Perturbations of SdS-type Solutions in the dRGT Massive Gravity Theory

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Based on the work

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Plan of the Talk

- Introduction
- dRGT theory
- SdS-type solutions
- Perturbations of the SdS-I solution
- Gauge-invariant formulation
- Comments on the SdS-II solution
- Summary and discussions

Introduction

Historical Background

Massive Gravity Theory

1939 **Fierz-Pauli theory**: linear massive gravity theory (5 DOFs).

1970 **vDVZ discontinuity** [van Dam H, Veltman MJG; Zakharov VI].

1972 **BD ghost**: Non-linear theory: 6 DOFs, one of them is ghost. [Boulware DG, Deser S]

1972 **Vainshtein mechanism**

2002 **Stueckelberg-type formulation** [Arkani-Hamed N, Georgi H, Schwartz MD]

2010 **Ghost free non-linear theory** (dRGT theory) [de Rham C, Gabadadze G, Tolley AJ]

2011 Full-nonlinear proof of the Ghost-freeness [Hassan SF, Rosen RA]

2012 **Bimetric gravity** [Hassan, Rosen]

Black Hole Solutions

2011- Various static BH solutions [Koyama ; …]

2011 **SdS BH solution for special parameters** [Berazhiani L et al; Niuwenhuizen Th M]

2013 **Instability of bi-SBH in the bimetric theory**: GL type instability of the bi-Schwarzschild solution. [Babichev A, Fabbri E]

2013 Massive graviton hair of BH [Brito R, Cardoso V, Pani P]

2014 **Perturbative stability and dynamical degeneracy of SdS-type BH solution in dRGT theory** [Kodama H, Arraut I]

\- Stability of non-bidiagonal BH solutions in the bimetric theory [Babichev, Fabbri]

dRGT Massive Gravity

Lagrangian

- Basic fields
  - Physical spacetime metric: $g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$
  - Strueckelberg fields: ϕ^{a} $(a=0,1,2,3)$

    => Reference metric: $f_{\mu\nu} dx^{\mu}\, dx^{\nu} = \eta_{ab} d\phi^{a}\, d\phi^{b}$

- Requirements
  - General covariant
  - Ghost free
  - Reduces to the Fierz-Pauli theory when $h=g-f$ is small.

- Action

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R(g)+m^2\mathscr{U}(g,\phi)\right],$$

$$\mathscr{U}(g,\phi)=\mathscr{U}_2+\alpha_3\mathscr{U}_3+\alpha_4\mathscr{U}_4,$$

where

$$g^*f_*=(I_4-\mathscr{Q})^2$$

$$\mathscr{Q}=-\sum_{n=1}^{\infty}\binom{1/2}{n}(g^*f_*-I_4)^n.$$

$$\mathscr{U}_2=Q^2-Q_2,$$

$$\mathscr{U}_3=Q^3-3QQ_2+2Q_3,$$

$$\mathscr{U}_4=Q^4-6Q^2Q_2+8QQ_3+3Q_2^2-6Q_4,$$

$$Q=Q_1,\quad Q_n=\mathrm{Tr}\mathscr{Q}^n$$

The potential functions satisfy the Cayley-Hamilton equation:

$$\mathscr{Q}^4-Q\mathscr{Q}^3+\frac{1}{2}\mathscr{U}_2\mathscr{Q}^2-\frac{1}{6}\mathscr{U}_3\mathscr{Q}+\frac{1}{24}\mathscr{U}_4=0$$

Field Equations

- Metric equations:

$$G_{\mu\nu} + m^2 X_{\mu\nu} = 0$$

$$\begin{aligned} X^{\mu}_{\nu} = g^{\mu\alpha} X_{\alpha\nu} \quad = \quad & -Q_1 + \mathscr{Q} \\ & +\alpha \left\{ \frac{1}{2}(Q_2 - Q_1^2) + Q_1 \mathscr{Q} \right\} - \mathscr{Q}^2 \\ & +\beta \Big\{ \frac{1}{6}(-Q_1^3 + 3Q_1 Q_2 - 2Q_3) + \frac{1}{2}(Q_1^2 - Q_2)\mathscr{Q} \\ & \qquad - Q_1 \mathscr{Q}^2 + \mathscr{Q}^3 \Big\} \end{aligned}$$

where

$$\alpha = 1 + 3\alpha_3, \quad \beta = 3(\alpha_3 + 4\alpha_4)$$

Consistency Equations

- Bianchi identities ⇔ diffeomorphism invariance

The diffeomorphism invariance of the mass terms of the action is:

$$\int d^4x' \sqrt{-g'}\,\mathscr{U}(g', \phi') = \int d^4x \sqrt{-g}\,\mathscr{U}(g, \phi).$$

