cos^2(k/2) \geq T\varphi''(h)$. Thus the most unstable wave number is $k \to 0$ and the neutral curve is given by $T_c = U'^2/\varphi''(h)$. At $T = T_c(1-\mu)$ the expansion of σ_+ around $k = 0$ is given by

$$\sigma_+(k) \simeq i[c_0 k - \frac{c_0}{6}k^3] + \frac{c_0\mu}{\zeta}k^2 - \frac{c_0^2}{4\zeta}k^4 + \cdots, \tag{3}$$

where $c_0 = U'(h)$. Thus, for $\mu > 0$ the uniform state is unstable due to the negative diffusion.

Adopting $U(r) = \tanh(r-2) + \tanh(2)$, $\varphi(r) = \mathrm{sech}^2(r)$, $\zeta = 2$, $N = 256$, $T_c = 3.95798\cdots$, and $a = 2$ at $t = 0$, we simulate (1) by the classical Runge-Kutta method until $t = 2^{11}$ with time interval $\Delta t = 1/2^4$ under the periodic boundary condition. We use the uniform random number distributed between $-X$ and X with $X = 9/1024$ for $f_n(t)$. Figure 1 and 2 display the power spectrum $P(f) = |\tilde{c}(f)|^2$ obtained from our simulation of (1) at $\mu = 1/64$ for Fig.1 and $\mu = -1/64$ for Fig.2, where $\tilde{c}(f)$ is the Fourier transform of the discretely

sampled data of the density $c(t) = \frac{1}{N}\sum_n \frac{1}{r_n(t)}$ with the interval 1. These figures clearly support $P(f) \sim f^{-4/3}$ as in our experiment. From Figs.1 and 2, the qualitative results are insensitive to the sign of μ when $|\mu| \ll 1$. This result is reasonable because near the neutral curve the time scale of relaxation or growth of fluctuations is much longer than the time scale induced by the noise $f_n(t)$. Our result suggests that the linear relaxation theory of fluctuations can be used to explain $P(f) \sim f^{-4/3}$.

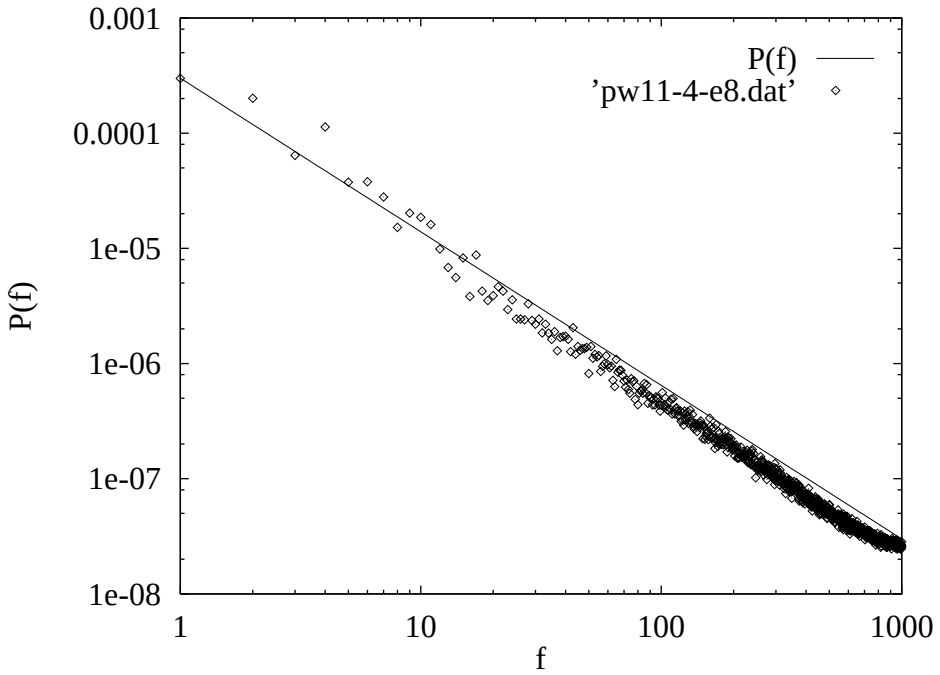

FIG. 1. Log-log plot of power spectrum $P(f)$ obtained from the simulation of (1), where the unit of f is $1/(2\pi)$. The solid line represents $f^{-4/3}$.

