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Holographic Holes in Higher Dimensions

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with

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“Holographic vistas on Gravity and Strings”

Introduction

Entropy of a black hole is expressed by the area of the event horizon.

$$S = \frac{A}{4G_{\mathrm{N}}}$$ Bekenstein-Hawking entropy

This expression is applied to any Killing horizon (dS, Rindler, …)

Recently, it was proposed that the BH entropy formula can be applied to more general region.

Spacetime Entanglement Conjecture

Bianchi & Myers (2012)

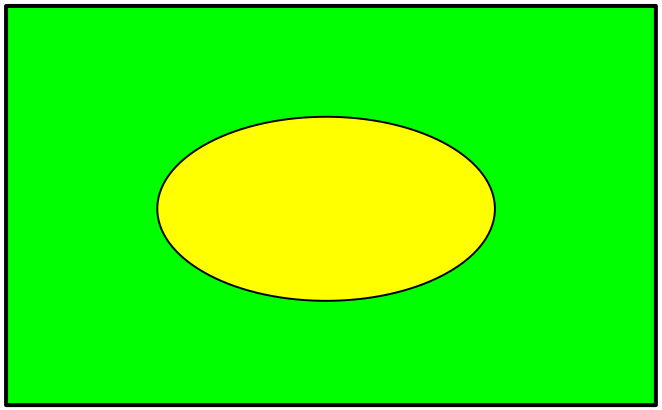
Spacetime Entanglement

Bianchi & Myers (2012)

More precisely, the spacetime entanglement conjecture proposed that
in a theory of quantum gravity, the short-range entanglement entropy between the degrees of freedom describing any large region and its complement is finite and described in terms of geometry of the entangling surface
and the leading contribution is given by the BH entropy formula.

$$S = \frac{A}{4G_{\rm N}} + \cdots$$

assuming that the Einstein-Hilbert action emerges as the low-energy effective gravitational action.



Holographic Entanglement Entropy

Ryu & Takayanagi (2006)

One support for the spacetime entanglement conjecture comes from the AdS/CFT correspondence.

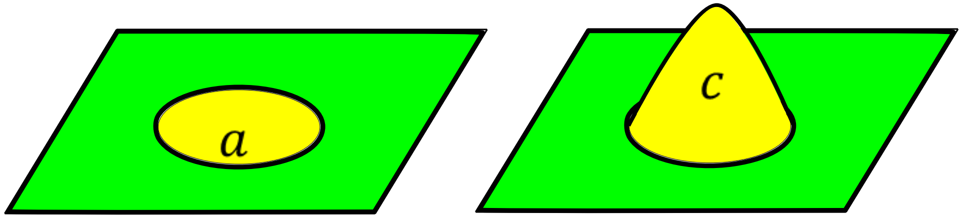
AdS/CFT correspondence

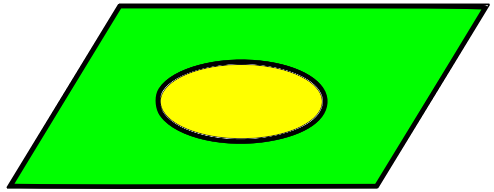
QG theory on AdS $\longleftrightarrow$ QFT on the bdry of AdS

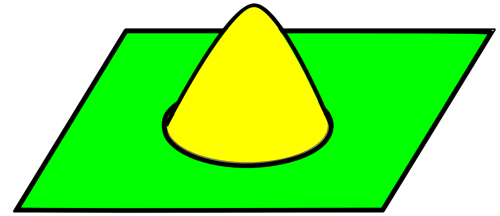
- holographic entanglement entropy

  The EE for a region in the bdry can be evaluated by the minimal area in the bulk.

$$S(a) = \operatorname*{ext}_{\partial c = \partial a} \frac{A(c)}{4G_{\mathrm{N}}}$$







entanglement entropy = the BH entropy for the extremal surface

The extremal surface is not horizon.

Can we evaluate the area of a more general surface in the bulk by a quantity of the boundary theory?

Recently, it was shown that the area of a general hole in AdS3 can be obtained by a combination of EE of CFT2 using HEE formula.

Balasubramanian, Chowdhury, Czech, de Boer, Heller (2013)

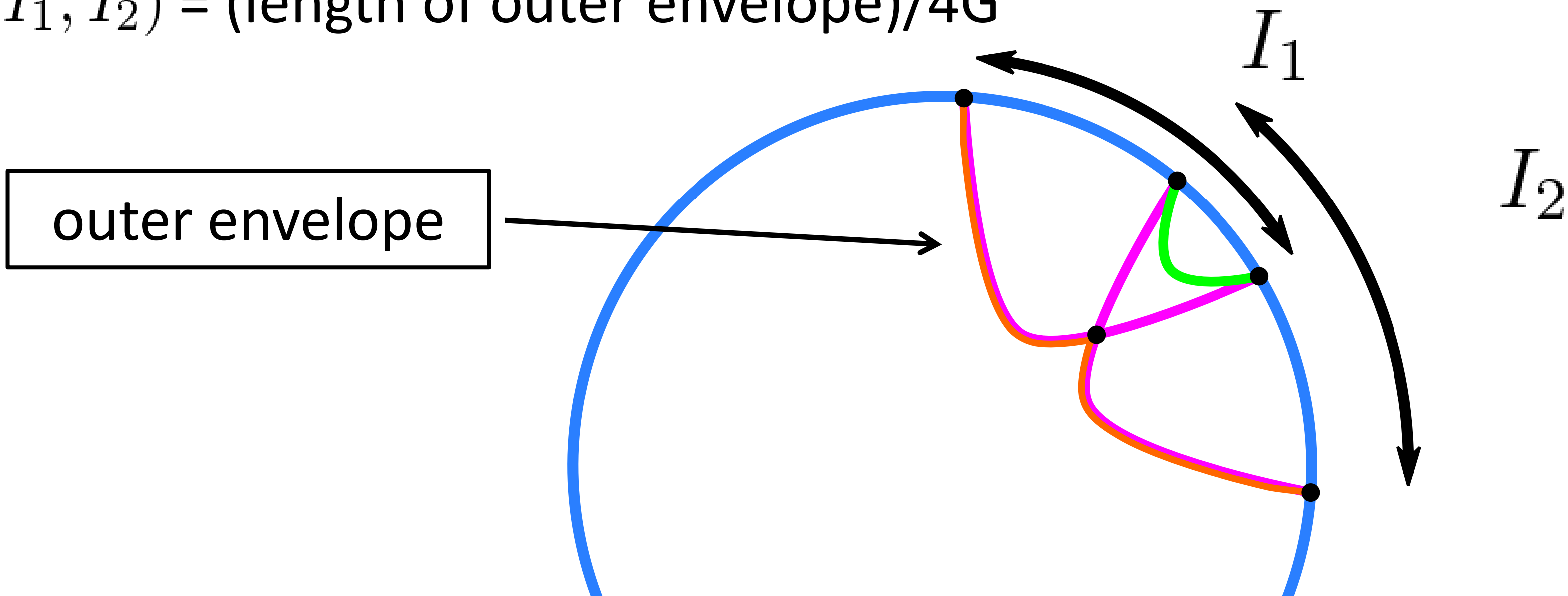
Geometric inequality

Consider two overlapping regions I_1& I_2 at the boundary of the global coordinates of AdS3. We have the following inequality.

