Charged Quantum Entanglement and Symmetry Protected Topological phases

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QUANTUM ENTANGLEMENT

Useful probe in quantum systems

- Quantum Information

- efficiency of information transmission

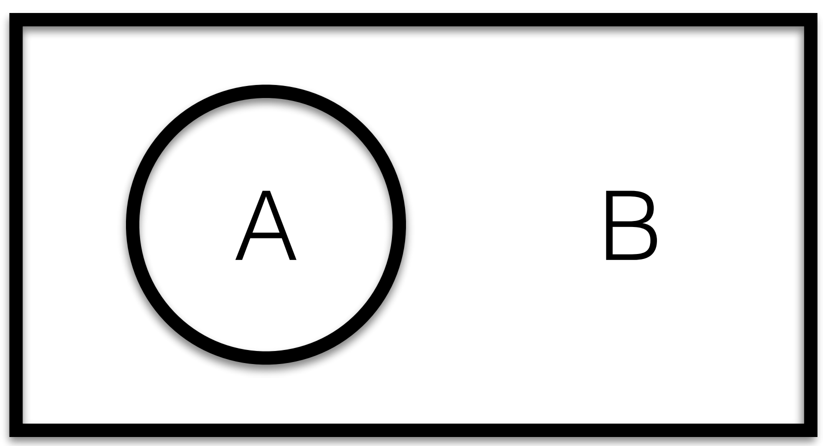
- High energy physics / Quantum Gravity

- emergent space time, black hole information

- Condensed Matter Physics

- topological order, quantum phase transition

QUANTUM ENTANGLEMENT



Reduced density matrix

$$\rho_A = \mathrm{Tr}_B\, \rho$$

Entanglement Hamiltonian

$$\rho_A = \frac{e^{-H_E/T_0}}{Z(T_0)}$$

Entanglement Entropy

$$S_{EE} = -\mathrm{Tr}_A\, \rho_A \log \rho_A$$

Number of states in B that are
consistent with all measurements in A

Entanglement vs Thermal

Entanglement Entropy

$$S_{EE} = -\mathrm{Tr}_A\, \rho_A \log \rho_A$$

$$\rho_A = \frac{e^{-H_E/T_0}}{Z(T_0)}$$

- finite at zero temperature
- one state
- in and out of equilibrium
- grand canonical ensemble

[Belin-Hung-Myers-SM-Sierens]

Thermal Entropy

$$S = -\,\mathrm{Tr}\, \rho_{therm} \log \rho_{therm}$$

$$\rho_{therm} = \frac{e^{-H/T}}{Z(T)}$$

- zero at zero temperature
- all states
- in equilibrium
- Laws of thermodynamics

$$dF = -S_{therm} dT + \mu dQ$$

Entanglement Chemical potential

Grand Canonical ensemble

$$\tilde{\rho}_{therm} = \frac{e^{-(H-\mu Q)/T}}{Z(T,\mu)}$$

Entanglement Chemical potential

Grand Canonical ensemble in Entanglement

$$\tilde{\rho}_A = \frac{e^{-(H_E - \mu_E Q)/T_0}}{Z(T_0, \mu_E)}$$

μ_E Entanglement chemical potential

Total system has a fixed charge (say Q_total=0)
while subsystems can have any charges (Q_A=-Q_B)

