Fermi Statistics of Weakly Excited Granular Materials

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ABSTRACT: In this paper, we briefly review our recent effort to develop a thermodynamic theory of weakly excited granular materials from the view point of elementary excitation of Fermions. Within the framework of the lattice gas model, particles and holes interact only by excluded volume interactions. We introduce a global temperature T that is associated with the acceleration amplitude Γ in vibrating beds. We then show that the configurational statistics obey the Fermi statistics and present an explicit relation between T and the external control parameters. We also explain the slow relaxation in the granular compaction process within our model and discuss its extension to surface fluidization and possibly to the Fermi liquid theory

1. INTRODUCTION

The system we are concerned with here is a dense and driven nonequilibrium dissipative granular system, where the mean free path of the grains is of order of a few particle diameters. Hence, each particle may be considered to be effectively confined in a cage as in the free volume theory of dense liquid (Hill, 1987). In such a case, we make an observation that the basic granular state is not a gas but a solid or a crystal and thus the effective thermodynamic theory may be more appropriate than the kinetic theory in studying the granular state. Such a theory, if successful, will then enable us to characterize a nonequilibrium steady state of the granular systems as the most probable state as in equilibrium thermodynamics. Recently we have been developing a thermodynamic theory of excited granular materials from the view point of elementary excitation of Fermions(Hayakawa and Hong 1995, 1996). Such thermodynamic approach may be further justified by the following two recent experiments.

1. Warr and Hansen[WH](Warr and Hansen, 1996) have performed an experiment on highly agitated vertically vibrating beds using steel balls that have small coefficient of restitution. They have found that the collective behaviors of this vibrated granular medium , in a stationary nonequilibrium state, exhibits strong similarities with those in an atomistic fluid in thermal equilibrium, at the corresponding particle packing fraction, in particular in the two-point correlation function.

2. Clement and Rajchenbach[CR] (Clement and Rajchenbach, 1991) have performed an experiment for a two dimensional vibrating bed with the inclined side walls to suppress convections. They have found that the ensemble-averaged density profile as a function of height from the bottom layer obeys a universal function that is independent of the phase of oscillations of the vibrating plate.

The results of both experiments indicate that one-to-one correspondence seems to exist between configurational statistics of nonequilibrium stationary state and the equilibrium thermal state. Further, the existence of distinctive configurational statistics in the density profile of CR (and also in WH in a special case) appears to be a fairly convincing evidence that kinetic aspects of the excited granular materials can be separated out from the statistical configurations. These observations are the basis of our thermodynamic theory to be described below. In this framework, one can use the method developed in equilibrium statistical mechanics to characterize some of the properties of excited granular materials, notably from the view point of elementary excitations such as the

Landau's Fermi liquid theory in condensed matter physics. We will also explain how to extend our theory to study the collective dynamics of granular materials, such as the problem of surface fluidization, and granular compaction.

2. THERMODYNAMIC MODEL

We first view the system of granular particles as a mixture of holes and particles as in the lattice gas or the diffusing void model(Caram and Hong, 1991), which is the simplest version of the free volume theory. We now assign virtual lattice points by dividing the vibrating bed of width L and the height μD into cells of $D \times D$ with the diameter of the grain D. Each row, i, is then associated with the potential energy $\epsilon_i = mgz$ with $z = (i-1/2)$D and m the mass of the grain. Note that the degeneracy, Ω, of the each row is simply the number of available cells, i.e: $\Omega = L/D$. For a weakly excited system with $\Gamma \simeq 1$, the kinetic energy may be neglected and the potential energy dominates, for which case the most probable configuration should be determined by the state that maximizes the entropy in the microcanonical ensemble approach.

Taking into account the excluded volume effect, which is the origin of Fermi statistics in dense granular systems, we find that the entropy, S, defined as the total number of ways of distributing N particles into the system is given by:

$$S = \ln W = \ln(\prod_i \frac{\Omega!}{N_i!(\Omega - N_i)!}), \qquad (1)$$

where N_i is the number of particles occupying the $i-$th cell. Maximizing S with constraints, $\sum_i N_i = N$ and $\sum_i N_i \epsilon_i =< U(T(\Gamma)) >$, the mean steady state system energy, we find that the density profile, $\phi(z)$, which is the average number of occupied cells at a given energy level, is given by the Fermi distribution:

$$\phi(z) = N_i/\Omega = 1/[1 + \exp(\beta(z-\mu))], \qquad (2)$$

where $\beta = mgD/T$, the height $z = z_i/D$ and the Fermi energy μ measured in units of D is the initial number of layers in the low temperature limit.