$$\boxed{S(I_1) + S(I_2) - S(I_1 \cap I_2) \geq \hat{S}(I_1, I_2)}$$

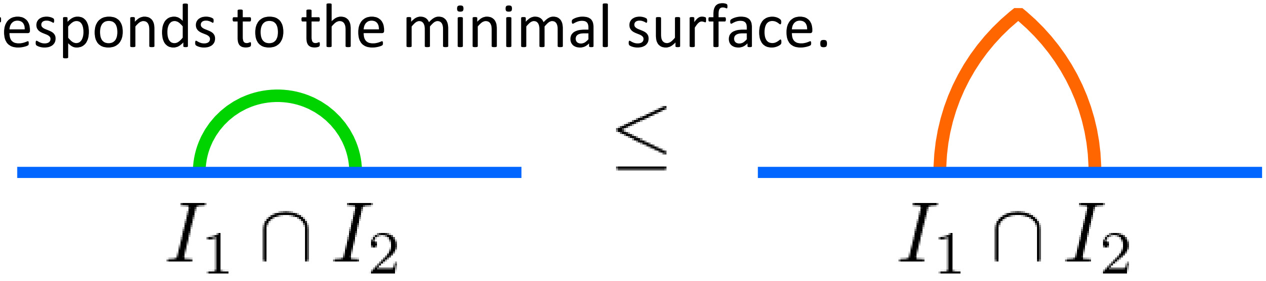
$S(I)$ = EE of I = (length of minimal curve)/4G

$\hat{S}(I_1, I_2)$ = (length of outer envelope)/4G

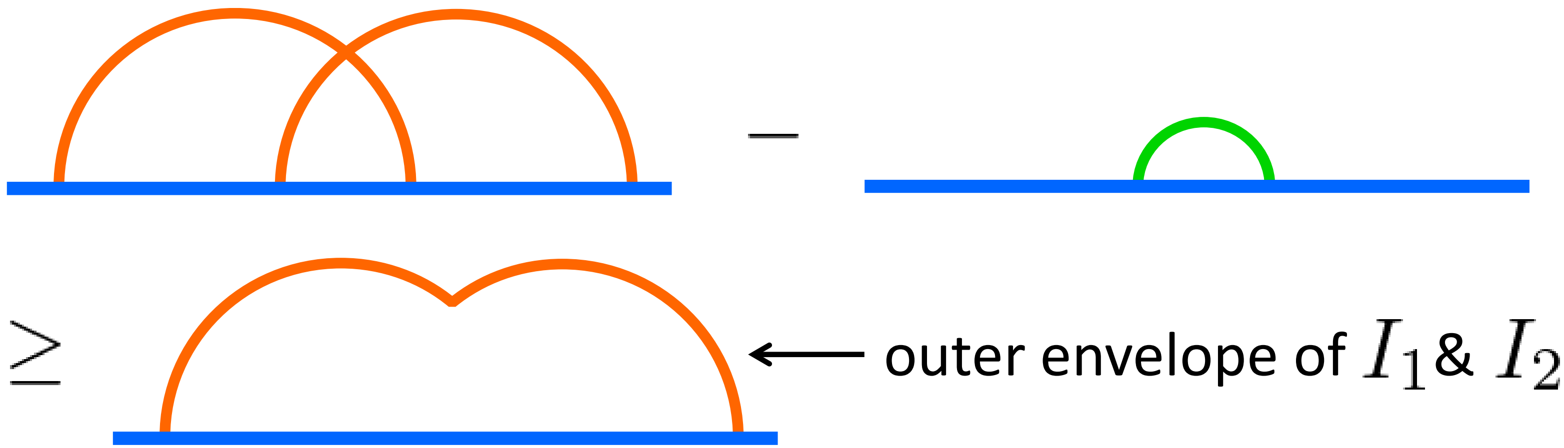


Proof of geometric inequality

The following inequality holds, since the left hand side corresponds to the minimal surface.



Therefore, we have the geometric inequality:



$$\Longleftrightarrow \quad S(I_1) + S(I_2) - S(I_1 \cap I_2) \geq \hat{S}(I_1, I_2)$$