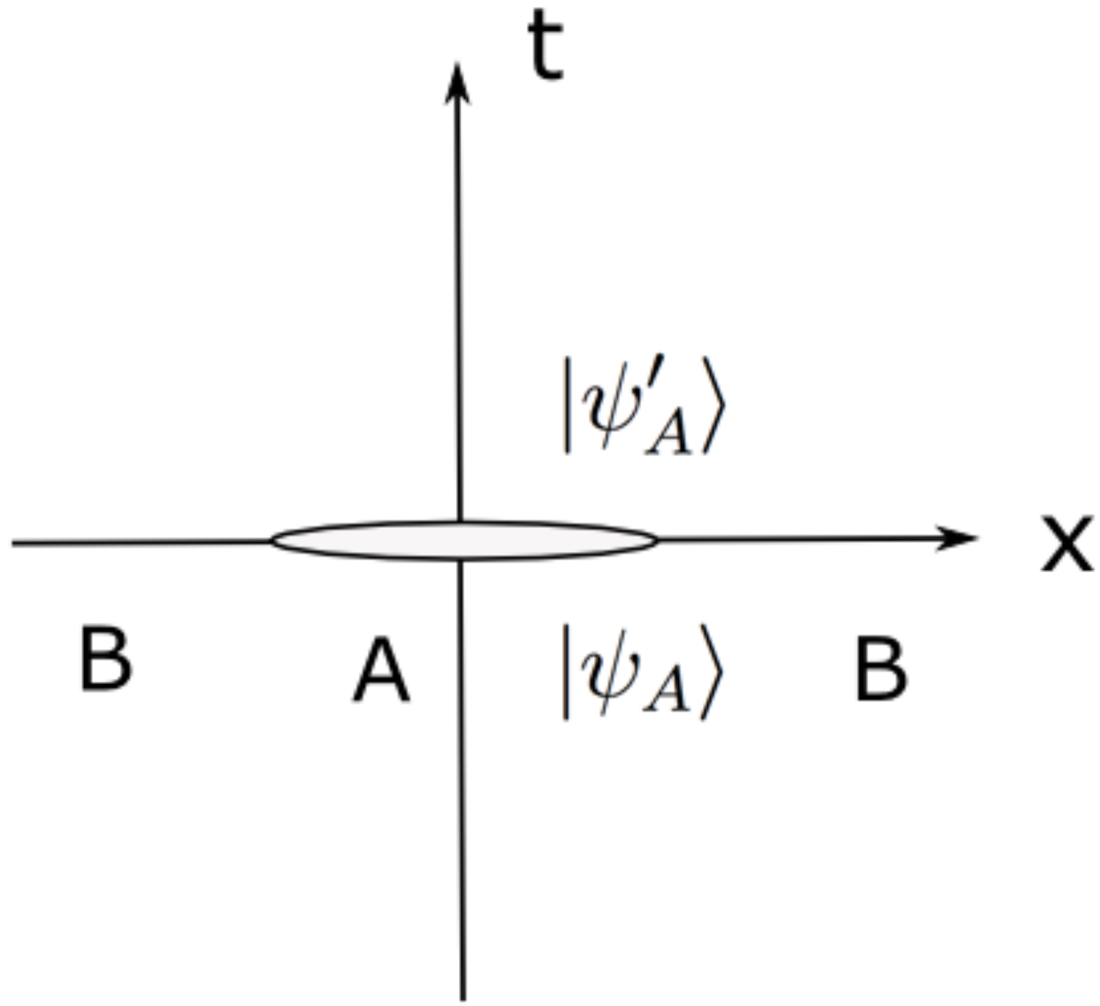
$$|\Psi\rangle = \sum_i a_i |A, Q_i\rangle \otimes |B, -Q_i\rangle$$

$$S_n(\mu, A) = S_n(-\mu, B)$$

Path Integral

Reduced density matrix of ground state

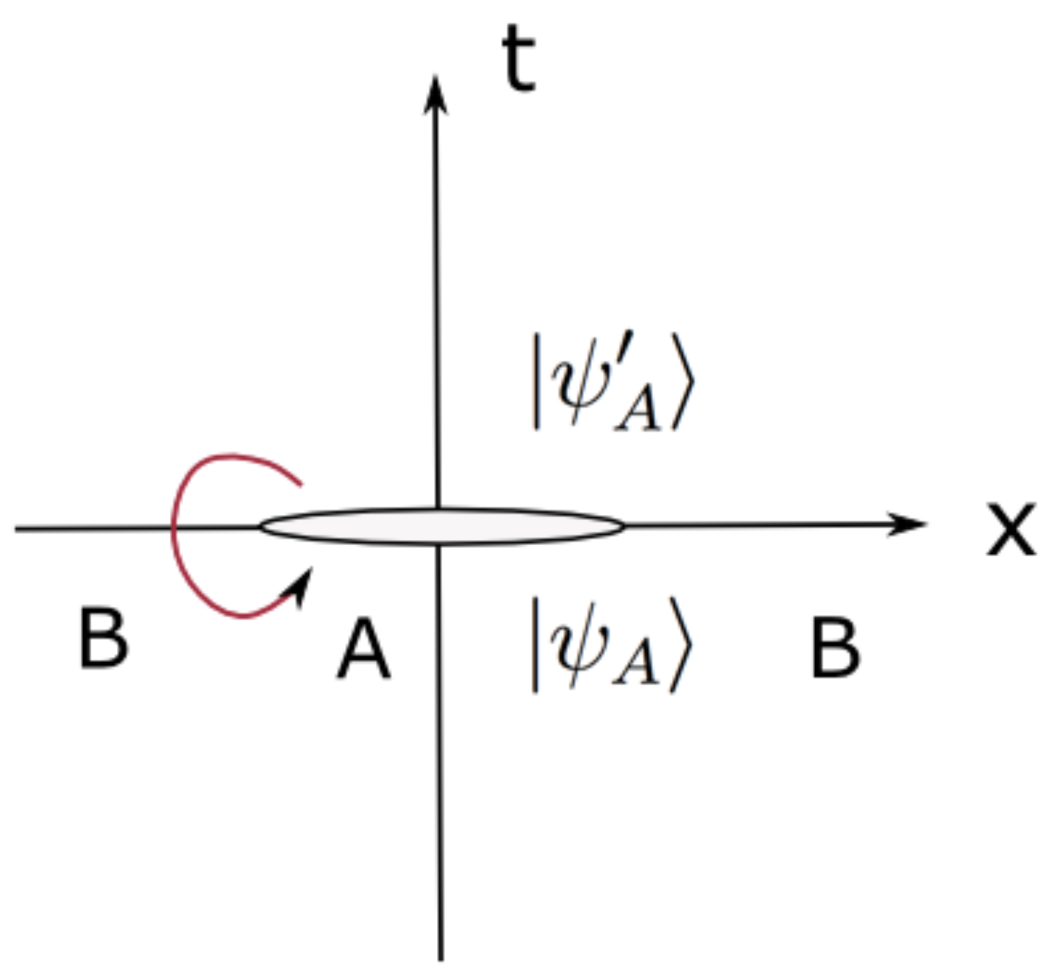
$$\langle \psi_A | \rho | \psi'_A \rangle = \frac{1}{Z} \int_{\substack{\psi(t=0_-, x\in A)=\psi_A \\ \psi(t=0_+, x\in A)=\psi'_A}} \mathcal{D}\psi \; e^{-S_E}$$



