Slow Viscous Flows in Micropolar Fluids

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Abstract

Systematic calculation of micropolar fluid flows around a sphere and a cylinder is presented. The explicit velocity fields and the drag forces exerted by the fluid flow in both two dimension and three dimension are obtained. The obtained solution of a steady micropolar fluid flow inside the cylinder is identical to the form observed in an experiment for granular vibrating beds.

PACS: 47.15.Gf, 45.70.Mg, 83.50.Jf

I. INTRODUCTION

Micropolar fluids are fluids with micro-structures. They belong to a class of fluid with a non-symmetric stress tensor. Micropolar fluids consist of rigid, randomly oriented (or spherical) particles which have own spins and micro-rotations suspended in a viscous medium. The concept of micro-rotation, originally, has been proposed by Cosserat brothers in the theory of elasticity [1]. Condiff and Dahler [2], and Eringen [3] applied its concept to describe fluids with micro-structures in the middle of 60s. Recently, a comprehensive textbook on micropolar fluids has been published [4].

Physical examples of micropolar fluids can be seen in ferrofluids [5], blood flows [6,7], bubbly liquids [8], liquid crystals [9] and so on. However, we believe that the most important application of micropolar fluid dynamics is to characterize granular flows [10–13]. In fact, the

granular flow is one of flows which have micro-structure and rotation of particles. Therefore, it is natural to introduce a fluid model which contains an equation for micro-rotation in addition to an equation of velocity. Along this idea, Kanatani [14] formulated micropolar fluid dynamics for granular flows, which contains the angle of repose within the frame work. Kano et al [15] have confirmed the quantitative validity of a micropolar fluid model in a chute flow of granular particles from the comparison of their simulation of micropolar fluids with their experience. Note that the velocity profile obtained from the micropolar fluid model [15] is far from the parabolic curve obtained from the conventional flow, but similar to a linear function when the slope is not large. For larger slopes, the profile becomes concave (i.e. power index is smaller than 1) with the surface slips.

Although we do not know how the micropolar fluid model is relevant in other situations of granular materials, it is worthwhile to investigate fundamental properties of micropolar fluids from the view of granular physics. Thus, micropolar fluid dynamics is not a nonsense generalization of the Navier-Stokes model, but is a physically relevant model which has many applications in physical systems.

In this paper, we focus on slow viscous flows of micropolar fluids. The purpose of this study is as follows. First of all, the main purpose of this study is to clarify the mathematical structure of the creep flow in a Newtonian micropolar fluid model. This motivation is independent of our interest in granular flows. The main reason why we adpot the Newtonian model is its simplicity. Of course we must admit the existence of gap between granular flows and such the kind of Newtonian model. In fact, Kanatani's micropolar granular fluid model [14,15] is a strong nonlinear and non-Newtonian fluid model which is not free from ambiguity of constitutive equations for the stress tensor. However, (i) the flow of granular materials can be smooth when the shear rate exceeds the yield value, and flows are metastable after the shear rate is a littile lower than the critical yield. (ii) The profile of granular flow in a pipe is similar to the solution of the Newtonian micropolar fluid model. For example, the exact solution of the Poiseuille flow between parallel plates for the Newtonian micropolar fluid under the nonslip boudary condition [4,16] can describe the plug flow which has been observed in classical phenomenology , experiments, and simulations [17–19]. The explicit expression of flow profile will be introduced in Discussion. (iii) However, many models under steady flows may be reduced to a linearized Newtonian model in viscous limit, which is also physically relevant as a creep flow in actual situations. (iv) The micropolar fluid model for granular flows by Kanatani [14,15] can actually reduce to the Newtoninan model in the linear approximation (i.e. when the velocity and microrotation are small).

If we admit the relevancy of the micropolar fluid mechanics, it is obvious that flows around a cylinder or a sphere and their drag force play a fundamental role. Thus, we will calculate the fluid flows under such the conditions. We also discuss a steady flow inside a cylinder. The result inside a cylinder is identical to that observed in an experiment on vibrating beds [20]. We also briefly comment on the steady flow inside a rectangular container, which also gives identical results to those in experiments and simulations [20,21].

There are some relevant previous investigations on creep flows of micropolar fluid mechanics [4]. In particular, Ramkissoon [22] has obtained the solution of a micropolar fluid flow around a sphere and the drag force exerted on the sphere. It seems, however, that Ramkissoon's calculation [22] contains minor mistakes. Thus, we may need a revised calculation. Although Buchukuri and Chichinadze [23] obtained a fundamental solution and a solution of the fluid flow as an integral form for the flow around a cylinder, they could not present the explicit form of fluid flow and the drag force. Here, we will give explicit expressions based on the algorithm by Kaplun and Lagerstrom [24–28].

The organization of this paper is as follows. In the next section, we explain the general framework for a steady viscous flow around a sphere or a cylinder. In section 3, we show the correct calculation of the Stokes flow around a sphere and the drag force. We correct the result obtained by Ramkissoon [22]. In section 4, we obtain an explicit solution of micropolar fluid model around a cylinder by the method of Kaplun and Lagerstrom [24–28]. We calculate the drag force exerted by fluid flow. I section 5 we will briefly calculate axial symmetric flow inside a cylinder. The flow field is surprisingly identical to that observed in vibrating beds [20]. In section 6, we will discuss the relevancy of our calculation in granular

physics and how to improve it. In section 7, we conclude our results.

II. GENERAL FRAMEWORK

Although a developed granular flow is a compressible and non-Newtonian fluid, the viscous limit of the flow can be a linearized, Newtonian and incompressible flow. Let us restrict our interest to steady viscous flow around/inside a sphere or a cylinder. For simplicity, let us use dimensionless quantities, which are normalized by the velocity of flow far from the sphere or the cylinder (in cases of outside a cylinder and a sphere), and the radius of the sphere or the cylinder.

Thus, we start from the following set of equations. The incompressible condition is given by

$$\mathrm{div}\mathbf{v} = 0, \tag{1}$$

where $\mathbf{v}$ is the velocity field. The equation for momentum conservation is [4]

$$R\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \triangle\mathbf{v} + \mu_r \mathrm{rot}\vec{\omega}, \tag{2}$$

where $\triangle$ is the Laplacian, R is the effective Reynolds number, p is the pressure, $\vec{\omega}$ is the micro-rotation and the dimensionless viscosity of micro-rotation field μ_r which is assumed to be less than 2. Note that the Reynolds number R and μ_r are represented by quantities with physical units as $R = \rho_0 Ua/(\eta+\eta_r)$ and $\mu_r = 2\eta_r/(\eta+\eta_r)$, where ρ_0, U, a, η and η_r are the density, the magnitude of the characteristic flow, e.g. the flow far from a sphere/cylinder, the radius of the sphere/cylinder, the conventional viscosity and the viscosity for micro-rotation, respectively. The flow of micro-rotation is governed by

$$\frac{RI}{\mu_r}\mathbf{v}\cdot\nabla\vec{\omega} = \mathrm{rot}\mathbf{v} - 2\vec{\omega} + \mu_A\nabla\mathrm{div}\vec{\omega} + \mu_B\triangle\vec{\omega}, \tag{3}$$

where I is the dimensionless micro-inertia coefficient [4]. In general, $\mathrm{div}\vec{\omega}$ is not equal to zero but it is easy to show

$$\mathrm{div}\vec{\omega} = 0 \tag{4}$$

for axial symmetric flows. For later discussion we will discuss axial symmetric situations and assume (4). Thus, we regard the micro-rotation field as incompressible, where the term in proportion to μ_A in (3) is zero in the later discussion.