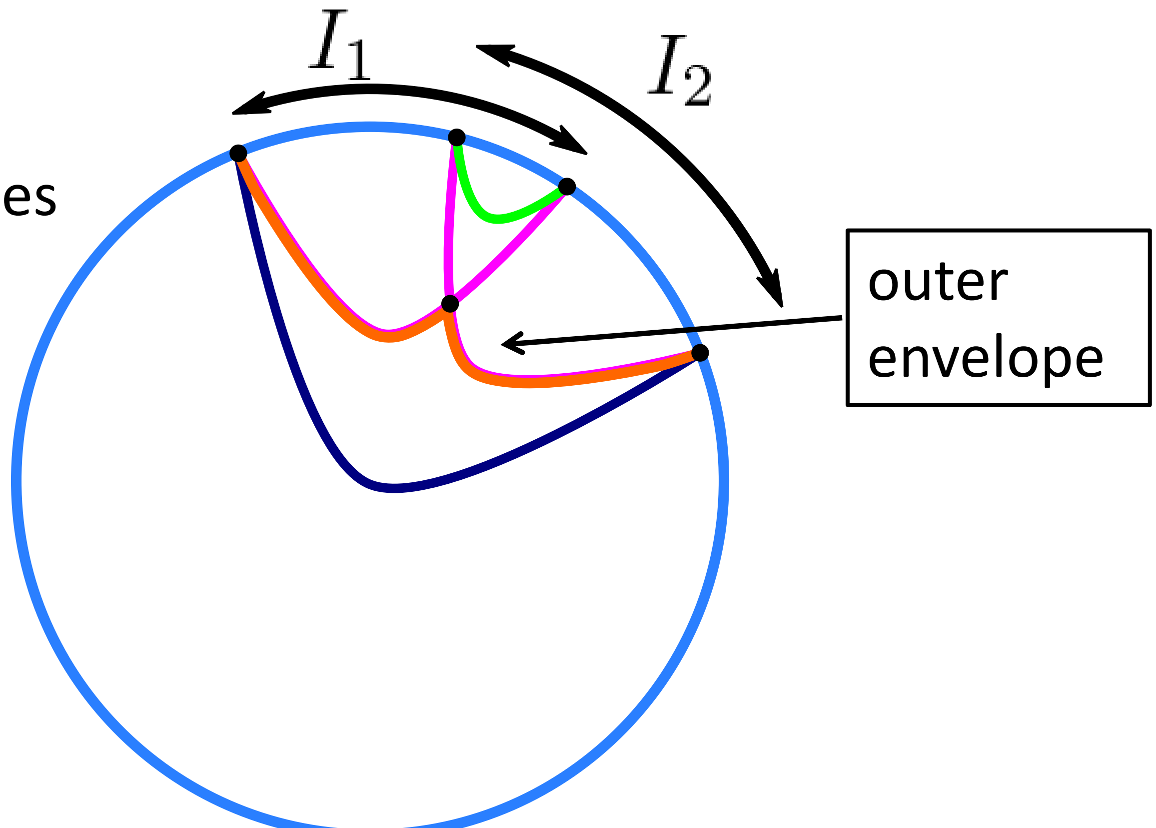
Geometric inequality

$$\boxed{\hat{S}(I_1, I_2) \le S(I_1) + S(I_2) - S(I_1 \cap I_2)}$$

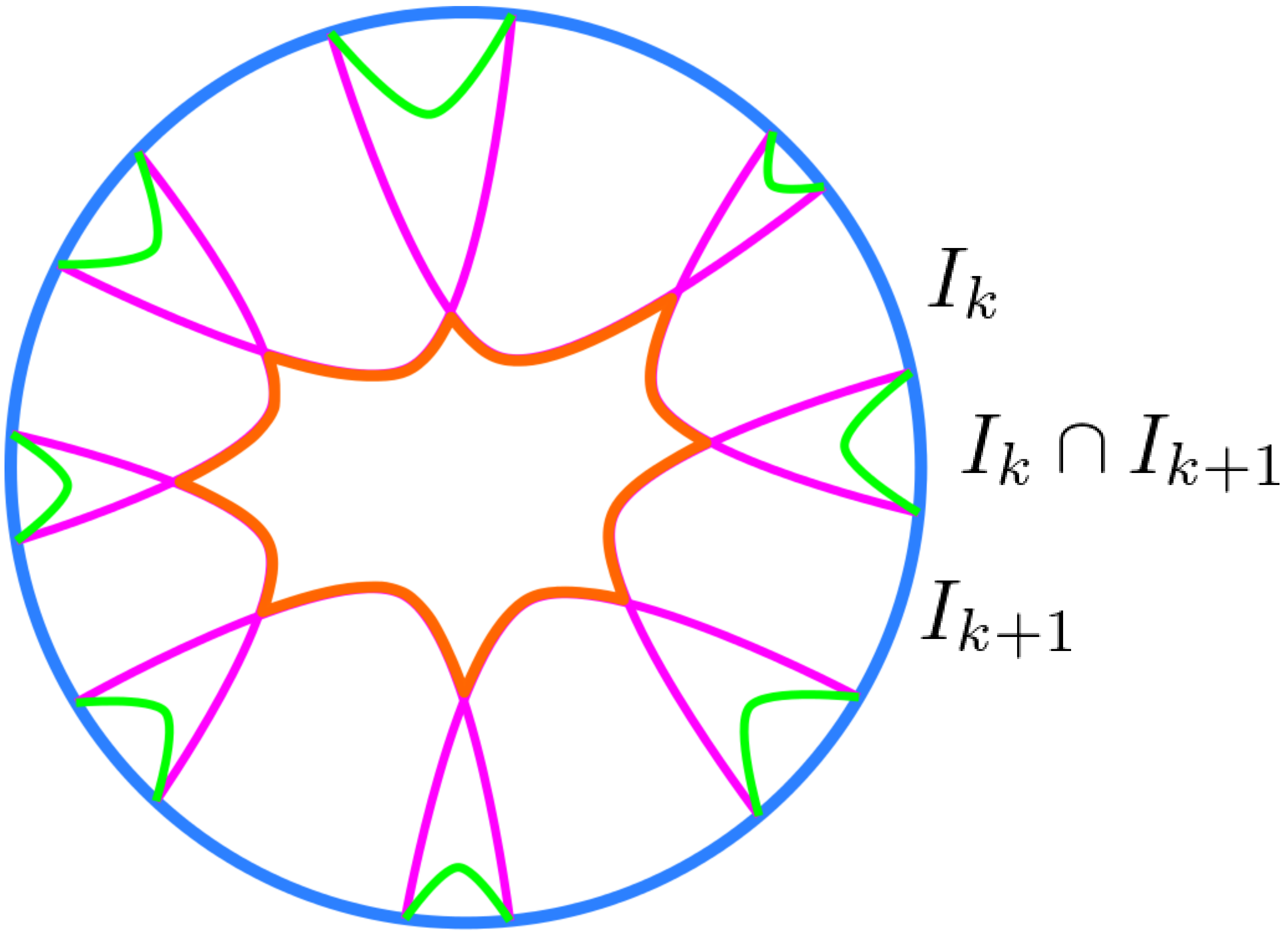
This inequality is stronger than strong subadditivity.

