# $AdS_5$ with two boundaries and holography of N=4 SYM theory.

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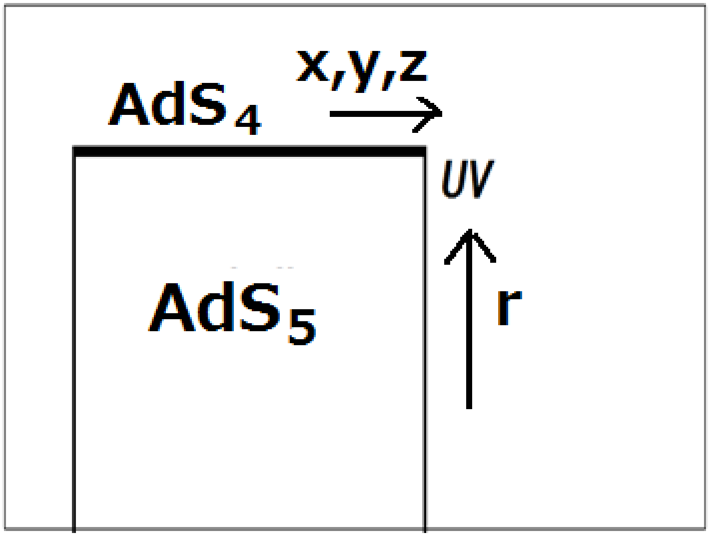
K.Ghoroku, M.I. and A.Nakamura Phys.Rev. D89 (2014) 066009 arXiv:1310.2007

K.Ghoroku, M.I. and A. Nakamura Phys.Rev. D75 (2007) 046005 hep-th/0612244

# Introduction

We consider $AdS_5$ metric whose 4D boundary at $r \to \infty$ (UV) has ($AdS_4$) metric.

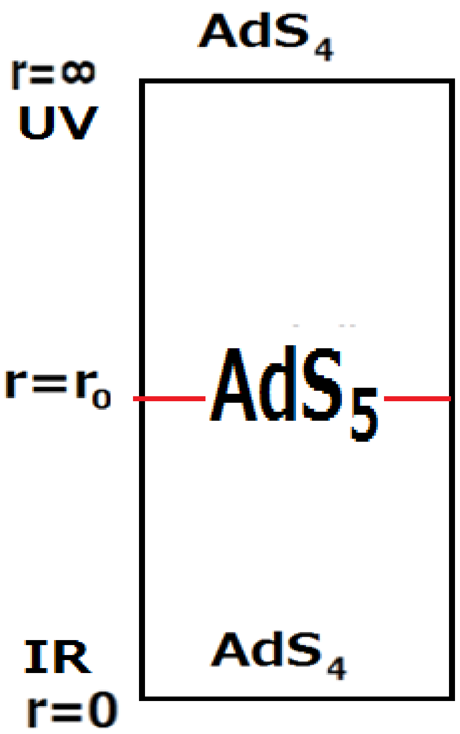



r : 5-dimensional direction

# Introduction

It is found that there is another 4D boundary at $r \to 0$ (IR).

UV and IR boundary field theories are described by common gravity dual but there is a domain wall in the 5D-bulk at some specific $\boldsymbol{r = r_0}$



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# Set Up

We consider the 10D metric in the following form

$$ds_{10}^2 = g_{MN} dx^M dx^N + R^2 d\Omega_5^2 \quad (M, N = 0 \cdots 5)$$

where the equation of motion for the non-compact 5-dimensional space-time is given as

$$R_{MN} = -\Lambda g_{MN} \ (M, N = 0 \cdots 5)$$

# Set up

Solution is obtained in the following form of the metric,

$$ds_5^2 = \frac{r^2}{R^2} A\,(r)^2\left(-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right) + \frac{R^2}{r^2} dr^2$$

such that 4-dimensional UV boundary ($r \to \infty$ ) metric becomes AdS4 metric.