We assume the boundary condition outside a sphere/cylinder as

$$\mathbf{v} = \vec{\omega} = 0 \quad \text{at} \quad r = 1 : \quad \mathbf{v} = \mathbf{e}_x, \quad \vec{\omega} \to \frac{1}{2}\mathrm{rot}\mathbf{v} \quad \text{as} \quad r \to \infty, \tag{5}$$

where r is the distance from the center of the sphere or the cylinder whose radius is unity, and $\mathbf{e}_x$ is the unit vector along x axis. The above boundary conditions are not always valid in micropolar fluids. We assume the non-slip boundary condition on the boundary surface, which ensures the continuity of flow in entire region. Effective slipperiness of particles is included as micro-rotation. The micro-rotation on the surface is assumed to be zero because the center of rotation cannot exist on the surface but exist at a position deviated with particles' radius. The micro-rotation coincides with the rotation of flow $\mathbf{v}$ if the place is enough away from the sphere or the cylinder.

On the other hand, inside the cylinder, we assume an axial symmetric flow $\mathbf{v} = (v_r(r,z), 0, v_z(r,z))$. The boundary condition is assumed to be

$$v_r = 0, \frac{\partial v_z}{\partial r} = 0 \quad \text{at } r = 0, \quad v_r = 0 \quad \text{at } r = 1. \tag{6}$$

Note that we do not impose $v_z = 0$ at $r = 1$, because this condition produces an over-complete problem. In addition to (6) we have a conservation law

$$\int_0^1 drrv_z(r,z) = 0. \tag{7}$$

The problem inside a cylinder, thus, is rather special than flow around a cylinder/sphere. There are some unclear points for the set up the problem for the flow inside a cylinder.

Let us remark on the micropolar fluid model. It is obvious that the model is reduced to the Navier-Stokes equation for $\mu_r \to 0$. Thus, the model is a simple generalization of fluid mechanics. We may have objections to apply the micropolar fluid model to granular

flows by the following reasons: As mentioned in Introduction the micropolar fluid model for granular flow is assumed to be a strong nonlinear and non-Newtonian model. Perhaps, in the application of micropolar fluid to granular flows, it is important to characterize a stick state when the flow is stagnant. However,(i) the flow is expected to be a Newtonian when the flow is slow and stagnation points do not exist, and (ii) we may believe that the stagnation is caused by an effect of boundary. Thus, we expect that the micropolar fluid model is useful for slow granular flows when the boundary effects are not important. When we accept the micropolar fluid model itself, most of regular fluid models including Kanatani's one [14] reduce to a Newtonian model in the viscous limit. Thus eqs.(2) and (3) are one of universal forms of micropolar fluid dynamics.

What we will perform is to carry out the systematic calculation of the flow field around a sphere or a cylinder. For this purpose, we adopt the matched asymptotic method developed by Kaplun and Lagerstrom [24–28] for 2D problem. There are several advantages of this scheme. First of all, this method can discuss the systematic expansion of the Reynolds number. The second, this method ensures the simple mathematical structure. For example, Buchukuri and Chichinadze [23] cannot obtain a explicit solution of the Oseen approximation of 2D fluid flow. However, as will be shown, even in 2D we do not have to solve the Oseen problem of the micropolar fluid model explicitly but need the Oseen solution for the standard fluid flow.

Let us explain the second advantage more explicitly. It is well known that the Stokes approximation ($R = 0$ and $\mu_r = 0$ in eq.(2)) is invalid far from cylinder, and a naive perturbative calculation from the Stokes approximation encounters a secular term for the flow around a sphere. Therefore, in micropolar fluids, we need careful treatments to calculate the flow around a cylinder/sphere. To remove such difficulties we introduce an appropriate contracted coordinate as

$$\tilde{x} = Rx, \quad \tilde{y} = Ry. \tag{8}$$

Let us introduce scaled variables

$$\mathbf{v} = \mathbf{e}_x + \alpha(R)\mathbf{u}(\tilde{\mathbf{r}}), \quad \vec{\omega} = \alpha(R)\tilde{\omega}\mathbf{e}_\perp, \quad p = R\alpha(R)\tilde{p}, \tag{9}$$

where $\mathbf{e}_\perp$ is the unit vector vertical to the flow direction. The function $\alpha(R)$ will be determined by the matching. Thus, e.g. in 2D case eqs.(2) and (3) are reduced to

$$\frac{\partial \mathbf{u}}{\partial \tilde{x}} = -\tilde{\nabla}\tilde{p} + \tilde{\triangle}\mathbf{u} + \mu_r \tilde{\mathrm{rot}}(\tilde{\omega}\mathbf{e}_z) \tag{10}$$

$$\frac{RI}{\mu_r}\frac{\partial \tilde{\omega}}{\partial \tilde{x}} = \frac{\partial u_y}{\partial \tilde{x}} - \frac{\partial u_x}{\partial \tilde{y}} - 2\tilde{\omega} + \mu_B R^2 \tilde{\triangle}\vec{\omega}, \tag{11}$$

where $\mathbf{e}_\perp$ is reduced to $\mathbf{e}_z$ in 2D. Thus, in the limit of $R \to 0$, we obtain the two relations

$$\tilde{\omega} = \frac{1}{2}\left(\frac{\partial u_y}{\partial \tilde{x}} - \frac{\partial u_x}{\partial \tilde{y}}\right) : \quad \mathrm{rot}(\tilde{\omega}\mathbf{e}_z) = -\frac{1}{2}\tilde{\triangle}\mathbf{u}. \tag{12}$$

The situation in 3D is almost the same as that in 2D. The outer equations of micropolar fluid, thus, are reduced to

$$\frac{\partial \mathbf{u}}{\partial \tilde{x}} = -\tilde{\nabla}\tilde{p} + \left(1 - \frac{\mu_r}{2}\right)\tilde{\triangle}\mathbf{u} + O(R) \tag{13}$$

in both 2D and 3D problems. The solution of (13) is regular even far from the cylinder. As a result, we do not have to solve the Oseen approximation of (2) and (3) which cannot be represented by an explicit form [23]. Equation (13) supports the validity of the boundary condition $\omega \to \mathrm{rot}\mathbf{v}/2$ as $r \to \infty$ in (5).

Since we know the solution of (13) for 2D and 3D, what we need to solve is the Stokes approximation of (2) and (3) in an axial symmetric flow $\vec{\omega} = \omega\mathbf{e}_\perp$ as

$$-\nabla p + \triangle\mathbf{v} + \mu_r \mathrm{rot}(\omega\mathbf{e}_\perp) = 0 \tag{14}$$

and

$$(\mathrm{rot}\mathbf{v})_\perp - 2\omega + \mu_B \triangle\omega = 0 \tag{15}$$

near the cylinder or the sphere. The solutions of (13) and (14),(15) will be connected with the aid of matching asymptotic technique.

III. FLOW PAST A SPHERE

In this section let us look for a solution to describe a three dimensional steady flow past a sphere. The vectors $\mathbf{v}$ and $\vec{\omega}$ can be represented by their element in the polar coordinate as $\mathbf{v} = (v_r, v_\theta, v_\varphi)$ and $\vec{\omega} = (\omega_r, \omega_\theta, \omega_\varphi)$, where the polar axis is identical to x-axis and its origin is the center of the sphere. Here $\mathbf{e}_\perp$ in the previous section is replaced by $\mathbf{e}_\varphi$. Under the assumption of axial symmetric flow with $v_\varphi = 0$, there is a stream function which is related to velocity field

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} : \quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \tag{16}$$

in three dimensional systems. On the other hand, e.g. ω_r and ω_θ in 3D are independent constants of $\mathbf{v}$. So we assume $\omega_r = \omega_\theta = 0$ and

$$\omega_\varphi = \frac{\Phi}{r\sin\theta} \tag{17}$$

in 3D.

In 3D polar coordinate there are useful relations

$$(\mathrm{rot}\mathbf{v})_\varphi = -\frac{1}{r\sin\theta}D^2\Psi \tag{18}$$

and

$$\triangle(\mathrm{rot}\mathbf{v})_\varphi = -\frac{1}{r\sin\theta}D^4\Psi, \tag{19}$$

where the differential operator D^2 is defined by

$$D^2 = \left\{\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right\}. \tag{20}$$

Thus, it is convenient to represent the flow by the stream function Ψ.

