Analysis of the Einstein equation in the Large D limit

Ryotaku Suzuki
Osaka City University (OCAMI)

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Motivation

Large D limit seems to be a successful analytical method in the linear analysis of Black holes

- Gregory-Laflamme instability
- QNMs of BH (AdS/rotating/brane)
- Inst. of MPBH (bar/axissym. mode)

etc...

↓

How about beyond the linear analysis?

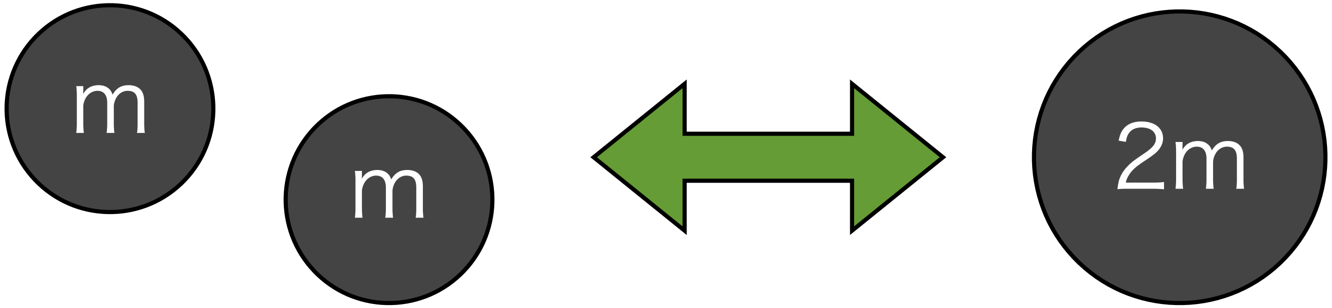
Let us solve the Einstein Eq with 1/D expansion.

