# Fuzzy Sphere and Hyperbolic Space from Deformation Quantization

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## § Introduction and Motivation

Recently noncommutative gauge theory have been investigated enthusiastically.

It is interesting not only as a model of nonlocal field theory but also as a low energy effective theory of strings on nonzero NS-NS $B$-field background.

| D-brane on *flat* and *constant $B$-field* background |
|---|
| $\|\|$ |
| Noncommutative gauge theory (*Moyal product*) |

Seiberg-Witten (1999) and its references and citations,...

With the nocommutative parameter:

$$\theta^{ij} = -(2\pi\alpha')^2 \left( \frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{ij},$$

this Moyal product is given by

$$f(x) * g(x) = f(x) \exp\left( \frac{i}{2} \frac{\overleftarrow{\partial}}{\partial x^i} \theta^{ij} \frac{\overrightarrow{\partial}}{\partial x^j} \right) g(x),$$

## More generic backgrounds?
## *curved* and *nonconstant* $B$*-field*...

⋆ **Deformation Quantization**

There are some techniques to construct noncommutative associative $*$ product as a generalization of Moyal product on $\mathbb{R}^{2n}$.

*Kontsevich, Fedosov, Omori-Maeda-Yoshioka, De Wilde-Lecomte,...*

Most general one is that on Poisson manifold.

⋆ **Nonlinear $\sigma$-Model of Strings**

There are some tractable cases by using CFT.

If the relation between them becomes clear on more generic backgrounds, deformation quantization may be useful to study string theory on nontrivial backgrounds.

Formally there are prescriptions to construct general $*$ products, but to investigate the relation concretely, *explicit form* of $*$ product is more useful.

⋆ Here we construct $*$ product explicitly in *tractable but nontrivial* case:

on 2 dimensional constant curvature space
$$S^2, H^2$$

by using *Fedosov's deformation quantization*.

*B. V. Fedosov, "Deformation quantization and index theory," Berlin, Germany: Akademie-Verl.(1996).*

The resulting $*$ products form $su(2), su(1,1)$ algebra which is known as fuzzy sphere, hyperbolic space algebra respectively.

⋆ Fuzzy Sphere in String Theory (an example)

Strings on $S^3$ (radius $R_3$) with $H = dB$

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$SU(2)$ WZW model at level $k$ ($\sim R_3^2$)

D-brane in $SU(2)$ WZW at $k \to \infty$

*V. Schomerus et al,...*

$\downarrow$

OPE among boundary fields

$||$

Fuzzy sphere algebra $M_{N+1}(\mathbb{C})$

cf. *Madore*

## § Fedosov's $*$ Product

**Fedosov's procedure to construct $*$ product:**

1. **Weyl algebra bundle $(W, \circ)$ on $(M, \Omega_0)$**
   **$\leftarrow$ input: $\nabla, \theta$ with parameter $\hbar$**

Its section is given by

$$a(x,y,\hbar) = \sum_{2k+p\geq 0, k\geq 0} \hbar^k \sum_{q=0}^{2n} \frac{1}{p!q!} a_{k,i_1\cdots i_p,j_1\cdots j_q}(x) y^{i_1}\cdots y^{i_p}\theta^{j_1}\wedge\cdots\wedge\theta^{j_q}.$$

The $\circ$ product is Moyal type with respect to $y^i$:

$$\circ = \exp\left(-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial y^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial y^j}\right) \text{ where } \Omega_0 = -\frac{1}{2}\omega_{ij}\theta^i\wedge\theta^j, \omega_{ik}\omega^{kj} = \delta_i^j.$$

2. **Abelian connection $D$ on $W$**
   **$\leftarrow$ input: $\mu, \Omega_1$**

General connection $D$ on $W$ is :

$$Da = \nabla a - \delta a + \frac{i}{\hbar}(\mathrm{r}\circ a - (-1)^{|a|} a \circ \mathrm{r}) \text{ where } \delta = \theta^i \frac{\partial}{\partial y^i}.$$

$D$ is called Abelian iff $D^2 = 0$.

