A Simple Model on Granular Friction

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Abstract

We propose a simple phenomenological model to describe the motion of a plate on granular layers pushed with a constant speed. The model consists of the order parameter characterizing the transition from a solid-like state to a liquid-like state and the position of the plate. The model reproduces the hysteresis in frictional force and the universal profile of the slippage velocity, which are observed in an experiment. Our model predicts that the area of hysteresis loop depends on the pushing speed.

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Recently much attention has been paid to physics of granular materials. Unlike usual solids, liquids or gases, granular materials are known to show complex dynamical behaviors [1], such as convection [16], size segregation [3], standing waves and localized excitations under vertical vibration [5], and $1/f^{4/3}$ spectra in a narrow tube [6].

Friction in granular layers is also an interesting subject as a fundamental physics. This subject can be regarded as a part of physics of earthquakes [7,8]. The simplest situation to think of granular frictions is the following: Let a plate place on lubricants such as granular particles. Push the plate with a constant speed. When the speed is low, we can observe stick-slip motion of the plate, while the speed is high we can see a steady sliding of the plate. Similar beahviors have been observed even in atomic dry friction and and melt fractures in polymers [9,10]. Recently, Nasuno et al [11] have reported a sensitive measurements of frictional forces produced by granular particles in the above situation. They found that the frictional force is a multi-valued and exhibits a hysteresis, where the frictional force is almost a constant when the slip velocity is decreased. They also reported a universal form in instantaneous velocity profile during slippage in stick-slip regime.

In this Communications, we propose a simple phenomenological model to describe granular friction. The situation under the consideration corresponds to the experimental one by Nasuno et al. [11]. We will demonstrate that the universal form in slippage velocity profile can be observed in our model, which is quite similar to experimental one can be represented by an analytic function.

Our approach is inspired by the phenomenological model proposed by Carlson and Batista [12]. They assume that the friction dynamics is characterized by the coupled equations for the displacement of the plate and the order parameter which specifies the state of granular particles.

We also adopt a phenomenology which is a set of coupled equations of the displacement and the order parameter. However, our model contains two essential differences from theirs. First of all, we take into account some characteristics in distinct element method (DEM) which is a standard method for simulation of granular particles [13]. In particular, we

regard Coulomb's friction law during collision of particles as one of keep points for granular frictions. Namely, the shear friction in colliding two granular particles is proportional to relative velocity for small relative velocity and is a constant which is the Coulomb's law for large relative velocity in granular collisions. Therefore, our frictional term is not proportional to relative velocity of layers, but the friction is saturated when the relative velocity is large. This friction mechanism is thought to be important to recover the characteristic hysteresis observed in the experiment [11] in granular friction. Second, we introduce the fracture mechanism of the stress network in granular layers in our model. This idea reflects recent indications on the importance of bridging and its fractures in friction processes [15].

Here let us explain our model explicitly. Let the mass of the plate be M, and its center of mass be located at x. We assume that the friction between granular layers and the block is characterized by the order parameter θ, where $\theta = 1$ represents solid-like state, i.e. the grains are contacted with each other and there are stress chains in granular layers. The plate is pushed by a driving motor with the constant speed V, which is connected with the motor by a Hookian spring (the spring constant k).

Thus, the equations of block is given by

$$M\ddot{x} = k(Vt - x) - \frac{\partial}{\partial x}W - \mu MgFr\left(\frac{\dot{x}}{u_0(V)}\right), \tag{1}$$

where g and μ are respectively the gravitational acceleration and the dynamical friction constant. We adopt the friction force $Fr(x)$ as

$$Fr(x) = \tanh(x) \quad \text{for } x > 0, \quad Fr(x) = bx \quad \text{for } x < 0, \tag{2}$$

where we assume b is a large constant, which essentially inhibits the motion with negative velocity. Note that the saturation of the friction in $\tanh(x)$ for large velocity corresponds to the Coulomb's friction law inside the DEM model [13,14]. The potential W is given by

$$W = \frac{k\theta\xi^2}{2}\left[1 - \exp\left(-\frac{(Vt - x)^2}{\xi^2}\right)\right], \tag{3}$$

which plays an important role as a resistance force in solid-like state. We adopt here the Gaussian-Taguchi model [16] which has the stress $-\partial W/\partial x = \theta k y \exp(-y^2/\xi^2)$ with the

deflection of the spring $y = Vt - x$ from the natural length and the critical length ξ for the fracture. Namely, this model contains resistance force to cancel the elastic force from the deflection of the spring for small y, while no resitance force exists for large y because of fracture of stress network. The reason why we introduce a continuous fracture model is to represent precursory events before main slips.

