Trace Anomaly Matching and Exact Results For Entanglement Entropy

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Introduction

- Entanglement entropy is an important and useful quantity which finds applications in many branches of physics, starting from black holes to quantum critical phenomena.
- In general it is a difficult thing to compute even for free field theories.
- Many exact results are known for conformal field theories but non-conformal field theories are even more difficult to deal with.
- Some exact results are known for two dimensional non-conformal field theories and for strongly coupled theories via gauge-gravity duality (Ryu-Takayanagi formula).

Goal

- Our goal is to write down some exact results for the entanglement entropy of a non-conformal field theory in arbitrary even dimensions.
- It turns out that the techniques developed by Komargodski and Schwimmer to prove the a-theorem in four dimensions is useful for this purpose.
- This is a completely non-perturbative technique and gives us results for the entanglement entropy which does not depend on the weak coupling or strong coupling limits.

Results

- Let us take a four dimensional UV-CFT and deform it by a relevant (marginally relevant) operator O of dimension Δ so that in the deep IR it reaches another fixed point which is described by an IR-CFT.
- The action can be written as,

$$S = S_{CFT}^{UV} + \int d^4x\sqrt{h}\, g(\Lambda)\Lambda^{4-\Delta}O \tag{1}$$

  where $g(\Lambda)$ is the dimensionless coupling constant defined at scale Λ.
- We want to compute the coefficients of the logarithmically divergent term in the entanglement entropy for this theory. The answer is given by,

- $$S_E \supset -n\frac{\partial}{\partial n}|_{n=1} \int_{cone} d^4x\sqrt{h}\,(\frac{\Delta c}{16\pi^2}W^2 - 2\Delta a E_4)\; ln\frac{\Lambda_g}{\Lambda} \quad (2)$$

  The integral is done over a background having conical singularities with the angular excess at the vertices given by $2\pi(n-1)$. W^2 and E_4 are the Weyl squared term and the Euler density in four dimensions. Δa and Δc are the differences of the central charges between the UV and the IR, given by, $\Delta a = a_{UV} - a_{IR}$ and $\Delta c = c_{UV} - c_{IR}$, respectively. Λ is the UV-cutoff and Λ_g is the renormalization group invariant scale associated with the coupling constant g, given by,

  $$\Lambda_g = \Lambda e^{-\int \frac{dg}{\beta(g)}} \quad (3)$$

  $\beta(g)$ is the beta function defined as, $\beta(g) = \Lambda\frac{d}{d\Lambda}g(\Lambda)$.

- In four dimensions there is one more term which can occur as the coefficient of the logarithmically divergent term. The term can be written as,

$$S_E \supset a_2\ \Lambda_g^2\ A_\Sigma\ ln\frac{\Lambda_g}{\Lambda} \tag{4}$$

where a_2 is a coupling constant independent dimensionless number and A_Σ is the area of the entangling surface. Our method does not allow us to compute the number a_2, but it fixes the geometry dependence of this term completely and unambiguously.

Method

- Let us now calculate the logarithmically divergent terms in this deformed theory.
- The coefficient of the logarithmically divergent term is dimensionless and so let us write ,

$$S_E = A(g(\Lambda), \Lambda R)\ ln\Lambda + ... \tag{5}$$

  where R denotes the length scale associated with the background geometry or the entangling surface.
- Let us now change the cut-off to $\Lambda' = e^t \Lambda$.
- So

$$A(g(\Lambda'), \Lambda' R)\ ln\Lambda' = A(g(\Lambda'), \Lambda' R)\ ln\Lambda + t\ A(g(\Lambda'), \Lambda' R) \tag{6}$$

- When we say that the coefficient of the logarithm is a universal term, what we mean is actually the equation,

$$A(g(\Lambda'), \Lambda' R) = A(g(\Lambda), \Lambda R) \tag{7}$$

  which expresses the fact that A is renormalization group invariant.
- There is a unique function which satisfies this condition and can be written as,

$$A(g(\Lambda), \Lambda R) = A(\Lambda_g R) \tag{8}$$

  where Λ_g is the unique RG invariant scale given by,

$$\Lambda_g = \Lambda e^{-\int \frac{dg}{\beta(g)}} \tag{9}$$

- Eqn-6 and 7 give us a strategy for computing the universal coefficient of the logarithmically divergent term.
- The strategy is to compute the entanglement entropy along the RG trajectory parametrized by t and pick up the RG invariant terms linear in t.
- So

$$A(\Lambda_g R) = (\frac{d}{dt}|_{t=0} S_E(t))_{RG-Invariant} \tag{10}$$

- Along the RG trajectory the action can be parametrized as,

$$S(t) = S^{UV}_{CFT} + \int d^4x\sqrt{h}\, g(\Lambda e^t)\Lambda^{4-\Delta} O \tag{11}$$

- We want to compute the entanglement entropy for this one parameter family of actions.
- What makes the computation possible is the identification of t with a constant background dilaton field.

Brief introduction to the method Komargodski and Schwimmer

- Our deformed field theory is not conformal but it can be made conformally invariant by coupling to a background dilaton field.
- The dilaton couples to the deformed theory as,

$$S = S_{CFT}^{UV} + \int d^4x\sqrt{h}\, g(e^{\tau(x)}\Lambda)\Lambda^{4-\Delta}O \tag{12}$$

- This is conformally invariant if the metric and the background field are transformed as,

$$h_{ab} \to e^{2\sigma}h_{ab},\ \ \tau(x) \to \tau(x) + \sigma \tag{13}$$

- Since we are interested in a constant rescaling we can couple to a constant dilaton background field.