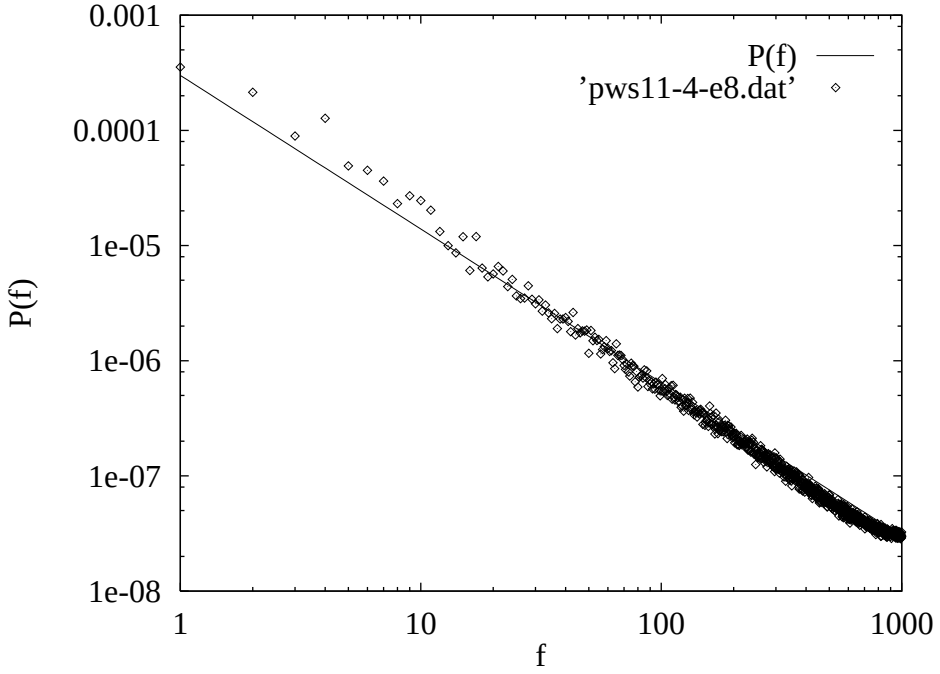

FIG. 2. Log-log plot of power spectrum $P(f)$ obtained from the simulation of (1), where the unit of f is $1/(2\pi)$. The solid line represents $f^{-4/3}$.

A TWO-LANE TRAFFIC MODEL

The most crucial simplification of the Langevin model is to introduce the white noise to represent passing motion of one particle to another in a quasi 1D flow. Although the result is consistent with experiments [19–21] and a LGA simulation [5], readers may regard this agreement as an accidental one. To demonstrate the structure stability of $f^{-4/3}$ law, we show that a simple two-lane traffic model in a highway which does not include the white noise can reproduce $f^{-4/3}$ law.

The model is as follows:

- Each vehicle is characterized by the position x_i and the spin variable s_i ($s_i = 1$ or -1). The index of the vehicle is not fixed on the vehicle but represents the order in the projected line along a highway. The spin variable corresponds to the lane index.

- The motion of vehicle in each lane obeys the generalized OV model [14]

$$\ddot{x}_i = a[W(x_{i+j} - x_i)V(x_i - x_{i-k}) - \dot{x}_i] \tag{4}$$

  where j and k is the minimum numbers which satisfy $s_{i+j} = s_i$ and $s_{i-k} = s_i$, respectively. For explicit calculations, we adopt $W(x) = \tanh(x-2) + \tanh(2)$ and $V(x) = 1 + f_0[1 - \tanh(x-2)]$ with $f_0 = 0.4$, although we have checked that the qualitative results are independent of the choice of U, V and f_0.

- We adopt the periodic boundary condition for x_i and s_i.

- The probability of the lane exchange is governed by the Glauber dynamics [24]:

$$w_{s_k \to -s_k} = \frac{1}{\tau_0}[1 - \tanh(2K)s_k(s_{k-1} + s_{k+1})] \tag{5}$$

  where τ_0 is the characteristic time of spin flip, K is a parameter of spin flip. For $K \to \infty$, each vehicle cannot change its lane, while for $K = 0$, the lane change occurs at random.

Figure 3 shows the result of our simulation for number of vehicles $N = 128$, the average distance $\bar{h} = 0.849$, $a = 1.572$ and $K = 0.10$. The dashed guide line is $f^{-4/3}$ law. The best fit value of α in $P(f) \sim f^{-\alpha}$ is given by $\alpha = 1.3345 \pm 0.0098$. This result also supports the universality of $f^{-4/3}$ law.