For an infinitesimal diffeomorphism

$$\delta x = \xi^\mu, \quad \delta g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)}, \quad \delta\phi = -\xi^\mu \partial_\mu \phi,$$

this equation reads

$$0 = \int d^4x \sqrt{-g}\left(-m^2 \nabla_\nu X^{\mu\nu}\xi_\mu - \frac{\delta\mathscr{U}}{\delta\phi}\nabla_\mu\phi\xi^\mu\right),$$

Hence we botain

$$m^2 \nabla_\nu X_\mu{}^\nu = -\partial_\mu\phi^a \frac{\delta\mathscr{U}}{\delta\phi^a} = \partial_\mu\phi^a \nabla_\nu\left(\frac{\partial\mathscr{U}}{\partial(\partial_\nu\phi^a)}\right).$$

From the Bianchi identity, this is equivalent to the Euler equation for ϕ^a:

$$\nabla_\mu\left(\frac{\partial\mathscr{U}}{\partial(\partial_\mu\phi^a)}\right) = 0.$$

Schwarzschild-de Sitter-type Solutions

Strategy

- Consider a general spherically symmetric metric in the spherical coordinates:

$$ds^2 = g_{tt}(t,r)dt^2 + 2g_{tr}(t,r)dtdr + g_{rr}(t,r)dr^2 + r^2 S(t,r)^2 d\Omega_2^2$$

- Find the algebraic condition on the metric under which the equation

$$m^2 X^\mu_\nu = \Lambda \delta^\mu_\nu$$

  is satisfied. (Λ condition)

- Then, for a given Stueckelberg field, any spherically symmetric metric satisfying this algebraic condition becomes an exact solution to the field equation of the dRGT theory, if the metric is diffeomorphic to the Schawrzschild-de Sitter/anti-de Sitter solution.

- This exhausts all solutions to the dRGT theory that are diffeomorphic to the SdS/adS solution and spherically symmetric w.r.t. $g_{\mu\nu}$ and ϕ^a simultaneously.

Solutions to the Λ condition

- Unitary gauge on the Stueckelberg field ϕ^a :

$$\phi^0 = t, \quad \phi^i = x^i = r\Omega^i \ (i = 1, 2, 3),$$
$$f_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} d\phi^a d\phi^b = -dt^2 + dr^2 + r^2 d\Omega_2^2.$$

- Representation for the Q matrix

$$g^* f_* = \left(\begin{array}{cc|c} -g^{tt} & g^{tr} & \\ -g^{tr} & g^{rr} & \smash{0} \\ \hline \multicolumn{2}{c|}{0} & S^{-2} I_2 \end{array} \right) \quad \Rightarrow \quad \mathscr{Q} = \left(\begin{array}{cc|c} a & c & \\ -c & b & \smash{0} \\ \hline \multicolumn{2}{c|}{0} & (1 - S^{-1}) I_2 \end{array} \right)$$

$$\frac{g_{tt}}{g_{(2)}} = (1-b)^2 - c^2, \quad \frac{g_{rr}}{g_{(2)}} = -((1-a)^2 - c^2), \quad \frac{g_{tr}}{g_{(2)}} = c(2 - a - b)$$
$$(-g_{(2)})^{-1/2} = c^2 + (1-a)(1-b)$$

- The Λ condition

$$
\begin{aligned}
X^t_t &\equiv -bF_3 - (F_1+1)\frac{S-1}{S} = \Lambda/m^2, \\
X^t_r &\equiv cF_3 = 0, \\
X^t_t - X^r_r &\equiv (a-b)F_3 = 0, \\
X^t_t - X^\theta_\theta &\equiv F_1\left(a-1+\frac{1}{S}\right) + F_2\left\{ab + c^2 - b\frac{S-1}{S}\right\} = 0,
\end{aligned}
$$

where

$$
F_1 = \alpha + 1 - \frac{\alpha}{S}, \quad F_2 = \alpha + \beta - \frac{\beta}{S},
$$

$$
F_3 = F_1 + \frac{S-1}{S}F_2
$$

- Important special cases
  - $F_1=F_2=0 \;\Rightarrow\;$ S=const, $\beta=\alpha^2$.
  - $F_3=0$, $F_1\neq 0 \;\Rightarrow$ S=const, a relation among a, b and c.

