3 Day Workshop "Holographic vistas on Gravity and Strings"
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Quantum Entanglement of Local Operators

Masahiro Nozaki

Yukawa Institute for Theoretical Physics (YITP), Kyoto University

1. Based on arXiv:1401.0539 [hep-th] (Phys. Rev. Lett. 112, 111602 (2014)) with Tokiro Numasawa, Tadashi Takayanagi

2. Based on arXiv:1405.5875 [hep-th]

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- It is useful to study the distinctive features of various quantum state in condensed matter physics. (***Quantum Order Parameter***)
- (Renyi) entanglement entropy is expected to be an important quantity which may shed light on the mechanism behind the AdS/CFT correspond .(***Gravity*** $\leftrightarrow$ ***Entanglement***)

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

It is important to study the properties of (Renyi) entanglement entropy.

- (Renyi) entanglement entropy is expected to be an important quantity which may shed light on the mechanism behind the AdS/CFT correspond .(***Gravity*** $\leftrightarrow$ ***Entanglement***)

The Definition of (Renyi) Entanglement Entropy

- Definition of Entanglement Entropy

We divide the total Hilbert space into A and B: $H_{tot} = H_A \otimes H_B$.

The reduced density matrix ρ_A is defined by $\rho_A \equiv Tr_B \rho_{tot}$

This means the D O F in B are traced out.

The entanglement entropy is defined by von Neumann entropy S_A.

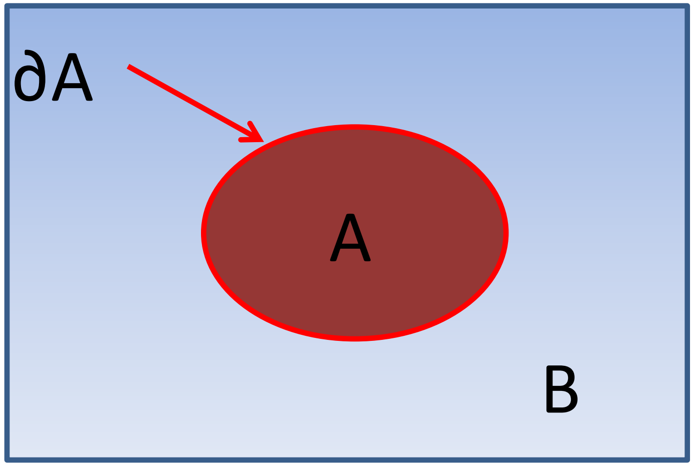
(Renyi) Entanglement Entropy (REE)

$$\boxed{S_A^{(n)} = \frac{\log \mathrm{tr}[\rho_A^n]}{1-n}}$$

$\Downarrow$ ***n→1***

Entanglement Entropy (EE)

$$\boxed{S_A = -\mathrm{tr}_A \rho_A \log \rho_A}$$



on a certain time slice

A new class of excited state

Quantum Quench: Prepare the ground state $|\Psi\rangle$ for H_0

$$H_0 \longrightarrow H_1$$

$|\Psi\rangle$ is ***not*** the ground state for H_1

[Refer to Lopez' talk and Das' talk]

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$$H_0 \to H_1$$

$|\Psi\rangle$ is ***not*** the ground state for H_1

[Refer to Lopez' talk and Das' talk]

Locally Excited state: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle$

A new class of excited state

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[Refer to Lopez' talk and Das' talk]

Locally Excited state: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle$

Excitation is milder than ***Quantum Quench.***

$\Delta S_A^{(n)}$ is finite even for the size of subsystem is infinite.

Motivation

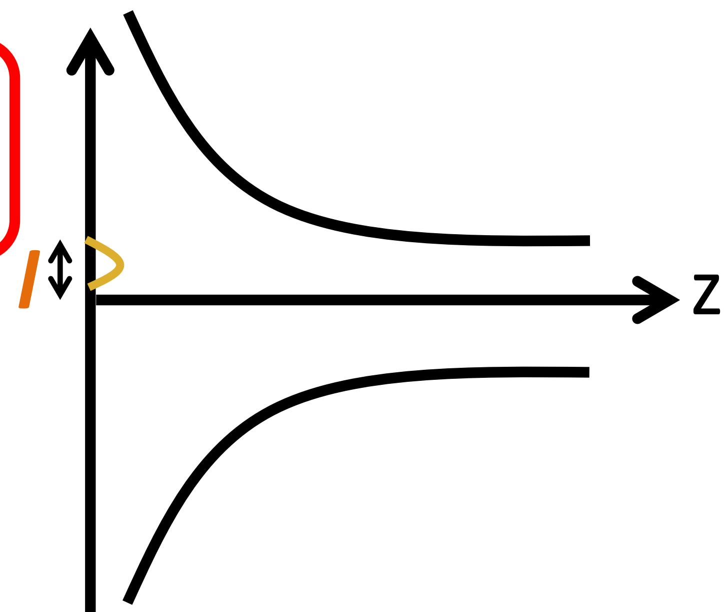
Previously, we studied the property of EE for the subsystem whose size (l) is ***very small*** in d+1 CFT.