In this model the static friction is not introduced explicitly as in DEM model but is replaced by the fracture mechanism of stress network. Correspondingly, the equation of the order parameter is assumed to obey

$$\tau\dot{\theta} = \theta(a(\dot{x}) - \theta)(1 - \theta) - \frac{\partial W}{\partial \theta} - \zeta(V)\theta\dot{x}, \tag{4}$$

where $a(\dot{z}) = \tanh^2(\dot{z}/v_0)$ with a characteristic velocity v_0. This model has a stable fixed point at $\theta = 1$ corresponding to the solid state and $\theta = 0$ is an inflection point when the velocity is zero. While $\theta = 1$ becomes an inflection point and a stable fixed point exists at $\theta = 0$ which corresponding to the liquid state, when the velocity is infinite. For finite velocity, both two states become locally stable, and $a(\dot{x})$ is an unstable fixed point. Note that the model by Carlson and Batista [12], the liquid state is described by an unstable fixed point. The last term in (4) represents shear-induced melting force which is the same form as introduced in Carlson and Batista [12].

Let us analyze our model. At the first stage we nondimensionalize the model. The units of space, time, and velocity are respectively given by Mg/k, $\sqrt{M/k}$, and $g\sqrt{M/k}$. We restrict our interest to the case for $k = 135$ N m^{-1}, $M = 1.1 \times 10^{-2}$ kg, $g = 9.8$ m/s^2, and $\mu = 0.45$. All of these parameters correspond to the typical set up of the experiment by Nasuno et al [11]. Thus, the unit length, the unit time and the unit speed in the dimensionless unit are about Mg/k =0.8 mm, $\sqrt{M/k} = 9.0 \times 10^{-3}$ sec, $g\sqrt{M/k}$ =9 cm/sec in the actual scale. To achieve the stick-slip motion in the very low velocity V region we introduce the V dependence as $\zeta(V) = \zeta_0/\tanh(V)$, and $u_0(V) = \bar{u}\tanh(V)$ with $\zeta_0 = \bar{u} = 0.1$ in the dimensionless unit. Our choice of $\zeta(V)$ and $u_0(V)$ corresponds to the non-existence of a characteristic velocities in low V limit, because ζ^{-1} and u_0 can be characteristic velocities

to the transition from stick-slip motion to creep if they are constants. The critical length ξ in (3) and the time scale τ in (4) are assumed to be 0.7 and 3.0×10^{-2} in the dimensionless unit. The dimensionless parameters v_0 in $a(\dot{z})$ in (4) and b in (2) are set to be $v_0 = 1.0$ and $b = 10^4$ respectively. We adopt the classical fourth order Runge-Kutta method for numerical integration with the time interval $\Delta t = 10^{-5}$.

At first we simulate the model for relatively low V, where typical stick-slip motion of the block can be observed. In Fig.1, we plot the profile of slippage velocity, where $t = 0$ is fixed at the end of slip events. All the data between $V = 2.5 \times 10^{-4}$ and 2.5×10^{-3} seem to be on a universal curve as in the experiment. Although we can observe stick-slip motion for larger V, the profile depends on V in such regions. Even in universal region, we can find a non-universal part of the profile near the start of slippage, where the velocity has the larger for large V value than that for small V. In Fig.2, we plot the frictional force ($M\ddot{x} - k(Vt - x)$ in phyical unit) as a function of slippage velocity. As in Fig.2 we obtain hysteresis loops for the friction which are similar to that observed in the experiment. Hysteresis loops, however, depends on V in our model. In our simulation, the area of the hysteresis becomes narrower, when the driving velocity becomes slower. To check quantitative behavior of the area of hysteresis loop $S(V)$ as a function of V, we evaluated area of the loop for each V as follows: First, we distinguish the data for $\dot{x} > 10^{-3}$ from others. For one loop, we evaluated the area of the loop S by the subtraction from the area of the upper branch S_u defined through the time derivative of the frictional force $\dot{f}_i > 0$ to that of the lower branch S_l defined by $\dot{f}_i < 0$, $S = S_u - S_l$, based on Trapezoid's rule as $S_m = \sum_i (f_i + f_{i+1})|v_{i+1} - v_i|/2$, where $m = u$ or l, f_i and v_i are respectively the force and the velocity of the $i-$th data. The data between $V = 2.5 \times 10^{-4}$ and 1.0×10^{-2} can be fitted by a power law (see Fig.3)

$$S \simeq cV^{\alpha}, \quad c = 0.248, \quad \alpha = 0.547 \pm 0.033. \tag{5}$$

