Simulation of the impact of the elastic disk

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Abstract. The impact of an elastic disk with an elastic wall is numerically studied. We introduce a numerical model of the elastic disk and the elastic wall by mass points and linear springs. From our simulation, we find that the coefficient of normal restitution depends on the thickness of the elastic wall. We also calculate the coefficient of tangential restitution.

1 Introduction

The collision of particles with the internal degrees of freedom is inelastic in general. Examples can be seen in collisions of atoms, molecules, elastic materials, balls in sports, and so on. The study of inelastic collisions will be able to be widely accepted as one of fundamental subjects in physics, because they are almost always discussed in textbooks of elementary classical mechanics.

Physicists realize that inelastic collisions can be a fashionable subject in physics from recent extensive interest in granular materials[1,2]. In fact, granules consist of macroscopic dissipative particles. Therefore, the decision of interaction among particles is obviously important. We believe that static interactions among granular particles can be described by the theory of elasticity[3,4]. For example, the normal compression may be described by the Hertzian contact force[5] and the shear force may be represented by the Mindlin force[6]. The dynamical part related to the dissipation, however, cannot be described by any reliable physical theory. Thus, the distinct element method[7] which is one of the most popular models to simulate collections of granular particles contains some dynamical undetermined parameters. In other words, to determine such the parameters is important for both granular physics and fundamental physics.

When elastic materials collide each other, a part of translational energy is transformed into various modes of vibrations. In our previous papers, we introduced two different numerical models to investigate impact of an elastic disk with a structureless wall[8,9]. One of them is the lattice model consisted of mass particles and linear springs. Another is the continuum model which is identical to that by Gerl and Zippelius[10]. With these two models, we investigated the relation between the impact velocity and the coefficient of normal restitution, etc[8,9]. These two models have common features in that (i) they have no dissipative mechanism and (ii) the wall has no internal degrees of freedom. In actual situations of collision between a disk and a wall, however, a part of initial energy of the material is transfered into the wall. Thus we have to introduce the internal degrees of freedom of the wall to simulate impact of the disk with the wall.

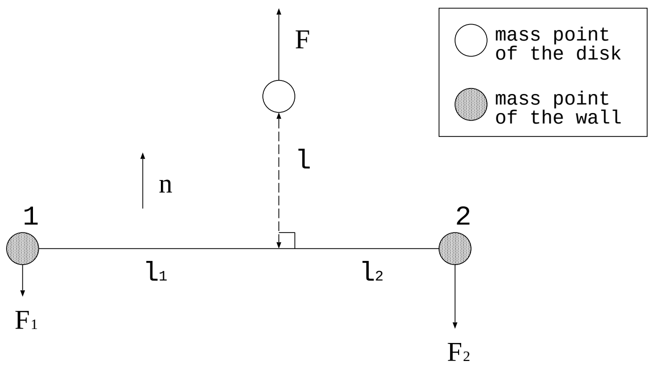

Fig. 1. The interaction between the mass points of the disk and the mass points of the wall.

To understand the macroscopic feature of impact phenomenon, two known parameters are introduced: the coefficient of normal restitution e and the coefficient of tangential restitution β. As for the normal impact, Sondergaard *et. al.* demonstrated that the coefficient of restitution depends on the thickness of the wall[11]. On the other hand, in the case of the oblique impact of the sphere, the coefficient of tangential restitution depends on the incident angle[12]. They also reported that there are two types of oblique collisions, slip and rotation.

In this paper, we perform simulations of the normal and the oblique impacts of the elastic disk with the elastic wall. In the next section, we will introduce our two-dimensional model. In section 3, we will show the results of our simulation. In the last section, we will discuss and summarize our results.

2 Our model

Let us introduce our numerical model. We construct an elastic disk of 1459 mass particles on a triangular lattice. In the continuum limit, Young's modulus E and Poisson's ratio σ correspond to $2\kappa/\sqrt{3}$ and $1/3$ respectively, where κ is the spring constant. We also introduce the elastic wall of 6400 mass particles on a square lattice. We introduce two spring constants: k_1 for nearest neighbor interaction and k_2 for next-nearest neighbor interaction for the wall. Controlling k_1 and k_2, we can change the material constant of the elastic wall[13]. We set k_1 and k_2 as the Young's modulus of the elastic wall is equal to that of the elastic disk when the force is applied to the wall in the normal direction. The width of the wall is 4 times as long as the diameter of the disk. Two sides of the wall are fixed. We can choose any thickness of the wall. The interaction between the disk and the wall during collision is introduced as follows. Fig. 1. is the schematic figure of the interaction of mass points of the disk and the wall. When the distance l between the mass point of the bottom edge of the disk and two nearest points of the surface of the wall is less than the cutoff length(we set it equal to the length of the linear spring), the points of the disk are rebounded by the force, $\boldsymbol{F}(l) = aV_0 \exp(-al)\hat{\boldsymbol{n}}$, where a is $100/d_0$, V_0 is $amc^2 d_0/2$, m is the mass of the particle, d_0 is the lattice constant, $c = \sqrt{E/\rho}$, and ρ is the density. $\hat{\boldsymbol{n}}$ is the unit vector perpendicular to the line between the two surface points. The reaction forces applied to the two points of the surface of the wall (point 1 and 2) are decided by the balance of the torques as $\boldsymbol{F_1}(l) = -F(l)\hat{\boldsymbol{n}}/(1 + l_1/l_2)$ and $\boldsymbol{F_2}(l) = -F(l)\hat{\boldsymbol{n}}/(1 + l_2/l_1)$, where l_1 and l_2 are the distances from the foot of the perpendicular line respectively.