l << (The Excitation Energy)$^{-d}$,

$$\boxed{E_A = \underline{T_{ent}} \cdot \Delta S_A}$$

This temperature is universal.

[Bhattacharya-MN-Takayanagi-Ugajin,
Blanco-Casini-Hung-Myers]



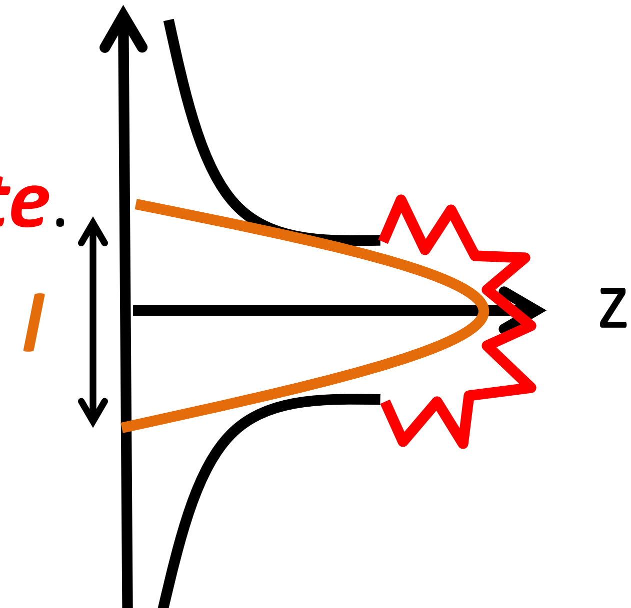
Motivation

We study the property of **(R)EE** for

1. The size of subsystem is ***infinite***.

   A half of the total system:

$$x^1 \geq 0$$



2. A state is defined by acting a local operator on the ground state:

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle\,.$$

Motivation

We study the property of **(R)EE** for

1. The size of subsystem is ***infinite***.

$$\Delta S_A^{(n)}$$

At late time

Some Constants

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle\,.$$

Motivation

We study the property of **(R)EE** for

1. The size of subsystem is ***infinite***.

$$\Delta S_A^{(n)}$$ $\Rightarrow$ **Some Constants**

At late time

Unique Behavior

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle\,.$$

Results

We compute $\Delta S_A^{(n)}$ for a new class of excited states.

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle \quad \text{for} \quad \mathcal{O} =: \phi^k :$$

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(Renyi) Entanglement Entropies of Local Operators

$$\Delta S_{A,k}^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_m^k \left({}_kC_m\right)^n\right)$$

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(Renyi) Entanglement Entropies of Local Operators

$$\boxed{\Delta S_{A,k}^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_m^k \left({}_kC_m\right)^n\right)}$$

They measure the D.O.F of operators and ***characterize*** the operators from the viewpoint of quantum entanglement.

(not conformal dim.)

The definition of $\Delta S_A^{(n)}$

$\Delta S_A^{(n)}$ is defined by the excess of REE:

$$\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G},$$

where

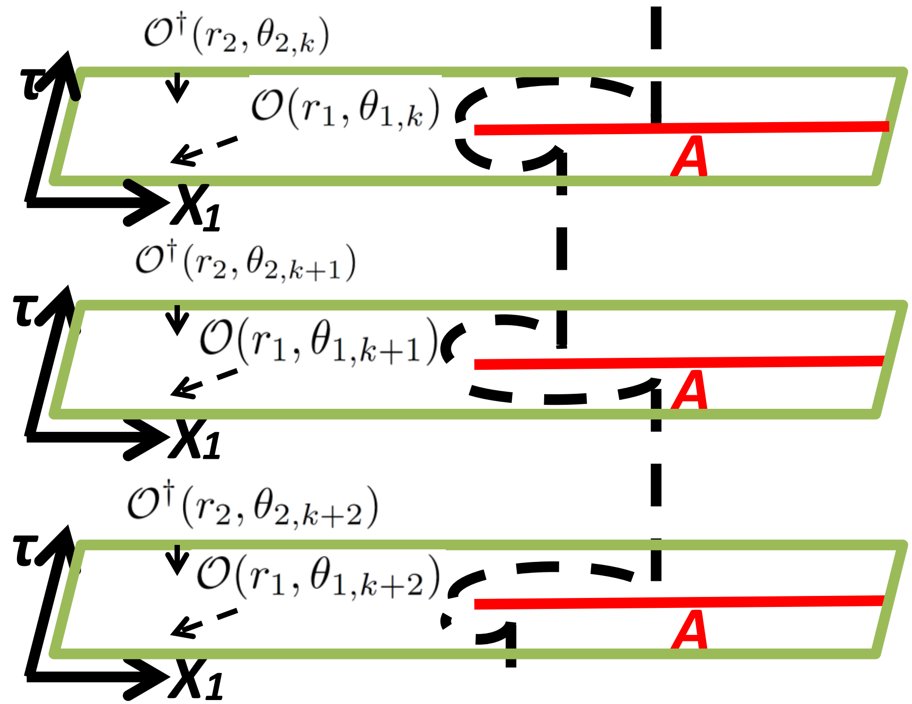
- REE for $\left|\Psi\right\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\left|0\right\rangle$:

$$S_A^{(n)Ex} = \frac{1}{1-n}\log\left[\frac{\int D\Phi\mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1})\cdots\mathcal{O}^\dagger(r_1,\theta_{1,n})\mathcal{O}(r_2,\theta_{2,n})e^{-S}}{\left(\int D\Phi\mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1})e^{-S}\right)^n}\right]$$

- REE for Ground State:

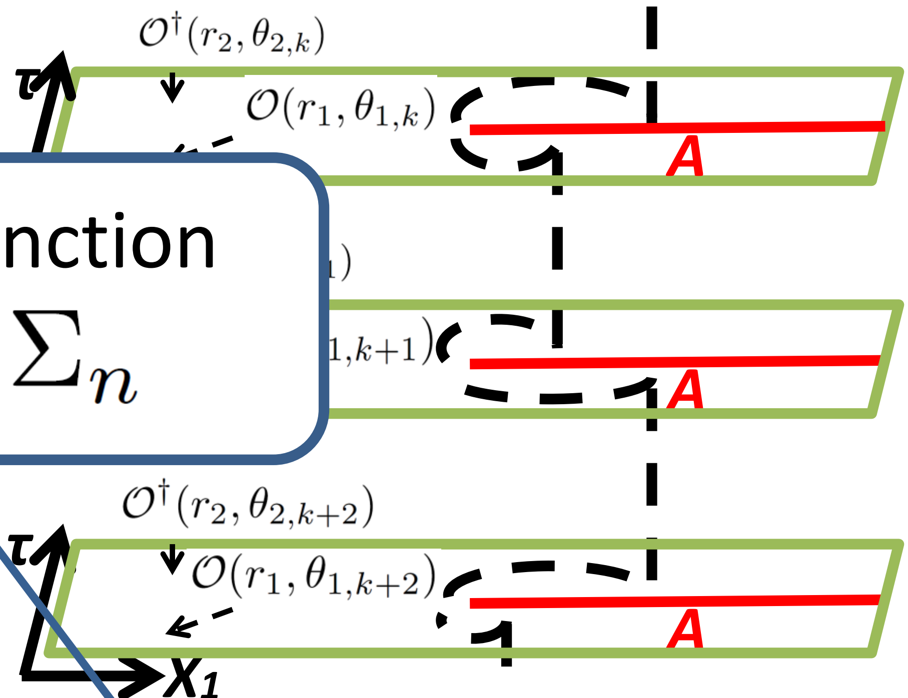
$$S_A^{(n)G} = \frac{1}{1-n}\log\left[\frac{Z_n}{Z_1^n}\right]$$