$$ds_4^2 = -dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j$$

where $\gamma_{ij}(x) = \delta_{ij}\left(1 - \frac{1}{4}\Sigma_{i=1}^3\left(x^i\right)^2\right)^{-2}$

$$a_0(t) = sin\left(\sqrt{-\lambda}t\right)/\sqrt{-\lambda}$$

$\lambda(< 0)$: 4D negative cosmological constant

We will find $\mathbf{A}(r)$ which satisfies

$$A(r) \to 1 \quad \text{for } r \to \infty.$$

# Set up

From the Einstein equation , $\boldsymbol{A(r)}$ is determined as

$$\boldsymbol{A(r) = 1 + \left(\frac{r_0}{r}\right)^2} \text{ where } \boldsymbol{r_0^2 = -\frac{\lambda R^4}{4}}$$

We can get the metric as follows

$$\boldsymbol{ds^2 = \frac{r^2}{R^2}\left(1 + \frac{r_0^2}{r^2}\right)^2 \left(-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5}$$

$$\boldsymbol{\gamma_{ij}(x) = \delta_{ij}\left(1 - \frac{1}{4}\Sigma_{i=1}^3\left(x^i\right)^2\right)^{-2}}$$

The boundary ( $\boldsymbol{r} \to \infty$) 4D metric becomes $\boldsymbol{AdS_4}$

# Domain wall

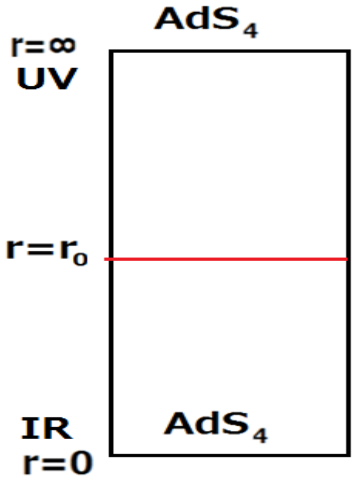
In this metric, there are two boundaries for $(\boldsymbol{r} \to \mathbf{0})$ and $(\boldsymbol{r} \to \infty)$

By changing $\boldsymbol{r} = \boldsymbol{r}_0^2/\boldsymbol{z}$, the metric becomes

$$ds^2 = \frac{z^2}{R^2}\left(1 + \frac{r_0^2}{z^2}\right)^2 \left(-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right) + \frac{R^2}{z^2}dz^2$$

Thus, the IR boundary metric at $\boldsymbol{z} \to \infty(\boldsymbol{r} \to \mathbf{0})$ is also $\boldsymbol{AdS_4}$ which is the same form as the UV boundary metric at $\boldsymbol{r} \to \infty$.

The bulk metric are separated at $\boldsymbol{r} = \boldsymbol{r}_0$ .



# Trace anomaly by the $AdS_4$ field theory

The trace anomaly of the field theory on the 4D curved space-time with $\boldsymbol{n_s}$ scalars, $\boldsymbol{n_f}$ fermions and $\boldsymbol{n_v}$ vectors is given as

$$\langle T^{\mu}_{\mu}\rangle = -\frac{n_s + 11n_f + 62n_v}{90\pi^2}E_4 - \frac{n_s + 6n_f + 12n_v}{30\pi^2}I_{(4)}$$

$$E_{(4)} = \frac{1}{64}(R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2)$$

$$I_4 = -\frac{1}{64}(R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2) \quad (\mu, \nu = t, x, y, z)$$

*(N.J.Duff 1994) (M. Henningson and K. Skenderis, 1998)*

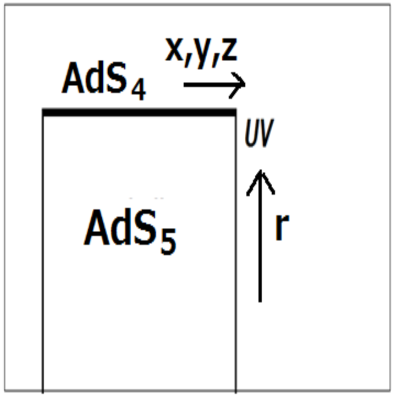
For 4D N=4 SYM theory on $\mathrm{AdS}_4$ with cosmological constant $\lambda(< 0)$ , it becomes

$$\langle T^{\mu}_{\mu}\rangle = -\frac{3\lambda^2}{8\pi^2}N^2 \qquad (n_s = 6, n_f = 2,\ n_v = 1)$$