Next, we check the validity of our approach. For strongly excited systems, Fermi exclusion principle does not apply. The Fermi analogy is valid when the zero temperature Fermi energy satisfies $\mu \gg n_l$, where n_l is the number of fluidized layer. Now, since the injected energy, E_i, to the system is of order $mA^2\omega^2/2$ and the potential energy, E_p, to fluidize particles on the top n_l layers is of order mgn_lD, by equating these two, we find a necessary condition for the fluidization of the top n_l layers, namely: $n_l \sim \Gamma(A/2D)$. Hence, for $\Gamma \sim 1$, the Fermi statistics will be valid, if $\mu \gg A/D$.

3. RELATION BETWEEN T AND Γ

We now relate the temperature T to the external control parameters such as Γ. First, a thermal expansion. We determine the energy *per* particle $\bar{u}(T) = U(T)/N = \int_0^\infty \phi(z) mzgdz/N$, from which we can determine the shift in the center of mass per particle; $\bar{h}(T) = \bar{u}(T)/mg$, which is given by:

$$\bar{h}(T) = h(0)[1 + \frac{\pi^2}{3}(\frac{T}{mgD\mu})^2] + \cdots \qquad (3)$$

with $h(0) = D\mu^2/2$. Second, a kinetic expansion. We make an observation that for a weakly excited granular system, most excitations occur near the Fermi surface, and thus the volume expansion may be effectively well represented by maximum height, $H_0(\Gamma)$ of a single particle bouncing on the vibrating plate assuming that the Fermi surface is in contact with the vibrating plate. The kinetic expansion, $H_0(\Gamma)$, is then determined by the maximum of $\Delta(t)$ in the following eq. that describes the trajectory of a single ball on a vibrating plane with the intensity Γ:

$$\Delta(t) = \Gamma(\sin(t_0) - \sin(t)) + \Gamma\cos(t_0)(t-t_0) - \frac{1}{2}(t-t_0)^2 \qquad (4)$$

in units of $g = \omega = 1$,where $t_0 = \sin^{-1}(1/\Gamma)$. Since the Fermi distribution near $T = 0$ can be approximated by a piecewise linear function and $H_0(\Gamma)$ is thought to be the edge of the function, we expect $H_0(\Gamma) \approx \Delta h/2 = (\bar{h}(T) - h(0))/2$. By equating the thermal expansion,(3), to the kinetic expansion, $H_0(\Gamma)g/\omega^2$, in physical units, we now complete our thermodynamic formulation by presenting the explicit relation between T and Γ:

$$T = \frac{mg}{\pi\omega}(3DgH_0(\Gamma))^{1/2}. \qquad (5)$$

Now we compare our theoretical prediction with an experimental result of CR (Clement and Rajchenbach 1991). Figure 1 shows the fitting of the experimental density profile for $\Gamma = 4$ of CR.

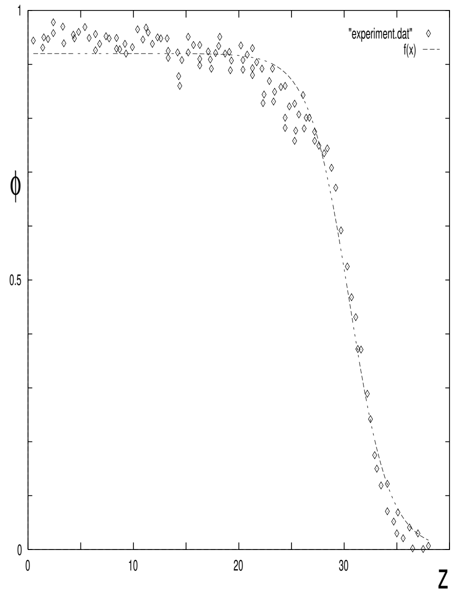

Figure 1: Density $\phi(z)$ as a function of the height z. The circles are the data by Clement and Rajchenbach and the dotted line is the Fermi distribution function(eq.(2)).