To perform the numerical simulation, we scale all parameters by the radius of the disk R, the speed of sound $c = \sqrt{E/\rho}$, where ρ is the density of the disk. We adopted the fourth order symplectic numerical method with the time step $\Delta t = 10^{-2}R/c$ (R is the radius of the disk) as the numerical scheme of the integration.

3 Simulation

Sondergaard *et. al.* investigated the influence of the ratio of the diameter of sphere to the plate thickness on the coefficient of restitution resulting in normal impacts[11]. They dropped ball bearings(steel or bronze) and glass spheres without any significant rotational velocity from various heights onto plates of lucite or aluminum of which two sides were solidly clamped to a relatively rigid support structure. From their experiment, the coefficient of restitution decreases as the ratio of the particle diameter to the plate thickness increase.

At first, we will carry out the simulation of the normal impact. We set the situation corresponding to the experiment by Sondergaard *et. al.*[11]. Our situation is as follows. We control the ratio of the disk diameter D to the wall thickness T, D/T, from 0.4 to 1.0. We make the disk hit the wall with various initial velocity ranging from $0.02c$ to $0.18c$ and calculate the coefficient of restitution (Fig. 2). When the impact velocity has same value, the coefficient of restitution of small D/T is larger than that of large D/T. The curve labeled model A is the result of the case that the lattice model of the disk and the wall with no internal degrees of freedom[9].

Second, we carry out the simulation of the oblique impact. Many experiments of the oblique impact focuses on the relation between the coefficient of tangential restitution and the incident angle.

Now let us introduce the coefficient of tangential restitution. We will consider the situation like Fig. 3. The coefficient of tangential restitution is defined by the ratio of the tangential component of the velocity of the contact point of the disk after collision to that before collision. The velocity of the contact point $\boldsymbol{v_c}$ is defined as

$$\boldsymbol{v_c} = \boldsymbol{v_i} - R \times \boldsymbol{\omega}, \tag{1}$$

where $\boldsymbol{v_i}$ is the velocity of the center of mass of the disk, R is the radius of the disk, and $\boldsymbol{\omega}$ is the angular velocity. Using the velocity of the contact point, the coefficient of tangential restitution β is defined as

$$\boldsymbol{v_c}' \cdot \boldsymbol{t} = -\beta \boldsymbol{v_c} \cdot \boldsymbol{t}, \tag{2}$$

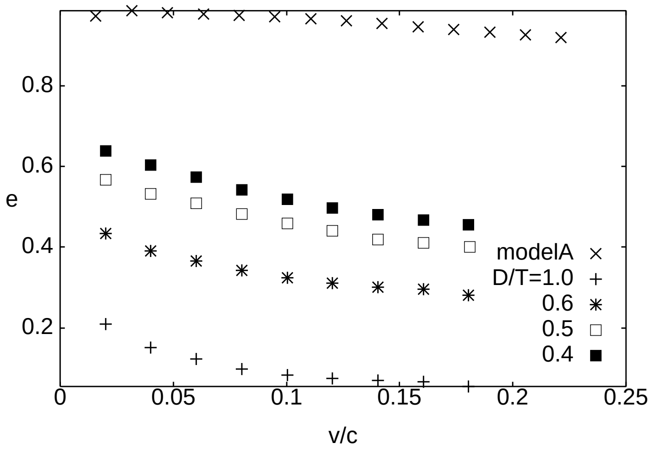

Fig. 2. The relation between the normal impact velocity of the disk and the coefficient of restitution. Each curve has different value of D/T. The curve labeled modelA is the result in the situation when the wall has no internal degrees of freedom [9].

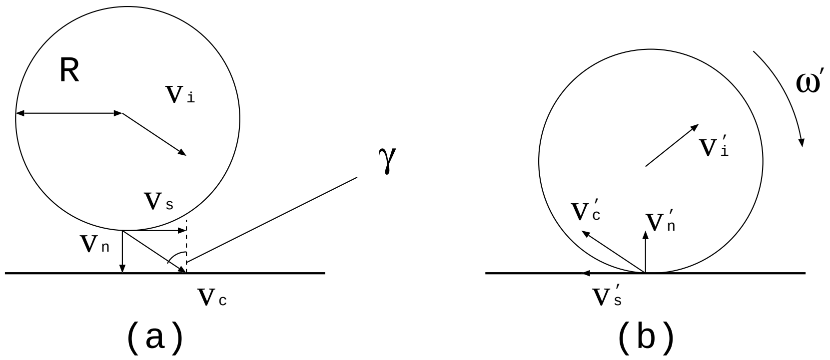

Fig. 3. The geometry of a collision to the wall. (a) the image before collision and (b) the image after collision. All the parameters after collision are denoted by primes.

where $\boldsymbol{v_c}'$ and $\boldsymbol{t}$ are the velocity of the contact point after collision and the unit vector in the tangential direction respectively. It is known that β is a function of the angle of incidence γ, with possible values lying in the range between -1 and 1 [14,15]. The incident angle γ is defined as $\gamma = \arctan(v_s/v_n)$, where v_n and v_s are the normal component and the tangential component of the initial velocity of the center of mass respectively.