$$S(I_1 \cup I_2) \le \hat{S}(I_1, I_2)$$

global coordinates of AdS3



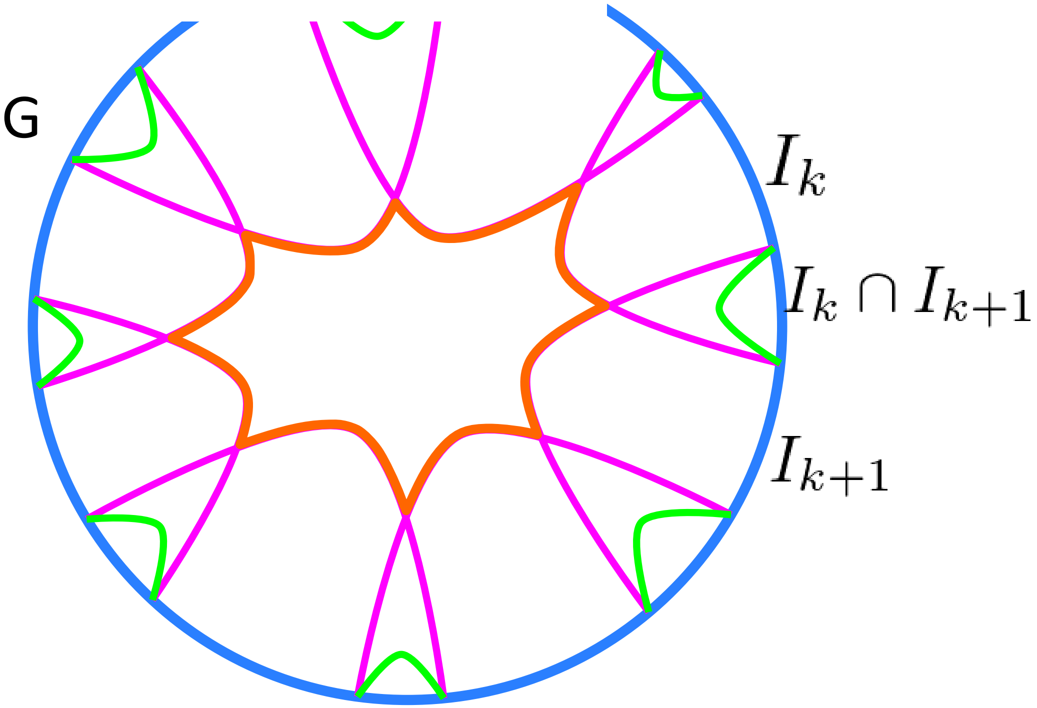
We consider the case where the intervals are chosen to cover the entire boundary.



In this case, the geometric inequality also holds

$$\hat{S}(\{I_k\}) \leq \sum_k [S(I_k) - S(I_k \cap I_{k+1})]$$

(length of outer envelope)/4G

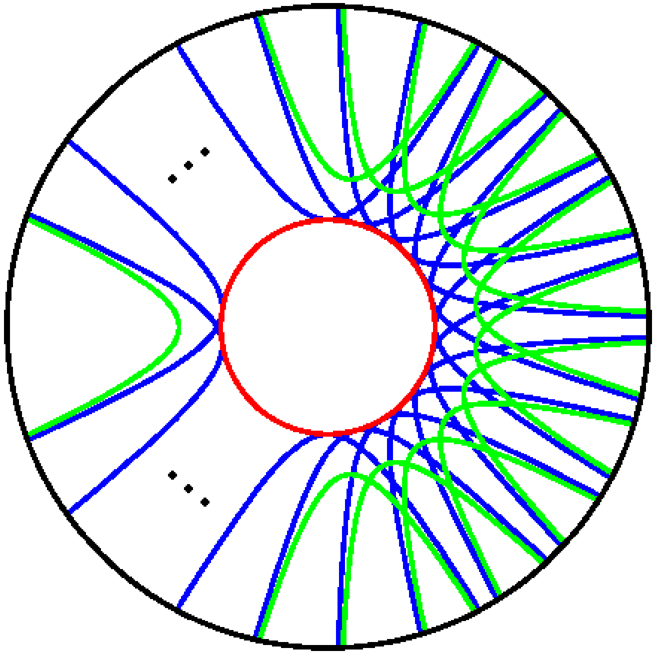


When the bulk is dual to any pure state in the bdry, $S(\cup I_k) = 0$

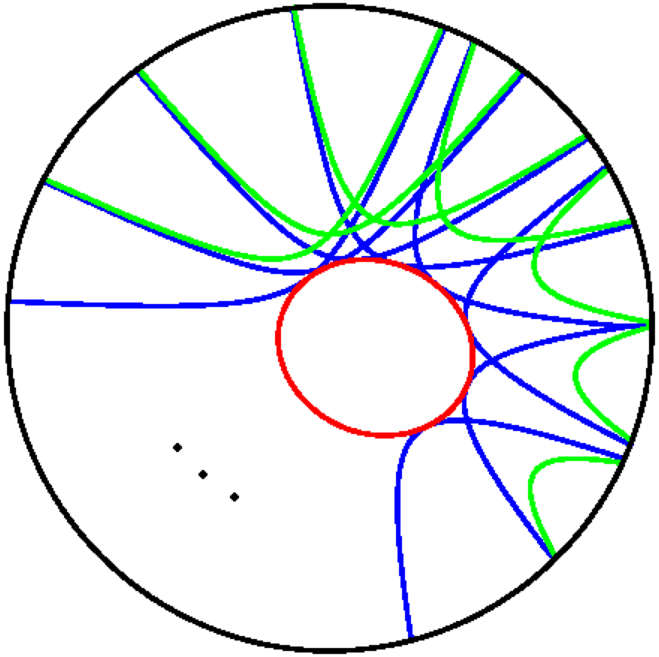
In contrast, the length of the outer envelope is finite, $\hat{S}(\{I_k\}) \neq 0$

We take a continuum limit, i.e. take the number of intervals to infinity.
In this limit, the outer envelope becomes a smooth curve.
For example, when the individual intervals have same length, then the outer envelope is a smooth circle.

circle



general curve



We take a continuum limit, i.e. take the number of intervals to infinity.
In this limit, the outer envelope becomes a smooth curve.
For example, when the individual intervals have same length, then the outer envelope is a smooth circle.

In this limit, the geometric inequality is saturated:

$$\frac{(\mathrm{length})}{4G_{\mathrm{N}}} = \sum_{k}^{\infty}[S(I_k) - S(I_k \cap I_{k+1})]$$

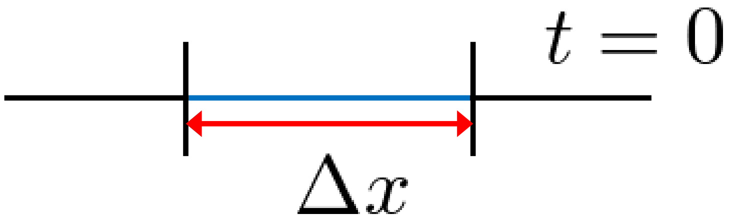
Balasubramanian, Chowdhury, Czech, de Boer, Heller (2013)
for global coordinates of AdS3

BH entropy can be evaluated in terms of a combination of EE.
We call this combination of EE "differential entropy".