A. Stokes flow

Let us consider the solution of the Stokes equations (14) and (15). The method of calculation itself is essentially the same as that by Ramkissoon [22]. In such the case from the divergence of (14) we obtain

$$\triangle p = 0. \tag{21}$$

Namely, the pressure is a harmonic function. On the other hand, the rotation of (14) is

$$\triangle \mathrm{rot}\mathbf{v} + \mu_r \mathrm{rotrot}\omega = 0. \tag{22}$$

Substituting (17) and (19) into (22) with the aid of an elementary formula of vector analysis we obtain

$$D^4\Psi + \mu_r D^2\Phi = 0. \tag{23}$$

On the other hand, substituting (18) into (15) we obtain

$$D^2\Psi + (2 - \mu_B D^2)\Phi = 0. \tag{24}$$

From (19) and (23) eq.(24) can be rewritten as

$$D^2(D^2 - \xi^{-2})\Phi = 0, \tag{25}$$

where

$$\xi = \sqrt{\frac{\mu_B}{2 - \mu_r}}. \tag{26}$$

Note that eq.(2.5) in Ramkissoon's paper [22] which can be translated into

$$\Phi = \frac{1}{2}[D^2\Psi + \frac{\mu_B}{\mu_r}D^4\Psi] \tag{27}$$

is not correct, but the correct correspondence from (23) and (24) is given by

$$\Phi = -\frac{1}{2}[D^2\Psi + \frac{\mu_B}{\mu_r}D^4\Psi]. \tag{28}$$

As a result, Ramissoon's calculation [22] leads to not accurate results.

The boundary conditions (5) now reduce to

$$\Psi(1,\theta) = 0, \quad \frac{\partial\Psi}{\partial r}(1,\theta) = 0, \quad \Phi(1,\theta) = 0 \tag{29}$$

on the surface of the sphere, and asymptotic conditions of $r \to \infty$ are

$$\Psi \to \frac{1}{2}r^2\sin\theta : \quad \Phi \to -\frac{1}{2}D^2\Psi, \tag{30}$$

where we use (17) and (18). Thus, the problem is reduced to a boundary value problem of (23) and (25) under the boundary conditions (29) and (30).

Equations (23) and (25) are forth order partial differential equations (PDEs). A reduced variable $\phi = D^2\Phi$ satisfies

$$(D^2 - \xi^{-2})\phi = 0. \tag{31}$$

Assuming the separation of variables as $\phi = R_\phi(r)\Theta_\phi(\theta)$ we obtain

$$r^2(\frac{R''_\phi}{R_\phi} - \xi^{-2}) = -\frac{\sin\theta}{\Theta_\phi}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\Theta_\phi}{d\theta}\right) = \lambda_\phi, \tag{32}$$

where λ_ϕ is a separation constant. The equation for Θ_ϕ is reduced to

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + (\lambda_\phi - \frac{1}{1-z^2})w = 0, \tag{33}$$

where $z = \cos\theta$ and $\Theta_\phi = \sin\theta w(\cos\theta)$. Equation (33) is the equation for Legendre's bipolynomial $P_l^1(z)$. Thus we obtain

$$\Theta_{\phi,l}(\theta) = \sin\theta P_l^1(\cos(\theta)); \quad \lambda_\phi = l(l+1), \tag{34}$$

where l is a positive integer. The radial equation in (32) obeys

$$\frac{d^2R_{\phi,l}}{dr^2} - (\xi^{-2} + \frac{l(l+1)}{r^2})R_{\phi,l} = 0, \tag{35}$$

where we write an explicit dependence of R_ϕ on l as $R_{\phi,l}$. This equation can be converted into the equation for modified Bessel function. Taking into account the regularity of $r \to \infty$, the solution $R_{\phi,l}$ must be proportional to the modified Bessel function $K_{l+1/2}(r/\xi)$. Thus, the solution of ϕ can be written as

$$\phi(r,\theta) = \sum_{l=1}^{\infty} C_l\sqrt{\frac{r}{\xi}}K_{l+\frac{1}{2}}\left(\frac{r}{\xi}\right)\sin\theta P_l^1(\cos\theta). \tag{36}$$

Note that $K_{l+1/2}(r/\xi)$ can be represented by an elementary function because modified Bessel functions of fractional order are proportional to spherical modified Bessel functions.

Let us obtain Φ from the equation $D^2\Phi = \phi$. It is easy to show that $\Phi(r,\theta) = \sum_{l=1}^{\infty} R_{\Phi,l}(r)\Theta_{\Phi,l}(\theta)$ has the solution

$$\Theta_{\Phi,l} = \Theta_{\phi,l} = \sin\theta P_l^1(\cos\theta). \tag{37}$$

Thus, the radial equation for $R_{\Phi,l}$ is reduced to

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right) R_{\Phi,l} = C_l\sqrt{\frac{r}{\xi}}K_{l+\frac{1}{2}}\left(\frac{r}{\xi}\right). \tag{38}$$

The general solution of (38) is given by a linear combination of homogeneous solutions and a special solution of the inhomogeneous equation. The independent homogeneous solutions of (38) are r^{l+1} and r^{-l}. From the direct substitution, it is easy to show a special solution $R_{\Phi,l}^{NHS}$

$$R_{\Phi,l}^{NHS} = \xi^2 C_l\sqrt{\frac{r}{\xi}}K_{l+1/2}(\frac{r}{\xi}). \tag{39}$$

Thus, $\Phi(r,\theta)$ can be represented by

$$\Phi(r,\theta) = \sum_{l=1}^{\infty}\left(a_{\Phi,l}r^{l+1} + \frac{b_{\Phi,l}}{r^l} + C_l\xi^2\sqrt{\frac{r}{\xi}}K_{l+\frac{1}{2}}\left(\frac{r}{\xi}\right)\right)\sin\theta P_l^1(\cos\theta). \tag{40}$$

Similarly, from (23) and (40) we obtain

$$\Psi(r,\theta) = \sum_{l=1}^{\infty} R_{\Psi,l}(r)\sin\theta P_l^1(\cos\theta), \tag{41}$$

where

$$R_{\Psi,l}(r) = a_{\Psi,l}r^{l+1} + \frac{b_{\Psi,l}}{r^l} + c_{\Psi,l}r^{l+3} + d_{\Psi,l}r^{2-l} - \mu_r C_l\xi^4\sqrt{\frac{r}{\xi}}K_{l+1/2}(\frac{r}{\xi}). \tag{42}$$

These are general results for outer solutions without explicit considerations of boundary conditions.

From the boundary condition (30) there are the following relations

$$a_{\Phi,l} = c_{\Psi,l} = 0 \text{ (for all } l\text{)}: \quad a_{\Psi,1} = \frac{1}{2}, \quad a_{\Psi,l} = 0 \ (l \geq 2). \tag{43}$$

From the boundary conditions (29) and (30) we can determine $b_{\Psi,l}$, $d_{\Psi,l}$, C_l and $b_{\Phi,l}$. However, for $l \geq 2$, we have vector equations

$$A_l\mathbf{x}_l = \mathbf{0} \quad l \geq 2, \tag{44}$$

where A_l is the matrix determined through the boundary conditions, and the transverse of $\mathbf{x}_l$ is ${}^T\mathbf{x}_l = (b_{\Psi,l}, d_{\Psi,l}, C_l, b_{\Phi,l})$. Since eq.(44) should be held for any R and other parameters, $detA_l$ cannot be zero in general. Namely we have

$$b_{\Psi,l} = d_{\Psi,l} = C_l = b_{\Phi,l} = 0 \tag{45}$$

for $l \geq 2$. Equations (40)-(42) are now reduced to

$$\Psi(r,\theta) = \left(\frac{1}{2}r^2 + \frac{b}{r} + dr - \mu_r C\xi^4 e^{-r/\xi}(1+\frac{\xi}{r})\right)\sin^2\theta, \tag{46}$$

and

$$\Phi(r,\theta) = \left(\frac{b_\Phi}{r} + C\xi^2 e^{-r/\xi}(1+\frac{\xi}{r})\right)\sin^2\theta, \tag{47}$$

where $b = b_{\Psi,1}$, $d = d_{\Psi,1}$, $C = (\sqrt{\pi/2})C_1$ and $b_\Phi = b_{\Phi,1}$. Since $D^2\Psi \to -\frac{2d}{r}\sin^2\theta$ as $r \to \infty$, the second relation of (30) leads to

$$b_\Phi = d. \tag{48}$$

(Note that Ramkissoon's result becomes $b_\Phi = -d$). The other three boundary conditions in (29) and (30) determine b,d and C as

$$\begin{aligned} b &= -\frac{1}{2} + \frac{3(1+\xi)(1+\mu_r\xi^2)}{2\{2(1+\xi) - \mu_r\xi\}}, \\ d &= -\frac{3(1+\xi)}{2\{2(1+\xi) - \mu_r\xi\}}, \\ C &= \frac{3e^{\xi^{-1}}}{2\xi^2\{2(1+\xi) - \mu_r\xi\}}. \end{aligned} \tag{49}$$

Thus, the problem has been solved.