3. **one to one map between $W_D$ and $C^\infty(M)[[\hbar]]$**
   **$\Rightarrow$ map $\sigma, Q$**

$W_D$ is flat section for an Abelian connection $D$, i.e,,
$Da = 0$ for $a \in W_D$.

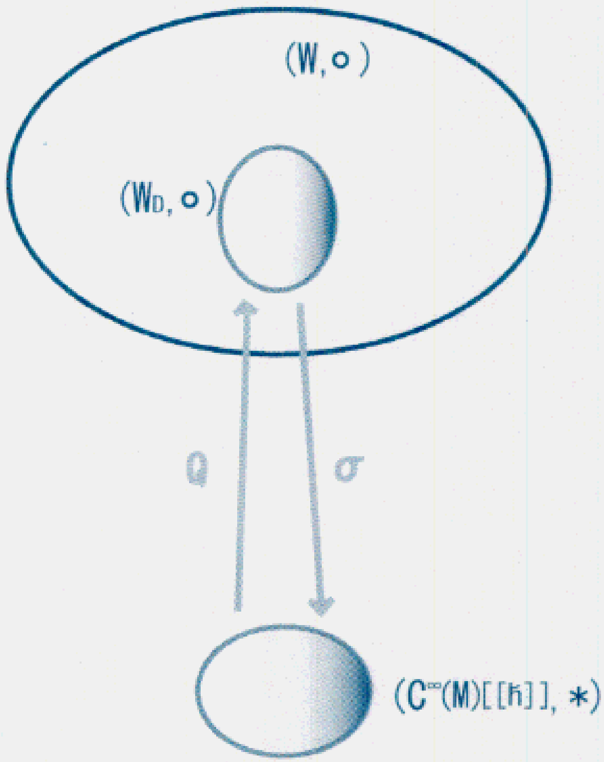
**Fedosov's $*$ product is defined by**

$$a_0 * b_0 := \sigma(Q(a_0)\circ Q(b_0)), a_0, b_0 \in C^\infty(M)[[\hbar]].$$

In fact this is *noncommutative* and *associative* product and

$$[a_0, b_0]_* = i\hbar\{a_0, b_0\} + \mathcal{O}(\hbar^2),$$

where $\{\ ,\ \}$ is Poisson bracket with respect to $\Omega_0$.




We can calculate this $*$ product order by order in $\hbar$ at least *formally* for general symplectic manifold $(M, \Omega_0)$.

⋆ **Difficulties to obtain *explicit* formula of this $*$ product to *full order* in $\hbar$ :**

- **Construction of an Abelian connection $D$.**
  **Exact solution of iteration equation for $\mathrm{r}$:**

$$\mathrm{r} = \delta\mu + \delta^{-1}\left(\nabla(\omega_{ij}y^i\theta^j) + R - \Omega_1 + \nabla\mathrm{r} + \frac{i}{\hbar}\mathrm{r}\circ\mathrm{r}\right)$$

- **Construction of the map $Q$.**
  **Exact solution of flat section equation: $Da = 0$, i.e.,**

$$\nabla a - \delta a + \frac{i}{\hbar}(\mathrm{r}\circ a - a\circ\mathrm{r}) = 0,\ \ a \in W$$

More concretely,

$$\mathrm{r}_{k,i_1\cdots i_p,j} = \mathrm{r}^0_{k,i_1\cdots i_p,j} + \frac{p}{2(p+1)}(\nabla_{(i_1}\mathrm{r}_{|k|,i_2\cdots i_p),j} - \nabla_j\mathrm{r}_{k,(i_1\cdots i_{p-1},i_p)})$$
$$+\sum\frac{i}{m!p_1!p_2!}\frac{p!}{2(p+1)}\frac{\omega^{l_1l_1'}}{2i}\cdots\frac{\omega^{l_ml_m'}}{2i}[\mathrm{r}_{k_1,l_1\cdots l_m(n_1\cdots n_{p_1},j'},\mathrm{r}_{|k_2,l_1'\cdots l_m'|n_1'\cdots n_{p_2}'),j}\}$$