Path Integral

Reduced density matrix of ground state

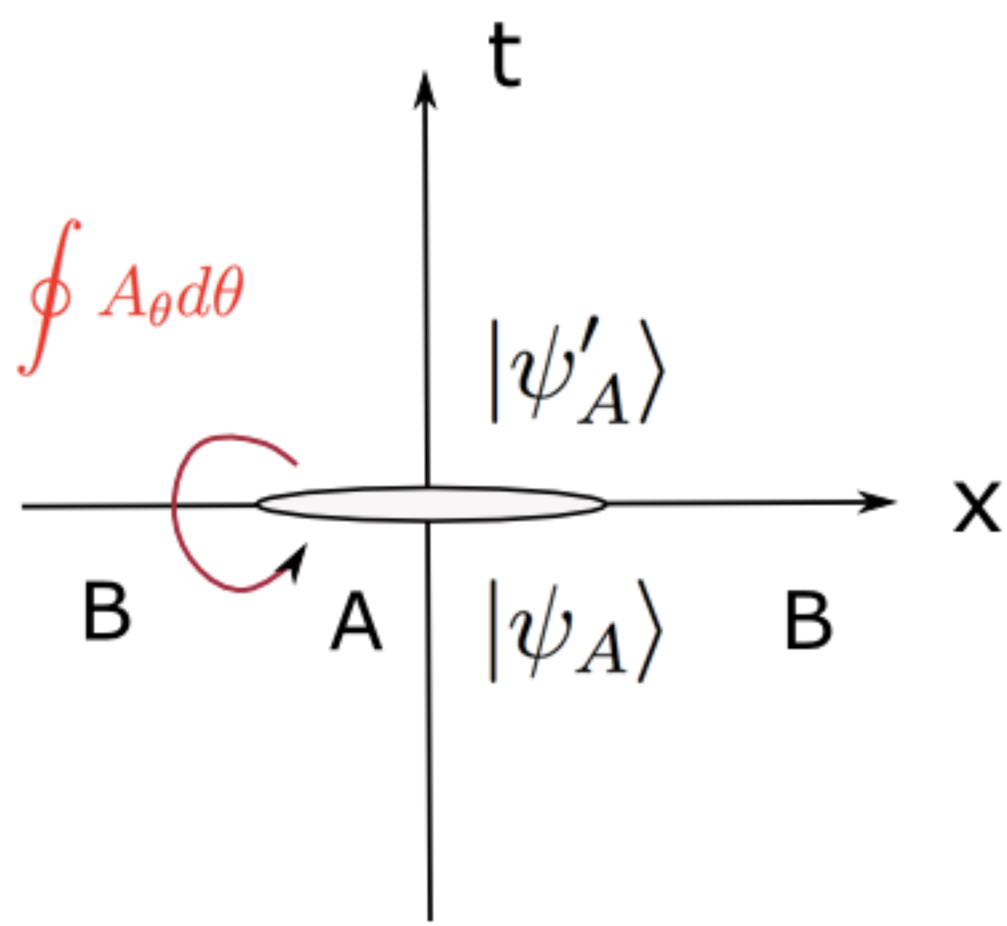
$$\tilde{\rho}_A = \frac{1}{Z(T_0)} \langle \psi_A | e^{-H_E/T_0} | \psi'_A \rangle$$