No interaction of BHs

Entropy (M:fixed) **Emparan, RS, Tanabe (2013)**

$$(\quad) \sim \quad \frac{D-2}{D-3} \xrightarrow{D\to\infty}$$

Additive

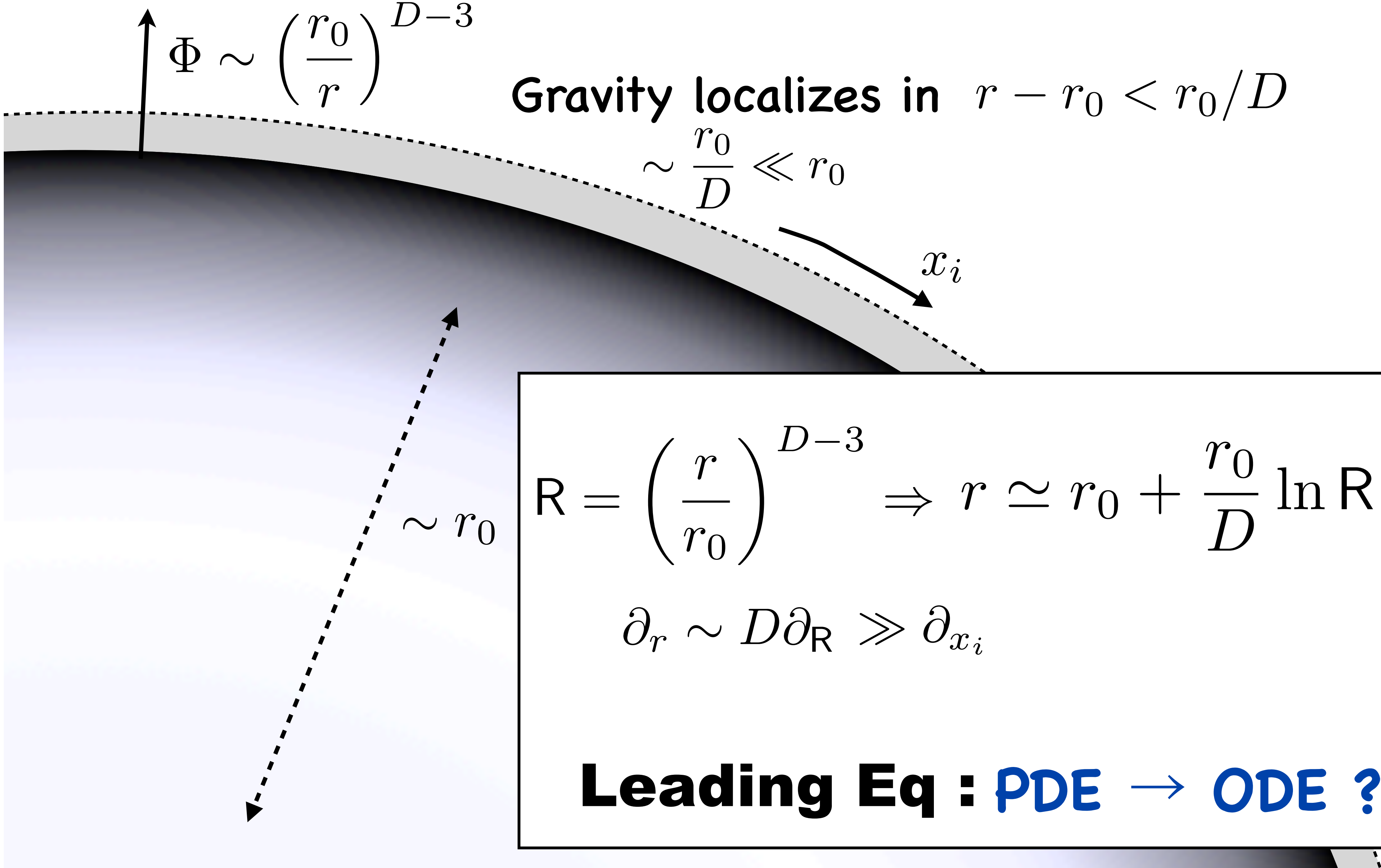


S=m+m=2m **=** S=2m

Horizon can be in the arbitrary shape

→ **Tractable beyond the linear regime ?**

Hierarchy at Large D



$$\mathsf{R} = \left(\frac{r}{r_0}\right)^{D-3} \Rightarrow r \simeq r_0 + \frac{r_0}{D}\ln \mathsf{R}$$

$$\partial_r \sim D\partial_{\mathsf{R}} \gg \partial_{x_i}$$

Leading Eq : PDE $\rightarrow$ ODE ?

Setup

D=n+4 Static Cylindrical Ansatz

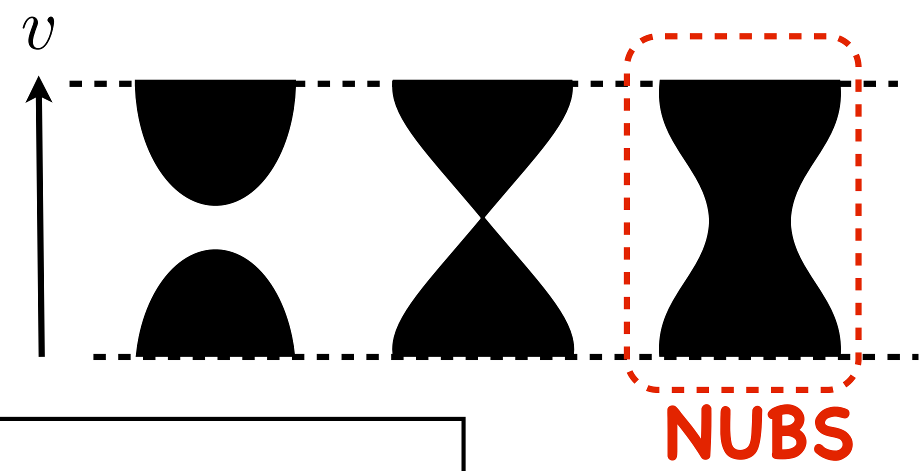
original var.
$B = K^{n+1}$

Written by Two variables : A(r,v), B(r,v) (HarmarkObers'02)

$$\boxed{ds^2 = -f(r)dt^2 + \frac{A(r,v)}{f(r)}dr^2 + \frac{A(r,v)}{B(r,v)n}dv^2 + r^2 B(r,v)^{\frac{1}{n+1}} d\Omega_{n+1}^2}$$

We already know $k_{GL} \simeq \sqrt{n}$ from the linear analysis.

$$f(r) = 1 - r_0^n / r^n$$



Einstein Eq.