Thus, the area may become zero in the slow driving limit, which means that the transition from the static friction state to the dynamic friction state is instantaneously as in classical Amontons-Coulomb law. The result in (5) is interesting one which has not been reported

in experiments. We are not sure whether the actual experiments reproduce such a power law or how this law is structural stable. This result should be checked by experiments and direct simulation based on DEM to check the validity of our picture. At present we do not know the mechanism to produce such a power law. We also found that the force f_i in the upper branch of the loop defined above obeys

$$f_i = F(\dot{x}/V^{\beta}); \quad \beta \simeq 0.78, \tag{6}$$

where $F(x)$ is a universal function (Fig.4). Since the exponent β is different from α, we believe that the power law in (5) comes from complicated combinations of several processes.

Our model may reduce to the classical Amontons-Coulomb model in the limit of $V \to 0$, which predicts the followings on stick-slip motion. The slip motion of the block is non-dissipative. The trajectory in the phase space is hemi-circle. Its period is $T = 2\{\pi - \sin^{-1}\left(\frac{y_1-y_2}{\sqrt{MV^2/k+(y_1-y_2)^2}}\right)\}\sqrt{M/k}+2(y_1-y_2)/V$, where $y_1 = \mu_s Mg/k$ with the static friction constant $\mu_s > \mu$ and $y_2 = \mu Mg/k$. The time evolution of velocity $\dot{x}$ is represented by a positive part of sinusoidal curve whose amplitude is $\sqrt{M/k}(\mu_s - \mu)g$ with the static friction constant μ_s in the low V limit, which satisfies $V^2 \ll k(y_1 - y_2)^2/m = mg^2(\mu_s - \mu)/k$. The solid line in Fig.1 is the dimensionless expression of Amontons-Coulomb law $\dot{x}(t) = -A\sin(t)H(\dot{x})$ with $A = 0.272$ in the slow V limit, where $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for otherwise. Note that the amplitude is almost equivalent to $\mu_s - \mu$ in this unit, though the actual μ_s is not a given parameter. In fact the frictional force which is equivalent to the static friction in this unit is almost 0.72, a little larger than $\xi = 0.7$, when the slip starts, and $\mu_s - \mu \simeq 0.27$ as in the fitting value of A. Thus, the universal behavior in low V is thought to be the state that the inertia of spring motion can be neglected.

The results reported here are consistent with experimental results on stick-slip motion [11]. For example, we need the condition $V^2 \ll mg^2(\mu_s - \mu)/k$ to reach universal law of the velocity profile of slippage. In Ref. [11] the universal law is satisfied below $V = 1137.27\mu$ m/s for $k = 135$ N/m. We, thus, obtain $V^2k/(mg^2(\mu_s - \mu)) \simeq 0.083$ where we adopt $\mu_s = 0.65$ and $\mu = 0.45$ evaluated from experimental data. Therefore, the universal behavior of the

slippage is only observable in slow velocity region as in our model simulation. Note that we can observe a little asymmetric velocity profile of slippage in the actual experiment [11]. We believe that this asymmetric behavior can be understood by the occurrence creep motion before main slip. In fact, even in our simulation, we can observe small asymmetry in the profile, where the velocity before main slip is larger than that after the main event.