Path Integral

Reduced density matrix of ground state

$$\tilde{\rho}_A = \frac{1}{Z(T_0, \mu_E)} \langle \psi_A | e^{-(H_E - i\mu_E Q)/T_0} | \psi'_A \rangle$$



Aharonov-Bohm effect around the entangling surface

Conformal Field Theories

Entanglement Hamiltonian of a spherical region

$$ds^2_{R^d} = dt_E^2 + dr^2 + r^2 d\Omega_{d-2}^2 \qquad (r < R)$$

$$= \Omega^{-2}(d\tau_E^2 + R^2(du^2 + \sinh^2 u d\Omega_{d-2}^2))$$

$$\Omega = \left|1 + \cosh\left(u + i\frac{\tau_E}{R}\right)\right| \qquad \exp\left[-\left(u + i\frac{\tau_E}{R}\right)\right] = \frac{R-(r+it_E)}{R+(r+it_E)}$$

$$\begin{aligned} H_E &= 2\pi \int_A d^{d-1}x \frac{R^2 - r^2}{2R} T^{00}(x) \\ &= \partial_{\tau_E} \end{aligned}$$

Reduced density matrix on a spherical region

= Thermal density matrix on a hyperbolic space

Conformal Field Theories

Entanglement chemical potential of a spherical region

$$ds^2_{R^d} = dt_E^2 + dr^2 + r^2 d\Omega^2_{d-2} \qquad (r < R)$$

$$= \Omega^{-2}(d\tau_E^2 + R^2(du^2 + \sinh^2 u d\Omega^2_{d-2}))$$

Wilson loop along τ_E direction $\displaystyle\int_A A_{\tau_E} J^{\tau_E}$

Finite density system on the hyperbolic space

Conformal Field Theories

Renyi entropy

$$S_n = \frac{1}{1-n} \log \frac{Z(T_0/n)}{Z(T_0)^n}$$

By using the following relations

$$F(T) = -T \log Z(T), \quad S_{therm} = -\frac{\partial F(T)}{\partial T}$$

we obtain

$$\begin{aligned} S_n &= \frac{n}{1-n} \frac{1}{T_0} (F(T_0) - F(T_0/n)) \\ &= \frac{n}{1-n} \frac{1}{T_0} \int_{T_0/n}^{T_0} S_{therm}(T) dT \end{aligned}$$