4 nonzero components

$$\boxed{R_{rr} = 0, \ R_{zz} = 0, \ R_{rz} = 0, \ R_{\Omega\Omega} = 0}$$

Large D limit

As the linear analysis,

$$\mathsf{R} = r^n / r_0^n$$

and

$$A = \frac{1}{r_0^2} \sum_{k \geq 0} \frac{A^{(k)}(\mathsf{R}, v)}{n^k}, \quad B = \frac{1}{r_0^{2n+2}} \sum_{k \geq 0} \frac{B^{(k)}(\mathsf{R}, v)}{n^k}.$$

Here after $r_0 = 1$

Also assume $\partial_v \sim \mathcal{O}(1)$

so that $\partial_{\bar{v}} = \sqrt{n}\partial_v \ll n\partial_{\mathsf{R}}$ here $v = \sqrt{n}\bar{v}$

Leading order equation

Master Equation

from $R_{rr} = 0,\ R_{zz} = 0,\ R_{\Omega\Omega} = 0$

$$
\begin{aligned}
&-\frac{3(\mathsf{R}-1)^2\mathsf{R}^2(\partial_{\mathsf{R}}B^{(0)})^4}{2B^{(0)2}} - \frac{3\mathsf{R}\left(2\mathsf{R}^2-3\mathsf{R}+1\right)(\partial_{\mathsf{R}}B^{(0)})^3}{2B^{(0)}} + \left(\frac{3(\mathsf{R}-1)^2\mathsf{R}^2(\partial_{\mathsf{R}}^2B^{(0)})}{2B^{(0)}} + 9\mathsf{R}^2 - 9\mathsf{R} + 3\right)(\partial_{\mathsf{R}}B^{(0)})^2 \\
&+\left(3\mathsf{R}\left(2\mathsf{R}^2-3\mathsf{R}+1\right)(\partial_{\mathsf{R}}^2B^{(0)}) - (\mathsf{R}-1)^2\mathsf{R}^2\partial_{\mathsf{R}}^3B^{(0)}\right)(\partial_{\mathsf{R}}B^{(0)}) + (\mathsf{R}-1)^2\mathsf{R}^2(\partial_{\mathsf{R}}^2B^{(0)})^2 \\
&-2\left(3\mathsf{R}^2-3\mathsf{R}+1\right)B^{(0)}(\partial_{\mathsf{R}}^2B^{(0)}) - \mathsf{R}\left(2\mathsf{R}^2-3\mathsf{R}+1\right)B^{(0)}\partial_{\mathsf{R}}^3B^{(0)} = 0.
\end{aligned}
$$

2nd order ODE for $\partial_{\mathsf{R}} \ln B^{(0)}$

A is expressed by B from $R_{\Omega\Omega} = 0$

$$
A^{(0)} = 1 - \frac{(\mathsf{R}-1)\mathsf{R}(\partial_{\mathsf{R}}B^{(0)})^2}{2B^{(0)2}} + \frac{(2\mathsf{R}-1)(\partial_{\mathsf{R}}B^{(0)})}{2B^{(0)}} + \frac{(\mathsf{R}-1)\mathsf{R}(\partial_{\mathsf{R}}^2B^{(0)})}{2B^{(0)}}
$$

Leading order Solution

LO Solution with appropriate BCs

$$\boxed{A^{(0)} = 1, \quad B^{(0)} = S(v)^2}$$

$S(v) \approx$ Horizon Cross Section

S(v) cannot be determined in the leading order !

⇒ **Non-interactive picture of BH horizons in the Large D limit**

Next to Leading Order

$$A = A^{(0)} + \frac{A^{(1)}}{n}, \quad B = B^{(0)} + \frac{B^{(1)}}{n}$$

- Ordinary linear analysis

with unknown function S(v)

$$R_{rr} = 0,\ R_{zz} = 0,\ R_{\Omega\Omega} = 0$$

$$\boxed{\begin{aligned} A^{(1)} &= a_1(v) = 2\ln S(v) + S'(v)^2 + 2S(v)S''(v), \\ B^{(1)} &= b^{(1)}(v) + 2S(v)^3 S''(v) \ln \mathsf{R} \end{aligned}}$$