The definition of $\Delta S_A^{(n)}$



$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

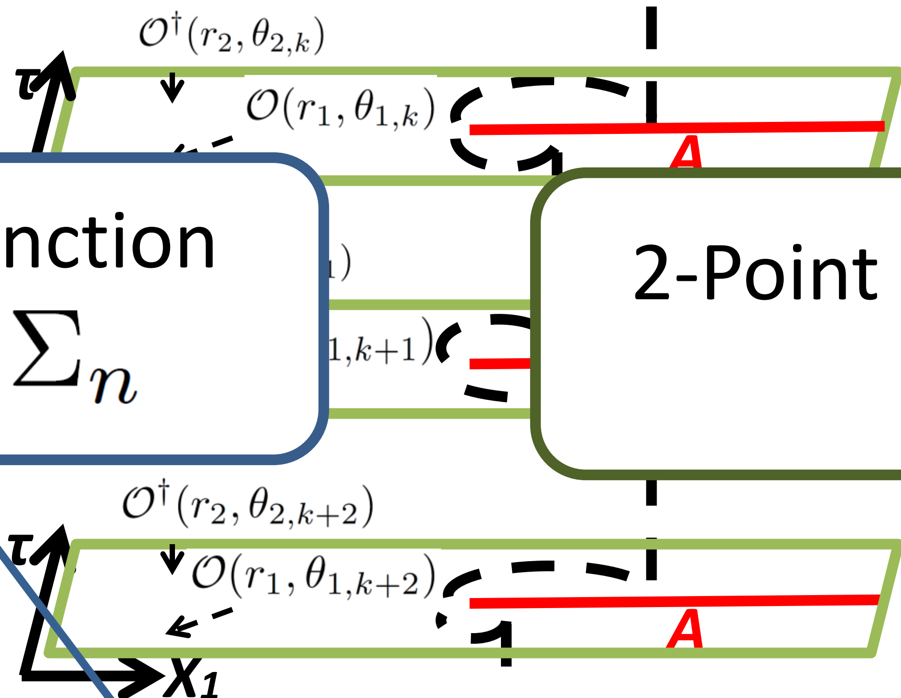
The definition of $\Delta S_A^{(n)}$



2n-Point Function on Σ_n

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

The definition of $\Delta S_A^{(n)}$



2n-Point Function on Σ_n

2-Point Function on Σ_1

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

Example

Let's compute $\Delta S_A^{(2)}$ for $|\Psi\rangle = \mathcal{N}^{-1}\phi(-t,-l,\mathbf{x})\,|0\rangle$
in 4-dimensional free massless scalar field theory.

$$\Delta S_A^{(2)} = -\log\left[\frac{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle_{\Sigma_1}^2}\right]$$

Green function:

$$\langle\phi\left(r,\theta,\mathbf{x}\right)\phi\left(s,\theta',\mathbf{x}\right)\rangle = \frac{1}{8\pi^2\left(r+s\right)\left(r+s-2\sqrt{rs}\cos\left(\frac{\theta-\theta'}{2}\right)\right)}$$

Example

$$\Delta S_A^{(2)} = -\log\left[\frac{\langle \phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle \phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle_{\Sigma_1}^2}\right]$$

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We compute $\Delta S_A^{(2)}$ by using Green function.

After that, we perform analytic continuation to real time.

Example

$$\Delta S_A^{(2)} = -\log\left[\frac{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle_{\Sigma_1}^2}\right]$$