# Energy momentum tensor (UV) by holography

The five dimensional metric is rewritten as

$$ds_5^2 = \frac{r^2}{R^2} A(r)^2\left(-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right) + \frac{R^2}{r^2}dr^2$$
$$= \frac{1}{\rho} A\,(r)^2\left(-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right) + \frac{d\rho^2}{4\rho^2}$$
$$\equiv \frac{1}{\rho}\hat{g}_{\mu\nu}dx^\mu dx^\nu + \frac{d\rho^2}{4\rho^2} \qquad \text{where } \rho \equiv \frac{r_0^2}{r^2} \quad (\text{and } R = 1)$$




By expanding 4D metric by powers of $\rho \left(= \frac{r_0^2}{r^2}\right)$ as

$$\hat{g}_{\mu\nu} = g_{(0)\mu\nu} + g_{(2)\mu\nu}\rho + \rho^2\left(g_{(4)\mu\nu} + h_{1(4)\mu\nu}\log\rho + h_{2(4)\mu\nu}(\log\rho)^2\right) + \cdots$$

the Energy momentum tensor of the **UV** boundary ($r \to \infty$) $\mathrm{Ad}S_4$ field theory are given by following formula.

$$\langle T_{\mu\nu}\rangle = \frac{4R^3}{16\pi G_N}\Big(\mathbf{g}_{(4)\mu\nu} - \frac{1}{8}\mathbf{g}_{(0)\mu\nu}\left(\left(\mathrm{Tr}\mathbf{g}_{(2)}\right)^2 - \mathrm{Tr}\mathbf{g}_{(2)}^2\right) - \frac{1}{2}\left(\mathbf{g}_{(2)}^2\right)_{\mu\nu}$$
$$+ \frac{1}{4}\mathbf{g}_{(2)\mu\nu}\mathrm{Tr}\mathbf{g}_{(2)}\Big)$$

*(S.de Haro, et al. 2000)*

# Trace anomaly by Holography

Then, we can get

$$\langle \boldsymbol{T_{\mu\nu}} \rangle = \frac{\boldsymbol{4R^3}}{\boldsymbol{16\pi G_N^5}}\left(\frac{\boldsymbol{3\lambda^2}}{\boldsymbol{16}}\left(-\boldsymbol{1}, \boldsymbol{g_{0ij}}\right)\right)$$

$\mathrm{g}_{(0)ij} \equiv \boldsymbol{a_0(t)^2\gamma_{ij}(x)}$ : boundary $AdS_4$ space metric

Trace anomaly is obtained as

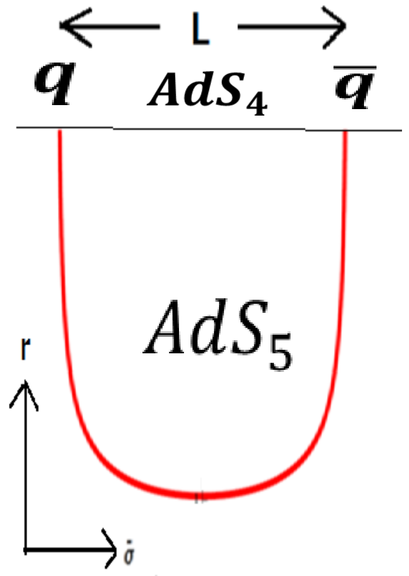
$$\langle \boldsymbol{T^{\mu}_{\mu}} \rangle = -\frac{\boldsymbol{3\lambda^2}}{\boldsymbol{8\pi^2}}\boldsymbol{N^2}$$

which is the same as the result of the field theory in $AdS_4$ space-time .

# quark-antiquark potential in $AdS_4$

Quark-Antiquark potential in the $AdS_4$ field theory is obtained by the energy of U-shaped string in the 10D bulk.

$$ds^2 = \frac{r^2}{R^2}\left(1+\frac{r_0^2}{r^2}\right)^2\left(-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2$$




The Nambu-Goto action of the U-shaped string :