In our simulation, we control the incident angle from $5.7°$ to $63.4°$ and make the disk collide the wall without any rotation. When controlling the incident angle, we set $v_n = 0.03c$ as a constant value and give v_s various values. The thickness of the wall is equal to the diameter of the disk (consisted of 6400 mass particles). Two sides of the wall are fixed. We will calculate β for various γ with

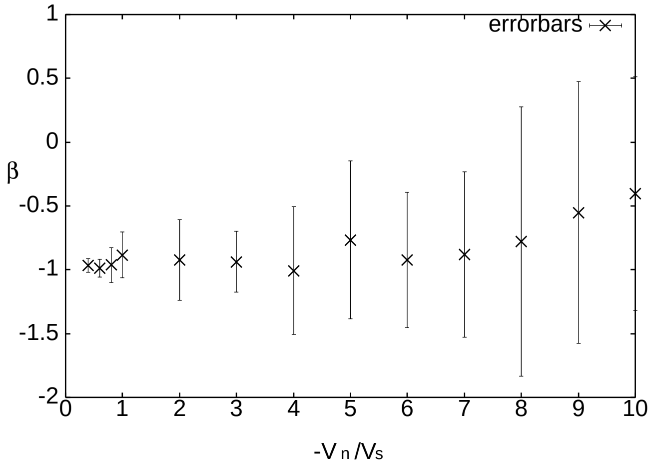

Fig. 4. The relation between the incident angle and the coefficient of tangential restitution β.

this model. To eliminate the anisotropy of the disk, we set the initial angle of the disk $\theta = \pi n/72 (n = 0, 1, 2, \cdots, 11)$ as the initial state and average these 12 data.

The relation between the cotangent of the incident angle, $-v_n/v_s$, and the coefficient of tangential restitution β is shown in Fig. 4. The cross points and the error bars represent the mean values and the standard deviations of the 12 data respectively. From this result, β is almost a constant negative value in the case of this model.

4 Discussion

We have performed both the normal and the oblique impact of the elastic disk with the wall. In the case of the normal impact, the coefficient of restitution e depends on the ratio of the diameter of the disk to the thickness of the wall D/T. If D/T is large, e is small. It is seen that the macroscopic vibration of the wall is so large that initial energy of the disk decreases. Sondergaard *et. al.* introduced the impact parameter made by rescaling D/T and showed that the coefficient of restitution is a function only of the impact parameter[11]. In our case, however, we could not rescale our data in order to get an universal relation between D/T and e. To clarify the reason will be one of future subject to be solved.

In the case of the oblique impact, we investigated the motion of the disk of each incident angle. According to Labous *et. al.*[12], β is believed to be represented by

$$\beta = \begin{cases} -1 - \mu_0 \left(1 + 1/K\right) (1 + e) v_n/v_s & (\gamma \geq \gamma_0) \\ \beta_0 & (\gamma \leq \gamma_0), \end{cases} \tag{3}$$

where $K = 4I/md^2$ is a constant equal to 1/2 for disks(I is the moment of inertia, m is the mass, d is the diameter) and μ_0 is the friction coefficient and e is the coefficient of restitution. If γ is larger than the critical angle γ_0, the disk slips on the surface of the wall according to the Coulomb's law. In that case, β is a function of the incident angle while μ_0 is given by the ratio of the tangential to the normal component. If γ is smaller than the critical angle γ_0, the disk begins to roll after the collision with a constant value β_0 as β. As for our model, the coefficient of tangential restitution β is negative value in the range of incident angle $5.7^\circ \leq \gamma \leq 63.4^\circ$. In an actual case, Labous *et. al.* showed that β took a positive constant value $\beta_0 = 0.5 \pm 0.1$ when the incident angle was greater than the critical angle(around 37.5°)[12]. This means that the spheres begin to rotate with the rotational velocity which has same order as the velocity of the center of mass after binary collision. In our case, however, the negative value of β_0 means that collision of the disk and wall almost could not produce the rotational motion of the disk but the collision is slippery. It can be seen that this is caused by the weak frictional force acting on the disk. We also need to consider initial rotation of the disk. How initial rotation affect the rotational motion of the disk after collision is an interesting problem. These are our near future tasks.

The results of this paper are as follows: (i) The coefficient of restitution depends on the ratio of the disk to the thickness of the wall. (ii) In the case of the oblique impact, the coefficient of tangential restitution β takes almost a negative constant value.

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