by the scaled Fermi distribution, $\phi(z) = \rho(z)/\rho_c$, with ρ_c the closed packing density. For hexagonal packing, $\rho_c \approx 0.92$. The fitting value of T/mg is $2.0[mm]$, while eq.(5) yields $T/mg \approx 2.6[mm]$. The agreement between the two is fairly good in spite of such a simple calculation. This expression also agrees with the simulation result (Gallas et al 1993). Note that the detailed expression of $H_0(\Gamma)$ depends on the manner by which the grains are excited and we expect that our main scaling prediction of eq.(5), namely $T \propto g^{3/2}D^{1/2}/\omega$, will hold even for systems driven not by sinusoidal waves. Further, T has a gap at T=0 because the time between the launching and landing of the ball is always finite for $\Gamma > 1$. Next, it is well known that the *specific heat* per particle, $C_v = d\bar{u}/dT$, can be written as the fluctuations in the energy, namely $<(\Delta\bar{u})^2> = <(\bar{u}(z)-\bar{u})^2> = T^2C_v$. Hence, our theory makes a nontrivial prediction:

$$<(\Delta\bar{u})^2> = \frac{\pi^2}{3}\frac{T^3}{mgD} \propto g^{7/2}D^{1/2}\omega^{-3}, \quad (6)$$

This prediction may be checked by experiments or large scale simulations of discrete element method, which we hope will further confirm the validity of our thermodynamic theory.

4. DISCUSSION

We now discuss an extension of our theory to other physical phenomena associated with excited grains, in particular the problem of compaction, and fluidization and the effect of dissipation.

First, slow relaxation in three dimensional compaction process. In three dimensional systems, many metastable configurations exist, which results in the hysteresis-dependent density profile as observed in the experiment of Knight et al (1995). Nevertheless, even in this case, we can explain the slow relaxation observed in the compaction experiment within the framework of our ideal Fermion picture. This is because the mean free path of the grains is of order of a few particle diameters and we thus may consider each particle to be effectively confined in a cage as in free volume theory of dense liquid. Within this framework, we can argue that the probability, $P(\mathbf{r})$, of the hopping of a particle from $\mathbf{r}$ to the nearest neighbor site is given by: $P(\mathbf{r}) \propto \exp(-A/T)$, where the activation energy $A(\mathbf{r}) \simeq a\phi(\mathbf{r})/(1-\phi(\mathbf{r}))$ with constant a. Therefore, the relevant time scale τ which determine the time evolution of the compaction is given by $\tau/\tau_0 \simeq \exp[b\phi/(1-\phi)]$ with $b = a/T$ and τ_0 some characteristic time scale. This simple argument predicts that the time evolution of ϕ should be logarithmic as $\phi(t) \simeq \ln(t/\tau_0)/(b + \ln(t/\tau_0))$, where we have replaced τ by t. This is consistent with the experimental result reported by Knight et al (1995). Unfortunately, the logarithmic time dependence of three dimensional systems prevents us from discussing further the configurational function as presented in (2). Although this kind of slow relaxation may be expected even in two dimensional systems, the strong geometrical constraint may suppress such the slow process.

Second, surface fluidization and interactions among particles. Our theory only considers the free particles and holes on a lattice. For the quantitative argument or discussion of dynamical behaviors of granular particles, our free Fermion picture needs modification, most importantly interactions among particles must be taken into account as in the Fermi liquid theory. It is known that even for hard core particles, an effective attractive interaction exists through the direct correlation function (Hansen and McDonalds, 1987), which will induce the curvature term in our free energy formulation. In the presence of such a curvature term, it is obvious that we can define surface tension for excited granular materials in a vibrating bed.

Third, the effect of dissipation. Although our formulation does not contain the dissipation explicitly, it is needed in order for the system to reach a steady state with energy supply. We have shown based on a simple one dimensional spring model that the effect of the inclusion of a small dissipation is a simple renormalization of the effective amplitude A_{eff} which is felt by the particle at the top in the limit of large N, namely: $A_{eff} = A/\sqrt{1+(\zeta/\omega)^2}$ where ζ is the dissipation constant, $\zeta = -(\omega/\pi)\ln e$ with the restitution constant e. This correction is not important for small e.

In summary we have explored simple consequences of the excluded volume effect for dense granular system, and proposed a thermodynamic theory for the dissipative nonequilibrium excited granular system from the view point of elementary excitation of non-interacting Fermions. Our theory works quite well for the steady state density profile of weakly excited granular materials and we expect that the presence of interaction may bring out some of the features of the Fermi liquid theory of dense electron gas. It is our hope that our proposed thermodynamic model may provide a useful paradigm around which unifying principles may emerge in the future. If successful, our approach may offer ways, for limited cases, to visualize microscopic classical and/or quantum behaviors through the manipulation of granular dynamics.

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