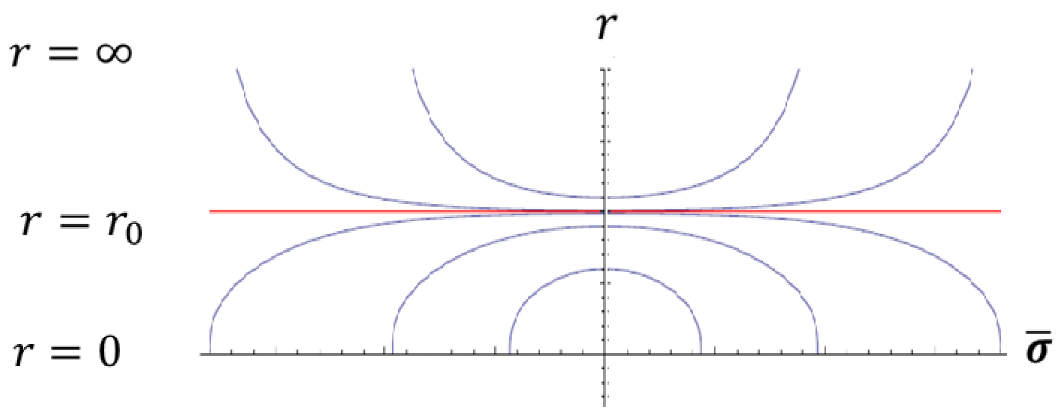
$$L_{NG} = -\frac{1}{2\pi\alpha'}\int d\sigma \left(\frac{rA}{R}\right)^2\sqrt{r'^2 + \left(\frac{r}{R}\right)^4\left(A(r)a_0(t)\gamma(x)\right)^2}$$

where we take the string world-volume coordinates $(\tau, \sigma)$ as

$$\tau = t,\ \sigma = x$$

# Numerical solution and domain wall

U-shaped strings given by solving the equation of motion numerically.




There are two classes of U-shaped strings.

・U-shaped strings whose endpoints are on the UV boundary ($r \to \infty$)

・U-shaped strings whose endpoints are on the IR boundary ($r \to 0$)

They are separated by domain walls (red line: $r = r_0$).

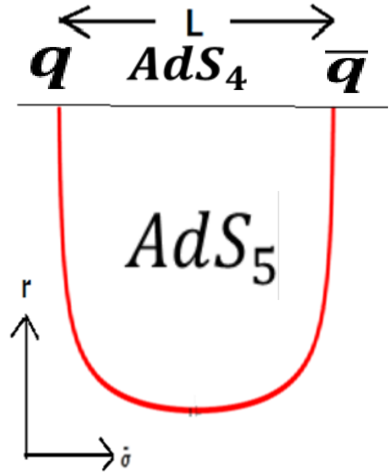
# The $q\bar{q}$ potential and confinement

We consider the quark-antiquark ($\boldsymbol{q\bar{q}}$) potential on UV-boundary

By introducing the proper distance as $\quad \bar{\sigma} \equiv a_0(t)\int d\sigma \frac{1}{1-\frac{\sigma^2}{4}}$

energy of U-shaped string becomes

$$E = -L_{NG} = \frac{1}{2\pi\alpha'}\int d\bar{\sigma}\left(\frac{rA}{R}\right)^2\sqrt{1+\left(\frac{R^2}{r^2A}\partial_{\bar{\sigma}}r\right)^2}$$

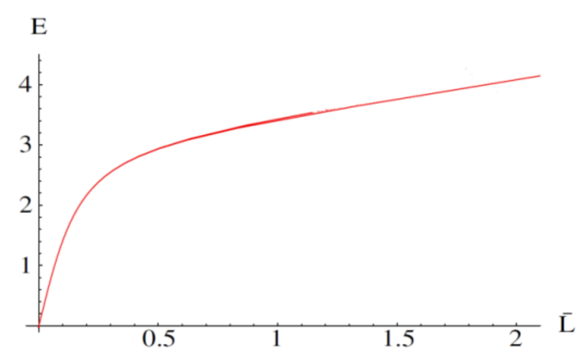



We can get the linear potential and $\boldsymbol{q\bar{q}}$ are confined.

$$E \sim \frac{(r_* A(r_*))^2}{2\pi\alpha' R^2}\int d\bar{\sigma} = \tau_{q\bar{q}}\, L \qquad (\partial_{\bar{\sigma}} r|_{r=r^*} = 0\ )$$

$L = \int_{\bar{\sigma}_{min}}^{\bar{\sigma}_{max}} d\bar{\sigma}$ the distance between quark ant antiquark