From (16) the explicit profile of velocity field is given by

$$\begin{aligned} v_r &= (1 + \frac{2b}{r^3} + \frac{2d}{r} - 2\mu_r C\frac{\xi^4}{r^2}(1+\frac{\xi}{r})e^{-r/\xi})\cos\theta, \\ v_\theta &= (-1 + \frac{b}{r^3} - \frac{d}{r} + \mu_r C\frac{\xi^3}{r}(1+\frac{\xi}{r}+\frac{\xi^2}{r^2})e^{-r/\xi})\sin\theta, \end{aligned} \tag{50}$$

and ω_φ is

$$\omega_\varphi = (\frac{d}{r^2} + C\frac{\xi^2}{r}e^{-r/\xi}(1+\frac{\xi}{r}))\sin\theta. \quad (51)$$

The pressure is determined from (14). Each component obeys

$$\frac{\partial p}{\partial r} = -\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\{D^2\Psi + \mu_r\Phi\}; \quad \frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\{D^2\Psi + \mu_r\Phi\}. \quad (52)$$

It is easy to show the terms proportional to the spherical modified Bessel function cancel with each other in (52) as expected the property of the harmonic function. The terms in proportion to $1/r$ and r^2 in Ψ disappear in (52) as in conventional cases. Thus, the pressure is determined by the Stokes pole of Ψ which is proportional to r as $\Psi_{st} = dr\sin^2\theta$ and $1/r$ in Φ as $\Phi_{st} = (d/r)\sin^2\theta$. From the relation $D^2\Psi_{st} + \mu_r\Phi_{st} = -\frac{d(2-\mu_r)}{r}\sin^2\theta$ and (52), we obtain

$$p = -\frac{d(2-\mu_r)}{r^2}\cos\theta + p_0, \quad (53)$$

where p_0 is an unimportant constant.

B. Drag force exerted on a sphere

Let us calculate the force exerted on a sphere in a micropolar fluid flow. For this purpose we explicitly write the stress tensor T_{ij} as

$$T_{ij} = -p\delta_{ij} + (1-\frac{\mu_r}{2})\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) + \frac{\mu_r}{2}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right) - \mu_r\epsilon_{mij}\omega_m, \quad (54)$$

where ϵ_{mij} is the alternative tensor of Levi-Civita. Now we decouple the stress tensor into two parts as

$$T_{ij} = \sigma_{ij} + \tau_{ij}: \quad \tau_{ij} = \mu_r\epsilon_{mij}(\Omega_m - \omega_m), \quad (55)$$

where $\Omega_m = (\mathrm{rot}\mathbf{v})_m/2$, and τ_{ij} is the asymmetric part of the stress tensor and σ_{ij} is the symmetric part.

Let us consider the contribution from τ_{ij} to the drag force. From the consideration of symmetry, the drag force from this part is

$$F_\tau = \int_S \tau_{rx}|_{r=1}dS = \mu_r\int_S(-\frac{y}{r}\Omega_z + \frac{z}{r}\Omega_y)dS \quad (56)$$

where dS represents the integration on the surface of the sphere. To derive the last expression in (56) we use (55) and $\vec{\omega} = 0$ on the surface of the sphere. From $x/r = \cos\theta$, $y/r = \sin\theta\cos\varphi$ and $z/r = \sin\theta\sin\varphi$ eq.(56) is reduced to

$$F_\tau = -\int_0^{2\pi} d\varphi\cos\varphi\int_0^\pi d\theta\Omega_\varphi(1,\theta)\sin\theta = 0. \quad (57)$$

Other contributions for the drag force correspond to those in the conventional Stokes problem. The contribution from the velocity field is given by

$$F_v = -2\pi(1-\frac{\mu_r}{2})\int_S dS\sin\theta\frac{\partial v_\theta}{\partial r}(1,\theta) = 2\pi(2-\mu_r)\frac{2(1+\xi)+\mu_r(1+3\xi+6\xi^2(1+\xi))}{2(1+\xi)-\mu_r\xi}, \quad (58)$$

where we use (50) and $\int_0^\pi \sin^3\theta = 4/3$. On the other hand, the contribution from the pressure term is

$$F_p = 2\pi\int_S dSp\cos\theta = \frac{2\pi(2-\mu_r)(1+\xi)}{2(1+\xi)-\mu_r\xi}, \quad (59)$$

where we use $\int_0^\pi d\theta\cos^2\theta\sin\theta = 2/3$, (53) and (49).

The total contribution $F_v + F_p + F_\tau$ is given by

$$F_{tot} = \frac{2\pi(2-\mu_r)\{3(1+\xi)+\mu_r(1+3\xi+6\xi^2(1+\xi))\}}{2(1+\xi)-\mu_r\xi}. \quad (60)$$

The result can be converted into that in the physical unit. The drag force of the sphere (radius a) exerted by a micropolar fluid whose viscosity η and η_r is given by

$$F_{tot} = \frac{2\pi(\eta+\eta_r)aU(2-\mu_r)\{3(1+\xi)+\mu_r(1+3\xi+6\xi^2(1+\xi))\}}{2(1+\xi)-\mu_r\xi}, \quad (61)$$

which is reduced to $6\pi\eta aU$ as $\mu_r = 2\eta_r/(\eta+\eta_r) \to 0$ and $\xi \to 0$.

C. Beyond the Stokes approximation

In this subsection we briefly present the procedure to calculate the flow and the drag beyond the Stokes approximation. As mentioned in Introduction, it is hopeless to obtain an explicit representation of flow field based on the Oseen approximation. Thus, it is natural to adopt the series expansion of Reynolds number R. For this purpose we rewrite the full equations (2) and (3) by Ψ and Φ as

$$\frac{R}{r^2\sin\theta}\left(\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial r}-\frac{\partial\Psi}{\partial r}\frac{\partial}{\partial\theta}+2\cot\theta\frac{\partial\Psi}{\partial r}-\frac{2}{r}\frac{\partial\Psi}{\partial\theta}\right)D^2\Psi=D^4\Psi+\mu_rD^2\Phi, \tag{62}$$

and

$$\frac{RI}{\mu_r r^2\sin\theta}[\frac{1}{r}\frac{\partial\Psi}{\partial\theta}(1-r\frac{\partial}{\partial r})+\frac{\partial\Psi}{\partial r}(\frac{\partial}{\partial\theta}-\cot\theta)]\Phi=D^2\Psi+2\Phi-\mu_BD^2\Phi. \tag{63}$$

The solution is able to be obtained by

$$\Psi=\Psi_0+R\Psi_1+\cdots,\quad \Phi=\Phi_0+R\Phi_1+\cdots, \tag{64}$$

where Ψ_0 and Φ_0 are the solutions in Stokes approximation in (46) and (47). To obtain the explicit forms of Ψ_1 and Φ_1 we need long and tedious calculation. Thus, we do not give the explicit expressions.