Holographic entanglement entropy of AdS3

- evaluate the holographic EE of an interval of width Δx

⟺ EE in 2-dim CFT



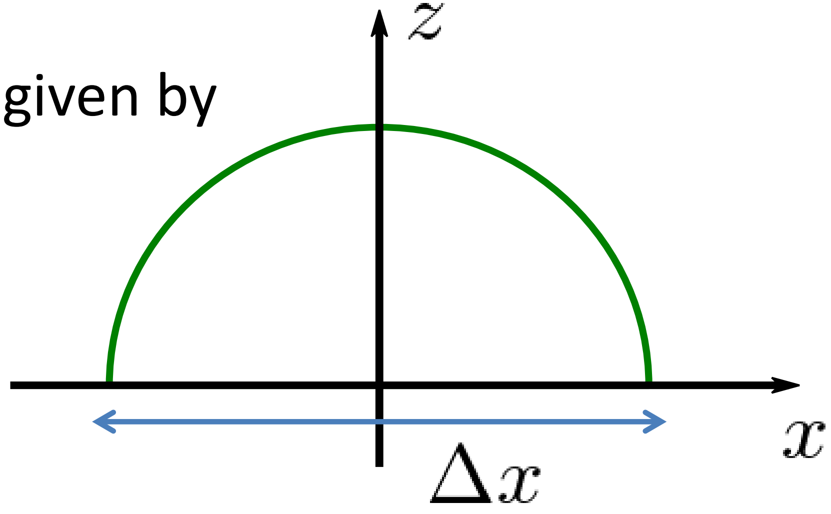
- Poincaré coordinates of AdS3

$$ds^2 = \frac{L^2}{z^2}(dz^2 - dt^2 + dx^2)$$ (L : AdS radius)

boundary: $z = 0$

- the extremal surface is given by a semi-circle:

$$z^2 + x^2 = \left(\frac{\Delta x}{2}\right)^2$$

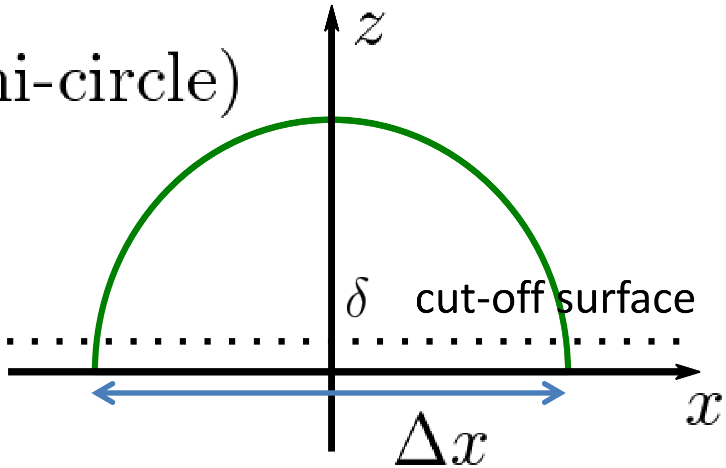


Holographic entanglement entropy of AdS3

- evaluating the length of the semi-circle

$$S(\Delta x) = \frac{1}{4G_{\mathrm{N}}}(\text{length of the semi-circle})$$
$$= \frac{L}{2G_{\mathrm{N}}} \log \frac{\Delta x}{\delta} + \mathcal{O}(\delta)$$

UV cut-off



If we use the relation, $c = \dfrac{3L}{2G_{\mathrm{N}}}$ Brown & Henneaux (1986)

this reproduces the famous result of 2-dim CFT,

$$\boxed{S(\Delta x) = \frac{c}{3} \log \frac{\Delta x}{\delta} + (\text{non universasal terms})}$$

Holzhey, Larsen, Wilczek (1994)
Calabrese, Cardy (2004)

We will confirm that the geometric inequality is saturated for general curves in AdS3.

$$\mathrm{d}s^2 = \frac{L^2}{z^2}(\mathrm{d}z^2 - \mathrm{d}t^2 + \mathrm{d}x^2)$$

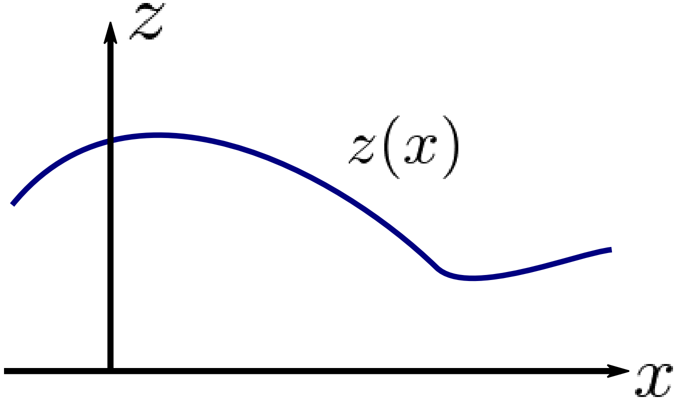
- impose periodic boundary condition in x-direction. (period ℓ_1)

single-valued function

- arbitrary closed curve: $z(x)$

- Bekenstein-Hawking entropy for this curve:

$$\frac{A}{4G_{\mathrm{N}}} = \frac{L}{4G_{\mathrm{N}}}\int_0^{\ell_1} \mathrm{d}x \frac{\sqrt{1+z'^2}}{z}$$

