Which surface maximizes entanglement entropy? *

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*work in progress (with Amin Faraji and Gary Gibbons)

Our main question in the talk:

Suppose that

• $\mathcal{M}_4$ is Minkowski spacetime

• Σ is a compact co-dimension 2 entangling surface in $\mathcal{M}_4$

• its area is fixed, $Area(\Sigma) = A$

• its topology is fixed

• UV cut-off ϵ in the theory is fixed

Then

For which surface Σ_0 entanglement entropy would take maximal value?

A natural guess

If such a surface Σ_0 exists it should be a special surface, and thus invariant under rotations and hence it should be a sphere

$$\Sigma_0 = S^2$$

A hint: geometric analogy with sandpile

Holographic entanglement entropy

We shall use a holographic interpretation of entanglement entropy in terms of area of minimal surface $\mathcal{H}$ in Anti-de Sitter space-time

$$S_{HE}(\Sigma) = \frac{Area(\mathcal{H})}{4\pi G_N}$$

Ryu-Takayanagi (06)

Statement (mathematical version):

Consider Minkowski spacetime $\mathcal{M}^4$ *as a boundary of* AdS_5. *Let* Σ *be a compact closed 2d surface in* $\mathcal{M}^4$ *and let* $\mathcal{H}_\Sigma$ *be a minimal surface in* AdS_5 *which bounds* Σ. *Suppose that area of* Σ *is fixed,* $Area(\Sigma) = A$. *So that area of minimal surface* $Area(\mathcal{H}_\Sigma)$ *is a functional of surface* Σ. *Then in the class of surfaces* Σ *with fixed area and fixed spherical topology this functional takes maximal value if* Σ *is 2-sphere* S^2.

Statement
(physics version):

Entanglement entropy calculated in Minkowski spacetime M^4 is maximal if the entangling surface Σ is 2-sphere S^2

Thus, spheres appear to be the most entropic surfaces!

We start with mathematical version

Area of minimal $(d-1)$-surface in AdS_{d+1}

$$Area(\mathcal{H}_\Sigma) = \frac{A(\Sigma)}{(d-2)\epsilon^{d-2}} - \frac{1}{32}\int_\Sigma (\mathrm{Tr}\, K)^2 C_d(\epsilon)$$

$$C_d(\epsilon) = \frac{1}{(d-4)\epsilon^{d-4}}, \ d > 4$$

$$C_4(\epsilon) = \ln\frac{1}{\epsilon}, \ d = 4$$

K is extrinsic curvature of Σ

Graham-Witten (1999)

Holographic EE

$$S_{HE}(\Sigma) = N^2 \left(\frac{A(\Sigma)}{4\pi\epsilon^2} - \frac{1}{16} \int_\Sigma (\mathrm{Tr}\, K)^2 \;\; \ln \frac{1}{\epsilon} \right)$$

$$\frac{1}{G} = \frac{2N^2}{\pi}$$

We do some re-writing (in $d=4$)

$$Area(\mathcal{H}_\Sigma) = \frac{A(\Sigma)}{2\epsilon^2} - \frac{1}{16}\int_\Sigma (R_\Sigma + K_\Sigma) \ln\frac{1}{\epsilon}$$