where

$$\begin{aligned}
\mathrm{r} &= \sum_{2k+p\geq 2,k\geq 0,p\geq 0}\hbar^k\frac{1}{p!}\mathrm{r}_{k,i_1\cdots i_p,j}y^{i_1}\cdots y^{i_p}\theta^j,\\
\mathrm{r}^0 &= \sum_{2k+p\geq 2,k\geq 0,p\geq 0}\hbar^k\frac{1}{p!}\mathrm{r}^0_{k,i_1\cdots i_p,j}y^{i_1}\cdots y^{i_p}\theta^j\\
&= \sum_{2k+p\geq 2,k\geq 0,p\geq 0}\hbar^k\frac{1}{p!}\mu_{k,i_1\cdots i_pj}y^{i_1}\cdots y^{i_p}\theta^j + \frac{1}{3}\omega_{im}T^m{}_{jk}y^iy^j\theta^k\\
&\quad+\frac{1}{8}R_{ijkl}y^iy^jy^k\theta^l - \frac{1}{2}(i\hbar R_{Ekl} + \Omega_{1kl})y^k\theta^l.
\end{aligned}$$

⋮

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⋮

## § 'Fuzzy Sphere and Hyperbolic Space'

In *special* case we can get explicit formula of Fedosov's $*$ product.

- **Flat space** $\mathbb{R}^{2n}$
  $\Rightarrow$ Moyal product:
  $a_0 * b_0 = a_0 \exp\left(\frac{i}{2}\frac{\overleftarrow{\partial}}{\partial x^i}\theta^{ij}\frac{\overrightarrow{\partial}}{\partial x^j}\right) b_0, \ \theta^{ij} = -\theta^{ji}$:*constant*
- **2 dimensional constant curvature space**
  positive curvature : sphere $S^2$
  negative curvature : hyperbolic space $H^2$

In the former case, we can carry out Fedosov's procedure rather trivially.

In the latter case, we get explicit formulae by *adjusting input parameters*, i.e.,
we select $\nabla, \theta, \mu, \Omega_1$ to be able to solve iteration equation for $\mathrm{r}$ easily.
In practice we required stronger conditions for $\mathrm{r}$ which gives an Abelian connection $D$:

$$\nabla \mathrm{r} + \frac{i}{\hbar}\mathrm{r} \circ \mathrm{r} = 0,$$
$$\mathrm{r} = \delta\mu + \delta^{-1}\left(\nabla(\omega_{ij} y^i \theta^j) + R - \Omega_1\right)$$

and solved them.

⋆ For *rotationally symmetric 2 dim. space* with a metric:

$$ds^2 = e^{\Phi(r)}(dr^2 + r^2 d\theta^2),$$

a symplectic form is given by

$$\Omega_0 = e^{\Phi(r)} r dr \wedge d\theta.$$

In this setup, we solved stronger conditions for $\mathrm{r}$, by adjusting input parameters:

$$\mathrm{r} = y^1 y^2 r^{-1} dr,$$

and obtained the map $Q$ by solving $Da = 0$ for this $\mathrm{r}$ :

$$a = Q(a_0(r,\theta)) = a_0\left(G(r,y^1), \theta + \frac{y^2}{r}\right),$$

where $G(r, y^1)$ is given by

$$\int_r^{G(r,y^1)} e^{\Phi(r')} r' dr' = y^1 r.$$

Then we have obtained a $*$ product:

$$a_0(r,\theta) * b_0(r,\theta) = \left(a_0\left(G(r,y^1), \theta + \tfrac{y^2}{r}\right)\right.$$
$$\left.\cdot \exp\left(-\tfrac{i\hbar}{2}\left(\tfrac{\overleftarrow{\partial}}{\partial y^1}\tfrac{\overrightarrow{\partial}}{\partial y^2} - \tfrac{\overleftarrow{\partial}}{\partial y^2}\tfrac{\overrightarrow{\partial}}{\partial y^1}\right)\right) b_0\left(G(r,y^1), \theta + \tfrac{y^2}{r}\right)\right)\Bigg|_{\substack{y^1=0,\\ y^2=0}}$$