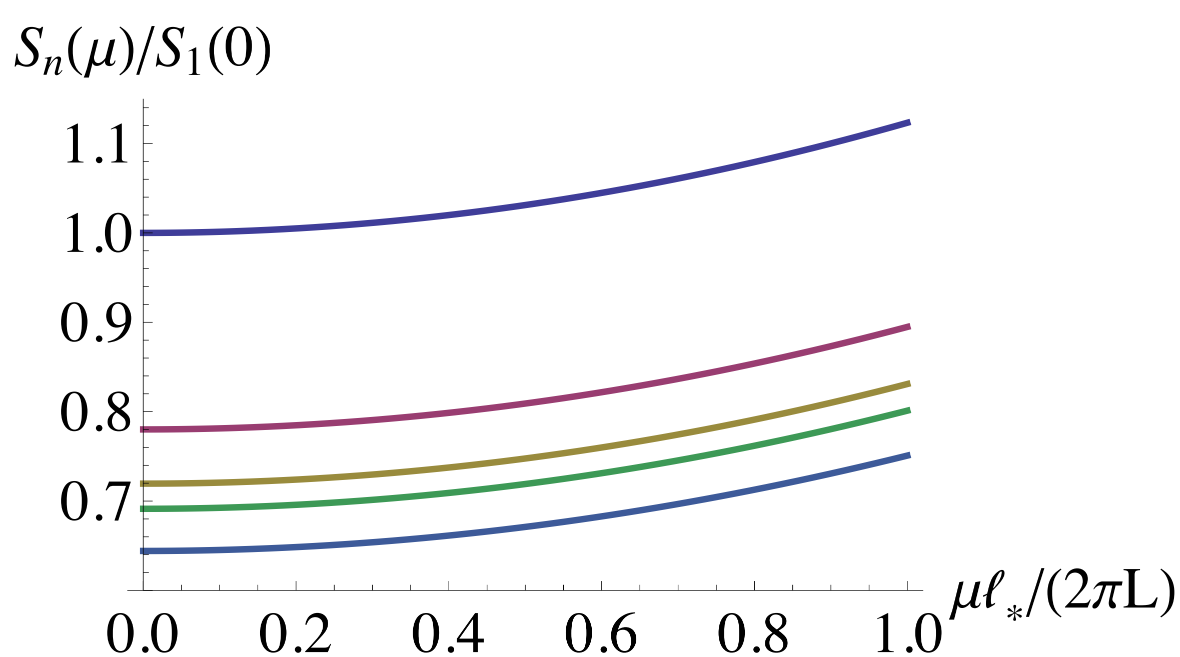
AdS/CFT

Charged hyperbolic black hole

$$I_{E-M} = \frac{1}{2\ell_{\rm P}^{d-1}} \int d^{d+1}x \sqrt{-g} \left(\frac{d(d-1)}{L^2} + \mathcal{R} - \frac{\ell_*^2}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

$$ds^2 = -f(r) \frac{L^2}{R^2} d\tau_E^2 + \frac{dr^2}{f(r)} + r^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2)$$

$$f(r) = \frac{r^2}{L^2} - 1 - \frac{m}{r^{d-2}} + \frac{q^2}{r^{2d-4}}$$



AdS/CFT

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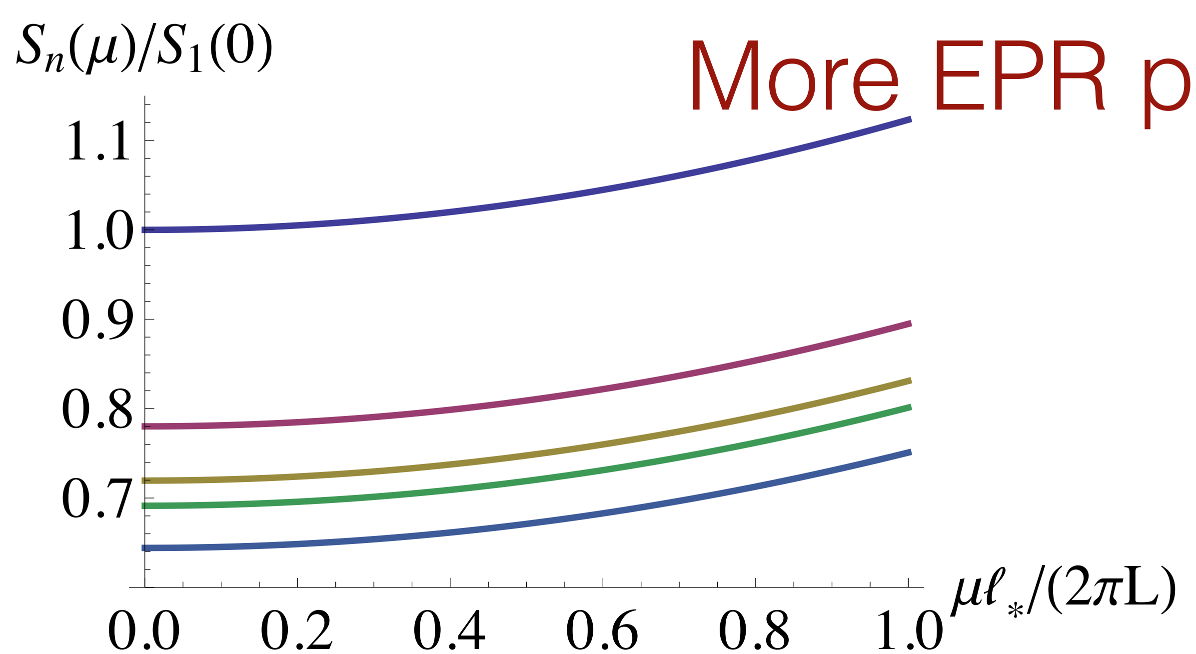


More EPR pairs

Phase diagram

[Dias,Monteiro,Real,Santos/Belin,Hung,Maloney,SM]

Einstein-Maxwell-scalar

$$I_{E-M} = \frac{1}{2\ell_{\rm P}^2}\int d^4x\sqrt{-g}\left(\frac{6}{L^2}+\mathcal{R}-\frac{\ell_*^2}{4}F_{\mu\nu}F^{\mu\nu}-V(|\Phi|)-\frac{1}{2}|\nabla\Phi-iqA\Phi|^2\right)$$

$$ds^2 = -f(r)\frac{L^2}{R^2}d\tau_E^2+\frac{dr^2}{f(r)}+r^2(du^2+\sinh^2 u d\Omega_{d-2}^2)$$