regularity@R=1

Horizon Equation

$$A^{(1)} = a_1(v) = 2\ln S(v) + S'(v)^2 + 2S(v)S''(v),$$

$$B^{(1)} = b^{(1)}(v) + 2S(v)^3 S''(v)\ln \mathsf{R}$$

$$R_{rz} = 0$$

ODE for S(v)

$$S'(v)(1 + 2S(v)S''(v)) + S(v)^2 S^{(3)}(v) = 0.$$

Integration

$$a_1(v) = 2a = 2\ln S(v) + S'(v) + 2S(v)S''(v).$$

Integration

$$S(v)(-2 - 2a + 2\ln S(v) + S'(v)^2) = 2b \geq -e^a$$

2 parameters a,b

Potential Problem

a can be set a = -1 (a is a scaling)

$$S(v) \to e^{\Delta a} S(e^{\Delta a} v), \quad v \to e^{\Delta a} v, \quad b \to e^{\Delta a} b$$

$$\boxed{\frac{1}{2} S'(v)^2 = -\ln S(v) + \frac{b}{S(v)} + \cancel{a+1}}$$

Solving S(v) $\to$ Potential Problem

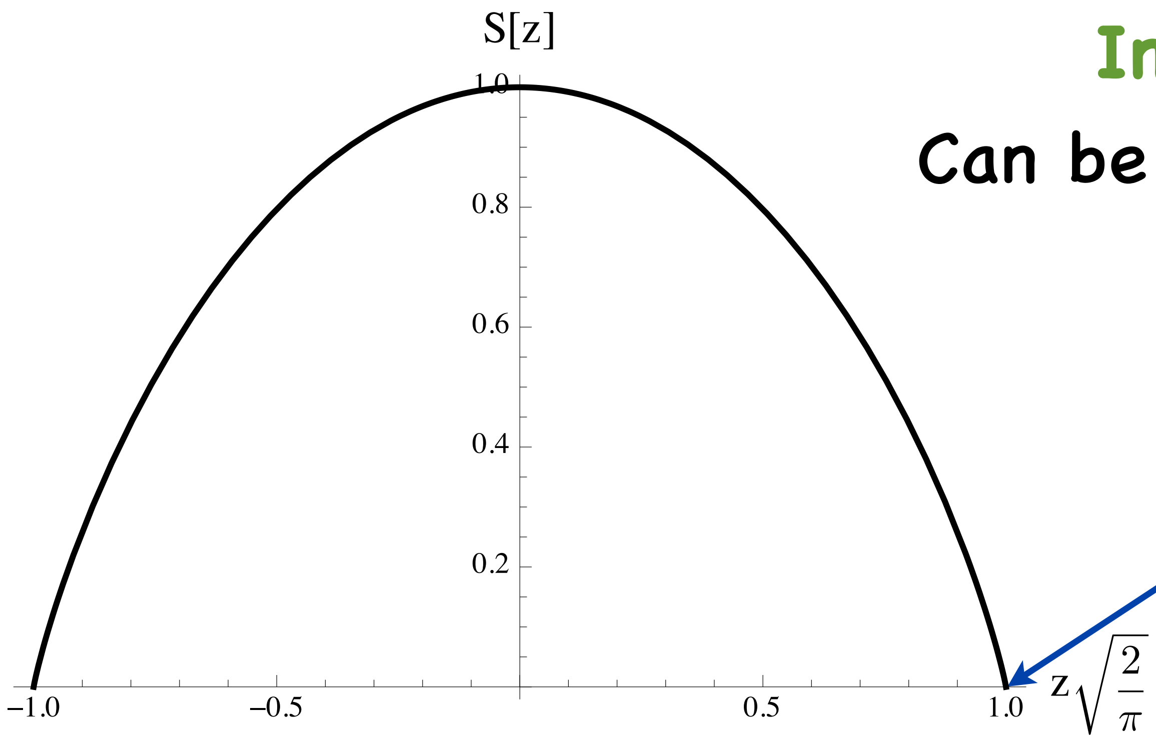
b = 0

If b=0, S(v) has an analytic form

$$S_{b=0}(v) = \exp\left[-\mathrm{erf}^{-1}\left(\sqrt{\frac{2}{\pi}}v\right)^2\right]$$

Inverse Error func.

Can be Shifted by $z \to z + C$



Horizon CS = 0

waist? BH?