## $S^2$ case

We embed $S^2$ in $\mathbb{R}^3$ as

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = R^2$$

and parameterize as

$$X^1 = \frac{2R^2 r}{r^2 + R^2}\cos\theta \ , X^2 = \frac{2R^2 r}{r^2 + R^2}\sin\theta, \ \ X^3 = R\frac{r^2 - R^2}{r^2 + R^2},$$

then we get the explicit formula of a $*$ product with

$$G(r, y^1) = \sqrt{\frac{r^2 + \frac{y^1}{2R^2} r(r^2 + R^2)}{1 - \frac{y^1}{2R^4} r(r^2 + R^2)}}.$$

Using this $*$ product we get

$$[X^i, X^j]_* = i\frac{\hbar}{R}\varepsilon^{ijk}X^k,$$
$$X^1 * X^1 + X^2 * X^2 + X^3 * X^3 = R^2\left(1 - \frac{\hbar^2}{4R^4}\right).$$

This is *fuzzy sphere algebra* ($\simeq su(2)$) with radius $R\sqrt{1 - \frac{\hbar^2}{4R^4}}$. Namely, we have obtained "fuzzy sphere" by deforming $S^2$ using our $*$ product!

## $H^2$ case

We embed $H^2$ in $\mathbb{R}^{1,2}$ as

$$-(Y^0)^2 + (Y^1)^2 + (Y^2)^2 = -R^2,$$

and parameterize as

$$Y^0 = R\frac{R^2 + r^2}{R^2 - r^2},\ \ Y^1 = \frac{2R^2 r}{R^2 - r^2}\cos\theta,\ \ Y^2 = \frac{2R^2 r}{R^2 - r^2}\sin\theta,$$

then we get the explicit formula of a $*$ product with

$$G(r, y^1) = \sqrt{\frac{r^2 + \frac{y^1}{2R^2} r(R^2 - r^2)}{1 + \frac{y^1}{2R^4} r(R^2 - r^2)}}.$$

Using this $*$ product we get

$$[Y^0, Y^1]_* = i\frac{\hbar}{R}Y^2, [Y^2, Y^0]_* = i\frac{\hbar}{R}Y^1, [Y^1, Y^2]_* = -i\frac{\hbar}{R}Y^0,$$
$$-Y^0 * Y^0 + Y^1 * Y^1 + Y^2 * Y^2 = -R^2\left(1 - \frac{\hbar^2}{4R^4}\right).$$

This is *fuzzy $H^2$ algebra* ($\simeq su(1,1)$) with radius $R\sqrt{1 - \frac{\hbar^2}{4R^4}}$. Namely, we have obtained "fuzzy $H^2$" by deforming $H^2$ with our $*$ product!

## Large $R$ limit of fuzzy $S^2, H^2$

For the complex coordinates $z := re^{i\theta}, \bar{z} := re^{-i\theta}$, we have *commutation relations* with our $*$ product :

$$[z, \bar{z}]_* = -\frac{\hbar}{2R^4}(R^2 + z * \bar{z})(R^2 + \bar{z} * z),$$

for 'fuzzy $S^2$,' and

$$[z, \bar{z}]_* = -\frac{\hbar}{2R^4}(R^2 - z * \bar{z})(R^2 - \bar{z} * z)$$

for 'fuzzy $H^2$.'

They are both reduced to fuzzy $\mathbb{R}^2$ (Heisenberg algebra) in the large $R$ limit,i.e.,

$$[z, \bar{z}]_* = -\frac{\hbar}{2} \quad \text{as } R \to \infty.$$

Using our $*$ product, we get

$$\begin{array}{ccc} S^2 & \xleftarrow{\hbar\to 0} & \text{fuzzy } S^2 \\ {\scriptstyle R\to\infty}\downarrow & & \downarrow{\scriptstyle R\to\infty} \\ \mathbb{R}^2 & \xleftarrow[\hbar\to 0]{} & \text{fuzzy } \mathbb{R}^2 \\ {\scriptstyle R\to\infty}\uparrow & & \uparrow{\scriptstyle R\to\infty} \\ H^2 & \xleftarrow[\hbar\to 0]{} & \text{fuzzy } H^2 \end{array}$$