This model obviously describes the transition from the stick-slip motion to a periodic oscillation (see Fig.5). Unfortunately, however, this model is not appropriate to describe a steady sliding motion. To describe the transition from the periodic motion to the steady sliding motion we may need to replace (1) by e.g.

$$M\ddot{x} = k(Vt - x) - \frac{\partial}{\partial x}W - \mu MgFr\left(\frac{\dot{x}}{u_0(V)}\right) - M\gamma\dot{x}, \tag{7}$$

where γ should be small for the consistence with the result for small V. The existence of such a friction term ensures the relax to a steady sliding motion. We may not be surprised by the existence of the term proportional to $\dot{x}$, because the collective motion of granular particles inside layers is complicated and crucial under the high shear rate. However, we do not discuss the details of this model, because at present the connection between collective motion of grains and the term $\gamma\dot{x}$ is not clear. The detailed discussion on what the collective mode under the shear is important will be reported elsewhere.

Our result reported here may crucially depend on our choice of $\zeta(V)$ and $u_0(V)$ to assume a steady stick-slip can be obtained for low V limit. If we change $\zeta(V)$ and $u_0(V)$ as constants, we observe creep motion in the low V limit, but disappears V dependence of hysteresis. Our model cannot predict the change of spatial structure during slippage and what the order parameter is . To remove unclear points and check the results reported here we should carry out DEM simulation, which is now in progress. The result of DEM simulation will be reported elsewhere.

In conclusion, we propose a simple phenomenological model for granular friction. Our model reproduces some characteristic results observed by Nasuno et al. [11] in stick-slip motion such as the universal velocity profile in slippage and hysteresis loop. Our model

shows that the area of hysteresis loop S depends on the driving velocity as $S \sim V^{0.547}$. Our model also can describe the transition to a oscillating region as the driving velocity increases, but is not appropriate for description of steady sliding motion.

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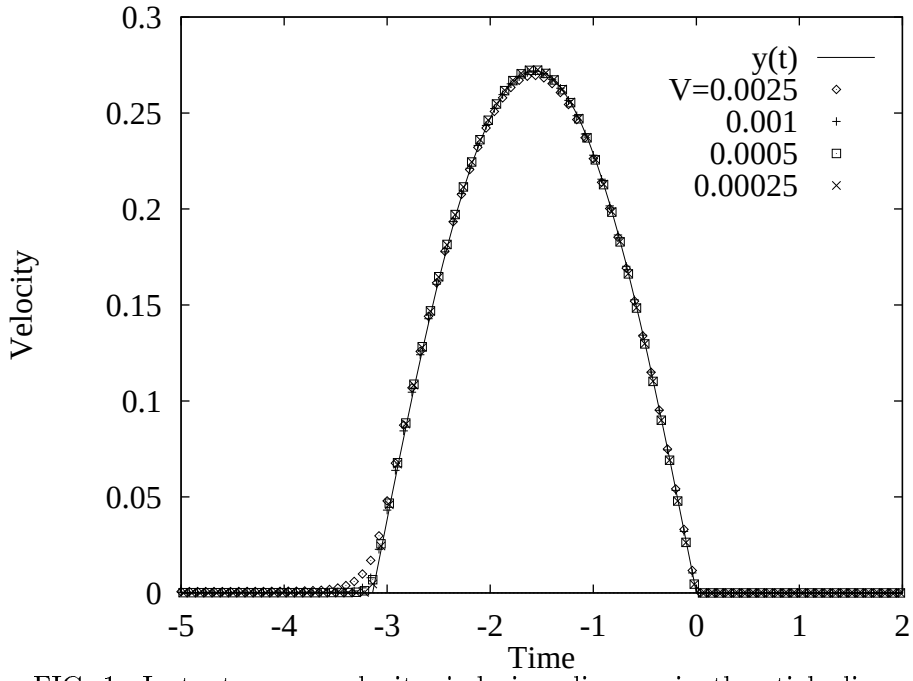

FIG. 1. Instantaneous velocity $\dot{x}$ during slippage in the stick-slip regime. The origins of the pulses are forced to agree at the end of each event. The solid curve is $-0.272\sin(t)$ in this unit.