Potential

No analytic form has found for any b ≠ 0

But, can be understood by the Potential

$$\boxed{\frac{1}{2}S'(v)^2 = -\ln S(v) + \frac{b}{S(v)}}$$

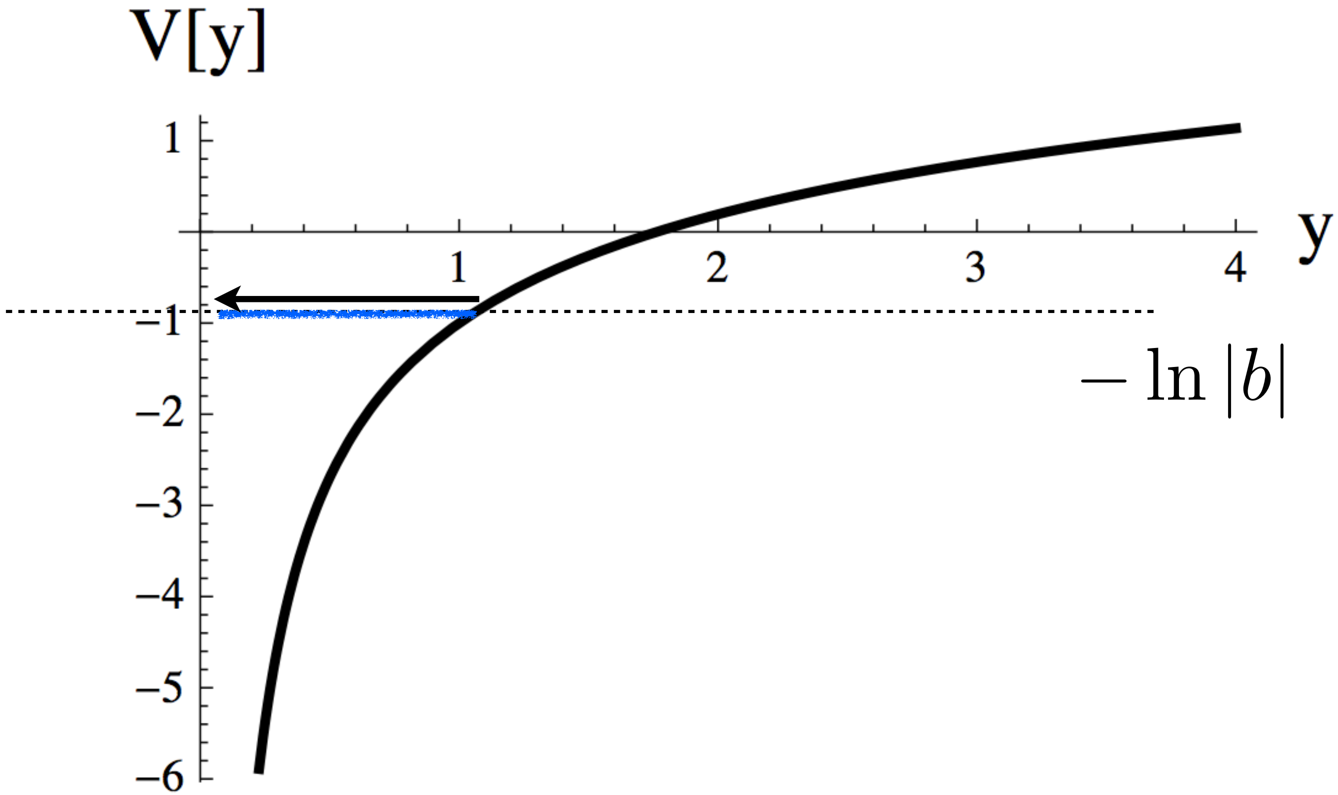
b≠0 -> rescaling by |b|

"Energy"