However, we note that at the first order approximation the velocity does not behave properly at infinity. The nonexistence of the second approximation to the Stokes solution in conventional flow is known as Whitehead's paradox [27]. To remove this difficulty, we use the matched asymptotics as introduced in Section 2. Namely, we use the outer solution determined from (13) far from the sphere, while we use the simple perturbative result as the inner solution. An unknown constant arising from the lack of the boundary condition far from the sphere is determined by the matching of the inner solution and the outer solution. Because (13) is solvable, it is straightforward to obtain the flow fields as series expansions of R. The explicit calculation will be discussed elsewhere.

IV. FLOW PAST A CYLINDER

As is well known, the conventional viscous flow in 2D, the solution based on the Stokes approximation does not satisfy the boundary condition far from the cylinder . This is known as Stokes' paradox which is more serious than Whitehead's paradox in 3D. Thus, we need to adopt the matched asymptotic method from the lowest order expansion of R.

In two dimensional systems, the stream function is defined through

$$v_r=\frac{1}{r}\frac{\partial\Psi}{\partial\theta}:\quad v_\theta=-\frac{\partial\Psi}{\partial r}. \tag{65}$$

These choices automatically ensure eq.(1). In 2D, the field variables are written as $\mathbf{v}=(v_r,v_\theta)$ and $\omega=\omega_z\hat{z}$ where $\hat{z}$ is the unit vector vertical to flow plane.

The procedure is as follows. In the next subsection we will give the explicit calculation of the Stokes approximation as an inner solution. The preceding subsection, we will obtain an outer solution and use the matched asymptotics. In subsection C, we will calculate the force exerted by the fluid flow.

A. Stokes approximation as an inner solution

At the first step, let us obtain the solution of (14) and (15) based on the Stokes approximation. From (65) and (22) we obtain an equation for ω_z and Ψ

$$\triangle^2\Psi+\mu_r\triangle\omega_z=0. \tag{66}$$

On the other hand, from (15) we obtain

$$\triangle\Psi+2\omega_z-\mu_B\triangle\omega_z=0. \tag{67}$$

From the operation of the Laplacian to (67) with the help of (66) we obtain

$$\triangle(\triangle-\xi^{-2})\omega_z=0, \tag{68}$$

where ξ is given in (26). The problem will be solved under the boundary condition on the cylinder. Although we cannot adopt the boundary condition far from the cylinder, we know that the leading singularity comes from a logarithmic divergent term which will be regularized by the matching with an outer solution. Thus, higher divergent terms which obey power laws will be omitted in discussion in this subsection. Thus, the boundary condition far from the cylinder is

$$\omega_z = -\frac{1}{2}\triangle\Psi, \tag{69}$$

which is equivalent to $\omega_z = \mathrm{rot}\mathbf{v}/2$.

Taking into account the boundary condition on the cylinder

$$\Psi(1,\theta) = \frac{\partial\Psi}{\partial r}(1,\theta) = \omega_z(1,\theta) = 0, \tag{70}$$

we obtain

$$\triangle\omega_z(r,\theta) = \sum_{n=1}^{\infty} C_n^{(2)} K_n\left(\frac{r}{\xi}\right)\sin n\theta, \tag{71}$$

where $C_n^{(2)}$ is a constant. Thus, it is easy to obtain the general form of ω_z as

$$\omega_z(r,\theta) = \sum_{n=1}^{\infty}\{a_{\omega,n}r^n + b_{\omega,n}r^{-n} + C_n^{(2)}\xi^2 K_n\left(\frac{r}{\xi}\right)\}\sin n\theta \tag{72}$$

It is obvious $a_{\omega,n} = 0$ for all n to satisfy the boundary condition far from the cylinder. From the last equation of (70) we obtain

$$b_{\omega,n} = -C_n^{(2)} K_n(\xi^{-1}). \tag{73}$$

Similarly, we can obtain Ψ as

$$\begin{aligned}\Psi(r,\theta) = \sin\theta\{a_\Psi^{(2)} r + \frac{b_\Psi^{(2)}}{r} + \alpha r\ln r - \mu_r\xi^4 C_1^{(2)} K_1\left(\frac{r}{\xi}\right)\}\\ + \sum_{n=2}^{\infty}\sin n\theta\{\frac{b_{\Psi,n}^{(2)}}{r^n} + d_{\Psi,n}^{(2)} r^{2-n} - \mu_r C_n^{(2)}\xi^4 K_n\left(\frac{r}{\xi}\right)\}.\end{aligned} \tag{74}$$

As in the case of the flow past a sphere, it is easy to show that the mode $n \geq 2$ will become zero. Thus, the problem can be simplified. From (69) and the first two relations of (70) we obtain

$$b_\omega = -\alpha, \quad a_\Psi^{(2)} = \alpha\hat{a}, \quad b_\Psi^{(2)} = \alpha, \quad C_1^{(2)} = \frac{\alpha}{K_1(\xi^{-1})}, \tag{75}$$

where

$$\hat{a} = \frac{1}{2}\left[\mu_r\xi^3\left(\xi + \frac{K_1'(\xi^{-1})}{K_1(\xi^{-1})}\right) - 1\right] : \quad \hat{b} = \frac{1}{2}\left[\mu_r\xi^3\left(\xi - \frac{K_1'(\xi^{-1})}{K_1(\xi^{-1})}\right) + 1\right]. \tag{76}$$

Thus, we obtain

$$\begin{aligned}\Psi(r,\theta) &= \alpha\sin\theta[\hat{a}r + \frac{\hat{b}}{r} + r\ln r - \mu_r\xi^4\frac{K_1(r/\xi)}{K_1(\xi^{-1})}],\\ \omega(r,\theta) &= \alpha\sin\theta[-\frac{1}{r} + \xi^2\frac{K_1(r/\xi)}{K_1(\xi^{-1})}].\end{aligned} \tag{77}$$

As usual Stokes problem in 2D the flow field produced from (77) has the logarithmic singularity in the limit of $r \to \infty$.

B. The outer solution and matched asymptotics

To resolve Stokes' paradox for $r \to \infty$, we adopt the matched asymptotics developed by Kaplun and Lagerstrom [25]. We reconsider (77)

$$\Psi(r,\theta) \sim \alpha(R)(\hat{a}r + \frac{\hat{b}}{r} + r\ln r - \mu_r\xi^4\frac{K_1(r/\xi)}{K_1(\xi^{-1})})\sin\theta, \tag{78}$$

where α is replaced by a multiplier $\alpha(R)$ which is allowed to depend upon Reynolds number, because our asymptotic sequence is unspecified. Although this approximation cannot satisfy the condition $\mathbf{v} = \mathbf{e}_x$ in (29) or $\Psi \to r\sin\theta$ for $r \to \infty$, it can be matched to the uniform stream, regarded as the first term of an Oseen expansion.

Now, introducing a new variable

$$\rho = \frac{R}{1-\mu_r/2}r = \epsilon r : \quad \epsilon = \frac{R}{1-\mu_r/2}, \tag{79}$$

the outer equation (13) is

$$\frac{\partial\mathbf{u}}{\partial\hat{x}} = -\hat{\nabla}\tilde{p} + \hat{\triangle}\mathbf{u}, \tag{80}$$

where $\hat{\nabla} = (1-\mu_r/2)^{-1}\tilde{\nabla}$. Then, the Oseen expansion begins with

$$\Psi \sim \frac{1}{\epsilon}\rho\sin\theta + \cdots \quad \text{as } \epsilon \to 0. \tag{81}$$

Writing the Stokes expression (78) by the Oseen variable (79) the leading term is now

$$\Psi \sim \frac{\alpha(\epsilon)}{\epsilon}\ln(1/\epsilon)\rho\sin\theta, \tag{82}$$

where $\alpha(\epsilon) = \alpha(R)$. This matches (81) if $\alpha(\epsilon) = (\ln(1/\epsilon) + k)^{-1}$ where k is a constant will be determined later.