Green f

$\langle\phi\,(r,\theta,\mathbf{x}$

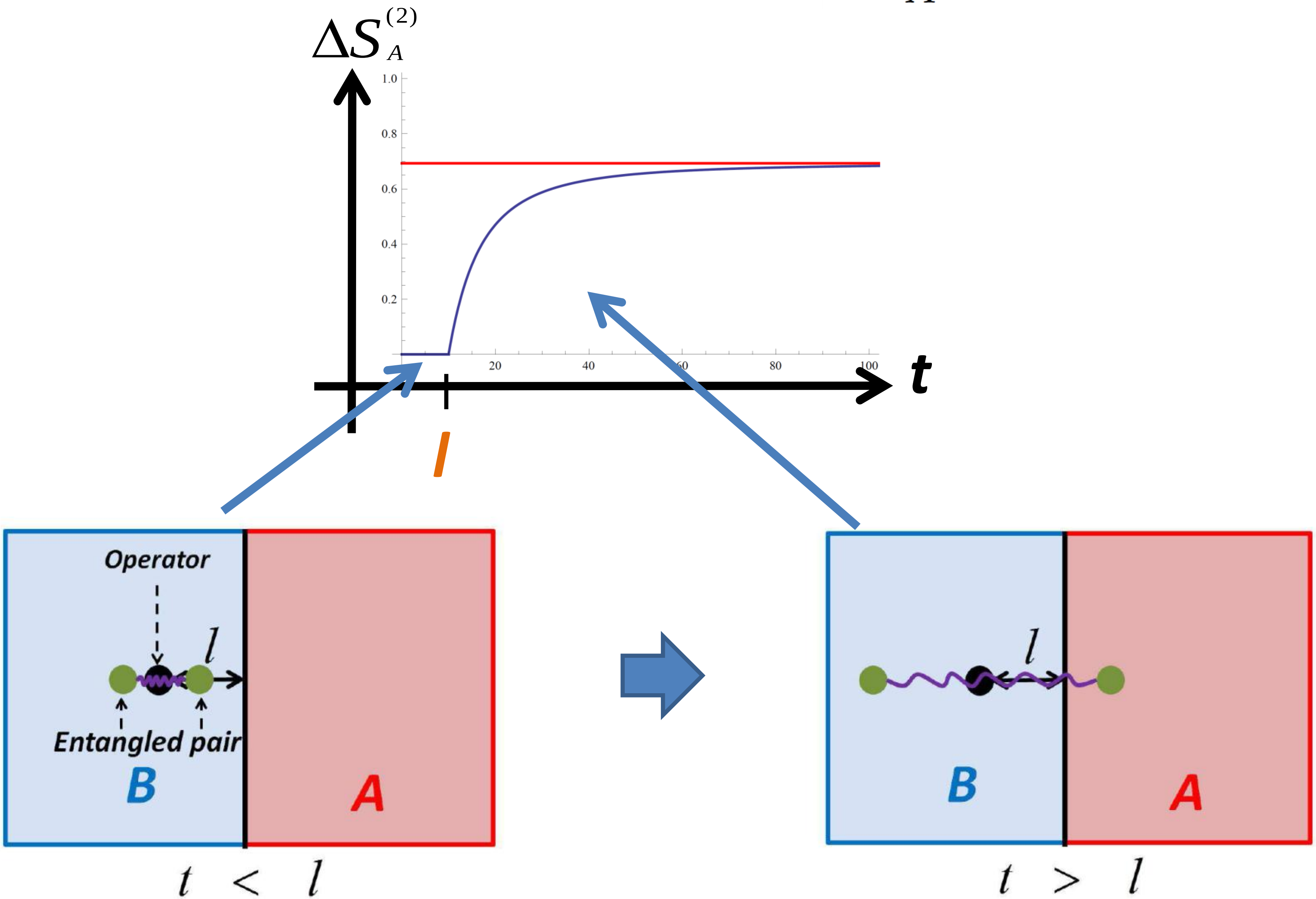
$\cos\left(\frac{\theta-\theta'}{2}\right)\Big)$

$$\Delta S_A^{(2)} = \log\left[\frac{2t^2}{t^2+l^2}\right]$$

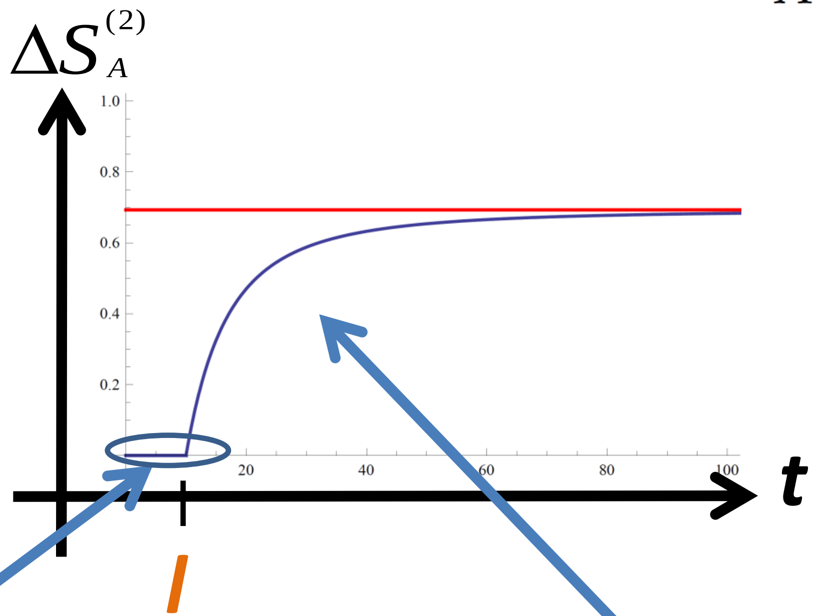
We compute $\Delta S_A^{(2)}$ by using Green function.