Exhaustive list of solutions

- Solution F (flat solutions): $\Lambda=0$

$$S=1,\quad \frac{3\alpha+2\beta\pm\sqrt{9\alpha^2-12\beta}}{2(3+3\alpha+\beta)}.$$

- Solution SdS-I: $F_1=F_2=0 \Rightarrow \alpha^2=\beta,\ \Lambda=m^2/\alpha$

$$ds^2=-f(Sr)dT_o(t,Sr)^2+\frac{S^2dr^2}{f(Sr)}+S^2r^2d\Omega^2,$$

$$S=\frac{\alpha}{\alpha+1},\qquad f(r)=1-\frac{2M}{r}-\frac{\Lambda}{3}r^2$$

- Solution SdS-II: $F_3=0,\ F_2\neq 0 \Rightarrow \alpha^2>\beta$

$$S=\frac{\alpha+\beta\pm\sqrt{\alpha^2-\beta}}{1+2\alpha+\beta},\qquad \Lambda=-m^2\left(1-\frac{1}{S}\right)\left(2+\alpha-\frac{\alpha}{S}\right)$$

$$S^2(T_0')^2=\frac{1-f(Sr)}{f(Sr)}\left(\frac{S^2}{f(Sr)}-\dot{T}_0^2\right)$$

Notes

- Solution SdS-I
  - The solution contains an arbitrary function of t and r, $T_0(t,r)$.
  - This implies the dynamical degeneracy of the field equation around this solution for the parameter space $\alpha^2=\beta$.
  - This solution contains the pure dS solution as a special case.
- Solution SdS-II
  - $T_0(t,r)$ has freedom of an arbitrary function of a single variable.
  - This function cannot be determined by the initial condition in general. Hence, dynamical degeneracy exists also for this solution.
  - The whole solution becomes regular at the future horizon for the choice

$$T_0 = St \pm \int^{Sr} \left(\frac{1}{f(u)} - 1 \right) du.$$

Perturbations of the SdS-I BH solution

Background Solution

- Stueckelberg field: the modified unitary gauge,

$$\phi^0 = t, \quad \phi^i = \frac{x^i}{S} = \frac{r}{S}\Omega^i \ (i = 1, 2, 3),$$

$$f_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} d\phi^a d\phi^b = -dt^2 + \frac{dr^2}{S^2} + \frac{r^2}{S^2} d\Omega_2^2.$$

In this part, we use this gauge condition also for perturbations.

$$\delta\phi^a = 0$$

- Metric: Generic spherically symmetric one in the spherical coordinates,

$$ds^2 = g_{ab}(y) dy^a dy^b + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$ds^2 = -f(r)(dT_0(t, r))^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad S = \frac{\alpha}{1+\alpha}$$

Vector Perturbations

- Metric perturbation

$$h_{ab} = 0, \quad h_{ai} = r f_a \mathbb{V}_i, \quad h_{ij} = 2r^2 H_T \mathbb{V}_{ij}.$$

- Perturbation of the mass term

$$\kappa^2 \tau^\mu_\nu := \kappa^2 \delta T^\mu_\nu = -m^2 \delta X^\mu_\nu,$$
$$\tau^a_b = 0, \quad \tau^a_i = r\tau^a \mathbb{V}_i, \quad \tau^i_j = \tau_T \mathbb{V}^i_j,$$

- Perturbation of the field equations for generic modes l>1:

$$\frac{1}{r^3} D^b \left(r^3 F^{(1)}_{ab} \right) - \frac{(l-1)(l+2)}{r^2} F_a = -2\kappa^2 \tau_a,$$
$$\frac{k_v}{r^2} D_a (r F^a) = -\kappa^2 \tau_T,$$

where

$$F_a := f_a + \frac{r}{k_v} D_a H_T, \quad F^{(1)}_{ab} = 2r D_{[a}(F_{b]}/r) = 2r D_{[a}(f_{b]}/r).$$