Expansion of the Stokes approximation (78) further by ρ and $\alpha(\epsilon)$ leads to

$$\Psi \sim \frac{1}{\epsilon}[1 + \alpha(\epsilon)(\ln\rho - k + \hat{a})]\rho\sin\theta, \tag{83}$$

where $\alpha(\epsilon) = \alpha(R)$. This requires the Oseen expansion (81) continues as

$$\Psi \sim \frac{1}{\epsilon}[\rho\sin\theta + \alpha(\epsilon)\psi(\rho,\theta) + \cdots]. \tag{84}$$

Substituting this into the full equation, ψ satisfies the linearized Oseen equation

$$(\hat{\triangle} - \frac{\partial}{\partial\hat{x}})\hat{\triangle}\psi = 0. \tag{85}$$

The appropriate solution for the stream function can be found as an infinite series [29].

The fundamental solution due to Oseenlet gives as the Cartesian velocity component

$$\begin{aligned} u_x &= \frac{\partial\psi}{\partial(\rho\sin\theta)} = 2c_2\{\frac{\partial}{\partial(\rho\cos\theta)}[\ln\rho + e^{\frac{1}{2}\rho\cos\theta}K_0(\rho/2)] - e^{\frac{1}{2}\rho\cos\theta}K_0(\rho/2)\}, \\ u_y &= -\frac{\partial\psi}{\partial(\rho\cos\theta)} = 2c_2\frac{\partial}{\partial(\rho\sin\theta)}[\ln\rho + e^{\frac{1}{2}\rho\cos\theta}K_0(\rho/2)], \end{aligned} \tag{86}$$

where c_2 is a constant which will be determined by the matching. The term in $\ln\rho$ at the origin cancels the term involving $K_0(\rho/2)$. For small ρ we obtain the integrated form of ψ as

$$\psi \sim -c_2(\ln\frac{4}{\rho} + 1 - \gamma)\rho\sin\theta + O(\rho^2\ln\rho), \tag{87}$$

where γ is Euler's constant $\gamma = 0.5772....$ Using this we find the Oseen expansion (84) behaves near the cylinder

$$\Psi \sim \frac{1}{\epsilon}\rho\sin\theta[1 + c_2\alpha(\epsilon)(\ln\frac{\rho}{4} + \gamma - 1)]. \tag{88}$$

This can match (83) with choosing

$$c_2 = 1: \quad k = \hat{a} - \gamma + 1 + \ln 4. \tag{89}$$

Thus, we obtain

$$\alpha(\epsilon) = (\ln\frac{4}{\epsilon} - \gamma + 1 + \hat{a})^{-1}. \tag{90}$$

This vanishes for $\epsilon \to 0$. The explicit expression near the cylinder, thus, is given by

$$\Psi(r,\theta) \simeq \alpha(\epsilon)\{\hat{a}r + \frac{\hat{b}}{r} + r\ln r - \mu_r\xi^4\frac{K_1(r/\xi)}{K_1(\xi^{-1})}\}\sin\theta. \tag{91}$$

This expression satisfies all of boundary conditions.

The explicit expression of the inner solution is thus given by

$$\begin{aligned} v_r &= \alpha(\epsilon)[\hat{a} + \frac{\hat{b}}{r^2} + \ln r - \mu_r\xi^4\frac{K_1(r/\xi)}{rK_1(\xi^{-1})}]\cos\theta, \\ v_\theta &= \alpha(\epsilon)[-\hat{a} + \frac{\hat{b}}{r^2} - \ln r - 1 + \mu_r\xi^3\frac{K_1'(r/\xi)}{K_1(\xi^{-1})}]\sin\theta. \end{aligned} \tag{92}$$

The pressure is similarly determined from (14) and (65) as

$$\frac{\partial p}{\partial r} = \frac{1}{r}\frac{\partial}{\partial\theta}\{\triangle\Psi + \mu_r\omega_z\} \quad \frac{1}{r}\frac{\partial p}{\partial\theta} = -\frac{\partial}{\partial r}\{\triangle\Psi + \mu_r\omega_z\}. \tag{93}$$

As in the case of 3D, the pressure is determined by the Stokes pole $\Psi \sim r\ln r$. The result is

$$p = p_0 - \frac{2-\mu_r}{r}\alpha(\epsilon)\cos\theta, \tag{94}$$

where p_0 is an unimportant constant. Thus, the explicit representations for the inner solution which can be connected with the outer solution are obtained.

C. Drag force exerted on a cylinder

As in the previous section, let us calculate the drag force exerted on a cylinder by a moving fluid. The drag force is calculated from three contributions as in the previous section. The force coming from asymmetric tensor is not zero in 2D. The result is

$$F_\tau = -\mu_r \int_0^{2\pi} \frac{y}{r}\Omega(1,\theta)d\theta = -\mu_r \int_0^{2\pi} \Omega(1,\theta)\sin\theta d\theta, \tag{95}$$

where $\Omega = (\mathrm{rot}\mathbf{v})_z/2$ and it is reduced to $\Omega(1,\theta) = \frac{1}{2}\partial v_\theta/\partial r$. From (92) we obtain

$$F_\tau = \pi\mu_r\alpha(\epsilon)\{1+\mu_r\beta(\epsilon)\} \tag{96}$$

where we use $\int_0^{2\pi}\sin^2\theta = \pi$ and

$$\beta(\xi) = \frac{\xi^2}{2K_1(\xi^{-1})}[K_1''(\xi^{-1}) - \xi K_1'(\xi^{-1}) + \xi^2 K_1(\xi^{-1})]. \tag{97}$$

The force coming from the normal stress is given by

$$F_p = -\int_0^{2\pi} p\cos\theta d\theta = \pi(2-\mu_r)\alpha(\epsilon) \tag{98}$$

and the force from the shear stress is

$$F_f = -(1-\frac{\mu_r}{2})\int_0^{2\pi} d\theta \sin\theta r\frac{\partial v_\theta}{\partial r}|_{r=1} = \pi(2-\mu_r)\alpha(\epsilon)(1+\mu_r\beta(\xi)). \tag{99}$$

Thus, the total force is given by

$$F_{tot} = \pi\alpha(\epsilon)(4-\mu_r+2\mu_r\beta(\xi)) \tag{100}$$

which can be rewritten as

$$F_{tot} = \pi(\eta+\eta_r)U\alpha(\epsilon)(4-\mu_r+2\mu_r\beta(\xi)) \tag{101}$$

in the physical unit. This is reduced to the result obtained from the Oseen approximation [30] as $F_{tot} \to 4\pi\eta U/(S+1/2)$ in the limit of $\mu_r \to 0$ where $S = \ln(4/R) - \gamma$.

V. STEADY FLOW INSIDE A CONTAINER

In this section we discuss steady circulation flows inside a container. Since we will not discuss the flow outside the cylinder, the Stoke approximation gives an appropriate solution for the slow viscous flow. Thus, the basic equations in this section are eqs.(14) and (15).

As mentioned in section 2, we do not impose the boundary condition $\mathbf{v} = 0$ but the slip boundary condition (6) because the problem becomes over-complete. This may be understood as follows. The rotation of particles without the slip boundary produces a fast flow near the boundary. The width of boundary layer as thin as the particles' radius. If we are able to normalize the flow by the fastest flow rate near the boundary, we may be able to solve the problem under the condition $\mathbf{v} = 0$ on the wall. It is, however, difficult to determine the position where the flow rate is maximum theoretically, and the width of the boundary layer becomes zero as the radius of particles becomes zero. Thus, we adopt the slip boundary condition in this section.