$$R_\Sigma = (\mathrm{Tr}\, K)^2 - \mathrm{Tr}\, K^2$$

is intrinsic curvature of Σ

$$K_\Sigma = \mathrm{Tr}\, K^2 - \frac{1}{2}(\mathrm{Tr}\, K)^2$$

is conformal invariant

Since topology of Σ is fixed

$$\int_\Sigma R_\Sigma = 8\pi$$

The other observation is that K_Σ is a complete square

$$K_\Sigma = (K_{ij} - \frac{1}{2}\gamma_{ij}\mathrm{Tr}\, K)^2$$

Thus, $Area(\mathcal{H}_\Sigma)$ is maximal if

$$K_\Sigma = 0$$

If $K_\Sigma = 0$ for some surface Σ_0 then

$$K_{ij} = \frac{1}{2}\gamma_{ij}\mathrm{Tr}\, K$$

and using the Gauss-Codazzi equations

$$\nabla^j K_{ij} = \nabla_i \mathrm{tr}\, K$$

we find that $\mathrm{Tr}\, K = const$

then intrinsic curvature

$$R_\Sigma = (\mathrm{tr}\, K)^2 - \mathrm{tr}\, K^2 = const > 0$$

so that the maximizer is round sphere, $\Sigma_0 = S^2$

Thus we proved that

$$S_{HE}(\Sigma) \leq S_{HE}(S^2)$$

Generic 4d CFT in Minkowski space-time

$$S_{CFT}(\Sigma) = \frac{Area(\Sigma)}{4\pi\epsilon^2} - \frac{1}{2\pi}\left(a\int_\Sigma R_\Sigma + b\int_\Sigma K_\Sigma\right)\ln\frac{1}{\epsilon}$$

where a and b are central charges related to conformal anomalies

$a \geq 0$ and $b > 0$ (for all fields except $s = 3/2$)

By same argument as before entropy is maximal for a surface Σ_0 for which $K_\Sigma = 0$ and thus $\Sigma_0 = S^2$ and we have a bound

$$S_{CFT}(\Sigma) \leq S_{CFT}(S^2)$$

A mass deformation of CFT

$$S(\Sigma) = \frac{Area(\Sigma)}{4\pi\epsilon^2} - \frac{1}{2\pi}\int_\Sigma \left(aR_\Sigma + bK_\Sigma + \frac{m^2}{12}D_s\right) \ln \frac{1}{\epsilon}$$

D_s is dimension of representation of spin s

This entropy is still bounded by entropy of round sphere $\Sigma_0 = S^2$

Curved space-time

For 4d CFT in curved space-time entanglement entropy

$$S(\Sigma) = \frac{Area(\Sigma)}{4\pi\epsilon^2}$$
$$-\ \frac{1}{2\pi}\int_\Sigma (aR_\Sigma + b(-W_{abab} + K_\Sigma))\ln\frac{1}{\epsilon}$$

Theorem still holds if

- space-time is conformally flat, $W = 0$
- Weyl tensor on surface Σ is negative

However, this is not the end of the story,
life is more interesting!

Willmore conjecture (1965)

• Willmore (bending) energy $W(\Sigma) = \frac{1}{4}\int_\Sigma (\mathrm{Tr}\, K)^2$

• Willmore energy satisfies $W(\Sigma) \geq 4\pi$ with equality if and only if Σ is sphere

• Willmore conjectured that for surfaces of higher genus there exists a better bound. For every torus in $\mathbf{R}^3$

$$W(\Sigma) \geq 2\pi^2$$

with equality if and only if Σ is Clifford torus with a certain ratio $\sqrt{2}$ of the radius of revolution to the radius of the circle being revolved.

Recent development

• Willmore conjecture proved by Fernando Codá Marques and André Neves (2012)

• They also proved that Clifford torus is minimazer of Willmore energy for any genus

$$W(\Sigma_g) \geq 2\pi^2$$

More geometry facts

• Conformal invariant Willmore energy

$$W(\Sigma) = \int_\Sigma (\frac{1}{4}(\mathrm{Tr}\, K)^2 - R_{abab} + \frac{1}{2} R_{aa})$$

• This problem can be considered for Σ embedded in S^3 of unite radius; surface in $\mathbf{R}^3$ is then obtained by stereographic projection.

The binding energy then is

$$W(\Sigma) = \int_\Sigma (\frac{1}{4}(\mathrm{Tr}\, K)^2 + 1)$$

For minimal embeddings $W(\Sigma) = Area(\Sigma)$

• Clifford torus in this case is square torus described by equations

$$x_1^2 + x_2^2 = \frac{1}{2} = x_3^2 + x_4^2$$

• *Lawson conjecture (1970)* (one of the Millennium Problems): Clifford torus is the only torus minimally embedded in S^3
proved by Simon Brendle (2012)

• For each genus g Lawson constructed a surface which is minimally embedded in S^3

• For higher genus Lawson surfaces are minimizers of Willmore energy (conjectured by Rob Kusner (1989))

We conclude that

After all spheres are not the most entropic surfaces!

Surprisingly, the Clifford tori are maximizers of entanglement entropy!

Questions

- more general curved spacetimes?
- higher dimensions $d > 4$ (work in progress)?
- and most importantly..

What is it good for?

THANK YOU!