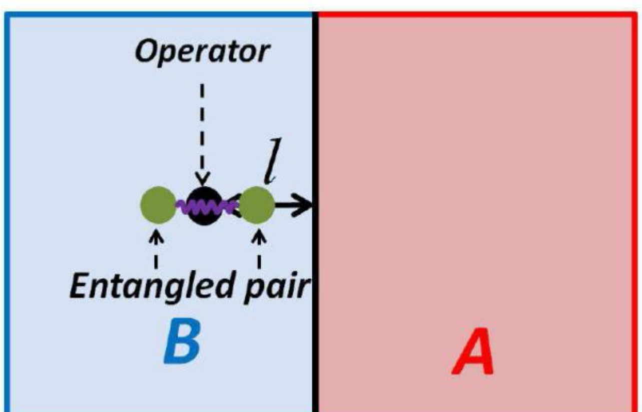
After that, we perform analytic continuation to real time.

Time Evolution of $\Delta S_A^{(2)}$



Time Evolution of $\Delta S_A^{(2)}$





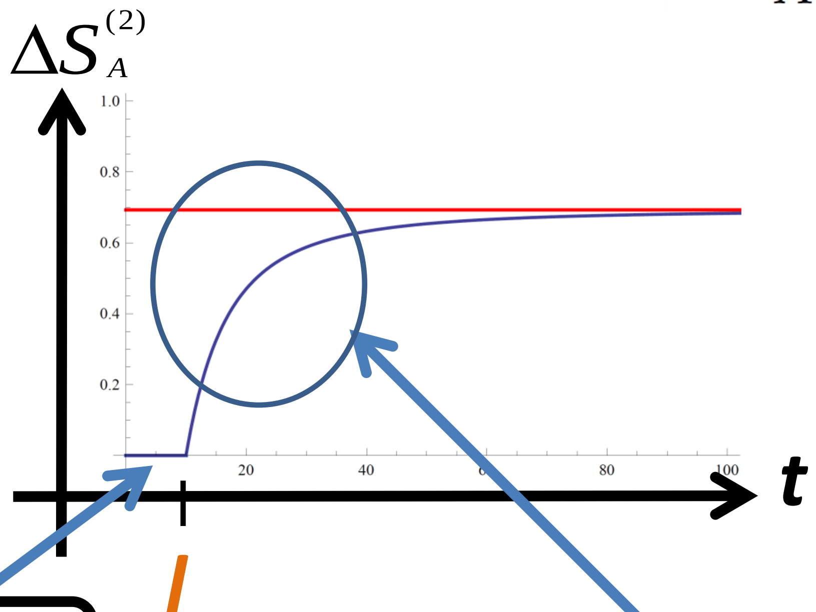
$t < l$

$t < l$

An entangled pair appears.

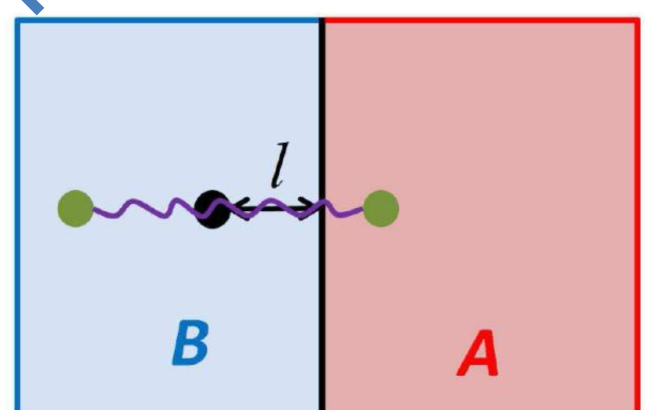
Each of pair is included in the region B.