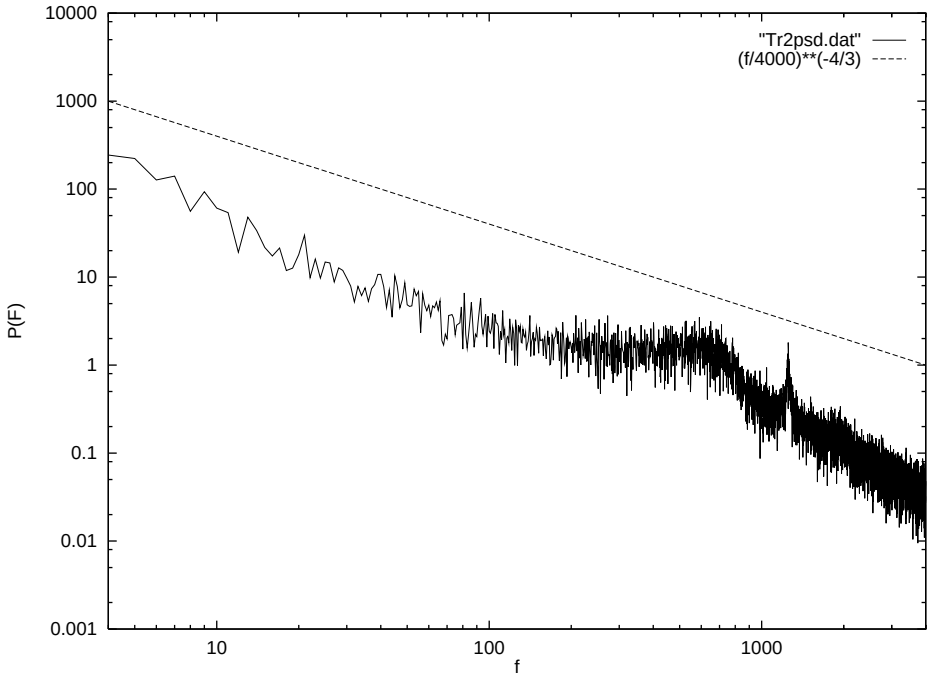

FIG. 3. Log-log plot of power spectrum $P(f)$ obtained from 2-lane traffic model.

ANALYTIC DERIVATION OF $f^{-4/3}$ LAW

Why can we get beautiful $f^{-4/3}$ law from simulations of different models? This result suggests that the origin of $f^{-4/3}$ law is simple. In this section, let us briefly explain how to appear $f^{-4/3}$ law from the behavior of structure factor

$$S_k(t) = \sum_{n,m} < \exp[ik(\delta r_n(t) - \delta r_m(0))] > \tag{6}$$

in weakly stable states, *i.e.*, $\mu < 0$ and $|\mu| \ll 1$. With the aid of the expansion of σ_+, (1) is reduced to

$$\partial_\tau \delta r(z,\tau) - \partial_z^3 \delta r(z,\tau) = \epsilon[\partial_z^2 - \partial_z^4]\delta r(z,\tau) + \xi(z,\tau), \tag{7}$$

where δr is the continuous version of δr_n, $\tau = \epsilon^3\beta t$, $z = \frac{2\zeta}{3c_0}\epsilon(x + c_0 t)$, $\xi(z,\tau) = \epsilon^3\beta f_n(t)$ with $\epsilon = \frac{3\sqrt{c_0}}{\zeta}\sqrt{-\mu}$ and $\beta = \frac{4}{3\sqrt{c_0}}$. The solution of (7) is given by $\tilde{r}_k(\tau) \simeq \int_0^\tau ds \exp[\lambda_k(\tau - s)]\tilde{\xi}_k(s)$, where $\lambda_k = ik^3 - \epsilon k^2(1 + k^2)$, and $\tilde{r}_k$ is the Fourier transform of δr. Thus, we obtain the correlation

$$< \tilde{r}_k(\tau)\tilde{r}_{-k}(0) >= \frac{D}{2\epsilon l k^2(1 + k^2)} \exp[\lambda_k \tau], \tag{8}$$

where l is the system size, and we use $< \tilde{\xi}_k(\tau)\tilde{\xi}_p(\tau') >= \frac{D}{l}\delta_{k+p,0}\delta(\tau - \tau')$. Thus, in principle, we can calculate $S_k(t)$ by substitution of (8) into (6).

The procedure to obtain the structure factor is parallel to that for polymer dynamics [25]. Skipping the calculation in details (see Ref. [18,22]), we obtain

$$S_k(\tau) \simeq \frac{\epsilon}{Dk^2} \exp[-\frac{Dk^2}{\epsilon\Gamma(1/3)}\tau^{1/3}] \tag{9}$$

in intermediate time range. In the limit of small τ, $S_k(\tau) \propto 1 - \frac{Dk^2}{\epsilon\Gamma(1/3)}k^2\tau^{1/3} + \cdots$. Thus its Fourier transform, which is nothing but the power spectrum $P_k(f) = |\tilde{c}_k(f)|^2$ obeys

$$P_k(f) \sim f^{-\alpha}, \quad \alpha = 4/3 \quad (\text{as} \quad f \to \infty), \tag{10}$$

where use was made of $\int_{-\infty}^{\infty} d\tau e^{i2\pi f\tau}|\tau|^{1/3} \propto f^{-4/3}$. The value 4/3 is identical to the one obtained by the experiment [19] and numerical simulations [5,19]. Thus our model (1) reproduces $\alpha = 4/3$. This result should be valid even when we start from fluid models [15,16] since the result is determined by the universal feature near the neutral curve. It should be noted that the appearance of this power-law form in the original model (1) is only for $f < \zeta$ since we eliminate the fast decaying mode σ_- in our analysis. This tendency is also observed as the higher-frequency cutoff in the experiment [19].

Thus, $f^{-4/3}$ law is determined by short time behavior of the dynamics of density waves induced by the noise, which is essentially determined by the linear dispersion relation $\lambda_k \sim ik^3$.

CONCLUSION

In this paper we have confirmed the universal law of $\alpha = 4/3$ as the power-law exponent in the frequency spectrum of density correlation function from the simulation of simple models and the theory. We have also clarified the mechanism to emerge $f^{-4/3}$ spectrum which is related to the critical slowing down of the density fluctuations. There is, however, discrepancy between our results with the one on the experiment in liquids [9]. The reason of this difference should be clarified in the future.

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