$$y = S/|b| \geq 0. \qquad \frac{|b|^2}{2}y'(v)^2 = -\ln|b| - V(y)$$

$$\boxed{V(y) = \ln y \pm \frac{1}{y}}$$

b > 0

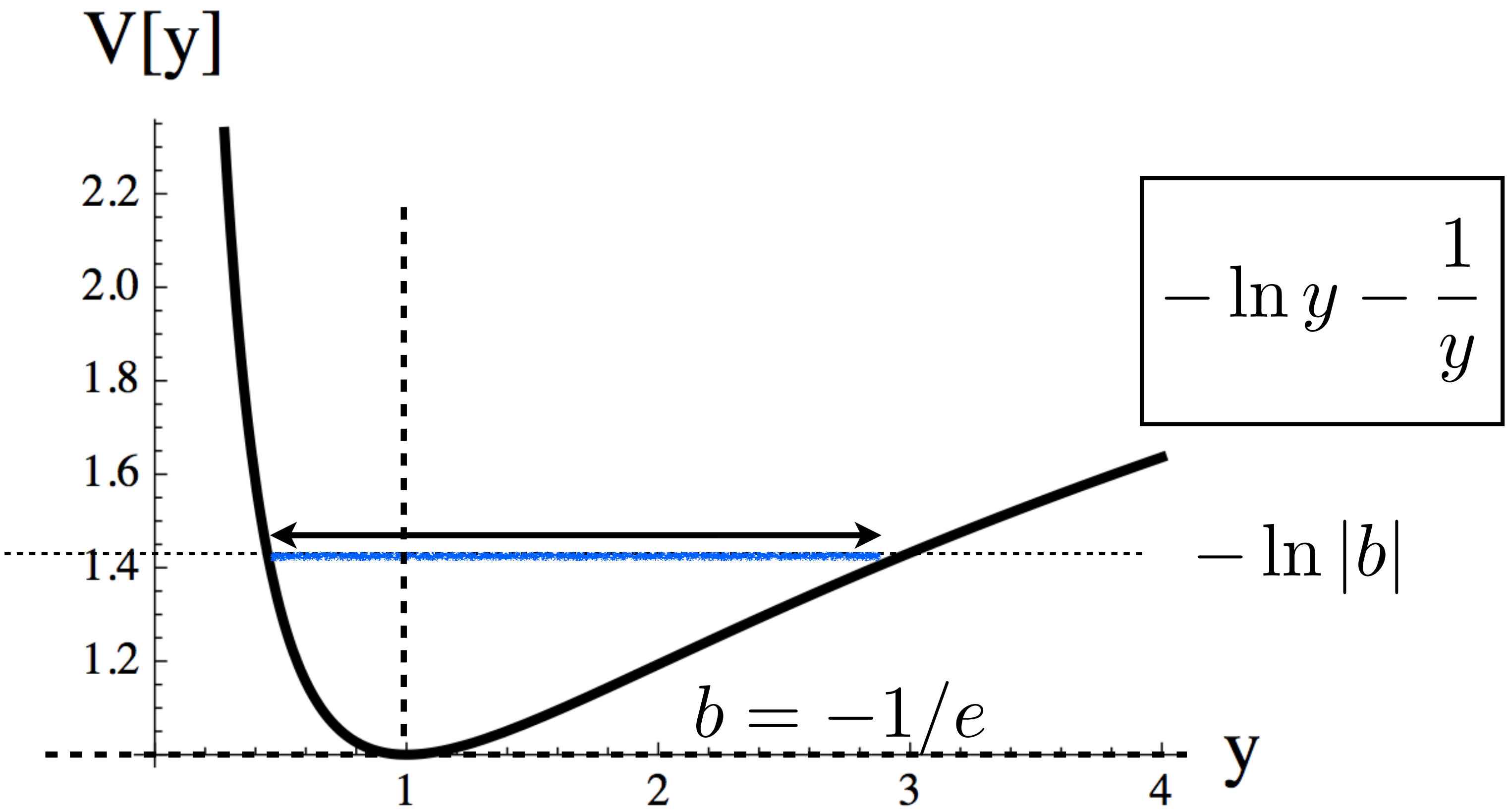


$$\boxed{-\ln y + \frac{1}{y}}$$

- One zero for every b>0
- collapse into y=0 ? (assumption not valid, though)

→ Caged BH or BH with a waist ?