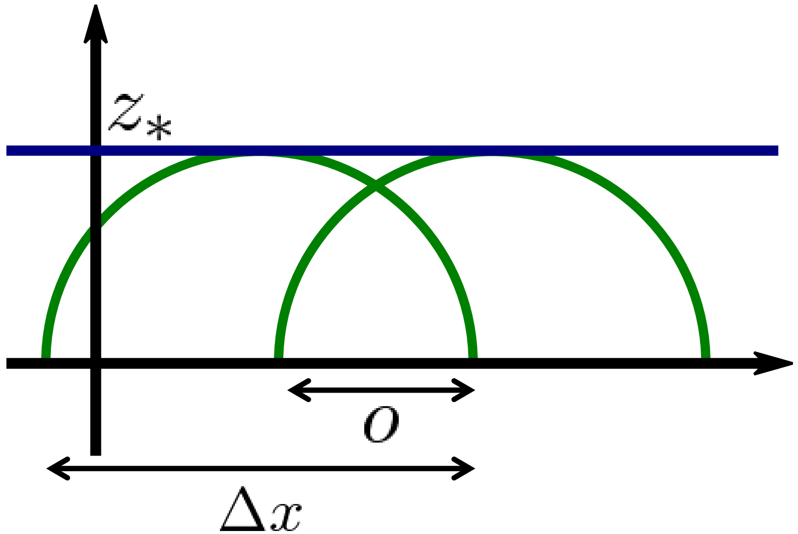


➢ constant profile $z(x) = z_*$

- consider a series of n equally spaced intervals with a width $\Delta x = 2z_*$
- the length of overlap: $o = \Delta x - \ell_1/n$

$$\sum_k^n [S(I_k) - S(I_k \cup I_{k+1})]$$
$$= \frac{nL}{2G_N}\left[\log\left(\frac{\Delta x}{\delta}\right) - \log\left(\frac{o}{\delta}\right)\right]$$
$$= -\frac{nL}{2G_N}\log\left(1 - \frac{\ell_1}{2nz_*}\right)$$

no UV divergence

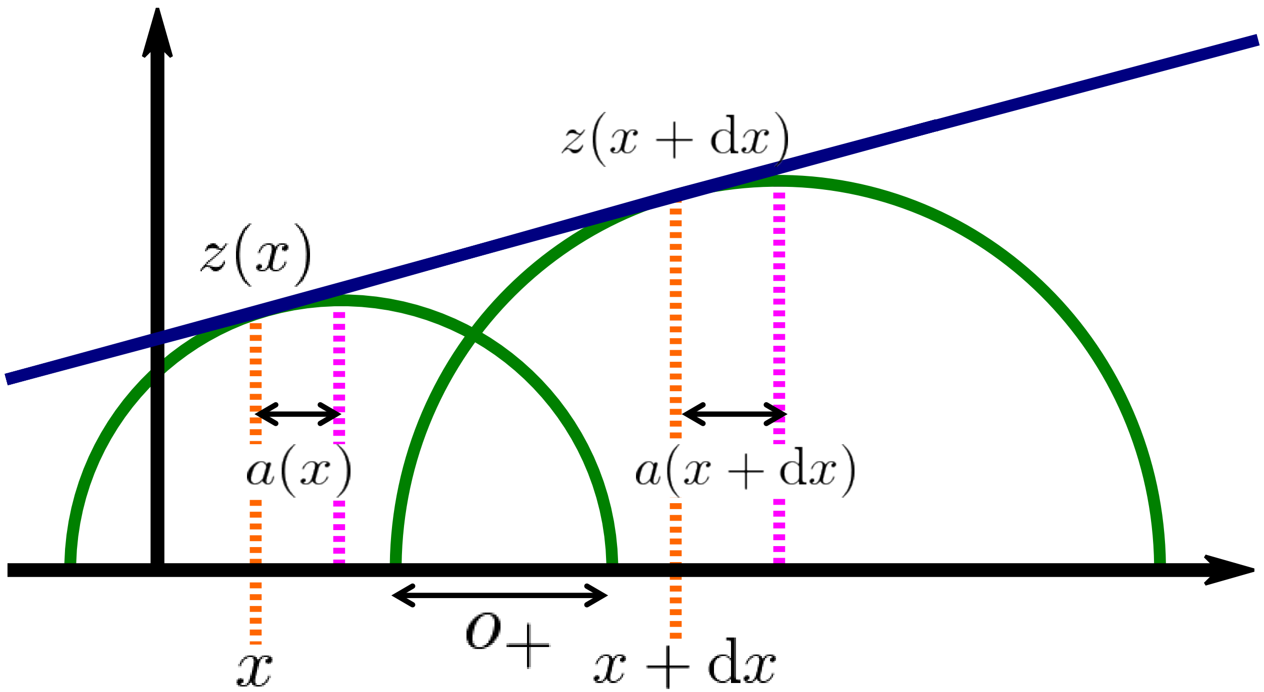


- take the "continuum" limit ($n \to \infty$)

$$\frac{L\,\ell_1}{4G_N\, z_*} = \frac{A}{4G_N}$$

geom. inequality is saturated

- arbitrary profile $z(x)$



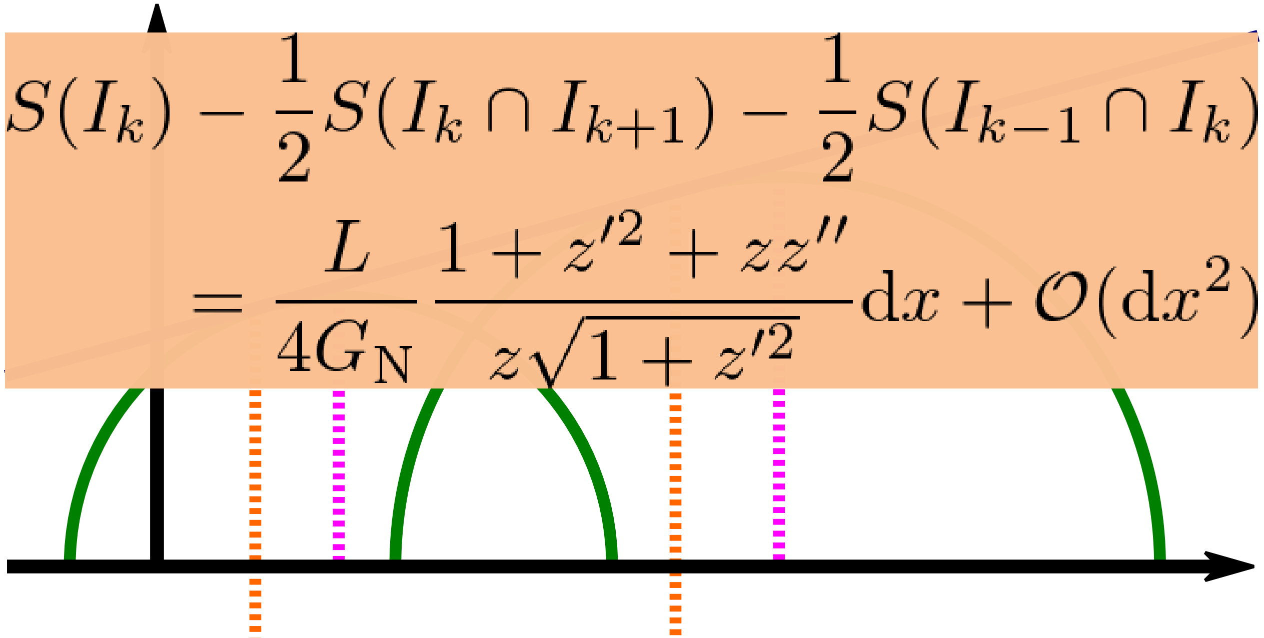
$a(x) = z(x)z'(x)$: shift from the midpt. of interval to the tangent pt.