## § Comparison with Conventional Fuzzy Sphere

There is a one to one mapping:

$su(2)$ representation *matrix* for spin $\frac{N}{2}$

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*function* expanded by $Y_{lm}(\vartheta, \varphi), l \leq N$

Conventionally we identify

$$M_{N+1}(\mathbb{C}) \simeq \text{fuzzy sphere algebra}$$

The induced noncommutative associative product between *functions* is

$$Y_{l_1 m_1} *_N Y_{l_2 m_2}$$
$$= \sum_{l=|l_1-l_2|}^{N} \sum_{m=-l}^{l} (-1)^m \sqrt{\frac{(2l+1)(2l_1+1)(2l_2+1)}{4\pi}} \begin{pmatrix} l & l_1 & l_2 \\ m & -m_1 & -m_2 \end{pmatrix}$$
$$\cdot (-1)^N \sqrt{N+1} \begin{Bmatrix} l & l_1 & l_2 \\ \frac{N}{2} & \frac{N}{2} & \frac{N}{2} \end{Bmatrix} Y_{l,m} \quad .$$

$N$ is noncommutative parameter and

$$Y_{l_1 m_1} *_N Y_{l_2 m_2} \longrightarrow Y_{l_1 m_1} Y_{l_2 m_2}$$

for $N \to \infty$.

This $*_N$ characterizes *fuzzy sphere*:

$$[X^i, X^j]_{*_N} = i\epsilon_{ijk}\frac{2R}{\sqrt{N(N+2)}}X^k$$
$$X^1 *_N X^1 + X^2 *_N X^2 + X^3 *_N X^3 = R^2$$

where $X^i, i = 1, 2, 3$ are given by

$$X^1 = R\sqrt{\frac{2\pi}{3}}(-Y_{1,1}(\vartheta, \varphi) + Y_{1,-1}(\vartheta, \varphi)) = R\sin\vartheta\cos\varphi,$$
$$X^2 = R\sqrt{\frac{2\pi}{3}}i(Y_{1,1}(\vartheta, \varphi) + Y_{1,-1}(\vartheta, \varphi)) = R\sin\vartheta\sin\varphi,$$
$$X^3 = R\sqrt{\frac{4\pi}{3}}Y_{1,0}(\vartheta, \varphi) = R\cos\vartheta.$$

Comparing $*_N$ with our $*$ product, we expect the correspondence between the noncommutative parameters as

$$\frac{\hbar}{2R^2} \sim \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

To understand the meaning of $\sim$, we should investigate $*$ and $*_N$ more precisely.

We can write down our $*$ product between spherical harmonic functions as:

$$Y_{l_1 m_1}(\vartheta,\varphi) * Y_{l_2 m_2}(\vartheta,\varphi) =$$
$$Y_{l_1 m_1}\left(\cos^{-1}\left(\cos\vartheta + \frac{m_2\hbar}{2R^2}\right),\varphi\right) Y_{l_2 m_2}\left(\cos^{-1}\left(\cos\vartheta - \frac{m_1\hbar}{2R^2}\right),\varphi\right).$$

The products $*_N$ and $*$ are expanded as:

$$Y_{l_1 m_1} *_N Y_{l_2 m_2} = Y_{l_1 m_1} Y_{l_2 m_2}$$
$$+\frac{1}{N}\sum_{l+l_1+l_2=\mathrm{odd}} C_{l,l_1 m_1 l_2 m_2} Y_{l,m_1+m_2} + \mathcal{O}\left(\frac{1}{N^2}\right),$$

$$Y_{l_1 m_1} * Y_{l_2 m_2} = Y_{l_1 m_1} Y_{l_2 m_2}$$
$$+\frac{\hbar}{2R^2}\sum_{l+l_1+l_2=\mathrm{odd}} C'_{l,l_1 m_1 l_2 m_2} Y_{l,m_1+m_2} + \mathcal{O}\left(\frac{\hbar^2}{4R^4}\right).$$

What is the relation between $C_{l,l_1 m_1 l_2 m_2}$ and $C'_{l,l_1 m_1 l_2 m_2}$ ?