Instability in the scalar field

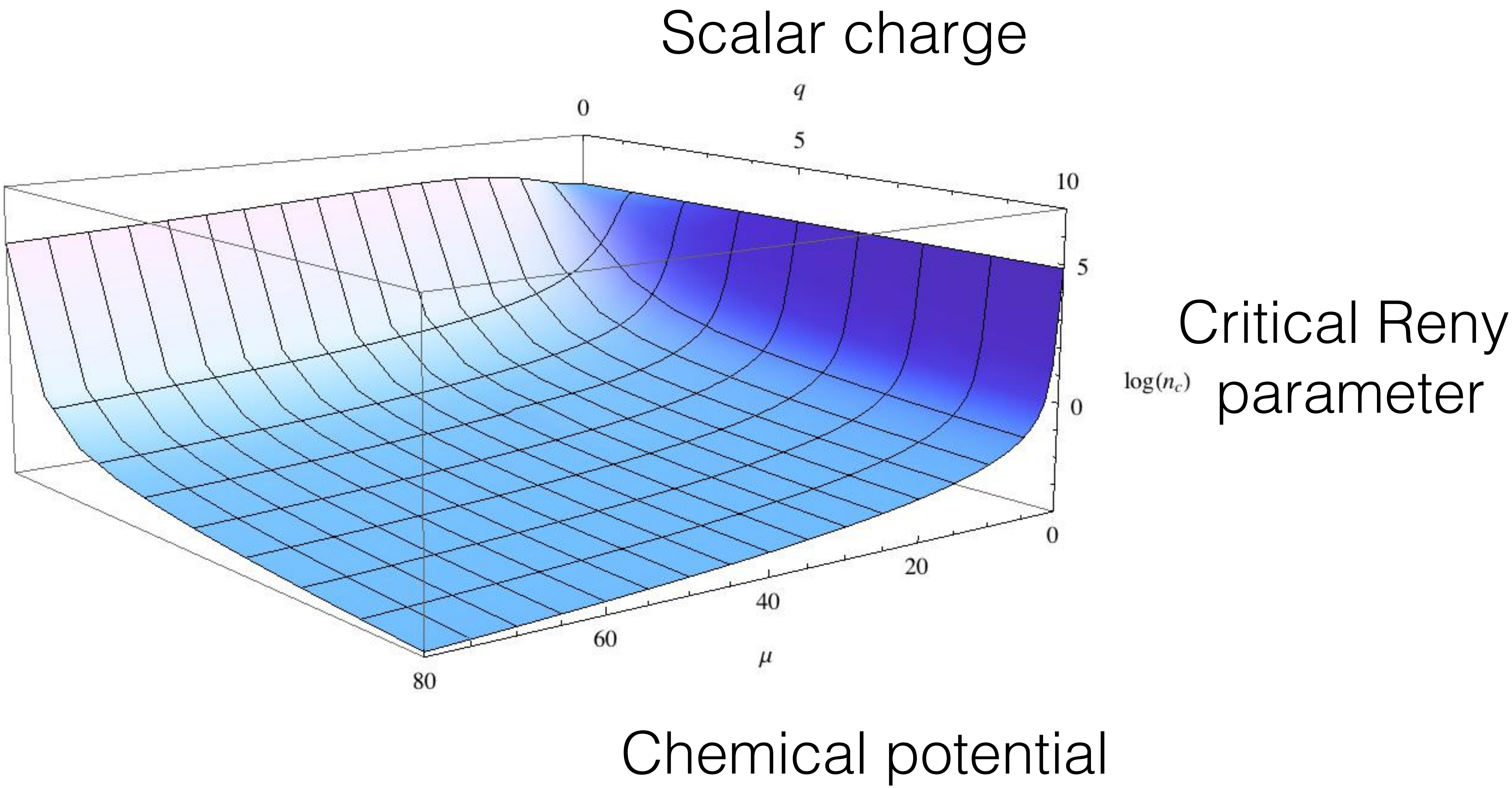
$$\left((\nabla^\mu - iqA^\mu)(\nabla_\mu - iqA_\mu) - m_\Phi^2\right)\Phi = 0$$

Time dependent solution

$$\Phi = \frac{\phi(r)e^{\omega t}Y(\sigma)}{r} \qquad \nabla_{H_2}^2 Y = -\lambda Y \qquad \lambda > 1/4$$

Effective mass/BF bound estimation

$$-\frac{d^2}{4} < m_\Phi^2 L^2 < -\frac{f''(r_H)}{8} - \frac{(d-2)^2}{4r_H^2} + \frac{2q^2\mu^2}{4\pi R^2 r_H^2 f''(r_H)}$$