$$\Delta x(x) = 2z(x)\sqrt{1 + z'(x)^2}$$

$$o_+ = \frac{1}{2}(\Delta x(x) + \Delta x(x + \mathrm{d}x)) + (a(x) - a(x + \mathrm{d}x)) - \mathrm{d}x$$

$$= 2z\sqrt{1 + z'^2} - (1 + z'^2 + zz'')\mathrm{d}x + \frac{\Delta x'}{2}\mathrm{d}x$$

- arbitrary profile $z(x)$



$$S(I_k) - \frac{1}{2}S(I_k \cap I_{k+1}) - \frac{1}{2}S(I_{k-1} \cap I_k)$$
$$= \frac{L}{4G_{\mathrm{N}}}\frac{1+z'^2+zz''}{z\sqrt{1+z'^2}}\mathrm{d}x + \mathcal{O}(\mathrm{d}x^2)$$

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$$= 2z\sqrt{1+z'^2} - (1+z'^2+zz'')\mathrm{d}x + \frac{\Delta x'}{2}\mathrm{d}x$$

- arbitrary profile $z(x)$

$$S(I_k) - \frac{1}{2}S(I_k \cap I_{k+1}) - \frac{1}{2}S(I_{k-1} \cap I_k)$$
$$= \frac{L}{4G_{\mathrm{N}}} \frac{1 + z'^2 + zz''}{z\sqrt{1+z'^2}} \mathrm{d}x + \mathcal{O}(\mathrm{d}x^2)$$

$$\sum_{k=1}^{n} \left[S(I_k) - \frac{1}{2}S(I_k \cap I_{k+1}) - \frac{1}{2}S(I_{k-1} \cap I_k)\right]$$

cont. limit

$$\frac{L}{4G_{\mathrm{N}}} \int_0^{\ell_1} \frac{1 + z'^2 + zz''}{z\sqrt{1+z'^2}} \mathrm{d}x$$

total derivative

$$= \frac{L}{4G_{\mathrm{N}}} \int_0^{\ell_1} \left[\frac{\sqrt{1+z'^2}}{z} + \frac{z''}{\sqrt{1+z'^2}}\right] \mathrm{d}x = \frac{A}{4G_{\mathrm{N}}}$$

The differential entropy reproduces the BH entropy.

Higher dimensions

We consider the following metric in (d+1)-dimensional background:

$$\boxed{\mathrm{d}s^2 = -g_0(z)\mathrm{d}t^2 + \sum_{i=1}^{d-1} g_i(z)(\mathrm{d}x^i)^2 + g_1(z)f(z)\mathrm{d}z^2}$$

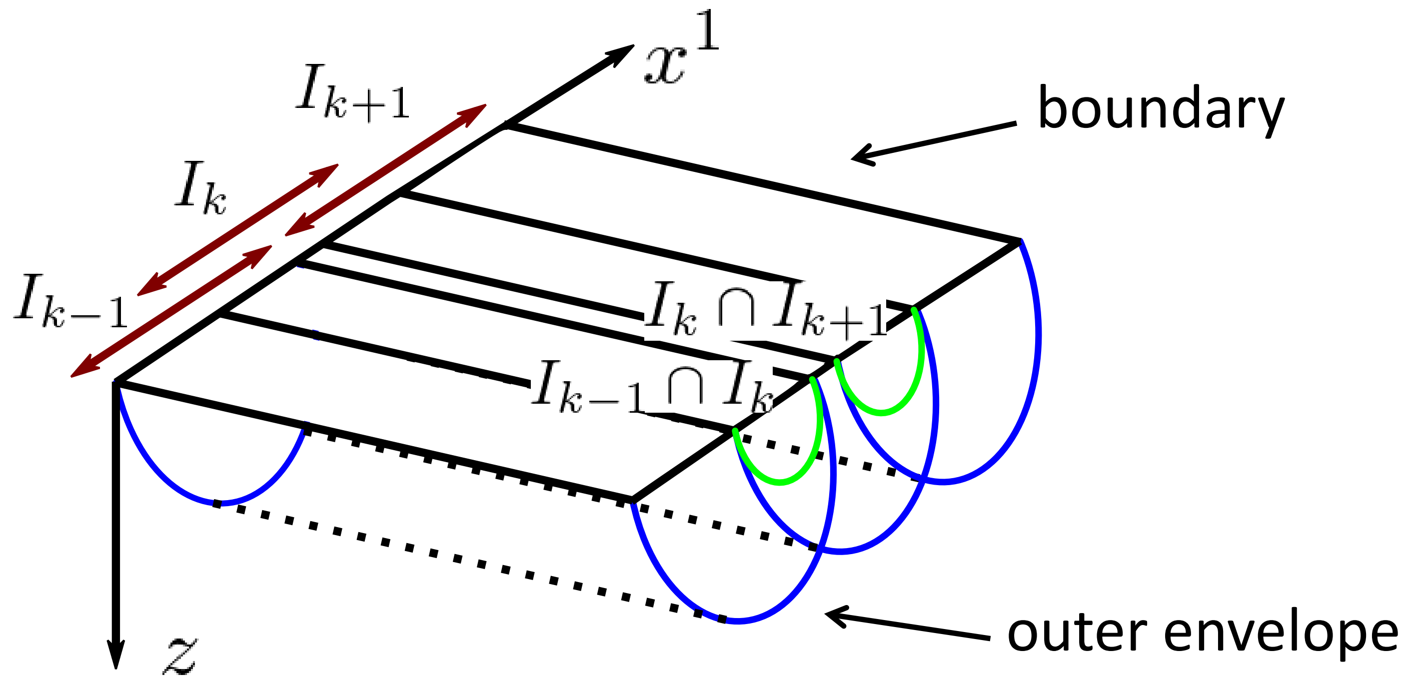
bdry: $z = 0$

e.g. planar AdS black hole

$$g_0(z) = \frac{L^2}{z^2}\left(1 - \frac{z^d}{z_h^d}\right),\ g_i(z) = \frac{L^2}{z^2},\ f(z) = \left(1 - \frac{z^d}{z_h^d}\right)^{-1}$$

assume that x^i-directions are periodic with periods ℓ^i

We only consider the case where
const. time slice in the bdry is partitioned by a set of
overlapping strips as the following fig.