$\tau_{q\bar{q}} \equiv \frac{(r_* A(r_*))^2}{2\pi\alpha' R^2}$ $\quad \boldsymbol{q\bar{q}}$ tension



# D7 brane embedding

D7-brane: mesons in dual field theory on $AdS_4$ *(A. Karch and E.Katz, 2002)* *(M.Kruczenski et al. 2003)*

We consider the D7-brane embedding in the 10D bulk with $AdS_4$ boundary.

$$ds^2 = \frac{r^2}{R^2}\left(1 + \frac{r_0^2}{r^2}\right)^2 \left(-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2$$

| | $X^0$ | $X^1$ | $X^2$ | $X^3$ | $X^4$ | $X^5$ | $X^6$ | $X^7$ | $X^8$ | $X^9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| D3-brane : | ○ | ○ | ○ | ○ | | | | | | |
| D7-brane : | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | | |

# D7 brane action

The induced metric for the D7-brane

$$ds_{D7}{}^2 = \frac{r^2}{R^2}\left(1+\frac{r_0^2}{r^2}\right)^2\left(-dt^2+a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right)$$
$$+\frac{R^2}{r^2}\left(\left(1+\left(\partial_\rho w\right)^2\right)d\rho^2+\rho^2 d\Omega_3^2\right)$$

where $\rho^2 = \sum_{i=4}^{7}\left(X^i\right)^2, \quad r^2 = \rho^2 + w(\rho)^2$

and we set $\left(X^8, X^9\right) \equiv (w(\rho),\ 0)$ due to the rotational invariance in the $X^8 - X^9$ plane

the DBI action of the D7-brane *:*

$$S_{D7} = -T_7 \int d^8\xi \sqrt{-\det(g_{ab})}$$
$$= -T_7 \int d^8\xi \rho^3 \gamma(x)^3 a_0(t)^3 \left(1+\left(\frac{r_0}{r}\right)\right)^4 \sqrt{1+w'(\rho)^2}$$

$w(\rho)$ determines the shape of the D7 brane.

# Equation of motion for D7

The equation of motion for $\boldsymbol{w(\rho)}$

$$\frac{w}{\rho + ww'}\left(\phi' - \sqrt{1+w'^2}(\phi + 4logA)'\right) + \left(w'\left(\frac{3}{\rho} + (\phi + 4log\ A)'\right) + \frac{w''}{1+w'^2}\right) = 0$$

From the profile solution $w(\rho)$, we can read the quark mass $m_q$ and the VEV of chiral condensate $c \equiv \langle\bar{\psi}\psi\rangle$

$\boldsymbol{w(\rho) = m_q + \frac{c}{\rho^2} + \cdots}$**(at large $\boldsymbol{\rho}$)**

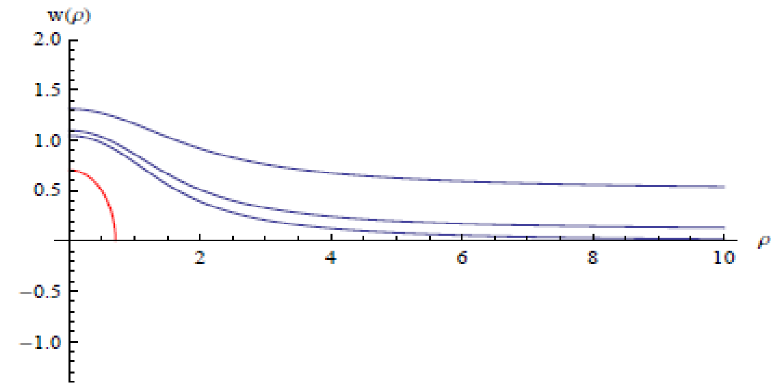
# D7 Solution and chiral symmetry breaking

Solution is given numerically

$$w(\rho) = m_q + \frac{c}{\rho^2} + \cdots$$

$c \equiv \langle \bar{\psi}\psi \rangle$: **VEV of chiral condensates**

$m_q$: **mass of a quark**




Solutions $w(\rho)$ with $m_q = 0$ $c > 0$ mean that Chiral symmetry is spontaneously broken

Red circle ($r = r_0$) is the domain wall.