- Since we want to differentiate with respect to the dilaton what we need is the effective action for the dilaton.
- KS have shown that this action consists of two terms. One is the Weyl non-invariant universal term which is completely determined by the conformal anomaly matching between the UV and the IR.
- The other part is the Weyl invariant part of the effective action which can be written as a functional of the Weyl invariant combination $e^{-2\tau} h_{ab}$.

Replica Trick

- Entanglement entropy is usually computed by replica trick.
- In replica trick the entanglement entropy is defined as,

$$S_E = n\frac{\partial}{\partial n}(F(n) - nF(1)) \mid_{n=1} \tag{14}$$

where $F(n)$ is the free energy of the Euclidean field theory on a space with conical singularities. The angular excess at each conical singularity is given by $2\pi(n-1)$. The detailed geometry of the space is determined by the geometry of the background space and the geometry of the entangling surface.

Final Formulae

- So we can write,

$$S_E(t) = n\frac{\partial}{\partial n}(F(n,t) - nF(1,t)) \mid_{n=1} \tag{15}$$

- $F(n,t)$ is the free energy or the effective action computed on the conical space in the presence of the constant background dilaton field t.
- Since we want to differentiate with respect to t we can replace F with the dilaton effective action.
- With this we can now compute the universal terms.

Two dimensions

- In two dimensions the universal (Weyl non-invariant) part of the dilaton effective action for a constant dilaton filed is given by,

$$F(n,t) = -\frac{c_{UV} - c_{IR}}{24\pi} \; t \int_{cone} \sqrt{h} R(h) \tag{16}$$

- For an infinite half-line this gives us the known answer derived by Calabrese and Cardy, in a different way

$$A = \frac{c_{UV} - c_{IR}}{6} \tag{17}$$

Four Dimensions

- In Four dimensions dimensions the universal (Weyl non-invariant) part of the dilaton effective action for a constant dilaton filed is given by,

$$F(n,t) = -t \int_{cone} d^4x\sqrt{h}\,(\frac{\Delta c}{16\pi^2}W^2 - 2\Delta a E_4) \qquad (18)$$

- This gives us the answer quoted in the beginning and it matches with both the perturbative answer for massive free fields and holographic answers obtained for field theories deformed by relevant operators.

Contributions from the Weyl invariant part of the dilaton effective action

- The dilaton effective action can be expanded in terms of tensors built out of $\hat{h}_{ab} = e^{-2t} h_{ab}$. Let us arrange these terms in order of increasing mass dimensions of the integrand.
- The first term is

$$\int_{cone} d^4x \sqrt{\hat{h}} = e^{-4t} \int_{cone} d^4x \sqrt{h} \tag{19}$$

  This term does not contribute to the entanglement entropy because the volume of the cone does not get any contribution from the tip.

- The second term is,

$$\int_{cone} d^4x\sqrt{\hat{h}}R(\hat{h}) = e^{-2t}\int_{cone} d^4x\sqrt{h}R(h) \tag{20}$$

- In order to be dimensionless and universal this term has to be multiplied by Λ_g^2 and so this gives rise to a universal term of the form,

$$a_2\ \Lambda_g^2\ A_\Sigma \tag{21}$$

where A_Σ is the area of the entangling surface.

- The dimension four terms can be written as linear combinations of $R^2(\hat{h})$, $R^2_{ab}(\hat{h})$ and $R^2_{abcd}(\hat{h})$. So a general dimension four term in the dilaton effective action has the structure,

$$\int_{cone} d^4x\sqrt{\hat{h}}(AR^2(\hat{h}) + BR^2_{ab}(\hat{h}) + CR^2_{abcd}(\hat{h})) \qquad (22)$$

  where A, B and C are dimensionless constants. Since this term is marginal, it does not couple to a constant dilaton and so does not contribute to the universal term.
- In fact this is the reason why the universal term of the first kind does not get any contribution from the Weyl invariant part of the dilaton effective action. This term is marginal and if this term contributed to the universal term then it would have changed our answer for the universal term of the first kind.

- The next term is dimension six and in order to be universal it has to be multiplied by negative powers of Λ_g.
- So this term cannot occur in the dilaton effective action because it will diverge in the conformal limit which corresponds to $\Lambda_g \to 0.$ and we do not expect the dilaton effective action to be singular in the CFT limit.
- This ends our proof that the universal coefficient of the logarithmically divergent term has only two terms in four dimensions, as promised in the introduction.
- This can be extended to arbitrary even dimensions.

Scale versus conformal invariance

- One can argue that the method we have used is equally applicable to field theories which are described by scale invariant but not conformally-invariant fixed points in the UV and the IR.
- The only change is in the form the universal term which comes from the anomalous part of the dilaton effective action.
- For theories like that we get an extra term given by,

$$n\frac{\partial}{\partial n}|_{n=1}\ \Delta e \int_{cone} d^4x\sqrt{h}\ R^2(h)\ ln\frac{\Lambda_g}{\Lambda} \tag{23}$$

  where e is the "central charge" corresponding to the R^2 term and $\Delta e = e_{UV} - e_{IR}$.
- So this term will not show up for example for a massive scalar field in four dimensions although it is allowed by naive dimensional analysis.