Let us demonstrate the existence of an interesting circulation flow inside a cylinder. The surface of micropolar fluid exists art $z = 0$ and z increases as depth from the surface increases. Let us consider the flow $\mathbf{v} = (v_r, v_\theta, v_z) = (v_r(r,z), 0, v_z(r,z))$. In this case the flow can be represented by a vector potential or a stream function $\mathbf{v} = \mathrm{rot}[\Psi(r,z)\mathbf{e}_\theta]$ or

$$v_r = -\frac{\partial\Psi}{\partial z} : \quad v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\Psi). \tag{102}$$

Since the micro-rotation field is represented by $\vec{\omega} = \omega_\theta\mathbf{e}_\theta$, the basic equations (14) and (15) can be rewritten as

$$\mathcal{D}^4\Psi + \mu_r\mathcal{D}^2\omega_\theta = 0 \tag{103}$$

and

$$\mathcal{D}^2(\mathcal{D}^2 - \xi^{-2})\omega_\theta = 0, \tag{104}$$

where $\mathcal{D}^2$ is defined by

$$\mathcal{D}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}. \tag{105}$$

The boundary condition (6) and the conservation law (7) are reduced to

$$\Psi(1,z) = \omega_\theta(1,z) = 0, \quad \frac{\partial\Psi}{\partial z} = \mathcal{D}^2\Psi = \omega_\theta = 0 \text{ at } r = 0. \tag{106}$$

Unfortunately the conditions at the center of cylinder will be identities, which will not be useful to determine unknown constants.

Assuming $\mathcal{D}^2\omega_\theta = R_\theta(r)Z(z)$, eq.(104) is reduced to

$$\frac{1}{R_\theta}\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right]R_\theta = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda, \tag{107}$$

where λ is a separation constant. In standard sense, λ should be negative, because a flow exists along the stream line (Ψ is a constant). In this case the flow becomes a circular flow or a convection. For $\lambda < 0$, Z is expected to be $\lambda = \lambda_n = (n/H)^2$ with a positive integer n, and

$$Z_n(z) = \sin(nz/H), \tag{108}$$

where H is the height of cylinder. The equation for R_θ is now solvable as $R_\theta \sim I_1(\frac{r}{(H/n)^2})$ where $I_1(z)$ is the modified Bessel function. Thus, we obtain

$$\mathcal{D}^2\omega_\theta = \sum_{n=1}^{\infty} c_{\theta,n} I_1\left(\frac{r}{(H/n)^2}\right)\sin\left(\frac{nz}{H}\right), \tag{109}$$

where $c_{\theta,n}$ is a constant. For later discussion we only discuss the mode of $n = 1$ because of its fundamental role. From (109) we obtain

$$\omega_\theta = C_0[C_1 I_1(r/H) + \xi^2 I_1(r/\zeta_0)]\sin(\frac{z}{H}) \tag{110}$$

where C_0, C_1 are constants, and $\zeta_0 = \xi/\sqrt{1+(\xi/H)^2}$. From (103) we obtain

$$\Psi = C_0\xi^4\mu_r[\frac{I_1(1/\zeta_0)}{I_1(1/H)}I_1(r/H) - I_1(r/\zeta_0)]\sin\left(\frac{z}{H}\right) \tag{111}$$

where we use the boundary condition (106). Note that we omit a formal special solution of (103) coming from the first term of (110), because it becomes singular at the center of the container.

From (102) the circular flow arising from (111) is given by

$$v_r = -\frac{1}{H}C_0\xi^4\mu_r[\frac{I_1(1/\zeta_0)}{I_1(1/H)}I_1(r/H) - I_1(r/\zeta_0)]\cos(\frac{z}{H}), \tag{112}$$

and

$$v_z = C_0\xi^4\mu_r[\frac{1}{H}\frac{I_1(1/\zeta_0)}{I_1(1/H)}I_0(r/H) - \frac{1}{\zeta_0}I_0(r/\zeta_0)]\sin(\frac{z}{H}), \tag{113}$$

where we use $(1/z)(d/dz)(zI_1(z)) = I_0(z)$. Equation (113) is reduced to

$$v_z \simeq C_0\xi^4\mu_r[2I_1(1/\xi) - \xi^{-1}I_0(r/\xi)]\sin(\frac{z}{H}) \tag{114}$$

for $H \gg 1$ (i.e. $\zeta_0 \to \xi$ and $I_1(1/H) \sim 1/(2H)$). Although Eq.(114) contains a similar $r-$dependence to that observed in the experiment of granular particles in vibrating beds [20], the $z-$dependence of the functional form is different.

Let us examine the case of $\lambda > 0$. This corresponds to the situation in which the flow is localized near the surface. A parallel procedure to previous section we obtain

$$\mathcal{D}^2\omega_\theta = c_0 I_1(r/\zeta)\exp[-\sqrt{\lambda}z], \tag{115}$$

where $I_1(x)$ is the modified Bessel function, λ is an undetermined separation constant in (107) assumed to be $\lambda < 1/\xi^2$, and $\zeta = \xi/\sqrt{1-\lambda\xi^2}$. Thus, we obtain

$$\omega_\theta = c_0[c_1 J_1(\sqrt{\lambda}r) + \xi^2 I_1(r/\zeta)]\exp[-\sqrt{\lambda}z], \tag{116}$$

where $J_1(z)$ is the Bessel function, c_0 and c_1 are constants which are determined by the boundary condition for ω_θ. From (103) and (106) we obtain

$$\Psi(r,z) = c_0\xi^4\mu_r[\frac{I_1(1/\zeta)}{J_1(\sqrt{\lambda})}J_1(\sqrt{\lambda}r) - I_1(r/\zeta)]\exp[-\sqrt{\lambda}z]. \tag{117}$$

This solution may not be a solution for smooth flow, because the flow along the stream line ($\Psi = const.$) cannot circulate as convection flows. Since the flow in vibrating beds is not smooth, we may examine this solution as a steady solution for granular flows.

The solution of the flow field described by (117) is given by

$$v_r = c_0\xi^4\mu_r\sqrt{\lambda}[\frac{I_1(1/\zeta)}{J_1(\sqrt{\lambda})}J_1(\sqrt{\lambda}r) - I_1(r/\zeta)]\exp[-\sqrt{\lambda}z] \tag{118}$$

and

$$v_z = c_0\xi^4\mu_r[\sqrt{\lambda}\frac{I_1(1/\zeta)}{J_1(\sqrt{\lambda})}J_0(\sqrt{\lambda}r) - \frac{1}{\zeta}I_0(r/\zeta)]\exp[-\sqrt{\lambda}z]. \quad (119)$$

The result (119) is reduced to

$$v_z \simeq c_0\xi^4\mu_r[2I_1(1/\xi) - \frac{1}{\xi}I_0(r/\xi)]\exp[-\sqrt{\lambda}z] \quad (120)$$

in the limit of $\lambda \to 0$, which is identical to that reported in an experiment for vibrating beds [20]. In fact, their v_z in their eq.(11b) is given by

$$v_z = \frac{l}{\tau}\{1 + \frac{1}{f_I(1/\xi)}(1 - I_0(r/\xi)\}e^{-z/l}, \quad (121)$$

where l and τ are respectively the characteristic length scale and the time scale, and $f_I(x) = 2I_1(x)/x - 1$ from the conservation law (7). Equation (121) can be rewritten as

$$v_z \propto (2\xi I_1(1/\xi) - I_0(r/\xi))e^{-\sqrt{\lambda}z} \quad (122)$$

with identifying $l = 1/\sqrt{\lambda}$. Thus, eq.(122) is identical to (120). On the other hand, eq.(118) cannot describe convection flows, because v_r in circulation flows must have turning point(s) to change the direction of flow. However, we expect that the flow is similar to that observed in experiments near the surface. Although we do not discuss the flow in a rectangular container explicitly, it is easy to derive eq.(11b) of Ref. [20] by the parallel discussion to that presented here.

The calculation in this section may not be matured when we compare it with those in the previous sections. However, the result is interesting, which is comparable with experimental observation of vibrating beds of granular flow. Therefore, we expect the existence of close relationships between the micropolar fluids and granular flows. We will need more systematic investigations taking into account the role of the gravity, the mechanism of the emergence of convections as well as the detail comparisons with experiments and simulations.

VI. DISCUSSION

In this paper, we illustrate systematic calculations on the viscous micropolar flows around cylinder and the sphere, and preliminary calculation on the steady flow inside the cylinder.