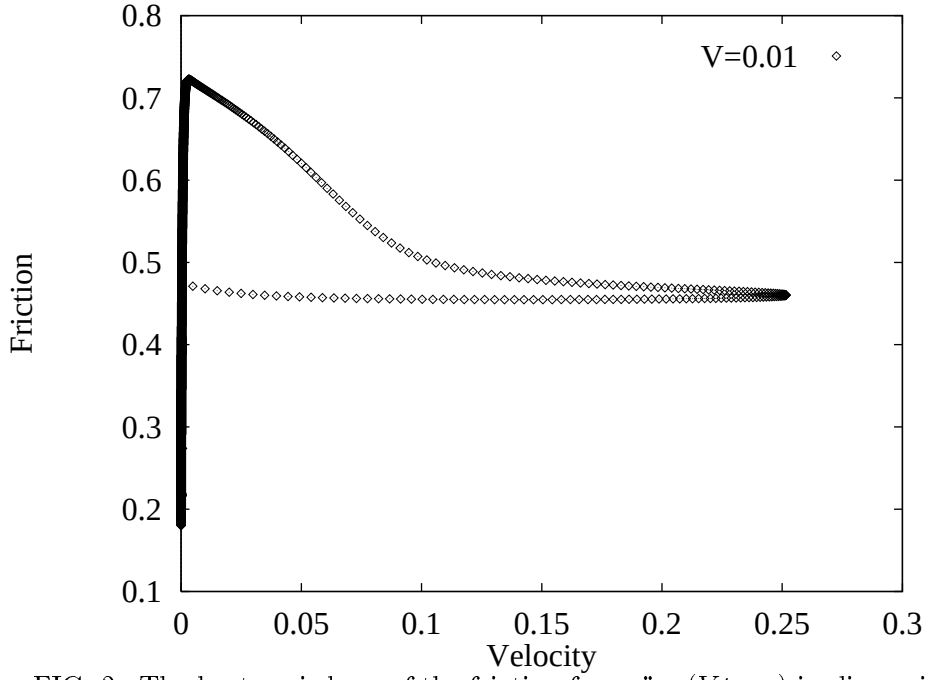

FIG. 2. The hysteresis loop of the friction force $\ddot{x}-(Vt-x)$ in dimensionless unit as a function of $\dot{x}$ for driving speed $V=0.01$.

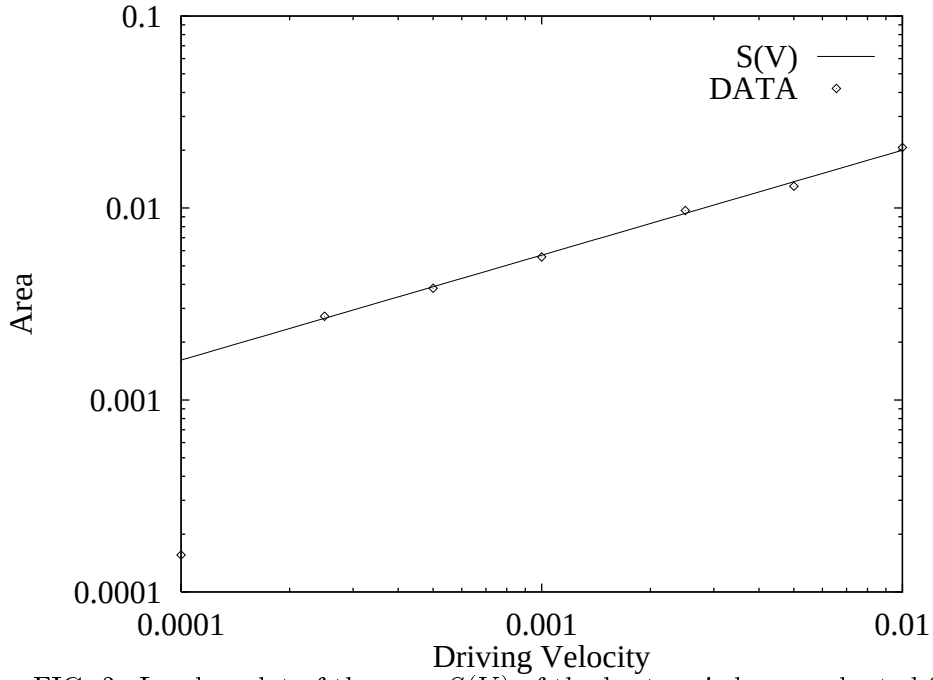

FIG. 3. Log-log plot of the area $S(V)$ of the hysteresis loop evaluated from Fig.2 as a function of V. The solid line represents $S(V) = 0.248V^{0.547}$.