SdS-I Background

- Calculation of the source term

$$\tau^a = 0, \quad \kappa^2 \tau_T = m^2 w(r) H_T;$$
$$w(r) = -\frac{1+\alpha}{\alpha}\{\beta(c^2+ab)+\alpha(a+b)+1\}$$

$$\beta = \alpha^2, \quad S = \frac{\alpha}{1+\alpha}$$

- Bianchi identities

$$D_a(r^3\tau^a) + \frac{(l+2)(l-1)}{2[l(l+1)-1]^{1/2}} r^2 \tau_T = 0 \Rightarrow \quad (l-1)w(r)H_T = 0.$$

- Stability for generic modes

All source terms in the perturbation equations for vector perturbations vanish.

$$\tau^a = 0, \quad \tau^T = 0.$$

This implies that the perturbation equations are identical to the gauge-invariant vacuum perturbation equations for a SdS black hole in the Einstein theory.

Because $H_T=0$, $f_a=F_a$ and no residual gauge freedom.

- Exceptional modes (l=1)

For exceptional modes, H_T does not exist, and we only have

$$D^b\left(r^3 F^{(1)}_{ab}\right) = 0$$

But, $F^{(1)}{}_{ab}$ does not uniquely determine f_a and leaves the freedom corresponding to the gauge transformation

$$\delta x^i = L\mathbb{V}^i \Rightarrow \quad \delta f_a = -rD_a L(y)$$

Because any solution to the above equation can be shown to be gauge equivalent to an addition of rotation in the Einstein theory, the general solution is given by

$$f_a = -rD_a L(t,r) - \frac{2aM}{r}\partial_a T_0(t,r).$$

This contains an arbitrary function L of t and r.

Hence, dynamics becomes degenerate in this sector.

Scalar Perturbations

- Perturbation variables

$$h_{ab} = f_{ab}\mathbb{S}, \quad h_{ai} = r f_a \mathbb{S}_i, \quad h_{ij} = 2r^2(H_L \gamma_{ij}\mathbb{S} + H_T \mathbb{S}_{ij}),$$

$$\delta T_{ab} = \tau_{ab}\mathbb{S}, \quad \delta T_i^a = r\tau^a \mathbb{S}_i, \quad \delta T_j^i = \tau_L \delta_j^i \mathbb{S} + \tau_T \mathbb{S}_j^i$$

- Gauge-invariant variables for generic modes (l>1)

$$F = H_L + \frac{1}{2}H_T + \frac{1}{r}D^a r X_a,$$

$$F_{ab} = f_{ab} + D_a X_b + D_b X_a,$$

$$\kappa^2 \Sigma_{ab} = \kappa^2 \tau_{ab} - 2\Lambda D_{(a}X_{b)}, \quad \tau_a, \quad \tau_L, \quad \tau_T,$$

where

$$X_a = \frac{r}{k}\left(f_a + \frac{r}{k}D_a H_T\right)$$

- Perturbation equations

$$E_{ab} = \kappa^2 \Sigma_{ab}, \quad E_a = \kappa^2 \tau_a, \quad E_L = \kappa^2 \tau_L, \quad E_T = \kappa^2 \tau_T$$

- Calculation of the source terms: $\alpha^2=\beta$, $\Lambda=m^2/\alpha$

$$\kappa^2\Sigma_{ab} = -\Lambda F_{ab}, \quad \tau_a = 0, \quad \kappa^2\tau_L = -m^2 w(r) H_L, \quad \kappa^2\tau_T = m^2 w(r) H_T$$

- Bianchi identities

$$\frac{1}{r^3}D_a(r^3\tau^a) - \frac{k}{r}\tau_L + \frac{(l-1)(l+2)}{2kr}\tau_T = 0,$$
$$\frac{1}{r^2}D_b\left[r^2(\Sigma_a^b + \kappa^{-2}\Lambda F_a^b)\right] + \frac{k}{r}\tau_a - 2\frac{D_a r}{r}\tau_L = 0,$$

$$k = [l(l+1)]^{1/2}$$

$\Rightarrow$ $H_L = 0, \quad H_T = 0(l > 1)$

Hence, the perturbation equations for generic modes are identical to those for the vacuum Einstein gravity with Λ .