## From explicit calculations, we get

$$
\begin{aligned}
C_{l,l_1m_1l_2m_2} &= (-1)^{m_1+m_2}\sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} \\
&\cdot\begin{pmatrix} l & l_1 & l_2 \\ m_1+m_2 & -m_1 & -m_2 \end{pmatrix} \\
&\cdot\frac{(-1)^{\frac{l_1+l_2+l+1}{2}}(l_1+l_2+l+1)\left(\frac{l_1+l_2+l-1}{2}\right)!}{\left(\frac{l_1+l_2-l-1}{2}\right)!\left(\frac{l_2+l-l_1-1}{2}\right)!\left(\frac{l+l_1-l_2-1}{2}\right)!} \\
&\cdot\sqrt{\frac{(l_1+l_2-l)!(l_2+l-l_1)!(l+l_1-l_2)!}{(l_1+l_2+l+1)!}},
\end{aligned}
$$

## and

$$
\begin{aligned}
&C'_{l,l_1m_1l_2m_2} \\
&= \frac{(m_1+m_2)!}{2^{l_1+l_2+l+1}}\sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}\frac{(l_1-m_1)!(l_2-m_2)!(l-m_1-m_2)!}{(l_1+m_1)!(l_2+m_2)!(l+m_1+m_2)!}} \\
&\cdot\sum_r\frac{(-1)^r(2r)!}{r!(l-r)!(2r-l-m_1-m_2)!}\Bigg( \\
&\sum_{p,q}\frac{(-1)^{p+q}(2p)!(2q)!m_1\Gamma\left(p+q+r-m_1-m_2-\frac{l_1+l_2+l}{2}\right)}{p!q!(l_1-p)!(l_2-q)!(2p-l_1-m_1)!(2q-l_2-m_2-1)!\Gamma\left(p+q+r-\frac{l_1+l_2+l}{2}\right)} \\
&-\sum_{p,q}\frac{(-1)^{p+q}(2p)!(2q)!m_2\Gamma\left(p+q+r-m_1-m_2-\frac{l_1+l_2+l}{2}\right)}{p!q!(l_1-p)!(l_2-q)!(2p-l_1-m_1-1)!(2q-l_2-m_2)!\Gamma\left(p+q+r-\frac{l_1+l_2+l}{2}\right)}\Bigg)
\end{aligned}
$$

for $m_1, m_2 \geq 0$.

## Generally,

$$C_{l,l_1m_1l_2m_2} \neq C'_{l,l_1m_1l_2m_2}.$$

## § Summary and Discussion

### ⋆Summary

- We have obtained ***explicit formulae of $*$ products*** on 2 dim. constant curvature spaces $S^2, H^2$ by completing calculations along the Fedosov's procedure for deformation quantization.

- We have shown that they form ***fuzzy $S^2, H^2$ algebra i.e., $su(2), su(1,1)$ algebra***.

- We applied them to our general formulation of [A-K] and solved $U(1)$ noncommutative BPS equation to $\mathcal{O}(\hbar^2)$.

- We compared our $*$ product on $S^2$ with conventional $*_N$ product for fuzzy sphere in the nearly commutative region. Naively $\frac{\hbar}{2R^2} \sim \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)$, but precisely their relation is rather complicated.

## ⋆Discussion

- We can get different $*$ products by *other choice* of the input parameters. If we explicitly get one which has simple relation with conventional $*_N$, it might give some suggestions to *string theory*.

  cf. *L. Freidel - K. Krasnov,* hep-th/0103070.

- More explicit examples?

  For Kähler coset space,

  *S. Aoyama - T. Masuda,* hep-th/0105271,

  *T. Masuda, Poster Presentation@Tohwa Symposium (2001).*

- Deformation quantization provides associative $*$ product by definition.
  In string theory, on *nonzero* $H = dB$ background, corresponding $*$ product from OPE is *nonassociative*.
  Therefore some generalization of deformation quantization is required in noncommutative context.

  *L. Cornalba - R. Schiappa,* hep-th/0101219.