The trivial solution $w(\rho) = 0$ ($m_q = 0, c = 0$) also exists. However, its energy is larger than nontrivial solution $\mathrm{w}(\rho)$ with $\mathrm{m_q} = 0$ , $c > 0$

# Meson spectrum

**Meson**: fluctuation of the D7-brane solution $\left(X^8, X^9\right) = (w(\rho), 0)$

$$X^8 = w(\rho) + \tilde{\phi}^8\left(t, x, \rho, S^3\right) \quad X^9 = 0 + \tilde{\phi}^9(t, x, \rho, S^3)$$

We write the functions in the following factorized form.

$$\tilde{\phi}^k = \psi^k(t, x)\phi_l(\rho)Y_l(S^3) \quad (k = 8, 9 \quad l\text{: anguler momentum})$$

$Y_l\left(S^3\right)$ : spherical harmonic function on $S^3$

$-\Box_4\psi^k(t, x) = -\mathrm{m}_\mathrm{k}^2\psi^k(t, x)$ **(k=8, 9)**

$m_k$ $(k = 8, 9)$ : The meson mass spectrum

# Meson spectrum

## Equation of motion for $\phi_l^9(\rho)$

$$\partial_\rho^2 \phi_l^9 + \frac{1}{L_0}\partial_\rho(L_0)\partial_\rho \phi_l^9 + (1+w'^2)\left(\frac{\left(\frac{R}{r}\right)^4 m_9^2}{A^2} - \frac{l(l+2)}{\rho^2} - 2K_{(1)}\right)\phi_l^9 + \frac{1}{r}\frac{\partial \Phi}{\partial r}\phi_l^9 = 0$$

$$L_0 = \frac{\rho^3 A^4}{\sqrt{1+w'^2}} \quad K_{(1)} = \frac{\partial_{r^2} A^4}{A^4}$$

## Equation of motion for $\phi_l^8(\rho)$

$$\partial_\rho^2 \phi_l^8 + \frac{1}{L_1}\partial_\rho(L_1)\partial_\rho \phi_l^8 \quad + (1+w'^2)\left(\frac{\left(\frac{R}{r}\right)^4 m_8^2}{A^2} - \frac{l(l+2)}{\rho^2} - 2(1+w'^2)\left(K_{(1)} + 2w^2 K_{(2)}\right)\right)\phi_l^8$$

$$+ (1+w'^2)^{\frac{3}{2}}\left(\left(2rK_{(1)}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2}\right)\frac{w^2}{r^2} + \frac{\partial \phi}{\partial r}\frac{\rho^2}{r^3}\right)\phi_l^8 = -2\frac{1}{L_1}\partial_\rho\left(L_0 w w' K_{(1)}\right)\phi_l^8$$

$$L_{(1)} = \frac{L_0}{1+w'^2} \quad K_{(2)} = \frac{1}{A^4}\partial_{r^2}^2 A^4$$

# Meson mass spectrum (simple case)

For simple case: Trivial solution $\boldsymbol{w(\rho) = 0}$

$\boldsymbol{\phi_l^8(\rho) = \phi_l^9(\rho) \equiv \phi_l(\rho)}$ **and** $\boldsymbol{m_8 = m_9 \equiv m}$

**Equation of motion for $\boldsymbol{\phi_l(\rho)}$**

$$\partial_\rho^2 \phi_l + \left(\frac{3}{\rho} - \frac{3r_0^2}{A\rho^3}\right)\partial_\rho \phi_l + \left(\frac{m^2 R^4}{A^2\rho^4} - \frac{l(l+2)}{\rho^2} + \frac{3r_0^2}{A\rho^4}\right)\phi_l = 0$$

Solution is obtained analytically in this case.

$$\phi_l = \left(r_0^2 + \rho^2\right)^{\frac{-3+\sqrt{9+\tilde{m}^2}}{2}} \times \left(c_1 \rho^{4+l} F\left(\alpha, \alpha + l + 1, l + 2, -\frac{\rho^2}{r_0^2}\right) + c_2 \rho^{2-l} F\left(\alpha, \alpha - l - 1, -l, -\frac{\rho^2}{r_0^2}\right)\right)$$