- Stability for generic modes
  - Because the SdS BH is stable in the Einstein gravity, it is also geometrically stable in the dRGT theory.
  - Under the coordinate transformation

$$\delta x^a = T^a \mathbb{S}, \quad \delta x^i = L \mathbb{S}^i$$

H_L and H_T transform as

$$\delta H_L = -\frac{k}{2} L - \frac{1}{r} T^r, \quad \delta H_T = kL$$

Hence, the equations $H_L = H_T = 0$ are preserved for any T^t

if $T^r = 0$ and $L = 0$.

This implies dynamical degeneracy of the dRGT theory, because the solution in the dRGT theory corresponding to each solution for the gauge-invariant perturbation equations in the Einstein gravity contains the arbitrary function T^t of t and r.

- Exceptional modes with l=1:
  - The general solution contains two arbitrary functions T^t and T^r of t and r:

$$f_{ab} = -D_a T_b - D_b T_a, \quad f_a = -r D_a L + \frac{k}{r} T_a, \quad H_L = 0,$$

where

$$L = -\frac{2}{kr} T^r.$$

- Exceptional models with l=0 (S-modes):
  - The general solution contains one arbitrary function T^t of t and r:

$$f_{ab} = -2D_{(a}T_{b)} + \delta M \partial_M g_{ab}, \quad H_L = 0,$$

where T^r=0.

Gauge-invariant Formulation

Gauge Invariants for $\delta\phi^\alpha$

- Perturbation variables of the Stueckelberg fields

$$\delta\phi^\alpha (\alpha = 0, \cdots, 3) \Rightarrow \quad \sigma^\mu := \frac{\partial x^\mu}{\partial X^\alpha}\delta\phi^\alpha \; (\mu = t, r, \theta, \phi)$$

- Gauge transformations

$$\delta_g \sigma^a = -\frac{T^a}{S}Y, \quad \delta_g \sigma^i = -\frac{L}{S}Y^i$$

- Vector perturbations

$$\sigma^a = 0, \quad \sigma^i = \sigma_T \mathbb{V}^i$$

  – For generic modes, the basic gauge invariants are

$$F_a = f_a + \frac{r}{k}D_a H_T, \quad \hat{\sigma}_T = \sigma_T + \frac{1}{kS}H_T \quad \Rightarrow \quad \hat{\sigma}_T = 0$$

  – For exceptional modes with l=1

$$\hat{F}_a = f_a - Sr\partial_a \sigma_T \quad \Rightarrow \quad \hat{F}_a = -rD_a L - \frac{2\alpha M}{r}\partial_a T_0$$

- Scalar perturbations
  - For generic modes, the basic gauge-invariants are

$$\hat{\sigma}^t = \sigma^t + X^t, \quad \hat{\sigma}^r = \sigma^r + \frac{X^r}{S}, \quad \hat{\sigma}_T = \sigma_T + \frac{1}{kS} H_T$$

In terms of these, our results can be expressed as

$$\hat{\sigma}^r = \frac{r}{S} F, \quad \hat{\sigma}_T = 0, \quad \hat{\sigma}^t : \text{ arbitrary}$$

  - For exceptional modes with l=1, the basic gauge invariants are

$$\hat{F} := F - \frac{Sr}{k} D^r \sigma_T - \frac{kS}{2} \sigma_T = -\frac{k}{2} L - \frac{r}{k} D^r L,$$

$$\hat{F}_{ab} := F_{ab} - \frac{2S}{k} D_{(a}(r^2 D_{b)} \sigma_T) = -\frac{2}{k^2} D_{(a} T_{b)},$$

$$\tilde{\sigma}^t := \hat{\sigma}^t - \frac{Sr^2}{k} D^t \sigma_T = T^t - \frac{r^2}{k} D^t L,$$

$$\tilde{\sigma}^r := \hat{\sigma}^r - \frac{r^2}{k} D^r \sigma_T = \frac{r}{S} \hat{F}, \quad T^r = -\frac{kr}{2} L.$$

– For exceptional modes with l=0, the basic gauge invariants are

$$\hat{F}_{ab} = f_{ab} - D_a\tilde{\sigma}_b - D_b\tilde{\sigma}_a,$$
$$\hat{F} = H_L - \frac{S}{r}\sigma^r,$$

where

$$\tilde{\sigma}^t = \sigma^t, \quad \tilde{\sigma}^r = S\sigma^r.$$

In terms of these gauge invariants, our results are expressed as

$$\hat{F}_{tt} = \delta M \partial_M g_{tt} + 2f\dot{T}^t,$$
$$\hat{F}_{tr} = \delta M \partial_M g_{tr} + f(h'\dot{T}^t + \partial_r T^t) - f'T^t,$$
$$\hat{F}_{rr} = \delta M \partial_M g_{rr} + 2h'f\partial_r T^t,$$
$$\hat{F} = 0,$$

where $T^t(t,r)$ is an arbitrary function.