Scalar charge
q
0
5
10
Critical Renyi parameter
log(n_c)
5
0
μ
0
20
40
60
80
Chemical potential

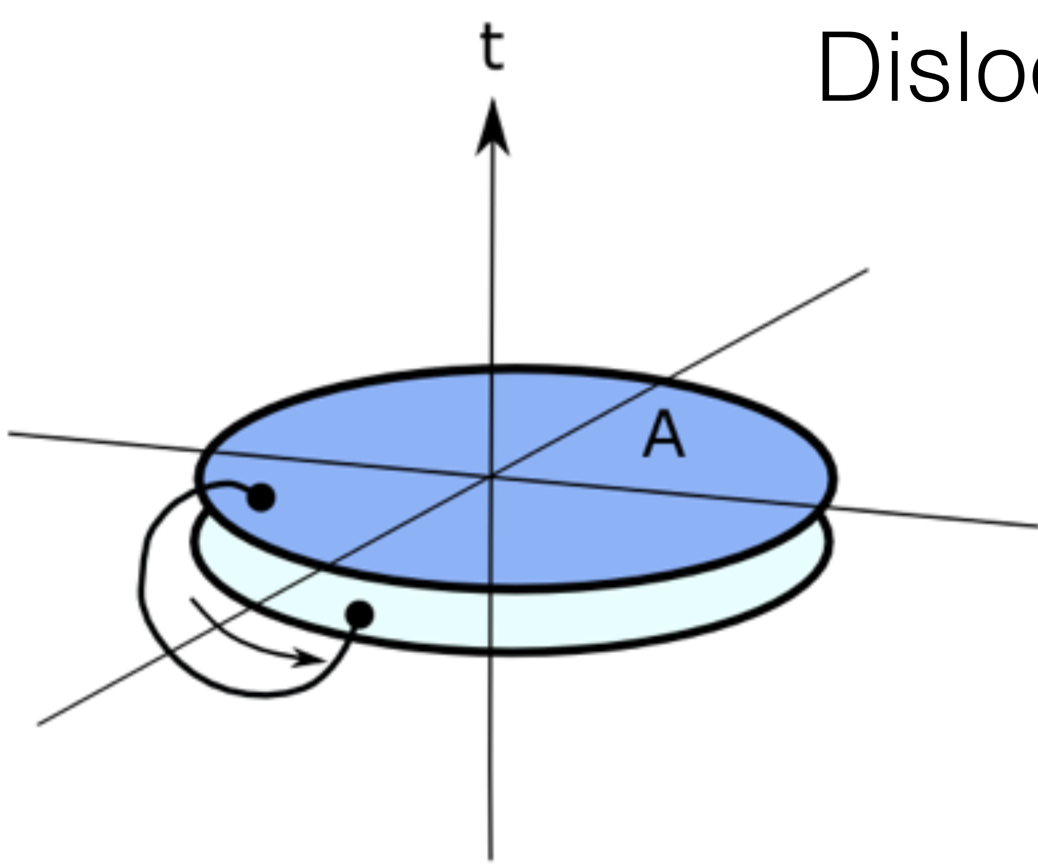
$\Delta = 2$

Rotating Entanglement entropy

$$\tilde{\rho}_A = \frac{e^{-(H_E - \Omega_E J)/T_0}}{Z(T_0, \Omega_E)}$$

Ω_E : Entanglement angular potential

Dislocation along the entangling surface



Finite temperature system on hyperbolic space with

$$\phi(0, \theta) = \phi(2\pi, \theta + 2\pi\Omega_E)$$

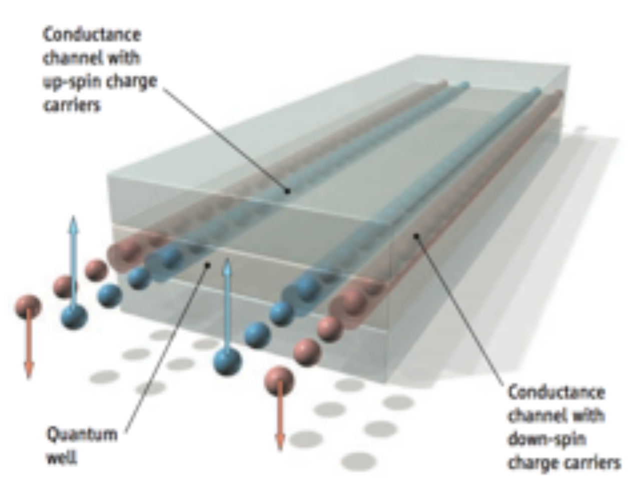
Rotating hyperbolic black holes

Application to Topological phases

Fractional Quantum Hall:

- Topological Order
- Ground State degeneracy
- Fractional charge
- Anyon statistics
- Long range entanglement

Quantum Spin Hall :



Schematic of the spin-polarized edge channels in a quantum spin Hall insulator.