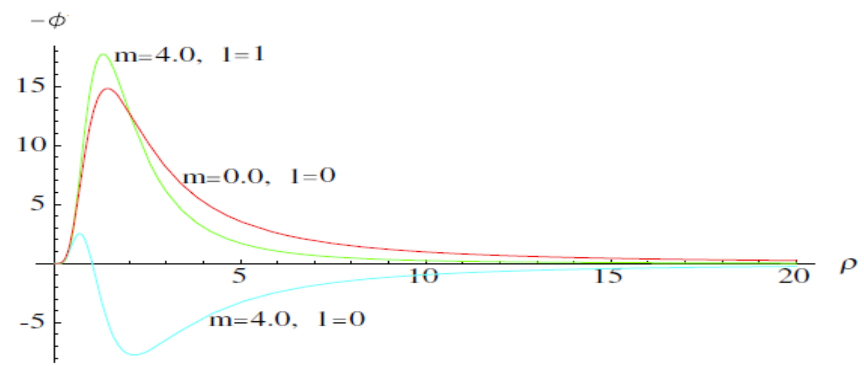
$$\alpha = \frac{1}{2}\left(1 - \sqrt{9 + \tilde{m}^2}\right) \qquad \tilde{m} = \frac{R^2 m}{r_0}$$

$c_1$ $c_2$ : arbitrary constants ***F***: hypergeometric function

# Meson mass spectrum (simple case)

From the regularity of $\boldsymbol{\phi_l(\rho)}$,

$\boldsymbol{c_2 = 0}$ and $\boldsymbol{\alpha + l + 1 = -n}$ $\boldsymbol{(n = 0, 1, 2 \cdots)}$




Then, the meson mass spectrum becomes

$$\boldsymbol{m_s^2 = -\lambda I(I-3)} \qquad \boldsymbol{I \equiv n + l + 3} \qquad \boldsymbol{(n, l = 0, 1, 2 \cdots)}$$

This is the same spectrum as the discrete mass spectrum of the scalar fields obtained by the field theory in $AdS_4$

($\boldsymbol{m_s^2 = -\lambda I(I-3)}$ $\boldsymbol{I = 3, 4 \cdots}$ *Avis, Isham and Storey 1978*).

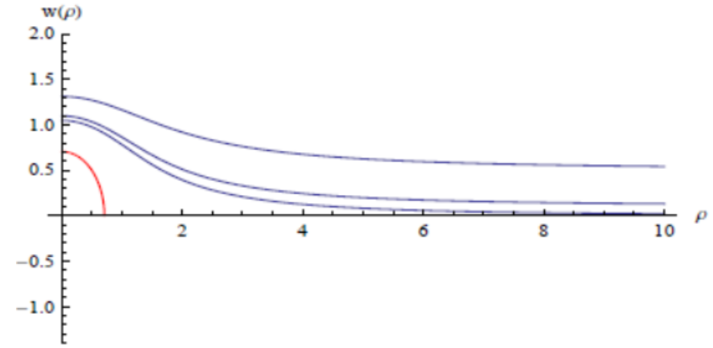
# Meson mass spectrum

Meson spectrums $\boldsymbol{m_9}$ for the non-trivial solutions

$$\boldsymbol{w(\rho) = m_q + \frac{c}{\rho^2} + \cdots}$$

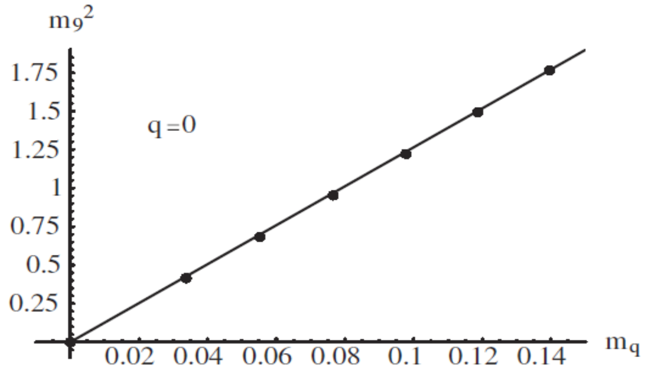
$$(\boldsymbol{c} \equiv \langle\bar{\boldsymbol{\psi}}\boldsymbol{\psi}\rangle \neq \boldsymbol{0})$$