0 > b > -1/e



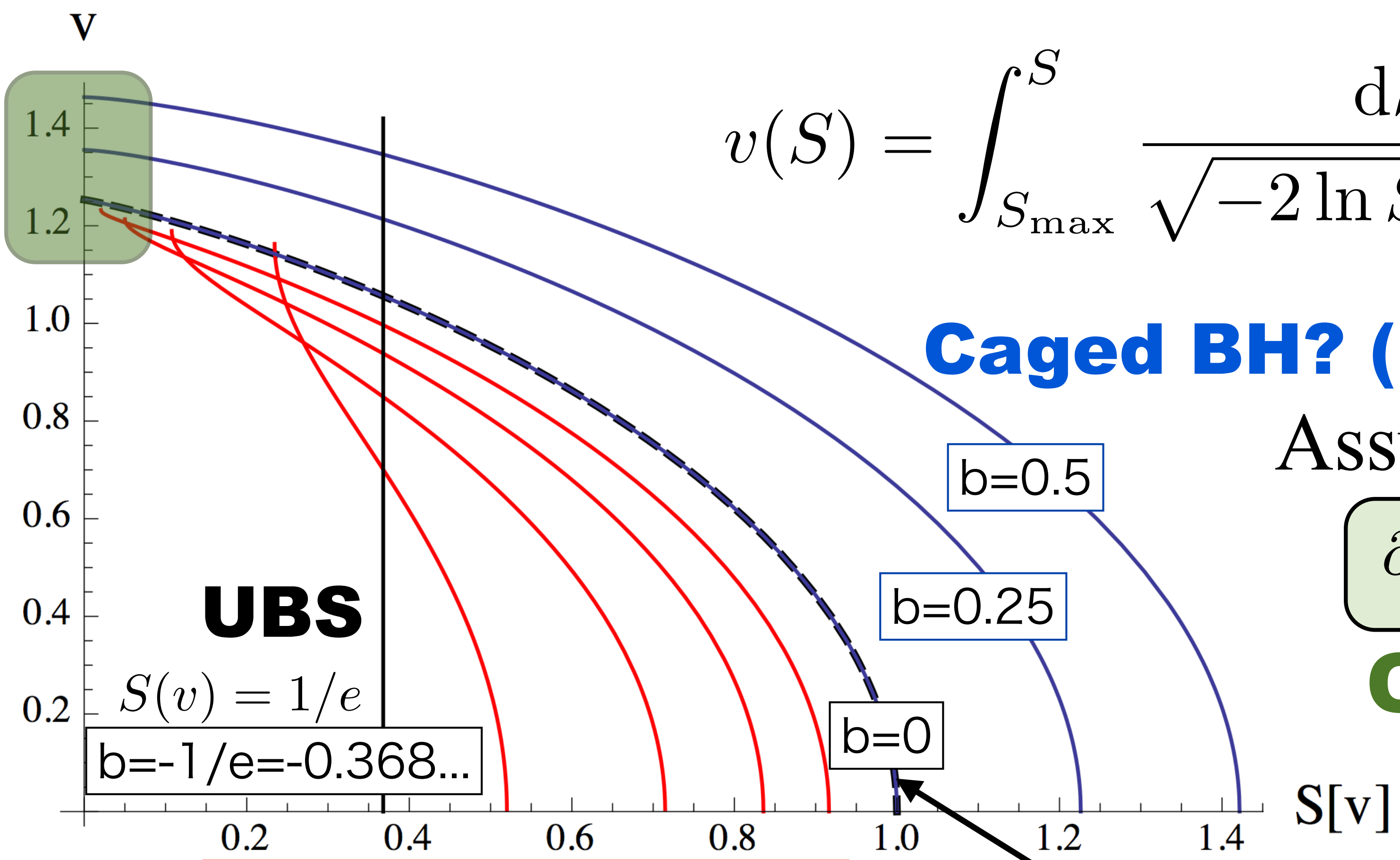
- **Minimum@y=1,(b=-1/e) : UBS**
- **2 zeros for 0>b>-1/e (y_min, y_max)**

  Oscillate between y_min and y_max : NUBS

Horizon Cross section

Numerically Integrating from $S_{\max}(b)$ to $S_{\min}(b)$

$$v(S) = \int_{S_{\max}}^{S} \frac{dS'}{\sqrt{-2\ln S' + b/S'}}$$



NUBS (-1/e<b<0)

b=0: BH with a waist ?

Caged BH? (b>0)

Assumptions

$$\partial_v S \sim S \sim \mathcal{O}(1)$$

Other patch at S~0 ?