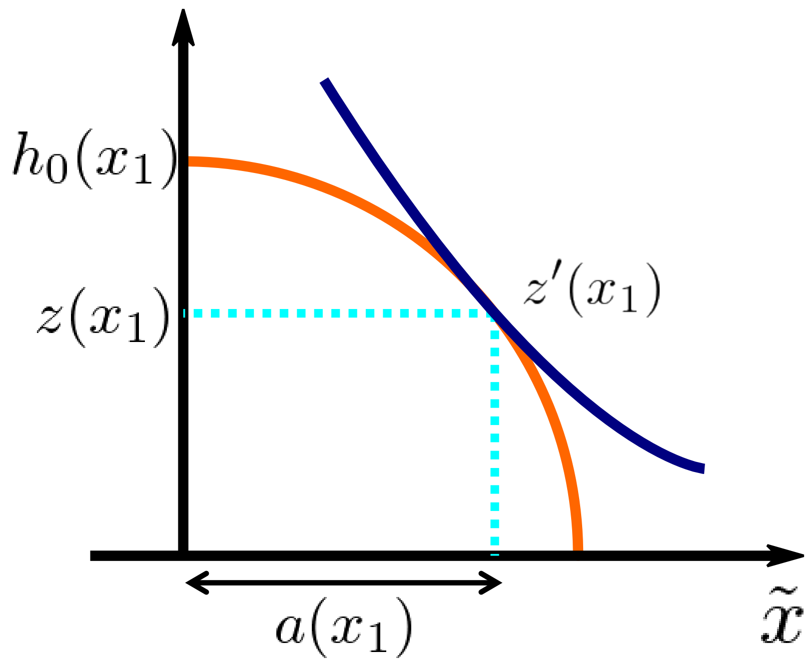


assume that bulk surfaces have translational sym. in
x^j-directions (j $= 2, \cdots, d-1$),
i.e. the bulk profiles have the form $z = z(x^1)$

We assume that there is the extremal surface which is tangent to the given profile $z(x_1)$ at each point x_1.

$z = h(\tilde{x}; x_1)$: the extremal surfaces which are tangent to the bulk surface at x_1.

$$\begin{cases} h(\tilde{x} = x_1; x_1) = z(x_1) \\ \dfrac{\mathrm{d}h}{\mathrm{d}\tilde{x}}(\tilde{x} = x_1; x_1) = z'(x_1) \end{cases}$$

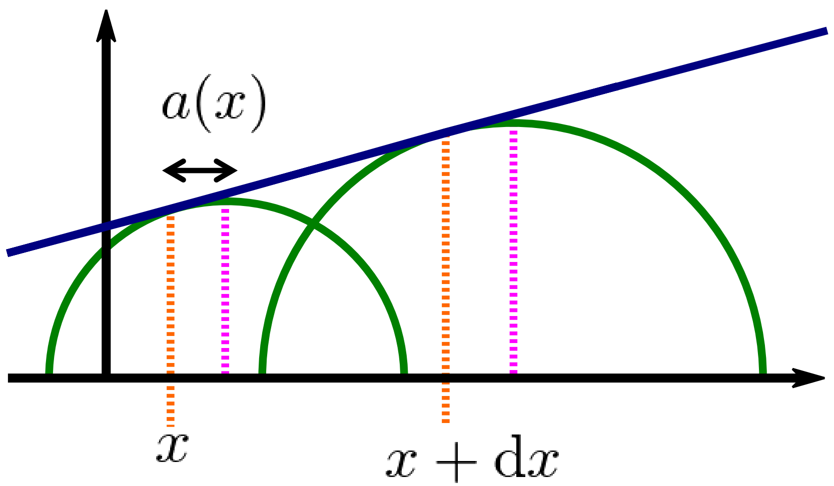


$h_0(x_1)$: the maximal height of $h(\tilde{x}; x_1)$

Holographic EE of a strip of width Δx

$$S(\Delta x) = \frac{\ell_2 \cdots \ell_{d-1}}{4G_{\mathrm{N}}} \int_0^{\Delta x} \mathrm{d}x \sqrt{G(h)(1 + f(h)h'^2)}$$

$$G(z) = g_1(z) \cdots g_{d-1}(z)$$



- differential entropy

$$E = \frac{\ell_2 \cdots \ell_{d-1}}{4G_{\mathrm{N}}} \int_0^{\ell_1} \mathrm{d}x \sqrt{G(h_0)}(1 + a')$$

- the differential entropy

$$E = \frac{\ell_2 \cdots \ell_{d-1}}{4G_{\mathrm{N}}} \int_0^{\ell_1} \mathrm{d}x \sqrt{G(h_0)}(1 + a')$$

- the BH entropy of the given surface

$$\frac{A}{4G_{\mathrm{N}}} = \frac{\ell_2 \cdots \ell_{d-1}}{4G_{\mathrm{N}}} \int_0^{\ell_1} \mathrm{d}x \sqrt{G(z)(1 + f(z) z'^2)}$$

We can show that the difference of the integrands is a total derivative.

$$\boxed{E = \frac{A}{4G_{\mathrm{N}}}}$$

The geom. inequality is saturated.

Summary and Discussion

- The spacetime entanglement conjecture proposes that the leading contribution of entanglement entropy in quantum gravity is given by the Bekenstein-Hawking entropy.

- In the context of AdS/CFT, we can interpret the Bekenstein-Hawking entropy of a special surface in terms of the boundary theory.

◆ black hole horizon $\longleftrightarrow$ thermal entropy

◆ extremal surface $\longleftrightarrow$ entanglement entropy

Summary and Discussion

- We have shown that BH entropy of general surfaces can be evaluated by the differential entropy.

- Our construction extends to Lovelock gravity.

- We should find a direct interpretation of the differential entropy in terms of the boundary theory.

Our results provide the relation between geometry and entanglement, which is a key concept in understanding quantum gravity.