$\boldsymbol{m_q}$: quark mass




$\boldsymbol{m_9}$ $(l = n = 0)$: mass of the NG boson by spontaneous chiral symmetry breaking

We can get the Gellmann-Oaks Renar relation ($\boldsymbol{m_9^2 \propto m_q}$) for small $\boldsymbol{m_q}$



# Glueball mass spectrum

Glueball spectrum can be obtained by the 5D bulk metric fluctuation $\boldsymbol{h_{ij}(t, x^i, r)}$ *(R.C. Brower, S.D.Mathur and C.I. Tan. 2003)*

$$\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}\,g^{MN}\partial_N h_{ij}\right) = 0$$

By decomposing $\boldsymbol{h_{ij}(x^\mu, r) = p_{ij}\chi(x^\mu)\phi(r)}$

The equation of 4D part $\boldsymbol{\chi(x^\mu)}$ is given by

$$\frac{1}{g_4}\partial_\mu\sqrt{g_4}\,g^{\mu\nu}\partial_\nu\chi(x^\mu) = m^2\chi(x^\mu)$$

$m$: Glueball mass

# Glueball mass spectrum

By defining $\boldsymbol{x} \equiv \frac{\boldsymbol{r}}{\boldsymbol{r_0}}$ , Equation for $\boldsymbol{\phi}(\boldsymbol{r})$ is given by

$$\boldsymbol{\partial_x^2 \phi + g_2(x) \partial_x \phi + \frac{R^4 m^2}{r_0^2 x^4 A^2(x)} \phi = 0}$$

Where $\boldsymbol{g_2(x) = \frac{1}{x}\left(5 - \frac{8}{x^2 A(x)}\right)}$ $\boldsymbol{A(x) = 1 + \frac{1}{x^2}}$

$\boldsymbol{\phi}$ becomes normalizable by choosing $\boldsymbol{m}$ as

$$\boldsymbol{m^2 = -\lambda(n+1)(n+4) \quad -\lambda = \frac{4r_0^2}{R^4} \quad n = 0, 1, 2 \cdots}$$

This is the similer spectrum to the discrete mass spectrum of the scalar fields obtained by the field theory in $AdS_4$

($\boldsymbol{m_s^2 = -\lambda I(I-3) \quad I = 3, 4 \cdots}$ *Avis, Isham and Storey 1978*).

However, the lowest glueball mass is different as $\boldsymbol{m^2 = -4\lambda(> 0)}$.

# Summary

We consider 10D bulk which has $AdS_4$ boundaries.

There is IR boundary at $r \to 0$ and UV and IR boundaries are separated by a domain wall in the bulk at $r = r_0$

From the holographic energy momentum tensor, we find the Weyl anomaly which is the same as the result of field theory in $AdS_4$ space-time.

By calculating the embedding solutions of strings and D7-branes in the 10D bulk, We found the quark confinement, spontaneous chiral symmetry breaking, meson mass spectrum which shows the Gellmann-Oaks Renar relation.

Glueball spectrums are also calculated by the bulk gravity fluctuations.

# Current and future works

We consider the effect both **4D cosmological constant** and **temperature** by solving the 5D Einstein equation in the following ansatz

$$ds_5^2 = \frac{r^2}{R^2}\left(-n\,(r)dt^2 + A^2(r)a_0^2(t)\gamma_{ij}(x)dx^i dx^j\right) + \frac{R^2}{r^2}dr^2$$

$n(r)$ and $A(r)$ as determined as

$$\mathbf{n(r)} = \frac{\left(\left(1+\left(\frac{r_0}{r}\right)^2\right)^2 - c_0\left(\frac{R}{r}\right)^4\right)}{A} \qquad A(r) = \left(\left(1+\left(\frac{r_0}{r}\right)^2\right)^2 + c_0\left(\frac{R}{r}\right)^4\right)^{1/2}$$

$c_0$ **: temperature of dual Yang-Mills theory (integration constant)**

$r_0 \equiv \frac{R^2}{2}\sqrt{-\lambda}$ **: cosmological constant of boundary 4D space-time.**

*(K.Ghoroku and A. Nakamura 2012) (J. Erdmenger K.Ghoroku and R. Meyer 2011)*

In this case, the field theory on the UV boundary ($r \to \infty$) are different from that of the IR boundary ($r \to 0$) .