Comment on the SdS-II solution

SdS-II Solution

- Metric: $\alpha^2 > \beta$

$$ds^2 = -f(r)(dT_0(t,r))^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$

$$\phi^0 = t, \quad \phi^i = \frac{x^i}{S} = \frac{r}{S}\Omega^i \ (i = 1,2,3),$$

$$f_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}d\phi^a d\phi^b = -dt^2 + \frac{dr^2}{S^2} + \frac{r^2}{S^2}d\Omega_2^2.$$

$$S = \frac{\alpha + \beta \pm \sqrt{\alpha^2 - \beta}}{1 + 2\alpha + \beta}, \quad \Lambda = -m^2\left(1 - \frac{1}{S}\right)\left(2 + \alpha - \frac{\alpha}{S}\right)$$

- Constraint

$$(T_0')^2 = \frac{1 - f(r)}{f(r)}\left(\frac{1}{f(r)} - \frac{\dot{T}_0^2}{S^2}\right)$$

- Static solution

$$T_0 = \mu S t + h(r)$$

Dynamical Degenracy?

- Hamiltonian system equivalent to the constraint:

$$H := \frac{1}{2}p_0^2 + \frac{1}{2}\frac{f(r)}{1-f(r)}p_1^2 - \frac{1}{2f(r)} = 0$$

$$\dot{t} = p_0, \quad \dot{r} = \frac{f}{1-f}p_1,$$

$$\dot{p}_0 = 0, \quad \dot{p}_1 = -\frac{f'}{2f^2}\left(\frac{f^2}{(1-f)^2}p_1^2 + 1\right),$$

$$\dot{T}_0 = \dot{t}p_0 + \dot{r}p_1 = \frac{1}{4f}$$

$$\Leftarrow \quad p_0 = \partial_t T_0, \quad p_1 = \partial_r T_0$$

- The general solution with $p_0^2 < 1$ for the pure dS solution :

$$T_0(t,r) = k(t_0) + \ell \ln\left[\frac{r^2 + (t-t_0)^2 \pm \sqrt{4\ell^2(t-t_0)^2 + \{(t-t_0)^2 - r^2\}^2}}{2(t-t_0)\sqrt{\ell^2 - r^2}}\right],$$

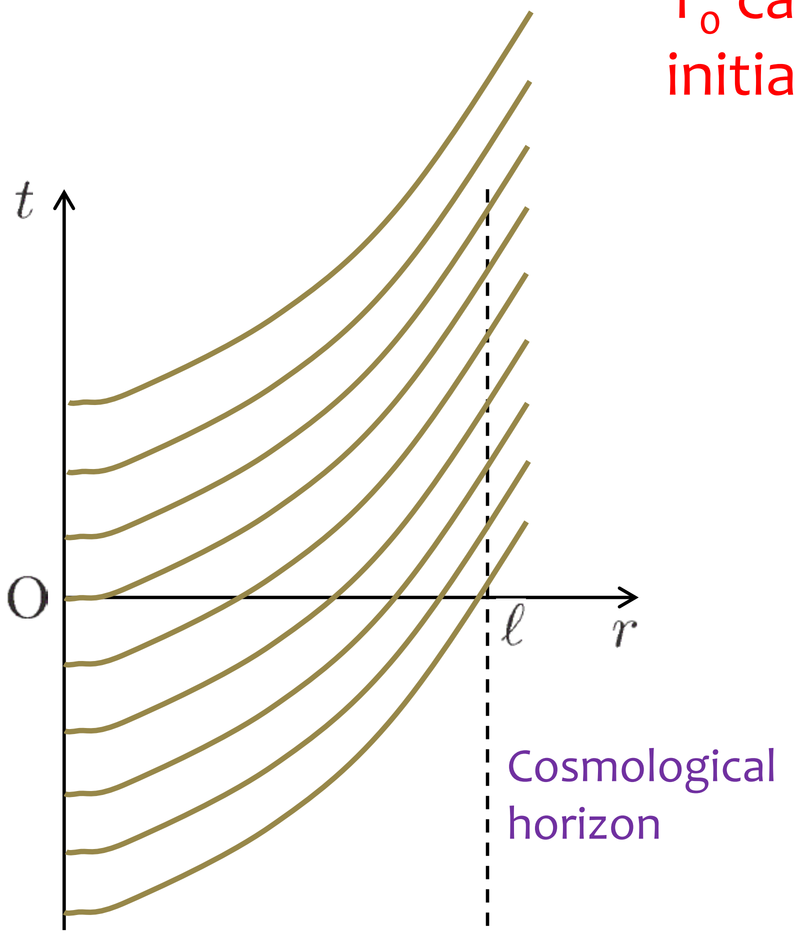
$$(t-t_0)^2 \pm 2\ell(t-t_0)\sqrt{(k'(t_0))^{-2} - 1} = r^2$$