Time Evolution of $\Delta S_A^{(2)}$



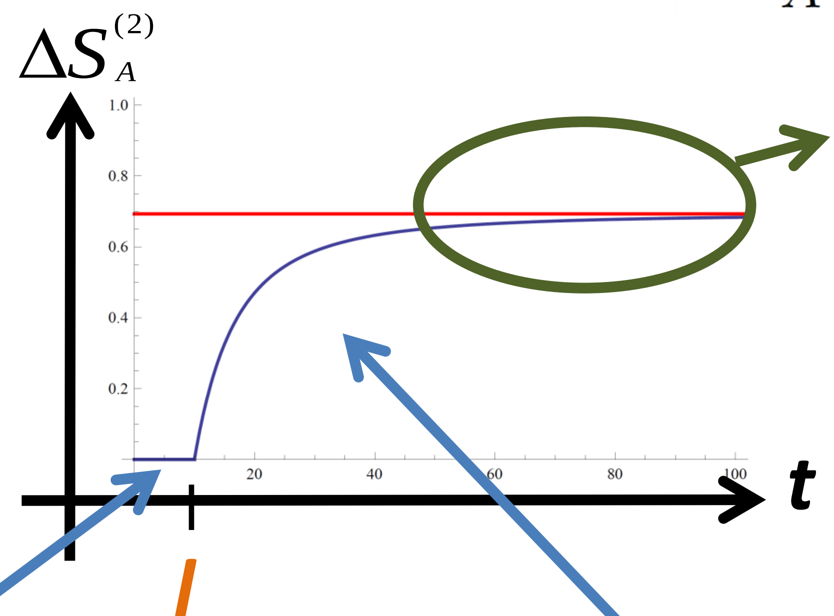
$t \geq l$

In this region, two quanta is included in A and B respectively .



Entanglement between quanta can contribute to $\Delta S_A^{(2)}$.

Time Evolution of $\Delta S_A^{(2)}$



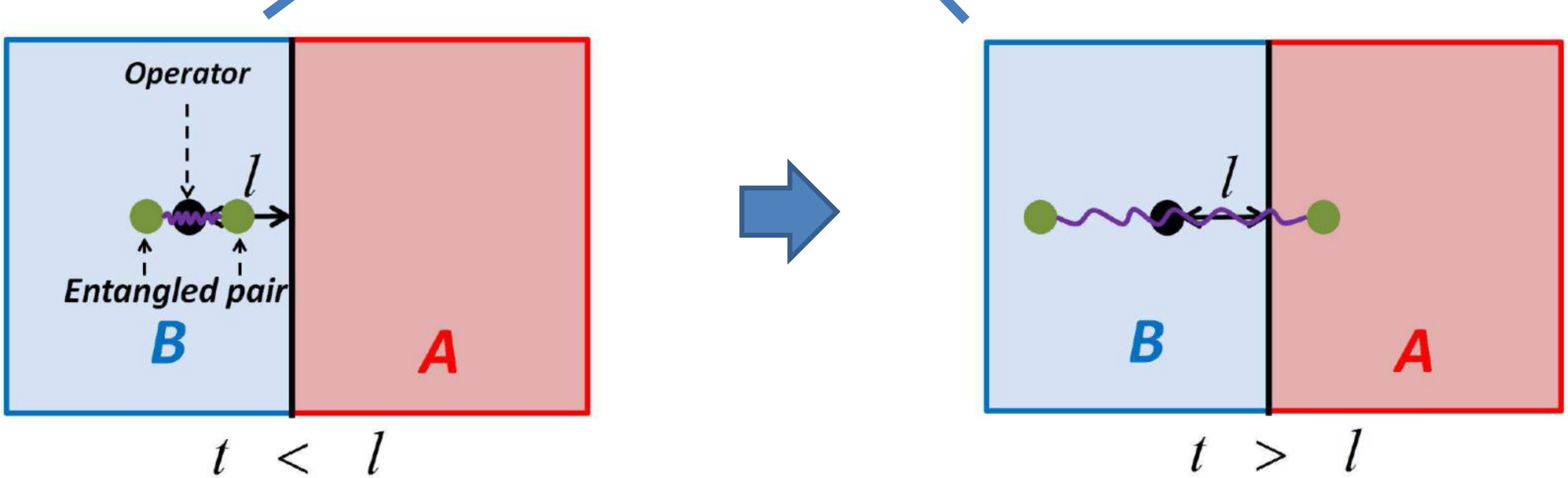
Subsystem

= a half of the

total space

$\Delta S_A^{(2)}$ approaches

to ***Constant!!***



Entangled Pair Interpretation

We derive $\Delta S^{(n)}_{A,k}$ for $|\Psi\rangle = \mathcal{N}^{-1} : \phi^k(-t, -l, \mathbf{x}) : |0\rangle$ from the entangled pair interpretation.