If the micropolar fluid is an important concept, the first two parts will play fundamental roles. On the other hand, the last part demonstrates that the solution of the micropolar fluid is identical to that observed in granular flows. In this discussion, at first let us discuss the relevancy of micropolar models in granular flows. In the second part, we will survey the feature direction of the investigation on the micropolar fluid model.

A. Is Micropolar fluid Model actually relevant in Granular Physics?

Now, we have to discuss whether the micropolar fluid mechanics is relevant in granular physics. We also need to check the relevancy of the Newtonian micropolar fluid model if we would accept the relevancy of the micropolar fluid model in granular physics.

First, as well known, granular materials contain grains. This may remind us of suspicion on the validity of all of continuous approaches. Although this suspicion is reasonable but not productive. Actually, it is hard to characterize flows by molecular dynamic simulations. In addition, if we expect the limitation of continuous description, we need quantitative checks. In fact, most of theoretical arguments on granular flows are based on continuous models.

Second, granular flows sometimes coexists with the stagnation regions, where the flow is influenced by interesting static properties. For example, the saturation of pressure is known as Janssen's law (derived by a continuous model with a suitable constituitve relation) which may suggest the difference between granular materials and usual fluids in which the pressure is a linear function of depth. Of course, statics is out of our scope at least in this paper, and we cannot characterize coexistence of statics and dynamics in granular materials in the present framework. However, as Janssen's law is reduced to the conventional fluid result if the width of the container is infinitely large, we realize that (i) if the boundary effects are not important, we may expect the situation that fluid models such as the miropolar fluid model is relevant. (ii) In such the case, granular flows cannot be stopped. (iii) After the flow is generated, the flow can be maintained easily. Thus, the static properties are suppressed in these cases. (iv) The experiment and simulation of Kano et al [15] strongly suggests the relevancy the micropolar fluid model in (non-stagnant) granular flows.

Third, if the fluid mechanical approach such as the micropolar fluid model is acceptable, readers may be skeptical of the validity of the Newtonian micropolar fluid model. To answer this objection, we have demonstrate the part of relevancy of the calculation based on the Newtoninan micropolar fluid model. We will compare its result with the result of the MD simulation elsewhere [21] but introduce the part of them. First we note that the comparision is only relevant in the region far from onset of convections, because below the onset the granular particles are not in a fluid-like state but in a solid-like state. Second, as mentioned in the previous section, we can predict not only the vertical velocity profile but the horizontal velocity profile. After the average of large number of cycles (e.g. 1000 in Wada and Hayakawa [21]), we can suppress the most of stagnant effects such as avelanches of arches and the flow caused by fractures in granular beds. Thus we can obtain smooth velocity profile as a function of time. The result of our simulation in the most of time regions supports the relevancy of the micropolar calculation in the previous section. Unfortunately, there remains the effect of the bottom boundary which produces the break down of the arches in the bottom part, which cause the completion of convective flow. This part cannot be described by our micropolar fluid calculation.

In simpler situations we may demonstrate the relevancy of Newtonian micropolar fluid model. Let us consider granular flows in a pipe. It is known that granular flow in a pipe becomes a plug flow, where the velocity profile is almost flat in the central part of flows. The velocity profile in 2D case (the Poiseille flow between parallel plates) can be fitted by hyperbolic cosine function [17,18]. If the boundary condition is assumed to be non-slip, the profile is continuously changes from parabolic to the plug flow depending on the coeeficient of restitution [19]. On the other hand, the Newtonian micropolar fluid model in the 2D Poiseuille flow between $y = [-1, 1]$ under the non-slip boundary condition has the following form [4]

$$v_x(y) = 1 - y^2 - 2\sqrt{\frac{2\mu_r}{\mu_B}}\frac{\cosh(k) - \cosh(ky)}{\sinh(k)} \tag{123}$$

where the flow direction is x and $k = \sqrt{\mu_r\mu_B/8}$. Note that we need the translation of parameters from eq.(3.1.9) into (123), because we adopt the dimensionless parameters introduced in this paper. Equation (123) is consistent with previous reports on granular pipe flows. In particular, the result is almost identical to that in the simulation by Peng and Ohta [19] which adopt the non-slip boundary condition. It is interesting that the simulation by Peng and Ohta is based on the lattice gas celleular automata which does not introduce the explicit rotation of particles. Thus, the microrotation may not have to be identified with the rotation of particles.

Even when we consider the case flow around a sphere or a cylinder, we may expect the relevancy of our calculation presented here in granular flows. In fact, I have already simulated steady flow aroud a circle (cylinder) based on the MD simulation, where the boundary conditions are periodic and the circle moves with a constant speed. My preliminary result produces a symmetric profile in front of the circle and behind the circle as explained in this paper. Details of the simulation and quantitative comparison with the theoretical calculation will be reported elsewhere.

Thus, we believe that our calculation presented here is relevant in granular physics, when the boundary effects are not important. Even when the boundary effects are crucial and the flow coexists with solid-like state , we believe that our calculation will play a fundamental role, because such the situation can be described by a sort of dynamical fluid-solid transition, and its flow part can be described by a relevant fluid model such as the micropolar fluid model.

Thus we may require further investigations on micropolar fluid models from the following reasons. (i) It is not easy to extract universal properties from molecular dynamic (MD) simulations and experiments. (ii) It is difficult to simulate large flow systems by MD but easy to simulate fluid models in large systems. (iii) As demonstrated in this paper, we can discuss physical properties based on the analytic calculations. (iv) The granular flow can be characterized by a simple generalization of the conventional flow.

B. General Discussion

One of important characteristics of micropolar fluids is the existence of a relaxation term of the micro-rotation to the rotation of flow in eq.(3). This is common in fluid models to describe fluidized beds and traffic flows [31–33]. Hong and his coworkers [34,35] proposed model B which has a relaxation term in momentum conservation to describe vibrating beds. Their model may not be acceptable because the existence of the relaxation term in momentum conservation means that the fluid cannot be described by stress tensor. However, the result of simulation of model B is obviously better than model A [36]. This puzzle may be solved to introduce the relaxation term in the micro-rotation flow. On the other hand, traffic flows and granular flows in a narrow tube have a beautiful mathematical structure [37–39]. Therefore, we expect that the compressible micropolar fluid flow may have beautiful mathematical structure. At least two dimensional flows or three dimensional flows on a slope will be interesting generalization of granular flows in a narrow tube and traffic flows.

Coming back to the flow outside the cylinder/sphere, we will have to compare results reported here with simulations and experiments. In particular, most of all experiments and simulation focused on stick-slip motion under slow shear forces [40–43], which need a solid-like state to describe stationary states under acting shear force. The existence of stick-slip region may lead the opinion that simple application of micropolar fluid to granular flow cannot be used. However, we believe that our calculation reported here will be comparable when the sphere and the cylinder are enough small to system sizes. The reasons of our expectation are as follows. Micropolar fluid dynamics is a useful concept to describe granular flow as indicated by Kano et al [15] and the discussion in the previous section. It is known that the force chains between the sheared material and the walls of container are needed to obtain sticking states. Thus, simulation of particles inside smooth walls cannot produce sticking states. In addition, creep flows are expected not to be removed when the force is enough small from rheological common sense. In any case, it will be important to clarify the friction mechanism of granular materials and the limitation of fluid dynamic description.

VII. CONCLUDING REMARKS

In this paper, we investigate fundamental properties of slow micropolar fluid flows. We present systematic calculation of flows around the sphere and the cylinder. We obtain the explicit forms of velocity fields (50) and (92), micro-rotation fields (51), the drag force exerted by the fluid flow (61) and (101). We also write the procedure to calculate contributions from the singular perturbation of Reynolds number. We found an interesting solution (120) of a steady flow inside the cylinder which is identical to eq.(121) observed in the experiment for granular vibrating beds.

The author thanks T. Ayukawa and Kim S. for useful discussion. This work was partially supported by the Grant-in-Aid for Science Research Fund from the Ministry of Education, Science and Culture (Grant No. 11740228).

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