where $k(t_0)$ is an arbitrary function and $\Lambda = 3/l^2$.

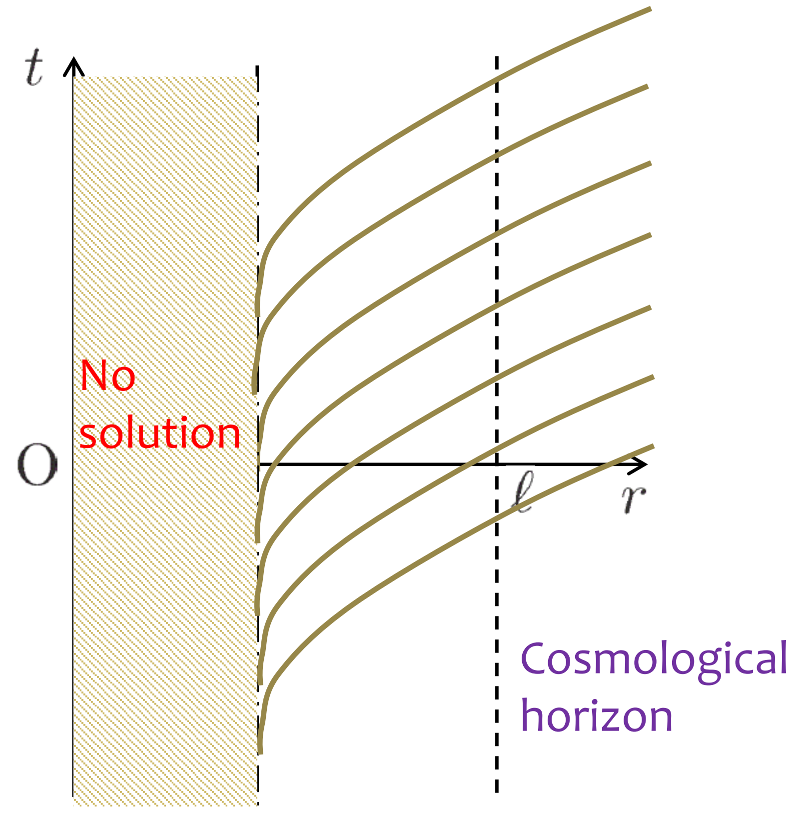
Example: the pure dS case

Flows along which the information of T_0 propagates

T_0 cannot be determined by the initial condition at t=0!!



$p_0^2 < 1$



$p_0^2 > 1$

Summary

Summary

- The dRGT massive gravity theory allows the SdS/adS BH metric as an exact solution for an open set of the theory parameters.
- For its sub family with the parameter choice $\beta=\alpha^2$, the solution is geometrically stable, but the dynamics contains degeneracy in the Stueckelberg sector.
- Even for the solutions corresponding to the parameter choic $\beta<\alpha^2$, there remains dynamical degeneracy partly. In particular, the system has dynamical degeneracy around a solution that represents a pure dS/adS spacetime geometrically.
- These results suggest that we have to be careful not only about the unitary but also the dynamical degeneracy when we discuss the viability of massive gravity theories.
- Note: all analyses can be easily extended to the massive gravity theory with cosmological constant, for which the Schwarzschild solution becomes an exact solution.