Summary

Summary

done

- Solved Einstein Eq. in static cylindrical ansatz in the large D limit (**only near horizon**)
- Obtained an equation of horizon
- Solution may describe NUBS to BH

work in progress

- Check 1st Law
- Match with Asymptotics ($\rightarrow$ Mass, Tension) near axis (for BH solution)
- Comparison with Numerics

Future Work

- Other Ansatz : Spherical Collapse($r,z \rightarrow r,t$),
  (A)dS, Rotating BH

- Generalized formulation

- Application to Holography

Appendix

Leading order solution : General

General solution

$b_1(v),\ b_2(v),\ b_3(v)$ as arbitrary functions

$$A^{(0)} = -\frac{b_1(v)^2 b_2(v)(\mathsf{R}-1)^{b_1(v)-1}\mathsf{R}^{b_1(v)-1}}{(\mathsf{R}^{b_1(v)} + b_2(v)(\mathsf{R}-1)^{b_1(v)})^2},$$

$$B^{(0)} = \frac{b_2(v)b_3(v)(\mathsf{R}-1)^{b_1(v)-1-\sqrt{b_1(v)^2-1}}\mathsf{R}^{b_1(v)-1+\sqrt{b_1(v)^2-1}}}{\left(\mathsf{R}^{b_1(v)} + b_2(v)(\mathsf{R}-1)^{b_1(v)}\right)^2}.$$

b_1, b_2, b_3 should satisfy $(\ R_{rz} = 0\ )$

$$\bigstar \quad \frac{b_2(z)\left(2b_3(z)b_1'(z) + b_1(z)\left(b_1(z)^2-1\right)b_3'(z)\right) + \sqrt{b_1(z)^2-1}\,b_1(z)^2 b_3(z) b_2'(z)}{(\mathsf{R}-1)\mathsf{R}b_1(z)b_2(z)b_3(z)\sqrt{b_1(z)^2-1}} = 0$$

Leading order : Asymptotics

★ cannot be solved directly.

Instead, we focus on the asymptotics

$$A^{(0)} = -\frac{b_1(z)^2 b_2(z)}{(1+b_2(z)^2)\mathsf{R}^2} + \mathcal{O}(\mathsf{R}^{-3}) \quad (b_2(z) \neq -1)$$

$$\boxed{A^{(0)} = 1 + \frac{1-b_1(z)}{\mathsf{R}} + \mathcal{O}(\mathsf{R}^{-2}) \qquad (b_2(z) = -1).}$$

looks better ?

$$\boxed{b_2(z) = -1}$$

Leading order Solution

Substituting $b_2(z) = -1$ into ★

$$\boxed{2b_3(z)b_1'(z) + b_1(z)(b_1(z)^2 - 1)b_3'(z) = 0}$$

$$\Rightarrow \boxed{b_3(z) = \frac{b_1(z)^2 C}{b_1(z)^2 - 1} \quad \text{or} \quad b_1(z) = 1.}$$

For now, we take $b_1(z) = 1.$

$$\boxed{A^{(0)} = 1, \;\; B^{(0)} = b_3(z) = S(z)^2}$$

NUBS from UBS

Expansion from UBS (b = -1/e) $\quad \bar{b} = eb + 1 \ll 1$

$$eS(v) = 1 + \bar{b} - \frac{\bar{b}^2}{12} + \left(1 - \frac{55\bar{b}}{144} + \frac{2347\bar{b}^2}{20736}\right) 2^{1/2}\bar{b}^{1/2}\cos(v/L)$$

$$\underline{- \left(\frac{2\bar{b}}{3} - \frac{5\bar{b}^2}{9}\right)\cos(2v/L) + \frac{17\bar{b}^{3/2}}{24\sqrt{2}}\cos(3v/L) - \frac{247\bar{b}^2}{540}\cos(4v/L)} + \mathcal{O}(\bar{b}^{5/2})$$

non-linear effect $\sim \bar{b}^{m/2}\cos(mv/L)$

Period

$$eL = 1 + \frac{\bar{b}}{12} + \frac{\bar{b}^2}{576} + \mathcal{O}(\bar{b}^{5/2})$$

Recover the scaling

$$S(v) \to (e/\lambda) S_{eb/\lambda}(ev/\lambda), \quad a \to -\ln \lambda$$