[C. Bickel/Science]

- No Topological Order
- Unique ground state
- robust gapless edge states
- Short range entanglement

Topological Insulators/Superconductors

Symmetry Protected Topological phases

Application to SPT phases

w/Janet Hung

Chern Simons theory

(Interacting bosons and fermions)

$$S = \frac{1}{4\pi} \int K_{IJ} a^I \wedge da^J$$

SPT phase: $|\det K| = 1$ No topological order

Simplest bosonic case: $K = \sigma_x$

Unstable without symmetry: Assume global symmetry Z_N

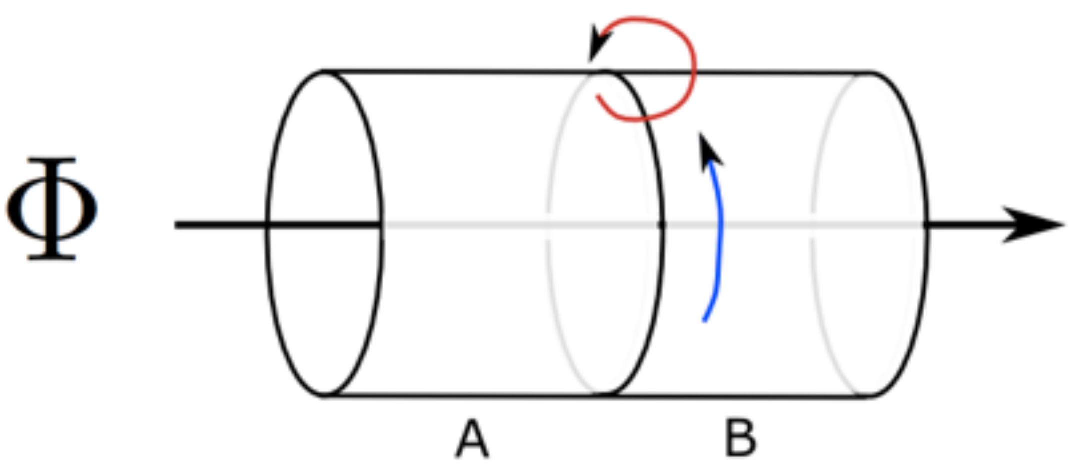
edge modes $\phi^I \to \phi^I + 2\pi \frac{t^I}{N}$ $t^T = (1, q), \quad q \in [1, N]$

Entanglement chemical potential = External gauge field

$$S = \frac{1}{4\pi}\int K_{IJ} a^I \wedge da^J + \frac{1}{2\pi}\int K_{IJ} t^I A da^J$$

$$A_0 = \mu \qquad \mu K_{IJ} t^I \int_A da^J = \mu K_{IJ} t^I \int_{\partial A} dx^i a_i^J$$

Entanglement chemical potential generates
Wilson loop **along** entangling surface



Flux threading (Laughlin)

$$A_y = \Phi$$

Flux threading generates
Wilson loop **around** entangling surface

CFT story

Partition function $$Z(\tau) = \mathrm{tr}\exp(-2\pi(\tau H - i\mu Q))$$

Hamiltonian and charge operator

$$\begin{aligned} H &= \frac{1}{2\pi}\int_0^{2\pi} dx \frac{1}{2}\left((\partial_x\phi_L)^2 + (\partial_x\phi_R)^2\right) \\ Q &= (\partial_x\phi_2 + q\partial_x\phi_1) \\ \phi_1(x+2\pi,t) &= \phi_1(x,t) + 2\pi(n + \frac{k_1}{N}) \\ \phi_2(x+2\pi,t) &= \phi_2(x,t) + 2\pi(m + \frac{l_1}{N}) \end{aligned}$$

$$k_1, l_1 \in [1, N], \quad l_1 = qk_1$$

Thermodynamical limit

$$\log Z(L/\tau \to \infty) = \frac{c}{24}\frac{L}{\tau} - \gamma(=0) + \text{Link number}\left(2\pi i\mu \frac{l_1 + qk_1}{N}\right)$$

Conclusion

Generalize quantum entanglement by twisting boundary conditions

CFT: central charge, current algebra,
SPT: generalized topological entanglement

Thermodynamical laws
Einstein/Maxwell equations
Phase diagrams
Quantum corrections
Discrete symmetries