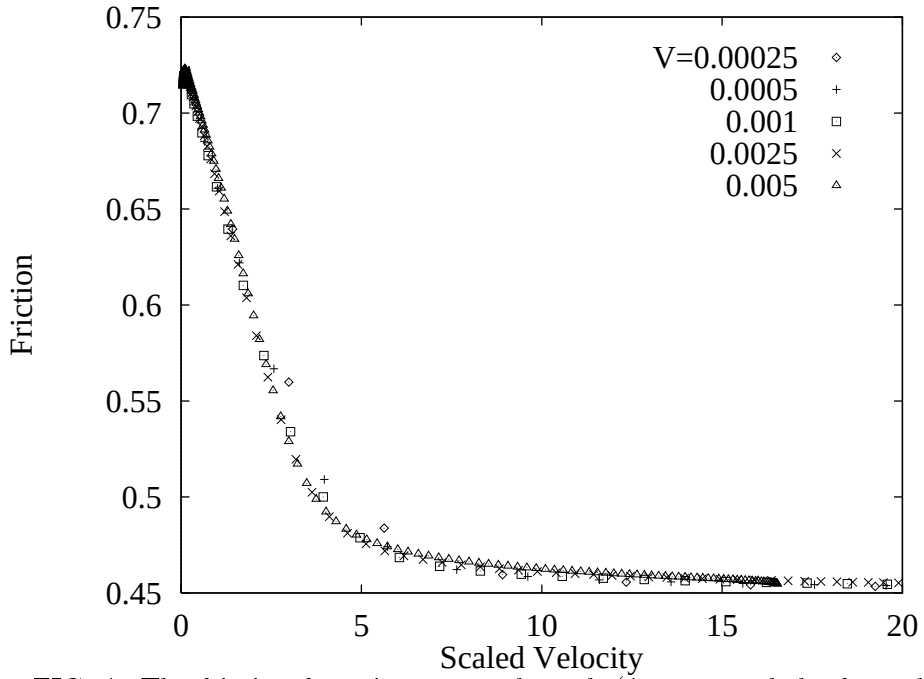

FIG. 4. The friction force in a upper branch ($\dot{x} > 0.3$ and the force decreases with time) as a function of scaled velocity $\dot{x}/V^{0.78}$

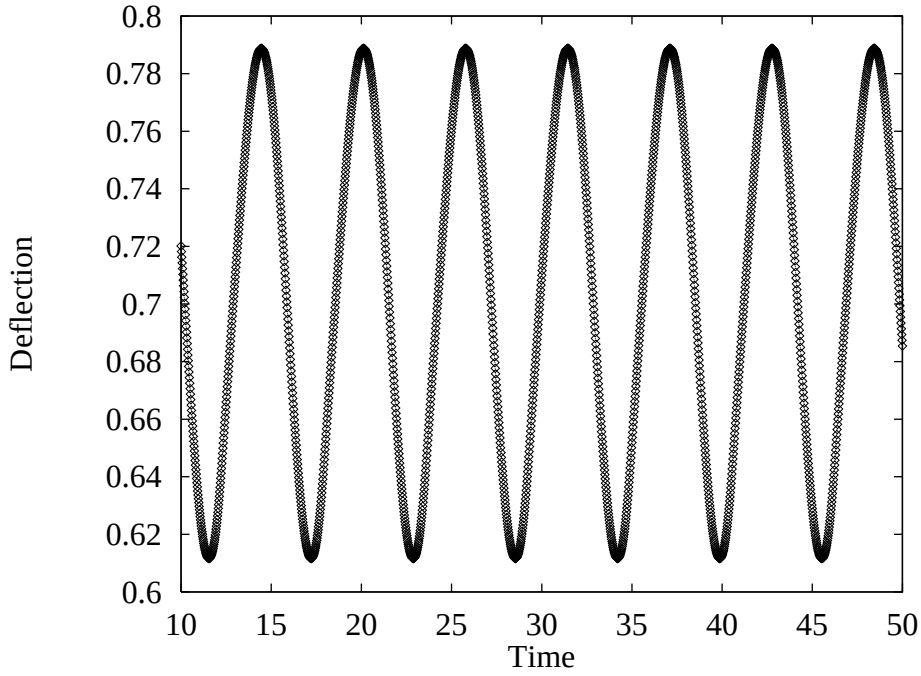

FIG. 5. The oscillation in deflection $Vt - x$ as a function of time ($V = 0.1$).