We decompose ϕ into the left moving mode and the right moving mode,

$$\boxed{\phi = \phi_L + \phi_R}$$

Entangled Pair Interpretation

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We decompose ϕ into the left moving mode and the right moving mode,

$$\boxed{\phi = \phi_L + \phi_R}$$

At late time, the d.o.f in the region B can be Identified with the d.o.f of left moving mode.

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$$\boxed{|\Psi\rangle = \frac{1}{2^{\frac{k}{2}}} \sum_{m=0}^{k} \sqrt{{}_kC_m}\, |m\rangle_A \otimes |k-m\rangle_B \,.}$$

Tracing out the d.o.f in B $\Rightarrow$ $\rho_A^f = 2^{-k}\left({}_kC_0\ ,\ {}_kC_1\ ,\ \cdots\ ,\ {}_kC_k\right)$

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$|\Psi$

$$\Delta S_A^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}\left({}_kC_j\right)^n\right).$$

$$\Delta S_A = k\cdot\log 2 - \frac{1}{2^k}\sum_{j=0}^{k}{}_kC_j\log{}_kC_j\,.$$

Tr

the d.o.f in B

$\rho_A = \ldots {}_kC_k)$

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

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$$\Delta S_A^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}\left({}_kC_j\right)^n\right).$$

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Tr

They agree with the results which we obtain by the Replica trick (See My paper!!).

Comments on Result

$$\Delta S_A^{(n)f} = \frac{1}{1-n} \log \left(\frac{1}{2^{nk}} \sum_{j=0}^{k} ({}_kC_j)^n \right).$$

$$\Delta S_A = k \cdot \log 2 - \frac{1}{2^k} \sum_{j=0}^{k} {}_kC_j \log {}_kC_j \, .$$

We defined ***the (Renyi) entanglement entropies of operators*** by them.

Large k limit: $$\Delta S_{A,k}^{(n)} \sim \frac{1}{2} \log k$$

Comments on Result

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$$\Delta S_A = k \cdot \mathrm{l} \quad \ldots \quad {}_{j=0}$$

They characterize the operators from the viewpoint of entanglement.

We defined ***the (Renyi) entanglement entropies of operators*** by them.

Large k limit: $\Delta S_{A,k}^{(n)} \sim \frac{1}{2} \log k$

Comments on Result

$$\Delta S_A^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k} ({}_kC_j)^n\right).$$

$$\Delta S_A = k \cdot \mathrm{l}$$

$$j=0$$

The REE of operators such as $:(\partial^m \phi)^k:$ obey this formula.

We defined ***the (Renyi) entanglement entropies of operators*** by them.

Large k limit: $$\Delta S_{A,k}^{(n)} \sim \frac{1}{2}\log k$$

Sum Rule

An excited state: $|\psi\rangle = \mathcal{N}^{-1}\mathcal{T}\prod_i^k \mathcal{O}^i(t^i, x^{1,i})\,|0\rangle$

$$\boxed{\Delta S_A^{(n)} = \sum_{m=1}^{k} \Delta S_A^{(n)m}}$$

They are given by the sum of the REE for the state defined by acting each operators on the ground state.

Summary

- We defined the (Renyi) entanglement entropies of local operators.

  -They characterize local operators from the viewpoint of quantum entanglement.

- These entropies of the operators (constructed of single-species operator) is given by the those of binomial distribution. (not depend on the spacetime demension)

  -The results we obtain in terms of entangled pair agree with the results we obtain by replica method.

- They obey the sum rule.

Future Problems

- The formula for the operators constructed of multi-species operators: $:\phi\partial_r\phi:$ (generally depend on the spacetime dimension).

- The (Renyi) entanglement entropies of operators in the interacting field theory . (also massive and finite temp.)

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- The formula for the operators constructed of multi-species operators: $:\phi\partial_r\phi:$ (generally depend on the spacetime dimension).

- The (Renyi) entanglement entropies of operators in the interacting field theory . (also massive and finite temp.)

- ***The (Renyi) entanglement entropies of operators in Large N, strongly coupled theory***

(Pawel's talk)